

Project 6 - Exam of Metric Embeddings

Student: Sara Asad

Theorem 1 (Assouad's embedding theorem). *For every $\varepsilon \in (0, 1)$ and $K > 0$, there exist $N = N(K, \varepsilon) \in \mathbb{N}$ and $D = D(K, \varepsilon) \in (1, \infty)$ such that for every separable metric space (X, d) with doubling constant K , the metric space $(X, d^{1-\varepsilon})$ admits a biLipschitz embedding into \mathbb{R}^N with distortion at most D .*

In the article by Assaf Naor and Ofer Neiman [2] it is proved that the dimension N of the Euclidean space into which the metric space $(X, d^{1-\varepsilon})$ embeds depends only on the doubling constant K , and not on the parameter $\varepsilon \in (0, 1/2)$:

Theorem 2. *For every $K > 0$ there exists $N = N(K) \in \mathbb{N}$, and for every $\varepsilon \in (0, 1/2)$ and $K > 0$ there exists $D = D(K, \varepsilon) \in (1, +\infty)$ such that for every separable metric space (X, d) with doubling constant K , the metric space $(X, d^{1-\varepsilon})$ admits a bi-Lipschitz embedding into \mathbb{R}^N with distortion at most D .*

Proof of the Theorem 2

Proof. 1. Setup, assumptions and constants

Considering a separable metric space (X, d) with doubling constant $K > 0$ and fixing $\varepsilon \in (0, 1/2)$, we define $\kappa := \log K$ and

$$N := \left\lceil \frac{c \log K}{\theta} \right\rceil = \left\lceil \frac{c\kappa}{\theta} \right\rceil,$$

where $\theta \in (0, 1)$ is fixed and $c > 0$ is a universal constant, chosen later sufficiently large. With this definition $N = N(K)$ is a natural number which does not depend on $\varepsilon \in (0, 1/2)$. Finally, we denote with τ the quantity

$$\tau := \frac{\varepsilon^\theta}{32\kappa^\theta}.$$

Firstly, we observe that it suffices to demonstrate the case of X finite. Indeed, N does not depend on $|X|$ and the Euclidean distortion of X is equal to the supremum of the Euclidean distortions of its finite subsets:

$$c_{\mathbb{R}^N}(X) = \sup \{c_{\mathbb{R}^N}(F) : F \subseteq X \text{ and } |F| < +\infty\}.$$

The last result is proved following the argument of Lemma 4.9 of [1]. Let D be the supremum of the distortions of all finite subsets $F \subseteq X$. For every such finite subset containing a fixed point $x_0 \in X$, there exists an (A, B) -Lipschitz embedding $g_F : F \rightarrow \mathbb{R}^N$ with $B/A \leq D$, such that $g_F(x_0) = 0$. We then define an application f_F from X to \mathbb{R}^N by setting $f_F(x) = g_F(x)$ if x belongs to F , and $f_F(x) = 0$ otherwise.

For each $x \in X$, we then set $f(x) = \lim_{F, \mathcal{U}} f_F(x)$, where \mathcal{U} is an ultrafilter finer than the filter of inclusion on the set of all finite subsets of X containing x_0 . We observe that $f_F(x)$ is, for any F , an element of the closed ball centered at 0 and of radius $Bd(x_0, x)$; its compactness ensures the existence of $f(x)$. The application f is then an embedding of X with distortion D .

Without loss of generality, we also assume that $\text{diam}(X) = 1$, since if $\text{diam}(X) = D$, we can rescale the metric as

$$d'(x, y) = \frac{d(x, y)}{D}, \quad \text{for every } x, y \in X,$$

obtaining that d' is still a metric. Moreover, the doubling constant K does not change, since

$$B_{d'}(x, r) = \{y \in X : d'(x, y) \leq r\} = \{y \in X : d(x, y) \leq rD\} = B_d(x, rD).$$

2. Nets and random radii

For every $i \in \mathbb{N}$ let $\{x_1^i, \dots, x_{n_i}^i\} \subseteq X$ be a $\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}$ -net of X , that is, for every $x \in X$ there exists $j \in \{1, \dots, n_i\}$ such that $d(x, x_j^i) \leq \frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}$. With this construction, the density of the nets increases as i increases.

We then fix a natural index $i \in \mathbb{N}$. For every point x_j^i in the net and for every $k \in \{1, \dots, N\}$, let $R_{ij}^k \in \left[\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}, \frac{1}{2}\tau^{\frac{i}{1-\varepsilon}}\right]$ be a random variable whose density Φ_i is defined for every $r \in \mathbb{R}$ as

$$\Phi_i(r) := \frac{16K^8 \log K}{\tau^{\frac{i}{1-\varepsilon}}(K^4 - 1)} K^{-\frac{16r}{\tau^{\frac{i}{1-\varepsilon}}}} \mathbf{1}_{\left[\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}, \frac{1}{2}\tau^{\frac{i}{1-\varepsilon}}\right]}(r).$$

This density Φ_i is well-defined since it is non-negative and its integral over its support equals 1. Indeed,

$$\begin{aligned} \int_{\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}}^{\frac{1}{2}\tau^{\frac{i}{1-\varepsilon}}} \Phi_i(r) dr &= \frac{16K^8 \log K}{\tau^{\frac{i}{1-\varepsilon}}(K^4 - 1)} \int_{\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}}^{\frac{1}{2}\tau^{\frac{i}{1-\varepsilon}}} K^{-\frac{16r}{\tau^{\frac{i}{1-\varepsilon}}}} dr \\ &= \frac{16K^8 \log K}{\tau^{\frac{i}{1-\varepsilon}}(K^4 - 1)} \cdot \frac{\tau^{\frac{i}{1-\varepsilon}}}{16 \log K} (K^{-4} - K^{-8}) \\ &= \frac{K^8}{K^4 - 1} (K^{-4} - K^{-8}) = 1. \end{aligned}$$

We will also use random variables

$$\{U_i^k(C) : i \in \mathbb{N}, k \in \{1, \dots, N\}, C \subseteq X\},$$

each of which is uniformly distributed on the interval $[0, 1]$. Thus, for each $i \in \mathbb{N}$ and $k \in \{1, \dots, N\}$, there are $2^{|X|}$ such random variables, since they are indexed by all subsets $C \subseteq X$.

We assume that the random variables

$$\{R_{ij}^k : i, k \in \mathbb{N}, j \in \{1, \dots, n_i\}\} \cup \{U_i^k(C) : i, k \in \mathbb{N}, C \subseteq X\}$$

are mutually independent and defined on a probability space (Ω, \mathbb{P}) .

3. Construction of random partitions The metric space is partitioned into disjoint clusters using the random variables previously defined.

We define the random partitions of $\bigcup_{j=1}^{n_i} B(x_j^i, R_{ij}^k)$

$$P_i^k := P_{R_{i1}^k, \dots, R_{in_i}^k}^{x_1^i, \dots, x_{n_i}^i}$$

as

$$P_i^k := \{B(x_1^i, R_{i1}^k)\} \cup \bigcup_{j=2}^{n_i} \left[B(x_j^i, R_{ij}^k) \setminus \bigcup_{l=1}^{j-1} B(x_l^i, R_{il}^k) \right] \setminus \{\emptyset\}.$$

For every element $x \in X$ we write $P_i^k(x)$ to denote the set in P_i^k to which x belongs. We observe that, since $\{x_1^i, \dots, x_{n_i}^i\}$ is a $\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}$ -net of X and every random variable R_{ij}^k belongs to $\left[\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}, \frac{1}{2}\tau^{\frac{i}{1-\varepsilon}}\right]$, the partitions P_i^k are $\tau^{\frac{i}{1-\varepsilon}}$ -bounded, meaning that the diameter of $P_i^k(x)$ is at most $\tau^{\frac{i}{1-\varepsilon}}$ for every $x \in X$.

For next purposes (Section 8), for every $y \in X$ we also introduce

$$J(i, y) := \left\{ j \in \{1, \dots, n_i\} : d(y, x_j^i) \leq 2\tau^{\frac{i}{1-\varepsilon}} \right\},$$

and we consider

$$j(i, k, y) := \min \{ j \in J(i, y) : y \in B(x_j^i, R_{ij}^k) \}.$$

The index $j(i, k, y)$ is guaranteed to exist because $\{x_1^i, \dots, x_{n_i}^i\}$ is a $\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}$ -net of X , and, since $R_{i1}^k, \dots, R_{in_i}^k \geq \frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}$, every point y is necessarily covered by at least one ball $B(x_j^i, R_{ij}^k)$. Moreover, from $R_{ij}^k \leq \frac{1}{2}\tau^{\frac{i}{1-\varepsilon}}$ it follows that necessarily $j \in J(i, y)$.

A point is assigned to the ball with the smallest index $j(i, k, y)$ that contains it, provided it was not already contained in any ball $B(x_l^i, R_{il}^k)$ with $l < j(i, k, y)$. From the definition of $P_i^k(y)$ it follows that

$$P_i^k(y) = B(x_{j(i,k,y)}^i, R_{ij(i,k,y)}^k) \setminus \bigcup_{l=1}^{j(i,k,y)-1} B(x_l^i, R_{il}^k). \quad (1)$$

We note that if there exists $z \in B(x_l^i, R_{il}^k) \cap B(x_{j(i,k,y)}^i, R_{ij(i,k,y)}^k)$, then

$$\begin{aligned} d(x_l^i, y) &\leq d(x_l^i, z) + d(z, x_{j(i,k,y)}^i) + d(x_{j(i,k,y)}^i, y) \\ &\leq R_{il}^k + 2R_{ij(i,k,y)}^k \\ &\leq \frac{1}{2}\tau^{\frac{i}{1-\varepsilon}} + \tau^{\frac{i}{1-\varepsilon}} \leq 2\tau^{\frac{i}{1-\varepsilon}}, \end{aligned}$$

which implies that $l \in J(i, y)$. It follows that (1) can be rewritten as

$$P_i^k(y) = B(x_{j(i,k,y)}^i, R_{ij(i,k,y)}^k) \setminus \bigcup_{l \in J(i,y) \cap \{1, \dots, j(i,k,y)-1\}} B(x_l^i, R_{il}^k). \quad (2)$$

4. Definition of the random embedding For every $i \in \mathbb{N}$ and $k \in \{1, \dots, N\}$ we define the random mapping

$$f_i^k : X \rightarrow \mathbb{R}$$

$$f_i^k(x) := U_i^k(P_i^k(x)) \cdot \min \left\{ \tau^i, 6ak\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1} d(x, X \setminus P_i^k(x)) \right\}$$

and then the random embedding

$$F : X \rightarrow \mathbb{R}^N$$

$$F(x) := \left(\frac{\sum_{i=1}^{\infty} f_i^1(x)}{\sqrt{N}}, \dots, \frac{\sum_{i=1}^{\infty} f_i^N(x)}{\sqrt{N}} \right).$$

Every sum converges geometrically since $\tau < 1/2$ and

$$0 \leq f_i^k(x) \leq \tau^i$$

for every $i \in \mathbb{N}$ and $k \in \{1, \dots, N\}$, leading that F is well-defined.

5. Lipschitz analysis of the local components f_i^k

We fix $i \in \mathbb{N}$, $k \in \{1, \dots, N\}$ and two elements $x, y \in X$ such that $f_i^k(x) > f_i^k(y)$; the opposite case where $f_i^k(x) \leq f_i^k(y)$ is analogous.

We consider the first case in which $P_i^k(x) \neq P_i^k(y)$. This means that $y \notin P_i^k(x)$ or, equivalently, $y \in X \setminus P_i^k(x)$. From the definition of f_i^k , it holds that $f_i^k(y) \geq 0$, therefore

$$\begin{aligned} f_i^k(x) - f_i^k(y) &\leq f_i^k(x) \\ &\leq \min \left\{ \tau^i, 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}} d(x, X \setminus P_i^k(x)) \right\} \\ &\leq \min \left\{ \tau^i, 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}} d(x, y) \right\}. \end{aligned}$$

The second-last inequality follows from the fact that $U_i^k(P_i^k(x)) \in [0, 1]$, while the last one follows from

$$d(x, y) \geq d(x, X \setminus P_i^k(x)),$$

by definition of the distance between a point and a subset.

Consider now the other case in which $P_i^k(x) = P_i^k(y) = C$. Since $f_i^k(x) > f_i^k(y)$, it cannot happen that $f_i^k(y) = U_i^k(C) \tau^i$, because this would give the contradiction $f_i^k(x) \leq U_i^k(C) \tau^i = f_i^k(y)$. Therefore one necessarily has $f_i^k(y) = U_i^k(C) 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1} d(y, X \setminus C)$. Then

$$\begin{aligned} f_i^k(x) - f_i^k(y) &= f_i^k(x) - U_i^k(C) 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1} d(y, X \setminus C) \\ &\leq U_i^k(C) 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1} [d(x, X \setminus C) - d(y, X \setminus C)]. \end{aligned}$$

Using that for every subset $A \subseteq X$ the inequality $|d(x, A) - d(y, A)| \leq d(x, y)$ holds, we obtain

$$|f_i^k(x) - f_i^k(y)| \leq 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1} d(x, y).$$

Since $f_i^k(x), f_i^k(y) \in [0, \tau^i]$, it is also true that

$$f_i^k(x) - f_i^k(y) \leq \tau^i.$$

Thus, for every $x, y \in X$

$$|f_i^k(x) - f_i^k(y)| \leq \min \left\{ \tau^i, 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1}d(x, y) \right\}. \quad (3)$$

6. Global extension of the Lipschitz bound By the definition of the random embedding F , we have

$$\begin{aligned} \|F(x) - F(y)\|_2 &\leq \left(\frac{1}{N} \sum_{k=1}^N \left(\sum_{i=1}^{\infty} |f_i^k(x) - f_i^k(y)| \right)^2 \right)^{1/2} \leq \left(\max_{k \in \{1, \dots, N\}} \left(\sum_{i=1}^{\infty} |f_i^k(x) - f_i^k(y)| \right)^2 \right)^{1/2} \\ &= \max_{k \in \{1, \dots, N\}} \sum_{i=1}^{\infty} |f_i^k(x) - f_i^k(y)| \leq \sum_{i=1}^{\infty} \min \left\{ \tau^i, 64\kappa\tau^{-\frac{i\varepsilon}{1-\varepsilon}-1}d(x, y) \right\}, \end{aligned}$$

where the last inequality follows from (3).

We observe that for every element $z \in X$

$$\|F(z)\|_2 \leq \sum_{i=1}^{\infty} \tau^i = \frac{\tau}{1-\tau} \lesssim \tau, \quad (4)$$

where the last inequality uses that $\tau < 1/2$. Therefore,

$$\|F(x) - F(y)\|_2 \leq \|F(x)\|_2 + \|F(y)\|_2 \lesssim \tau. \quad (5)$$

If $d(x, y) > \frac{\tau^{1+\frac{1}{1-\varepsilon}}}{64\kappa}$, by the definition of $\tau := \frac{\varepsilon^\theta}{32\kappa^\theta}$ it follows that

$$\|F(x) - F(y)\|_2 \lesssim \frac{\kappa^{(1+\theta)(1-\varepsilon)}}{\varepsilon^{1+\theta}} d(x, y)^{1-\varepsilon}.$$

Now we consider the opposite case when $d(x, y) \leq \frac{\tau^{1+\frac{1}{1-\varepsilon}}}{64\kappa}$. Let $m \in \mathbb{N}$ be such that

$$\frac{\tau}{64\kappa} \tau^{\frac{m+1}{1-\varepsilon}} < d(x, y) \leq \frac{\tau}{64\kappa} \tau^{\frac{m}{1-\varepsilon}}.$$

Then, splitting at m the sum in the previous estimate, we have

$$\begin{aligned} \|F(x) - F(y)\|_2 &\lesssim \frac{\kappa}{\tau} d(x, y) \sum_{i=1}^m \tau^{-\frac{i\varepsilon}{1-\varepsilon}} + \sum_{i=m+1}^{\infty} \tau^i \\ &\lesssim \frac{\kappa^{1+\theta}}{\varepsilon^\theta} d(x, y) \sum_{i=1}^m \tau^{-\frac{i\varepsilon}{1-\varepsilon}} + \sum_{i=m+1}^{\infty} \tau^i. \end{aligned}$$

Recalling that $0 < \varepsilon, \tau < 1/2$, the first sum can be estimated as

$$\begin{aligned} \sum_{i=1}^m \tau^{-\frac{i\varepsilon}{1-\varepsilon}} &= \frac{\tau^{-\frac{m\varepsilon}{1-\varepsilon}} - 1}{1 - \tau^{\frac{\varepsilon}{1-\varepsilon}}} \lesssim \frac{1}{\varepsilon} \tau^{-\frac{m\varepsilon}{1-\varepsilon}} \\ &\lesssim \frac{\tau^\varepsilon}{\varepsilon \kappa^\varepsilon d(x, y)^\varepsilon} \lesssim \frac{1}{\varepsilon \kappa^{(1+\theta)\varepsilon} d(x, y)^\varepsilon}. \end{aligned}$$

and similarly the second

$$\sum_{i=m+1}^{\infty} \tau^i = \frac{\tau^{m+1}}{1 - \tau} \lesssim \frac{\kappa^{1-\varepsilon}}{\tau^{1-\varepsilon}} d(x, y)^{1-\varepsilon} \lesssim \frac{\kappa^{(1+\theta)(1-\varepsilon)}}{\varepsilon^\theta} d(x, y)^{1-\varepsilon}.$$

This concludes the proof of the Lipschitz continuity of F with respect to the metric $d^{1-\varepsilon}$:

$$\begin{aligned} \|F(x) - F(y)\|_2 &\lesssim \frac{\kappa^{1+\theta}}{\varepsilon^\theta} d(x, y) \cdot \frac{1}{\varepsilon \kappa^{(1+\theta)\varepsilon} d(x, y)^\varepsilon} + \frac{\kappa^{(1+\theta)(1-\varepsilon)}}{\varepsilon^\theta} d(x, y)^{1-\varepsilon} \\ &\lesssim \frac{\kappa^{(1+\theta)(1-\varepsilon)}}{\varepsilon^{1+\theta}} d(x, y)^{1-\varepsilon} + \frac{\kappa^{(1+\theta)(1-\varepsilon)}}{\varepsilon^\theta} d(x, y)^{1-\varepsilon} \\ &\lesssim \frac{\kappa^{(1+\theta)(1-\varepsilon)}}{\varepsilon^{1+\theta}} d(x, y)^{1-\varepsilon} \lesssim \left(\frac{\log K}{\varepsilon}\right)^{1+\theta} d(x, y)^{1-\varepsilon}. \end{aligned} \tag{6}$$

7. Probabilistic groundwork In this section we define the subsets and the events necessary for studying the lower bound of the random embedding F . More specifically, they will be useful for using the Lovász Local Lemma in Section 12.

For each $i \in \mathbb{N}$, we denote by δ_i the quantity

$$\delta_i := \tau^{\frac{i+2}{1-\varepsilon}} \left(\frac{4\varepsilon}{c^* \kappa}\right)^{\frac{1}{1-\varepsilon}},$$

where c^* is the implied universal constant in the final inequality of (6).

We consider \mathcal{N}_i a δ_i -net of X , and define the set

$$M := \left\{ (i, u, v) \in \mathbb{N} \times \mathcal{N}_i \times \mathcal{N}_i : \tau^{\frac{i}{1-\varepsilon}} < d(u, v) < 3\tau^{\frac{i-1}{1-\varepsilon}} \right\}.$$

For every $(i, u, v) \in M$, we take the subsets

$$\begin{aligned} G(i, u, v) &:= \left\{ k \in \{1, \dots, N\} : \left| \sum_{j=1}^{\infty} f_j^k(u) - \sum_{j=1}^{\infty} f_j^k(v) \right| \geq \frac{\tau^{i+1}}{2} \right\} \subseteq \{1, \dots, N\} \\ L(i, u, v) &:= \left\{ k \in \{1, \dots, N\} : \left| \sum_{j=1}^i f_j^k(u) - \sum_{j=1}^i f_j^k(v) \right| \geq 2\tau^{i+1} \right\} \subseteq \{1, \dots, N\}, \end{aligned}$$

and we define the respective events

$$E(i, u, v) := \left\{ |G(i, u, v)| \geq \frac{N}{2} \right\} \subseteq \Omega$$

$$T(i, u, v) := \left\{ |L(i, u, v)| \geq \frac{N}{2} \right\} \subseteq \Omega.$$

We observe that for every $(i, u, v) \in M$ we have the inclusion $T(i, u, v) \subseteq E(i, u, v)$. Indeed, from (3) it follows that for all $k \in \{1, \dots, N\}$

$$\left| \sum_{j=i+1}^{\infty} f_j^k(u) - \sum_{j=i+1}^{\infty} f_j^k(v) \right| \leq \sum_{j=i+1}^{\infty} \tau^j \leq \frac{3}{2} \tau^{i+1}.$$

Hence, if $k \in L(i, u, v)$, then

$$\begin{aligned} \left| \sum_{j=1}^{\infty} f_j^k(u) - \sum_{j=1}^{\infty} f_j^k(v) \right| &\geq \left| \sum_{j=1}^i f_j^k(u) - \sum_{j=1}^i f_j^k(v) \right| - \left| \sum_{j=i+1}^{\infty} f_j^k(u) - \sum_{j=i+1}^{\infty} f_j^k(v) \right| \\ &\geq 2\tau^{i+1} - \frac{3}{2}\tau^{i+1} = \frac{\tau^{i+1}}{2}, \end{aligned}$$

which implies that $k \in G(i, u, v)$. Therefore, the condition $|L(i, u, v)| \geq \frac{N}{2}$ implies that $|G(i, u, v)| \geq \frac{N}{2}$, obtaining the desired inclusion.

Finally, we take the graph $H = (V, E_H)$, where

$$V := \{T(i, u, v) : (i, u, v) \in M\},$$

and the edges are defined as follows:

$$\{T(i, u, v), T(i', u', v')\} \in E_H \iff i = i' \text{ and } d(\{u, v\}, \{u', v'\}) \leq 4\tau^{\frac{i}{1-\varepsilon}}.$$

Moreover, we define the mapping

$$\rho : V \rightarrow \mathbb{N}, \quad \rho(T(i, u, v)) = i.$$

8. Dependence of the events on the random variables Fixing $(i, u, v) \in M$, we introduce two subsets of random variables, denoted by \mathcal{H} and \mathcal{Y} , and show that the intersection of all events $T(i', u', v')$ that are non-adjacent to $T(i, u, v)$ in the graph depends only on the variables in $\mathcal{H} \cup \mathcal{Y}$.

We consider a subset

$$Q \subseteq \{(i', u', v') \in M : i \geq i' \text{ and } \{T(i, u, v), T(i', u', v')\} \notin E_H\}.$$

Considering the following subset of random variables

$$\mathcal{H} := \{R_{i'j}^k : i' \in \{1, \dots, i-1\}, j \in \{1, \dots, n_{i'}\}, k \in \{1, \dots, N\}\} \cup \{U_{i'}^k(C) : i' \in \{1, \dots, i-1\}, k \in \{1, \dots, N\}, C \subseteq X\}.$$

If $(i', u', v') \in Q$ with $i' < i$, then the event $T(i', u', v')$ depends only on the random variables indexed by levels $j \leq i'$, and hence only on the variables in \mathcal{H} .

We now consider the case $i' = i$. If $(i, u', v') \in Q$ and $\{T(i, u, v), T(i, u', v')\} \notin E_H$, then it follows that $d(u', u), d(v', u) > 4\tau^{\frac{i}{1-\varepsilon}}$. Since $\text{diam}(P_i^k(u')) \leq \tau^{\frac{i}{1-\varepsilon}}$, for every $x \in P_i^k(u')$ we have

$$d(x, u) \geq d(u', u) - d(x, u') > 4\tau^{\frac{i}{1-\varepsilon}} - \tau^{\frac{i}{1-\varepsilon}} = 3\tau^{\frac{i}{1-\varepsilon}},$$

and we obtain the same argument for every $x \in P_i^k(v')$. Therefore,

$$P_i^k(u'), P_i^k(v') \subseteq X \setminus B\left(u, 2\tau^{\frac{i}{1-\varepsilon}}\right).$$

Hence, defining the subset \mathcal{Y} as

$$\mathcal{Y} := \{R_{ij}^k : j \in \{1, \dots, n_i\} \setminus J(i, u), k \in \{1, \dots, N\}\} \cup \left\{U_i^k(C) : k \in \{1, \dots, N\}, C \subseteq X \setminus B\left(u, 2\tau^{\frac{i}{1-\varepsilon}}\right)\right\},$$

we have that

$$U_i^k(P_i^k(u')), U_i^k(P_i^k(v')) \in \mathcal{Y}.$$

Moreover, from the previous observations it holds that

$$J(i, u) \cap J(i, u') = \emptyset \quad \text{and} \quad J(i, u) \cap J(i, v') = \emptyset,$$

and so, by their alternative definitions (2), $P_i^k(u'), P_i^k(v')$ depend only on the variables in \mathcal{Y} . Therefore, $f_i^k(u'), f_i^k(v')$ depend only on the variables in \mathcal{Y} , which implies that the event $T(i, u', v')$ depends only on $\mathcal{H} \cup \mathcal{Y}$. We conclude that the intersection

$$W := \bigcap_{(i', u', v') \in Q} T(i', u', v')$$

depends only on $\mathcal{H} \cup \mathcal{Y}$.

9. Local probability estimates We now estimate the probability of the events A_j , which describe how the random balls intersect a fixed neighborhood of a point u , namely $B\left(u, \frac{\tau}{64k} \tau^{\frac{i}{1-\varepsilon}}\right)$. We then derive a lower bound on the probability that this ball is contained in the random partition $P_i^k(u)$.

For every $j \in \{1, \dots, n_i\}$ consider the event A_j defined as

$$A_j := \left(\bigcap_{l=1}^{j-1} \left\{ B(x_l^i, R_{il}^k) \cap B\left(u, \frac{\tau}{64k} \tau^{\frac{i}{1-\varepsilon}}\right) = \emptyset \right\} \right) \cap \left\{ B(x_j^i, R_{ij}^k) \cap B\left(u, \frac{\tau}{64k} \tau^{\frac{i}{1-\varepsilon}}\right) \notin \left\{ \emptyset, B\left(u, \frac{\tau}{64k} \tau^{\frac{i}{1-\varepsilon}}\right) \right\} \right\}.$$

The event A_j occurs if j is the first index such that the ball $B(x_j^i, R_{ij}^k)$ has a non-trivial intersection with $B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right)$.

By the definition of the partition P_i^k , the union of such events corresponds to the event that the ball around u is not fully contained in $P_i^k(u)$:

$$\bigcup_{j \in J_u} A_j = \left\{ B\left(u, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}}\right) \not\subseteq P_i^k(u) \right\}.$$

For the aim of this section we define the subset

$$J_u := \left\{ j \in \{1, \dots, n_i\} : x_j^i \in B\left(u, \tau^{\frac{i}{1-\varepsilon}}\right) \right\}$$

and we observe that, since

$$R_{ij}^k, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}} \leq \frac{1}{2}\tau^{\frac{i}{1-\varepsilon}},$$

if a ball $B(x_j^i, R_{ij}^k)$ intersects $B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right)$, then by the triangle inequality its center x_j^i must lie in $B\left(u, \tau^{\frac{i}{1-\varepsilon}}\right)$, meaning that any such $j \in J_u$.

Since (X, d) is K -doubling, the ball $B\left(u, \tau^{\frac{i}{1-\varepsilon}}\right)$ can be covered by at most K^3 balls of radius $\frac{1}{8}\tau^{\frac{i}{1-\varepsilon}}$, and each of these balls contains at most one point of the $\frac{1}{4}\tau^{\frac{i}{1-\varepsilon}}$ -net $\{x_1^i, \dots, x_{n_i}^i\}$. Hence, $|J_u| \leq K^3$.

We must have $B(x_j^i, R_{ij}^k) \cap B\left(u, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}}\right) \neq \emptyset$ for some $j \in J_u$, and therefore, using the independence of $R_{i1}^k, \dots, R_{in_i}^k$, one has

$$\begin{aligned} 1 &= \sum_{j \in J_u} \mathbb{P} \left[B(x_j^i, R_{ij}^k) \cap B\left(u, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}}\right) \neq \emptyset \right. \\ &\quad \left. \wedge B(x_l^i, R_{il}^k) \cap B\left(u, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}}\right) = \emptyset \text{ for every } l < j \right] \\ &= \sum_{j \in J_u} \left(\prod_{l=1}^{j-1} \mathbb{P} \left[B(x_l^i, R_{il}^k) \cap B\left(u, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}}\right) = \emptyset \right] \right) \cdot \mathbb{P} \left[B(x_j^i, R_{ij}^k) \cap B\left(u, \frac{\tau}{64k}\tau^{\frac{i}{1-\varepsilon}}\right) \neq \emptyset \right]. \end{aligned} \quad (7)$$

We now fix $j \in J_u$. Denote by a and b the minimum and maximum distance from the center x_j^i to points in the ball $B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right)$:

$$a := \min_{z \in B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right)} d(x_j^i, z) \quad \text{and} \quad b := \max_{z \in B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right)} d(x_j^i, z).$$

By the triangle inequality, the distance between any two points in the ball $B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right)$ is at most twice its radius. Hence one has

$$b - a \leq 2 \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}}.$$

Then

$$\begin{aligned} & \mathbb{P} \left[B(x_j^i, R_{ij}^k) \cap B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \notin \left\{ \emptyset, B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \right\} \right] = \int_{\max \left\{ a, \frac{\tau^{\frac{i}{1-\varepsilon}}}{4} \right\}}^b \Phi_i(r) dr \\ & \leq \frac{K^8}{K^4 - 1} \left(K^{-\frac{16 \max \left\{ a, \frac{\tau^{\frac{i}{1-\varepsilon}}}{4} \right\}}{\tau^{\frac{i}{1-\varepsilon}}}} - K^{-\frac{16b}{\tau^{\frac{i}{1-\varepsilon}}}} \right) \leq \frac{K^8}{K^4 - 1} K^{-\frac{16 \max \left\{ a, \frac{\tau^{\frac{i}{1-\varepsilon}}}{4} \right\}}{\tau^{\frac{i}{1-\varepsilon}}}} (1 - K^{-32 \frac{\tau}{64\kappa}}). \end{aligned}$$

We observe that

$$\mathbb{P} \left[B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \cap B(x_j^i, R_{ij}^k) \neq \emptyset \right] = \int_{\max \left\{ a, \frac{\tau^{\frac{i}{1-\varepsilon}}}{4} \right\}}^{\frac{\tau^{\frac{i}{1-\varepsilon}}}{2}} \Phi_i(r) dr = \frac{K^8}{K^4 - 1} \left(K^{-\frac{16 \max \left\{ a, \frac{\tau^{\frac{i}{1-\varepsilon}}}{4} \right\}}{\tau^{\frac{i}{1-\varepsilon}}}} - K^{-8} \right),$$

obtaining that the last estimate can be rewritten as

$$\begin{aligned} & \mathbb{P} \left[B(x_j^i, R_{ij}^k) \cap B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \notin \left\{ \emptyset, B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \right\} \right] \\ & \leq (1 - K^{-32 \frac{\tau}{64\kappa}}) \left(\mathbb{P} \left[B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \cap B(x_j^i, R_{ij}^k) \neq \emptyset \right] + \frac{1}{K^4 - 1} \right). \end{aligned} \quad (8)$$

We now use (7) and (8), the observation that $|J_u| \leq K^3$ and the independence of $R_{i1}^k, \dots, R_{in_i}^k$ to conclude that

$$\begin{aligned} 1 - \mathbb{P} \left[B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \subseteq P_i^k(u) \right] & \leq \sum_{j \in J_u} \mathbb{P}[A_j] \\ & = \sum_{j \in J_u} \left(\prod_{l=1}^{j-1} \mathbb{P} \left[B(x_l^i, R_{il}^k) \cap B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) = \emptyset \right] \right) \\ & \quad \cdot \mathbb{P} \left[B(x_j^i, R_{ij}^k) \cap B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \notin \left\{ \emptyset, B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \right\} \right] \\ & \leq (1 - K^{-32 \frac{\tau}{64\kappa}}) + (1 - K^{-32 \frac{\tau}{64\kappa}}) \frac{|J_u|}{K^4 - 1} \\ & \leq (1 - K^{-32 \frac{\tau}{64\kappa}}) \left(1 + \frac{K^3}{K^4 - 1} \right) \\ & \leq 1 - K^{-64 \frac{\tau}{64\kappa}} \leq 1 - K^{-\frac{\tau}{\kappa}} = 1 - e^{-\tau}. \end{aligned}$$

10. Definition of the success events For every $(i, u, v) \in M$ and $k \in \{1, \dots, N\}$ we now define the event

$$S(i, u, v, k) := \{k \in L(i, u, v)\}.$$

and consequently

$$Z_k := S(i, u, v, k) \cap \left\{ B \left(u, \frac{\tau}{64\kappa} \tau^{\frac{i}{1-\varepsilon}} \right) \subseteq P_i^k(u) \right\}.$$

We observe that if $B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right) \subseteq P_i^k(u)$, then $f_i^k(u) = U_i^k(P_i^k(u)) \cdot \tau^i$. Indeed, the inclusion implies that the distance from u to the complement of its cluster satisfies $d(u, X \setminus P_i^k(u)) \geq \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}$. Recalling the definition of f_i^k given in Section 4, we observe that the second term in the minimum satisfies

$$64\kappa\tau^{-\frac{i}{1-\varepsilon}-1}d(u, X \setminus P_i^k(u)) \geq 64\kappa\tau^{-\frac{i}{1-\varepsilon}-1} \cdot \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}} = \tau^{-\frac{i}{1-\varepsilon}-1+1+\frac{i}{1-\varepsilon}} = \tau^i.$$

Therefore Z_k can be rewritten as

$$Z_k = \left\{ B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right) \subseteq P_i^k(u) \right\} \cap \left\{ \left| \sum_{j=1}^{i-1} (f_j^k(u) - f_j^k(v)) + U_i^k(P_i^k(u))\tau^i - f_i^k(v) \right| \geq 2\tau^{i+1} \right\}.$$

Dividing the above inequality by the positivity quantity τ^i , we can rewrite the second condition in Z_k as

$$\left| U_i^k(P_i^k(u)) + \tau^{-i} \left(\sum_{j=1}^{i-1} (f_j^k(u) - f_j^k(v)) - f_i^k(v) \right) \right| \geq 2\tau,$$

which, denoting

$$c := \tau^{-i} \left(\sum_{j=1}^{i-1} (f_j^k(u) - f_j^k(v)) - f_i^k(v) \right),$$

is equivalent to

$$U_i^k(P_i^k(u)) \notin (c - 2\tau, c + 2\tau).$$

From (2), the event $B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right) \subseteq P_i^k(u)$ depends only on the random variables determining the partition, and is therefore independent of the variables in $\mathcal{H} \cup \mathcal{Y}$.

We now claim that $U_i^k(P_i^k(u))$ is independent of c and of $\mathcal{H} \cup \mathcal{Y}$. Indeed, by the definition of M , $d(u, v) > \tau^{\frac{i}{1-\varepsilon}}$, while every $P_i^k(u)$ has diameter at most $\tau^{\frac{i}{1-\varepsilon}}$. Therefore, $P_i^k(u) \neq P_i^k(v)$. In particular, the random variables $U_i^k(P_i^k(u))$ and $U_i^k(P_i^k(v))$ are distinct, thus $U_i^k(P_i^k(u))$ is independent of $f_i^k(v)$ and hence of c .

Moreover, $U_i^k(P_i^k(u))$ is uniformly distributed on $[0, 1]$. Since the interval $(c - 2\tau, c + 2\tau)$ has length at most 4τ , we obtain

$$\mathbb{P}(U_i^k(P_i^k(u)) \notin (c - 2\tau, c + 2\tau)) \geq 1 - 4\tau.$$

Combining these observations, we conclude that

$$\begin{aligned} p &:= \mathbb{P}[Z_k \mid \mathcal{H} \cup \mathcal{Y}] \\ &= \mathbb{P}\left[B\left(u, \frac{\tau}{64\kappa}\tau^{\frac{i}{1-\varepsilon}}\right) \subseteq P_i^k(u)\right] \cdot \mathbb{P}[U_i^k(P_i^k(u)) \notin (c - 2\tau, c + 2\tau) \mid \mathcal{H} \cup \mathcal{Y}] \\ &\geq e^{-\tau}(1 - 4\tau) \geq 1 - 5\tau. \end{aligned}$$

11. Concentration of success events After fixing the random variables of $\mathcal{H} \cup \mathcal{Y}$, the events Z_1, \dots, Z_N are independent. Indeed, conditioning on $\mathcal{H} \cup \mathcal{Y}$ fixes all the variables on which different Z_k may depend jointly,

and therefore each Z_k depends only on variables indexed by k . In this section we aim to show that, with high probability, at least half of the events Z_k occur.

Applying a Chernoff bound to the independent Bernoulli variables $\mathbf{1}_{Z_k}$, we obtain

$$\begin{aligned} \mathbb{P} \left[\sum_{k=1}^N \mathbf{1}_{Z_k} \geq \frac{N}{2} \middle| \mathcal{H} \cup \mathcal{Y} \right] &= 1 - \mathbb{P} \left[\sum_{k=1}^N \mathbf{1}_{\Omega \setminus Z_k} > \frac{N}{2} \middle| \mathcal{H} \cup \mathcal{Y} \right] \\ &\geq 1 - \left(e^{p - \frac{1}{2}} \sqrt{2(1-p)} \right)^N \geq 1 - (10e\tau)^{\frac{N}{2}}. \end{aligned}$$

From the inclusion $Z_k \subseteq S(i, u, v, k)$, it holds that $\sum_{k=1}^N \mathbf{1}_{S(i, u, v, k)} \geq \sum_{k=1}^N \mathbf{1}_{Z_k}$, and hence

$$\mathbb{P} \left(\sum_{k=1}^N \mathbf{1}_{S(i, u, v, k)} \geq \frac{N}{2} \middle| \mathcal{H} \cup \mathcal{Y} \right) \geq \mathbb{P} \left(\sum_{k=1}^N \mathbf{1}_{Z_k} \geq \frac{N}{2} \middle| \mathcal{H} \cup \mathcal{Y} \right).$$

Since W depends only on the variables in $\mathcal{H} \cup \mathcal{Y}$, we can write

$$\mathbb{P}(W \cap T(i, u, v)) = \int_W \mathbb{P}(T(i, u, v) \mid \mathcal{H} \cup \mathcal{Y}) d\mathbb{P}.$$

Moreover, we observe that, by definition,

$$T(i, u, v) := \left\{ |L(i, u, v)| \geq \frac{N}{2} \right\} = \left\{ \sum_{k=1}^N \mathbf{1}_{S(i, u, v, k)} \geq \frac{N}{2} \right\}.$$

Using the previous estimate, we obtain

$$\begin{aligned} \mathbb{P}(W \cap T(i, u, v)) &= \int_W \mathbb{P} \left(\sum_{k=1}^N \mathbf{1}_{S(i, u, v, k)} \geq \frac{N}{2} \middle| \mathcal{H} \cup \mathcal{Y} \right) d\mathbb{P} \geq \int_W \mathbb{P} \left(\sum_{k=1}^N \mathbf{1}_{Z_k} \geq \frac{N}{2} \middle| \mathcal{H} \cup \mathcal{Y} \right) d\mathbb{P} \\ &\geq \int_W \left(1 - (30\tau)^{N/2} \right) d\mathbb{P} = \left(1 - (30\tau)^{N/2} \right) \mathbb{P}(W) \geq \left(1 - \left(\frac{\varepsilon}{\log K} \right)^{\frac{\theta N}{2}} \right) \mathbb{P}[W]. \end{aligned}$$

Therefore, for every $(i, u, v) \in M$ and for every $Q \subseteq M$, it holds that

$$\mathbb{P}(W \setminus T(i, u, v)) = \mathbb{P}(W) - \mathbb{P}(W \cap T(i, u, v)) \leq \mathbb{P}(W) - \left(1 - \left(\frac{\varepsilon}{\log K} \right)^{\frac{\theta N}{2}} \right) \mathbb{P}(W) = \left(\frac{\varepsilon}{\log K} \right)^{\frac{\theta N}{2}} \mathbb{P}(W). \quad (9)$$

Taking $Q = \emptyset$, we have

$$W = \bigcap_{(i', u', v') \in \emptyset} T(i', u', v') = \Omega.$$

Therefore, from (9),

$$\mathbb{P}(\Omega \setminus T(i, u, v)) = \mathbb{P}(W \setminus T(i, u, v)) \leq \left(\frac{\varepsilon}{\log K}\right)^{\frac{\theta N}{2}} \mathbb{P}(\Omega) = \left(\frac{\varepsilon}{\log K}\right)^{\frac{\theta N}{2}}.$$

We now estimate the degree d of the graph H associated to the events $T(i, u, v)$, recalling that two events $T(i, u, v)$ and $T(i, u', v')$ are adjacent whenever

$$d(u, u') \leq 4\tau^{\frac{i}{1-\varepsilon}}.$$

Therefore, we consider $(i, u', v'), (i, u, v) \in M$ such that $d(u, u') \leq 4\tau^{\frac{i}{1-\varepsilon}}$. From the definition of M , one has

$$d(u, v), d(u', v') \leq 3\tau^{\frac{i-1}{1-\varepsilon}},$$

and so the points v, u', v' all belong to the ball B of radius $r = 4\tau^{\frac{i-1}{1-\varepsilon}}$ centered at u . Hence, both u' and v' must belong to $B \cap N_i$, and so the number of such triples (i, u', v') is at most $|B \cap N_i|^2$.

Since (X, d) is K -doubling, the ball B can be covered by at most $K^{1+\log_2\left(\frac{2r}{\delta_i}\right)}$ balls of radius $\frac{\delta_i}{2}$. Each of these balls contains at most one point of the δ_i -net \mathcal{N}_i , hence

$$|B \cap \mathcal{N}_i| \leq K^{1+\log_2\left(\frac{2r}{\delta_i}\right)}.$$

We conclude that the maximal degree d of H is at most

$$|B \cap \mathcal{N}_i|^2 \leq K^{2\left(1+\log_2\left(\frac{2r}{\delta_i}\right)\right)} \lesssim K^{4+2\log_2\left(\frac{r}{\delta_i}\right)} = K^{O(\log \log K + \log(1/\varepsilon))},$$

where in the last equality we use that $\frac{r}{\delta_i} \sim \tau^{-\frac{1}{1-\varepsilon}}$, and hence $\log(r/\delta_i) = O(\log(1/\tau))$.

12. Application of Lovász Local Lemma We apply the Lovász Local Lemma to the finite family of events $\mathcal{A}_{(i,u,v)} := \Omega \setminus T(i, u, v)$, indexed by $(i, u, v) \in M$. We consider the graph G whose vertices are the events $\mathcal{A}_{(i,u,v)}$, and where two vertices are adjacent whenever the corresponding triples are adjacent in the graph H defined above. In particular, the maximum degree of G is bounded by d .

Lemma 1 (Lovász Local Lemma). *Fix $q \in (0, 1)$ and $d \in \mathbb{N}$. Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ be measurable sets in some probability space (Ω, \mathbb{P}) . Let $G = (V, E_G)$ be a graph on the vertex set $V = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\}$ with maximal degree d . Let $\rho : \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\} \rightarrow \mathbb{N}$ be a mapping that satisfies*

$$\{\mathcal{A}_i, \mathcal{A}_j\} \in E_G \implies \rho(\mathcal{A}_i) = \rho(\mathcal{A}_j).$$

Assume that for any $i \in \{1, \dots, n\}$ and for all $Q \subseteq \{j \in \{1, \dots, n\} : \{\mathcal{A}_i, \mathcal{A}_j\} \notin E_G \vee \rho(\mathcal{A}_i) \geq \rho(\mathcal{A}_j)\}$ we have

$$\mathbb{P}\left[\mathcal{A}_i \cap \bigcap_{j \in Q} (\Omega \setminus \mathcal{A}_j)\right] \leq q \mathbb{P}\left[\bigcap_{j \in Q} (\Omega \setminus \mathcal{A}_j)\right].$$

Assume also that

$$eq(d+1) \leq 1.$$

Then

$$\mathbb{P}\left[\bigcap_{i=1}^n (\Omega \setminus \mathcal{A}_i)\right] > 0.$$

Let Q be a set of indices corresponding to vertices which are not adjacent to $\mathcal{A}_{(i,u,v)}$. Then we have

$$\begin{aligned} & \mathbb{P}\left[\mathcal{A}_{(i,u,v)} \cap \bigcap_{(i',u',v') \in Q} (\Omega \setminus \mathcal{A}_{(i',u',v')})\right] = \mathbb{P}\left[(\Omega \setminus T(i,u,v)) \cap \bigcap_{(i',u',v') \in Q} T(i',u',v')\right] \\ & = \mathbb{P}[(\Omega \setminus T(i,u,v)) \cap W] = \mathbb{P}[W \setminus T(i,u,v)] \\ & \leq \left(\frac{\varepsilon}{\log K}\right)^{\frac{\theta N}{2}} \mathbb{P}[W] = q \mathbb{P}\left[\bigcap_{(i',u',v') \in Q} T(i',u',v')\right] = q \mathbb{P}\left[\bigcap_{(i',u',v') \in Q} (\Omega \setminus \mathcal{A}_{(i',u',v')})\right], \end{aligned}$$

where

$$q := \left(\frac{\varepsilon}{\log K}\right)^{\frac{\theta N}{2}}.$$

Thus, all the assumptions of the Lovász Local Lemma are satisfied. Therefore, provided that

$$eq(d+1) \leq 1,$$

we obtain

$$\mathbb{P}\left[\bigcap_{(i,u,v) \in M} (\Omega \setminus \mathcal{A}_{(i,u,v)})\right] = \mathbb{P}\left[\bigcap_{(i,u,v) \in M} T(i,u,v)\right] > 0.$$

Substituting the previous estimates, the condition becomes

$$e \left(\frac{\varepsilon}{\log K}\right)^{\frac{\theta N}{2}} \left(K^{O(\log \log K + \log(1/\varepsilon))} + 1\right) \leq 1,$$

which is verified for a sufficiently large choice of the constant c in the definition of $N = \lceil \frac{cN}{\theta} \rceil$.

We now indicate with E the intersection of the events $E(i,u,v)$

$$E := \bigcap_{(i,u,v) \in M} E(i,u,v) \subseteq \Omega,$$

and from the inclusion $T(i,u,v) \subseteq E(i,u,v)$ we obtain that the event E occurs:

$$\mathbb{P}[E] \geq \mathbb{P}\left[\bigcap_{(i,u,v) \in M} T(i,u,v)\right] > 0.$$

13. Final verification of the lower bound For this final section, we consider two points $x, y \in X$ and $i \in \mathbb{N}$ such that $\tau^{\frac{i}{1-\varepsilon}} < d(x, y) \leq \tau^{\frac{i+1}{1-\varepsilon}}$. Since \mathcal{N}_i is a δ_i -net, there exist $u, v \in \mathcal{N}_i$ such that

$$\max\{d(u, x), d(v, y)\} \leq \delta_i := \tau^{\frac{i+2}{1-\varepsilon}} \left(\frac{4\varepsilon}{c^* \kappa} \right)^{\frac{1}{1-\varepsilon}}.$$

Taking $k \in G(i, u, v)$, from the Lipschitz estimate of F (6) one has

$$\begin{aligned} & \max \left\{ \left| \sum_{j=1}^{\infty} f_j^k(u) - \sum_{j=1}^{\infty} f_j^k(x) \right|, \left| \sum_{j=1}^{\infty} f_j^k(v) - \sum_{j=1}^{\infty} f_j^k(y) \right| \right\} \\ & \leq c^* \left(\frac{\kappa}{\varepsilon} \right)^{1+\theta} \max\{d(u, x), d(v, y)\}^{1-\varepsilon} \\ & \leq c^* \left(\frac{\kappa}{\varepsilon} \right)^{1+\theta} \tau^{i+2} \left(\frac{4\varepsilon}{c^* \kappa} \right) \\ & = 4 \left(\frac{\kappa}{\varepsilon} \right)^{\theta} \tau^{i+2} = 4 \left(\frac{\kappa}{\varepsilon} \right)^{\theta} \tau \tau^{i+1} = \frac{\tau^{i+1}}{8}. \end{aligned}$$

Since $k \in G(i, u, v)$, it follows that

$$\begin{aligned} \left| \sum_{j=1}^{\infty} f_j^k(x) - \sum_{j=1}^{\infty} f_j^k(y) \right| & \geq \left| \sum_{j=1}^{\infty} f_j^k(u) - \sum_{j=1}^{\infty} f_j^k(v) \right| - \left| \sum_{j=1}^{\infty} f_j^k(u) - \sum_{j=1}^{\infty} f_j^k(x) \right| - \left| \sum_{j=1}^{\infty} f_j^k(v) - \sum_{j=1}^{\infty} f_j^k(y) \right| \\ & \geq \frac{\tau^{i+1}}{2} - 2 \frac{\tau^{i+1}}{8} = \frac{\tau^{i+1}}{4}. \end{aligned}$$

Since the event E occurs, this lower bound holds for at least $N/2$ values of $k \in \{1, \dots, N\}$. Therefore,

$$\|F(x) - F(y)\|_2 \gtrsim \tau^{i+1} = \tau^2 \tau^{i-1} \gtrsim \left(\frac{\varepsilon}{\kappa} \right)^{2\theta} d(x, y)^{1-\varepsilon}.$$

Together with the Lipschitz estimate proved above in Section 6, this shows that F is a bi-Lipschitz embedding of $(X, d^{1-\varepsilon})$ into \mathbb{R}^N :

$$\left(\frac{\varepsilon}{\kappa} \right)^{2\theta} d(x, y)^{1-\varepsilon} \lesssim \|F(x) - F(y)\|_2 \lesssim \left(\frac{\kappa}{\varepsilon} \right)^{1+\theta} d(x, y)^{1-\varepsilon}.$$

□

References

- [1] Patrice Assouad. “Plongements lipschitziens dans \mathbb{R}^n ”. In: *Bulletin de la Société Mathématique de France* 111.4 (1983), pp. 429–448.
- [2] Assaf Naor and Ofer Neiman. “Assouad’s theorem with dimension independent of the snowflake parameter”. In: *Revista Matemática Iberoamericana* 28.4 (2012), pp. 1123–1142.