

DIMENSION INDEPENDENT BERNSTEIN–MARKOV INEQUALITIES IN GAUSS SPACE

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ABSTRACT. We obtain the following dimension independent Bernstein–Markov inequality in Gauss space: for each $1 \leq p < \infty$ there exists a constant $C_p > 0$ such that for any $k \geq 1$ and all polynomials P on \mathbb{R}^k we have

$$\|\nabla P\|_{L^p(\mathbb{R}^k, d\gamma_k)} \leq C_p (\deg P)^{\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right)} \|P\|_{L^p(\mathbb{R}^k, d\gamma_k)},$$

where $d\gamma_k$ is the standard Gaussian measure on \mathbb{R}^k . We also show that under some mild growth assumptions on any function $B \in C^2((0, \infty)) \cap C([0, \infty))$ with $B', B'' > 0$ we have

$$\int_{\mathbb{R}^k} B(|LP(x)|) d\gamma_k(x) \leq \int_{\mathbb{R}^k} B(10(\deg P)^{\alpha_B} |P(x)|) d\gamma_k(x)$$

where $L = \Delta - x \cdot \nabla$ is the generator of the Ornstein–Uhlenbeck semigroup and

$$\alpha_B = 1 + \frac{2}{\pi} \arctan \left(\frac{1}{2} \sqrt{\sup_{s \in (0, \infty)} \left\{ \frac{sB''(s)}{B'(s)} + \frac{B'(s)}{sB''(s)} \right\} - 2} \right).$$

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1. INTRODUCTION

Let $d\gamma_k(x)$ be the standard Gaussian measure on \mathbb{R}^k , given by

$$d\gamma_k(x) = \frac{e^{-|x|^2/2}}{\sqrt{(2\pi)^k}} dx,$$

where $|x| = \sqrt{x_1^2 + \dots + x_k^2}$ is the Euclidean length of $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. Here and throughout, we will denote by φ_k the density of the Gaussian measure $d\gamma_k$ with respect to the Lebesgue measure on \mathbb{R}^k . For $1 \leq p < \infty$, define $L^p(\mathbb{R}^k, d\gamma_k)$ to be the space of those measurable functions on \mathbb{R}^k for which

$$\|f\|_{L^p(\mathbb{R}^k, d\gamma_k)} := \left(\int_{\mathbb{R}^k} |f|^p d\gamma_k \right)^{\frac{1}{p}} < \infty.$$

As usual, $L^\infty(\mathbb{R}^k, d\gamma_k)$ is defined by the condition $\|f\|_{L^\infty(\mathbb{R}^k, d\gamma_k)} = \text{esssup}_{x \in \mathbb{R}^k} |f(x)| < \infty$. For convenience of notation, we will abbreviate $\|f\|_{L^p(\mathbb{R}^k, d\gamma_k)}$ as $\|f\|_{L^p(d\gamma_k)}$.

1.1. Freud’s inequality in high dimensions. In his seminal paper [5], Freud obtained the following weighted Bernstein–Markov type inequality on the real line.

Theorem 1 (Freud’s inequality, [5]). *There exists a universal constant $C > 0$ such that for any $1 \leq p \leq \infty$ and all polynomials P on \mathbb{R} , we have*

$$(1) \quad \left(\int_{\mathbb{R}} |P'(x)\varphi_1(x)|^p dx \right)^{1/p} \leq C \sqrt{\deg P} \left(\int_{\mathbb{R}} |P(x)\varphi_1(x)|^p dx \right)^{1/p}.$$

After making a change of variables in (1), Freud’s inequality can be rewritten in terms of $\|\cdot\|_{L^p(d\gamma_1)}$ norms as

$$(2) \quad \|P'\|_{L^p(d\gamma_1)} \leq C \sqrt{\frac{\deg P}{p}} \|P\|_{L^p(d\gamma_1)},$$

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for all $1 \leq p < \infty$. Notice that (2) breaks down for $p = \infty$ as $\|P\|_{L^\infty(d\gamma_1)} = \infty$ for every non-constant polynomial P , nevertheless inequality (1) still persists.

After proving Theorem 1, Freud [6] extended his Gaussian estimates (1) to more general weights $e^{-Q(x)}$ on the real line, nowadays known as *Freud weights*, where the function $Q(x)$ satisfies certain growth and convexity assumptions. In this case, the bound $\sqrt{\deg P}$ in (1) is replaced by a certain quantity which depends on the so-called *Mhaskar–Rakhmanov–Saff numbers* of the weight $e^{-Q(x)}$. Since the works [5, 6] of Freud, several different proofs of such one-dimensional weighted Bernstein–Markov inequalities have been found (see, e.g., [7, 22, 14, 15, 16, 13]), in part due to important implications of such estimates in approximation theory (see, e.g., [5, Theorem 2] and [6, Theorems 4.1 and 5.1]). We refer the reader to the beautiful survey [18] of Lubinsky for a detailed exposition of results on this subject.

In relation to the “heat smoothing conjecture” [19] one can ask if a dimension independent discrete counterpart of Freud’s inequality holds on the Hamming cube $\{-1, 1\}^n$ equipped with uniform counting measure [4]. A positive answer by central limit theorem would imply the validity of Freud’s inequality in $L^p(\mathbb{R}^k, d\gamma_k)$ with constants independent of k . Therefore, it is of interest first to understand if Freud’s inequality can be extended to higher dimensions with a dimensionless constant. Throughout the ensuing discussion, for a smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and $0 < p < \infty$, we will denote

$$(3) \quad \|\nabla f\|_{L^p(d\gamma_k)} := \left(\int_{\mathbb{R}^k} \left(\sum_{j=1}^k (\partial_j f)^2(x) \right)^{p/2} d\gamma_k(x) \right)^{1/p}.$$

We first notice (see also Section 5) that the case $p = \infty$ of Freud’s inequality (1) easily extends in all \mathbb{R}^k with a constant independent of the dimension.

Proposition 2. *There exists a universal constant $C > 0$ such that for any $k \geq 1$, and all polynomials P on \mathbb{R}^k we have*

$$(4) \quad \|\varphi_k \nabla P\|_{L^\infty(\mathbb{R}^k)} \leq C \sqrt{\deg P} \|\varphi_k P\|_{L^\infty(\mathbb{R}^k)},$$

where $\deg P$ denotes the total degree of the multivariate polynomial P .

For finite values of p , the following question naturally arises, in analogy to (2).

Question 3 (Bernstein–Markov inequality in Gauss space). *Is it true that for each $1 \leq p < \infty$ there exists a constant $C_p > 0$ such that for any integer $k \geq 1$, and all polynomials P on \mathbb{R}^k the the dimension independent Gaussian Bernstein–Markov inequality*

$$(5) \quad \|\nabla P\|_{L^p(d\gamma_k)} \leq C_p \sqrt{\deg P} \|P\|_{L^p(d\gamma_k)}$$

holds true?

Remark 4. *Using (2), it is straightforward to obtain (5) with a dimension dependent constant $C_{p,k}$. Also, inequality (5) can easily be proven for $p = 2$ (and $C_2 = 1$) by expanding P in the Hermite basis and using orthogonality.*

Before moving to our main result, we mention that an elegant argument of Maurey and Pisier from [23], implies a weakening of Question 3 with a (suboptimal) linear bound on $\deg P$.

Proposition 5. *There exists a universal constant $C > 0$ such that for any $k \geq 1$, any $0 < p < \infty$ and all polynomials P on \mathbb{R}^k , we have*

$$(6) \quad \|\nabla P\|_{L^p(d\gamma_k)} \leq C \frac{\deg P}{\sqrt{p+1}} \|P\|_{L^p(d\gamma_k)}.$$

The main result of the present paper is that the linear bound on $\deg P$ in (6) can be improved.

Theorem 6. *For each $1 < p < \infty$ there exists a constant $C_p > 0$ such that for any $k \geq 1$, and all polynomials P on \mathbb{R}^k , we have*

$$(7) \quad \|\nabla P\|_{L^p(d\gamma_k)} \leq C_p (\deg P)^{\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right)} \|P\|_{L^p(d\gamma_k)}.$$

Notice that for each $p \in (1, \infty)$ we have $0 \leq \frac{1}{\pi} \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right) < \frac{1}{2}$, therefore (7) is worse than (5) but improves upon (6). Also, notice that for $p = 2$, inequality (7) recovers (5). To the extend of our knowledge, these are the best known bounds towards Question 5.

Our proof of Theorem 6, relies on a similar Bernstein–Markov type inequality for the generator of the Ornstein–Uhlenbeck semigroup (see Theorem 8 below) and Meyer’s dimension-free Riesz transform inequalities in Gauss space from [20].

1.2. Reverse Bernstein–Markov inequality in Gauss space. Our initial motivation to study Question 3 comes from a dual question that Mendel and Naor [19, Remark 5.5 (2)] asked on the Hamming cube. A positive answer to their question would, by standard considerations, imply its continuous counterpart in Gauss space, namely a *reverse Bernstein–Markov inequality*. To state the latter question precisely, let H_m be the probabilists’ Hermite polynomial of degree m on \mathbb{R} , i.e.,

$$(8) \quad H_m(s) = \int_{\mathbb{R}} (s + it)^m d\gamma_1(t).$$

For $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and a multiindex $\alpha = (\alpha_1, \dots, \alpha_k)$, where $\alpha_j \in \mathbb{N} \cup \{0\}$, we consider the multivariate Hermite polynomial on \mathbb{R}^k , given by

$$(9) \quad H_\alpha(x) = \prod_{j=1}^k H_{\alpha_j}(x_j).$$

The family $\{H_\alpha\}_\alpha$ forms an orthogonal system on $L^2(d\gamma_k)$. Denote by $|\alpha| = \alpha_1 + \dots + \alpha_k$ and let $L = \Delta - x \cdot \nabla$ be the generator of the Ornstein–Uhlenbeck semigroup. Then, one has

$$LH_\alpha(x) = -|\alpha|H_\alpha(x).$$

for every multiindex α . The operator L should be understood as the *Laplacian in Gauss space*. Now consider any polynomial P on \mathbb{R}^k which *lives on frequencies greater than d* , i.e., of the form

$$(10) \quad P(x) = \sum_{|\alpha| \geq d} c_\alpha H_\alpha(x),$$

where $c_\alpha \in \mathbb{C}$.

Question 7 (Mendel–Naor, [19]). *Is it true that for each $1 < p < \infty$ there exists a constant $c_p > 0$ such that for any $k \geq 1$, any $d \geq 1$, and all polynomials of the form (10) on \mathbb{R}^k , living on frequencies greater than d , we have*

$$(11) \quad \|LP\|_{L^p(d\gamma_k)} \geq c_p d \|P\|_{L^p(d\gamma_k)}.$$

In [4], we show that for every $1 < p < \infty$ there exists some $c_p > 0$ such that for all polynomials P which live on frequencies $[d, d + m]$, i.e. are of the form

$$P(x) = \sum_{d \leq |\alpha| \leq d+m} c_\alpha H_\alpha(x),$$

we have

$$(12) \quad \|LP\|_{L^p(d\gamma_k)} \geq c_p \frac{d}{m} \|P\|_{L^p(d\gamma_k)}.$$

For small values of m , (12) improves upon previously known bounds in Question 7 which follow from works of Meyer [19, Lemma 5.4] and Mendel and Naor [19, Theorem 5.10] on the Hamming cube for this smaller subclass of polynomials. In particular, when $m = O(1)$, (12) positively answers a special case of Question 7. We refer to [4] for further results on reverse Bernstein–Markov inequalities along with extensions for vector-valued functions on the Hamming cube.

1.3. Bernstein–Markov inequality with respect to L . In order to prove Theorem 6, it will be convenient to first study the analogue of Question 3 for the “second derivative” L , namely, is it true that for every polynomial P on \mathbb{R}^k , we have

$$(13) \quad \|LP\|_{L^p(d\gamma_k)} \leq C_p \deg P \|P\|_{L^p(d\gamma_k)} ?$$

The best result that we could obtain in this direction is the following theorem.

Theorem 8. *For any integer $k \geq 1$, any $p \geq 1$, and any polynomial P on \mathbb{R}^k , we have*

$$(14) \quad \|LP\|_{L^p(d\gamma_k)} \leq 10(\deg P)^{1+\frac{2}{\pi} \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right)} \|P\|_{L^p(d\gamma_k)}.$$

1.4. General function estimates. Our techniques for proving Theorem 8 allow us to replace p -th powers in L^p norms in (14) by an arbitrary convex increasing function in the spirit of Zygmund’s theorem (see [27], Vol 2., Ch. 10, Theorem (3.16)). We recall that Zygmund’s theorem asserts that if Φ is nondecreasing convex function on $[0, \infty)$, and

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos(kt) + b_k \sin(kt))$$

is a trigonometric polynomial of degree at most n , then for every $r \geq 1$, the sharp inequality

$$(15) \quad \int_0^{2\pi} \Phi(|f_n^{(r)}(t)|) dt \leq \int_0^{2\pi} \Phi(n^r |f_n(t)|) dt$$

holds true. In fact, by the results of Arestov [1, 2], the inequality holds for somewhat larger class of nondecreasing functions, i.e., $\Phi(t) = \psi(\ln t)$ for some convex ψ on $(-\infty, \infty)$. In particular, inequality (15) holds true for $\Phi(t) = t^p$ for every $p > 0$ (instead of just $p \geq 1$), thus implying the usual L^p Bernstein–Markov inequality for trigonometric polynomials.

One straightforward way to obtain an analog of (15) in Gauss space is to invoke the rotational invariance of the Gaussian measure. Indeed, we will shortly show the following estimates.

Theorem 9. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be an increasing convex function. For any $k \geq 1$, and all polynomials P on \mathbb{R}^k we have*

$$(16) \quad \int_{\mathbb{R}^k} \int_{\mathbb{R}} \Phi(|t \nabla P(x)|) d\gamma_1(t) d\gamma_k(x) \leq \int_{\mathbb{R}^k} \Phi((\deg P) |P(x)|) d\gamma_k(x);$$

and

$$(17) \quad \int_{\mathbb{R}^k} \Phi(|LP(x)|) d\gamma_k(x) \leq \int_{\mathbb{R}^k} \Phi((\deg P)^2 |P(x)|) d\gamma_k(x).$$

Our main result of this section is that under mild assumptions on Φ , one can further improve (17).

Theorem 10. *For any $k \geq 1$ and all polynomials P on \mathbb{R}^k , we have*

$$(18) \quad \int_{\mathbb{R}^k} B(|LP(x)|) d\gamma_k(x) \leq \int_{\mathbb{R}^k} B(10(\deg P)^{\alpha_B} |P(x)|) d\gamma_k(x)$$

for any function $B \in C([0, \infty)) \cap C^2((0, \infty))$ with $B', B'' > 0$, such that for every $x > 0$

$$\max\{|B(x + \varepsilon)|, B'(x + \varepsilon), B''(x + \varepsilon)\} < C(1 + x^{2N})$$

for each fixed $\varepsilon > 0$ and some $C = C(\varepsilon), N = N(\varepsilon) > 0$. Here

$$(19) \quad \alpha_B := 1 + \frac{2}{\pi} \arctan \left(\frac{1}{2} \sqrt{\sup_{s \in (0, \infty)} \left\{ \frac{sB''(s)}{B'(s)} + \frac{B'(s)}{sB''(s)} \right\} - 2} \right).$$

A straightforward optimization shows that Theorem 8 follows from Theorem 10 by considering $B(t) = t^p$, where $p \geq 1$.

The rest of the paper is structured as follows. In Section 2, we present the proof of Theorem 9 and its consequence, Proposition 5. In Section 3, we prove our main result, Theorem 10 from which we also deduce Theorem 8. Finally, Section 4 contains the deduction of Theorem 6 from Theorem 8 and Section 5 contains the proof of Proposition 2.

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2. PROOF OF THEOREM 9

We first prove Theorem 9. The argument is inspired by an idea of Maurey and Pisier [23] and only uses the rotational invariance of the Gaussian measure $d\gamma_k$ and Zygmund's inequality (15).

Proof of Theorem 9. Let

$$X = (X_1, \dots, X_k) \quad \text{and} \quad Y = (Y_1, \dots, Y_k)$$

be two independent multivariate standard Gaussian $\mathcal{N}(0, Id_k)$ random variables on \mathbb{R}^k . Take any polynomial $P(x)$ on \mathbb{R}^k of degree n , and consider the trigonometric polynomial

$$t(\theta) := P(X \cos(\theta) + Y \sin(\theta)).$$

Clearly $\deg(t(\theta)) \leq n$. It follows from Zygmund's inequality (15) that

$$(20) \quad \int_0^{2\pi} \Phi(|t'(\theta)|) d\theta \leq \int_0^{2\pi} \Phi(n|t(\theta)|) d\theta.$$

Since $t'(\theta) = \nabla P(X \cos(\theta) + Y \sin(\theta)) \cdot (-X \sin(\theta) + Y \cos(\theta))$, and the random variables

$$X \cos(\theta) + Y \sin(\theta) \quad \text{and} \quad -X \sin(\theta) + Y \cos(\theta)$$

are also independent multivariate standard Gaussians, we see that after taking the expectation of (20) with respect to X, Y , we obtain

$$\begin{aligned} 2\pi \mathbb{E}\Phi(|\nabla P(X)| |Y_1|) &= \int_0^{2\pi} \mathbb{E}\Phi(|\nabla P(X \cos(\theta) + Y \sin(\theta)) \cdot (-X \sin(\theta) + Y \cos(\theta))|) d\theta \\ &\leq \int_0^{2\pi} \mathbb{E}\Phi(n|P(X \cos(\theta) + Y \sin(\theta))|) d\theta = 2\pi \mathbb{E}\Phi(|P(X)|). \end{aligned}$$

This finishes the proof of (16). To prove (17) notice that

$$\begin{aligned} t''(\theta) &= \langle \text{Hess}P(X \cos(\theta) + Y \sin(\theta))(-X \sin(\theta) + Y \cos(\theta)), (-X \sin(\theta) + Y \cos(\theta)) \rangle \\ &\quad - \nabla P(X \cos(\theta) + Y \sin(\theta)) \cdot (X \cos(\theta) + Y \sin(\theta)), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product in \mathbb{R}^k . Therefore following the same steps as before and using Zygmund's inequality (15), we obtain

$$\mathbb{E}\Phi(|\langle \text{Hess}P(X)Y, Y \rangle - \nabla P(X) \cdot X|) \leq \mathbb{E}\Phi(n^2|P(X)|).$$

Finally, it remains to use convexity of the map $x \mapsto \Phi(|x|)$ and Jensen's inequality to get

$$\mathbb{E}_Y \Phi(|\langle \text{Hess}P(X)Y, Y \rangle - \nabla P(X) \cdot X|) \geq \Phi(|\mathbb{E}_Y(\langle \text{Hess}P(X)Y, Y \rangle - \nabla P(X) \cdot X)|) = \Phi(|LP(X)|),$$

which completes the proof of (17). \square

Deriving Proposition 5 is now straightforward.

Proof of Proposition 5. It follows from the proof of Theorem 9 that (16) holds true under the sole assumption $\Phi(t) = \psi(\ln t)$ with ψ nondecreasing and convex on $(-\infty, \infty)$. Thus, applying (16) to $\Phi(t) = t^p$, where $p > 0$, we deduce that

$$(21) \quad (\mathbb{E}|g|^p)^{1/p} \|\nabla P\|_{L^p(d\gamma_k)} \leq \deg P \|P\|_{L^p(d\gamma_k)},$$

where g is a standard Gaussian random variable. Inequality (6) then follows from (21) since

$$(\mathbb{E}|g|^p)^{1/p} = \sqrt{2} \left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \right)^{1/p} \asymp \sqrt{p+1} \text{ for } p > 0. \quad \square$$

3. PROOF OF THEOREM 10

In this section, we prove the main result of this paper, Theorem 10, and its consequence, the Bernstein–Markov inequality of Theorem 8.

3.1. Step 1. A general complex hypercontractivity. Any polynomial $P(x)$ on \mathbb{R}^k admits a representation of the form

$$(22) \quad P(x) = \sum_{|\alpha| \leq \deg(P)} c_\alpha H_\alpha(x),$$

for some coefficients $c_\alpha \in \mathbb{C}$. Next, given $z \in \mathbb{C}$, we define the action of the second quantization operator (or Mehler transform) T_z on $P(x)$ as

$$(23) \quad T_z P(x) = \sum_{|\alpha| \leq \deg(P)} z^{|\alpha|} c_\alpha H_\alpha(x).$$

Clearly $T_0 P(x) = c_0 = \int_{\mathbb{R}^k} P(x) d\gamma_k(x)$, and $T_1 P(x) = P(x)$.

In what follows we will be working with a real-valued function $R \in C^2((0, \infty)) \cap C([0, \infty))$ (and sometimes we will further require $R \in C^2([0, \infty))$) such that

$$(24) \quad |R(x)|, |R'(x)|, |R''(x)| \leq C(1 + x^N)$$

for some constants $C, N > 0$ and every $x \geq 0$. These assumptions are sufficient to avoid integrability issues.

Lemma 11. *Fix $z \in \mathbb{C}$, and let $R \in C^2([0, \infty))$ be a real-valued function such that $R' \geq 0$. Assume that*

$$(25) \quad (1 - |z|^2)R'(x)|w|^2 + 2xR''(x)((\Re w)^2 - (\Re zw)^2) \geq 0 \quad \text{for all } w \in \mathbb{C} \text{ and } x \geq 0.$$

Then, for all $k \geq 1$, and for all polynomials $P(x)$ on \mathbb{R}^k we have

$$(26) \quad \int_{\mathbb{R}^k} R(|T_z P(x)|^2) d\gamma_k(x) \leq \int_{\mathbb{R}^k} R(|P(x)|^2) d\gamma_k(x).$$

Remark 12. *We will see later (see Section 3.4) that the reverse implication also holds, i.e., inequality (26) implies (25) under the additional assumption that $R \in C^3((0, \infty))$.*

Proof. Denote the scaled Gaussian measure on \mathbb{R}^k of variance s by

$$d\gamma_k^{(s)}(x) = \frac{1}{\sqrt{(2\pi s)^k}} e^{-|x|^2/2s} dx, \quad s \in (0, 1].$$

Take any polynomial $g : \mathbb{R}^k \rightarrow \mathbb{R}$. We will denote partial derivatives by lower indices, for example

$$g_{x_i x_j}(x) := \frac{\partial^2}{\partial x_i \partial x_j} g(x).$$

Fix a complex number $z \in \mathbb{C}$ satisfying (25), and consider the map

$$(27) \quad g(x, u, s) := \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} g((u + iv) + z(x + iy)) d\gamma_k^{(s)}(v) d\gamma_k^{(1-s)}(y),$$

where for $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ we denote $zx = (zx_1, \dots, zx_k)$. An analysis done in [11, 10, 12] suggests that one should study the monotonicity of the following map,

$$(28) \quad [0, 1] \ni s \mapsto r(s) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(s)}(u) d\gamma_k^{(1-s)}(x),$$

where $d\gamma_k^{(s)}(u)$ at $s = 0$ (or $d\gamma_k^{(1-s)}(x)$ at $s = 1$) should be understood as delta measure at zero, i.e.,

$$\lim_{s \rightarrow 0} \int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(s)}(u) \stackrel{u = \tilde{u}\sqrt{s}}{=} \lim_{s \rightarrow 0} \int_{\mathbb{R}^k} R(|g(x, \tilde{u}\sqrt{s}, s)|^2) d\gamma_k(\tilde{u}) = R(|g(x, 0, 0)|^2)$$

by Lebesgue's dominated convergence theorem.

First notice that if $s \mapsto r(s)$ is increasing then (26) follows. Indeed, consider any polynomial on \mathbb{R}^k of the form $P(x) = \sum_{\alpha \leq \deg P} c_\alpha H_\alpha(x)$ and define $g(x) = \sum_{\alpha \leq \deg P} c_\alpha x_1^{\alpha_1} \cdots x_k^{\alpha_k}$. Then,

$$r(1) = \int_{\mathbb{R}^k} R\left(\left|\int_{\mathbb{R}^k} g(u + iv) d\gamma_k(v)\right|^2\right) d\gamma_k(u) \stackrel{(8) \wedge (9)}{=} \int_{\mathbb{R}^k} R(|P(u)|^2) d\gamma_k(u)$$

and, similarly, also

$$r(0) = \int_{\mathbb{R}^k} R\left(\left|\int_{\mathbb{R}^k} g(z(x+iy)) d\gamma_k(y)\right|^2\right) d\gamma_k(x) \stackrel{(8)\wedge(9)}{=} \int_{\mathbb{R}^k} R(|T_z P(x)|^2) d\gamma_k(x).$$

Therefore, (26) can be rewritten as $r(0) \leq r(1)$.

Next, we show that the monotonicity of (28) follows from (25). Notice that for any function $Q \in C^2(\mathbb{R}^k)$ such that $|Q(x)| < C(1 + |x|^N)$ for some $C, N > 0$ we have

$$(29) \quad \frac{d}{ds} \left(\int_{\mathbb{R}^k} Q(x) d\gamma_k^{(s)}(x) \right) = \int_{\mathbb{R}^k} \frac{\Delta Q(x)}{2} d\gamma_k^{(s)}(x).$$

Indeed by making change of variables $x/\sqrt{s} = y$ we obtain $\int_{\mathbb{R}^k} Q(x) d\gamma_k^{(s)}(x) = \int_{\mathbb{R}^k} Q(y\sqrt{s}) d\gamma_k(y)$. Therefore

$$\frac{d}{ds} \left(\int_{\mathbb{R}^k} Q(x) d\gamma_k^{(s)}(x) \right) = \frac{d}{ds} \left(\int_{\mathbb{R}^k} \frac{Q(y\sqrt{s})}{2} d\gamma_k(y) \right) = \int_{\mathbb{R}^k} \frac{\nabla Q(y\sqrt{s}) \cdot y}{2\sqrt{s}} d\gamma_k(y)$$

and

$$\int_{\mathbb{R}^k} \frac{\Delta Q(x)}{2} d\gamma_k^{(s)}(x) = \int_{\mathbb{R}^k} \frac{\Delta Q(y\sqrt{s})}{2} d\gamma_k(y).$$

Notice that if we denote $v(y) := Q(y\sqrt{s})$ then (29) simply means that $\int_{\mathbb{R}^k} (\Delta - y \cdot \nabla)v(y) d\gamma_k(y) = 0$. The latter follows from integration by parts. Therefore, we have

$$r'(s) = \int_{\mathbb{R}^k} \left[\frac{\Delta u}{2} \left(\int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(1-s)}(x) \right) + \left(\int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(1-s)}(x) \right)_s \right] d\gamma_k^{(s)}(u).$$

To compute the first term, one differentiation gives

$$\left[\int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(1-s)}(x) \right]_{u_j} = \int_{\mathbb{R}^k} R'(|g(x, u, s)|^2) [g(x, u, s)]_{u_j} d\gamma_k^{(1-s)}(x),$$

which implies that

$$\begin{aligned} & \left[\int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(1-s)}(x) \right]_{u_j u_j} = \\ & \int_{\mathbb{R}^k} R''(|g(x, u, s)|^2) ([g(x, u, s)]_{u_j})^2 + R'(|g(x, u, s)|^2) [g(x, u, s)]_{u_j u_j} d\gamma_k^{(1-s)}(x). \end{aligned}$$

For the second term, we have

$$\left(\int_{\mathbb{R}^k} R(|g(x, u, s)|^2) d\gamma_k^{(1-s)}(x) \right)_s = -\frac{1}{2} \int_{\mathbb{R}^k} \Delta_x R(|g(x, u, s)|^2) - 2[R(|g(x, u, s)|^2)]_s d\gamma_k^{(1-s)}(x).$$

Thus we get that $r'(s) = \frac{1}{2} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \tilde{r}(s) d\gamma_k^{(1-s)}(x) d\gamma_k^{(s)}(u)$, where

$$\tilde{r}(s) = \Delta_u R(|g(x, u, s)|^2) - \Delta_x R(|g(x, u, s)|^2) + 2[R(|g(x, u, s)|^2)]_s.$$

Now, compute

$$g(x, u, s)_{u_j} = g_j(x, u, s) := g_j \quad \text{and} \quad g(x, u, s)_{u_j u_j} = g_{jj}(x, u, s) := g_{jj},$$

where g_j is the j -th partial derivative of g and we denote

$$g_j(x, u, s) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} g_j((u+iv) + z(x+iy)) d\gamma_k^{(s)}(v) d\gamma_k^{(1-s)}(y),$$

and similarly $g_{jj}(x, u, s)$ means that we first differentiate the polynomial g twice in the j -th coordinate and then we apply *the flow* (27). Similarly, we have

$$g(x, u, s)_{x_j} = z g_j, \quad g(x, u, s)_{x_j x_j} = z^2 g_j \quad \text{and} \quad g(x, u, s)_s = \frac{z^2 - 1}{2} \sum_{j=1}^k g_{jj}.$$

Next, further abusing the notation, we will denote $g := g(x, u, s)$. We have

$$\begin{aligned}
(|g(x, u, s)|^2)_{u_j} &= g_j \bar{g} + g \bar{g}_j = 2\Re\left(\frac{|g|^2 g_j}{g}\right); \\
(|g(x, u, s)|^2)_{u_j u_j} &= g_{jj} \bar{g} + g \bar{g}_{jj} + 2g_j \bar{g}_j = 2\Re\left(\frac{|g|^2 g_{jj}}{g}\right) + 2|g_j|^2; \\
(|g(x, u, s)|^2)_{x_j} &= z g_j \bar{g} + g \bar{z} \bar{g}_j = 2\Re\left(\frac{|g|^2 z g_j}{g}\right); \\
(|g(x, u, s)|^2)_{x_j x_j} &= 2\Re\left(\frac{|g|^2 z^2 g_{jj}}{g}\right) + 2|z|^2 |g_j|^2; \\
(|g(x, u, s)|^2)_s &= \Re\left(\frac{|g|^2 (z^2 - 1) \sum_{j=1}^k g_{jj}}{g}\right).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\Delta_u R(|g(x, u, s)|^2) - \Delta_x R(|g(x, u, s)|^2) + 2[R(|g(x, u, s)|^2)]_s = \\
&R'(|g(x, u, s)|^2) (\Delta_u(|g(x, u, s)|^2) - \Delta_x(|g(x, u, s)|^2) + 2(|g(x, u, s)|^2)_s) + \\
&R''(|g(x, u, s)|^2) \left(\sum_{j=1}^k (|g(x, u, s)|^2)_{u_j}^2 - (|g(x, u, s)|^2)_{x_j}^2 \right) = \\
&2 \sum_{j=1}^k (1 - |z|^2) R'(|g|^2) |g_j|^2 + 2R''(|g|^2) ((\Re g_j \bar{g})^2 - (\Re z g_j \bar{g})^2).
\end{aligned}$$

Notice that if $|g| = 0$ then by (25) we have $(1 - |z|^2)R'(0) \geq 0$, and hence $r' \geq 0$ in this case. So assume that $|g| > 0$. Then, denoting $w = g_j \bar{g}$ and $x = |g|^2$, we see that $r'(s) \geq 0$ follows from

$$(1 - |z|^2)R'(x)|w|^2 + 2xR''(x)((\Re w)^2 - (\Re zw)^2) \geq 0, \quad x \geq 0, \quad w \in \mathbb{C}.$$

The latter condition is exactly (25) and the proof is complete. \square

Lemma 13. *Let $R \in C^2((0, \infty)) \cap C([0, \infty))$, be such that $R' \geq 0$, and $R(x, \varepsilon) := R(x + \varepsilon)$ satisfies (24) for each fixed $\varepsilon > 0$. Take any $z \in \mathbb{C}$, $|z| \leq 1$ such that the inequality*

$$(30) \quad (1 - |z|^2)R'(x)|w|^2 + 2xR''(x)((\Re w)^2 - (\Re zw)^2) \geq 0$$

holds for all $w \in \mathbb{C}$, and all $x > 0$. Then for all polynomials $P(x)$ on \mathbb{R}^k we have

$$(31) \quad \int_{\mathbb{R}^k} R(|T_z P(x)|^2) d\gamma_k(x) \leq \int_{\mathbb{R}^k} R(|P(x)|^2) d\gamma_k(x).$$

Proof. For each $\varepsilon > 0$ we consider the function $R(x, \varepsilon) := R(x + \varepsilon)$. We claim that $R(x, \varepsilon)$ satisfies (25). Indeed, applying (30) at points $x + \varepsilon$, we obtain

$$\frac{(1 - |z|^2)}{x + \varepsilon} R'(x + \varepsilon)|w|^2 + 2R''(x + \varepsilon)((\Re w)^2 - (\Re zw)^2) \geq 0.$$

Since $R'(x + \varepsilon) \geq 0$, $1 - |z|^2 \geq 0$, and $\frac{1}{x + \varepsilon} \leq \frac{1}{x}$ for $x > 0$, we deduce that

$$\frac{(1 - |z|^2)}{x} R'(x + \varepsilon)|w|^2 + 2R''(x + \varepsilon)((\Re w)^2 - (\Re zw)^2) \geq 0.$$

The latter means that $x \mapsto R(x, \varepsilon)$ satisfies (25) for all $\varepsilon > 0$ (the case $x = 0$ in (25) is trivial because $R'(\varepsilon) \geq 0$ by the assumption in the lemma). Therefore using Lemma 11, we get

$$\int_{\mathbb{R}^k} R(|T_z P(x)|^2) d\gamma_k(x) \leq \int_{\mathbb{R}^k} R(\varepsilon + |T_z P(x)|^2) d\gamma_k(x) \leq \int_{\mathbb{R}^k} R(\varepsilon + |P(x)|^2) d\gamma_k(x).$$

Next, we take $0 < \varepsilon < 1$ and $\varepsilon \rightarrow 0$. Notice that $\lim_{\varepsilon \rightarrow 0} R(\varepsilon + |P(x)|^2) = R(|P(x)|^2)$ and $R(\varepsilon + |P(x)|^2) \leq R(1 + |P(x)|^2) \in L^1(\mathbb{R}^k, d\gamma_k)$. Therefore we can apply Lebesgue's dominated convergence theorem and this finishes the proof of the lemma. \square

Proposition 14. Let $B \in C([0, \infty)) \cap C^2((0, \infty))$ be such that $B', B'' > 0$. Assume $x \mapsto B(\sqrt{x + \varepsilon})$ satisfies (24) for each fixed $\varepsilon > 0$. Then

$$(32) \quad \int_{\mathbb{R}^k} B(|T_z P(x)|) d\gamma_k(x) \leq \int_{\mathbb{R}^k} B(|P(x)|) d\gamma_k(x),$$

holds for all polynomials $P(x)$ on \mathbb{R}^k , and all $k \geq 1$, if z belongs to the lens

$$(33) \quad |2z \pm i\sqrt{c_B - 2}| \leq \sqrt{c_B + 2},$$

where $c_B := \sup_{s \in (0, \infty)} \left\{ \frac{sB''(s)}{B'(s)} + \frac{B'(s)}{sB''(s)} \right\}$.

Proof. By Lemma 13 applied to $R(x) = B(\sqrt{x})$ we deduce that (32) holds provided that (30) holds. Next, condition (30) for $R(x)$ is equivalent to

$$\frac{sB''(s)}{B'(s)} ((\Re w)^2 - (\Re wz)^2) + (\Im w)^2 - (\Im wz)^2 \geq 0,$$

holding for all $s > 0$ and $w \in \mathbb{C}$. The latter means that if we set $z = x + iy$, and $A(s) := \frac{sB''(s)}{B'(s)}$ then we must have

$$\begin{pmatrix} A(s) - A(s)x^2 - y^2 & xy(A(s) - 1) \\ xy(A(s) - 1) & 1 - A(s)y^2 - x^2 \end{pmatrix} \geq 0.$$

The trace of the matrix is $(1 - x^2 - y^2)(A(s) + 1)$, which is nonnegative if and only if $|z| \leq 1$. So it remains to study the sign of the determinant. If $y = 0$ then there is nothing to check, so assume that $y \neq 0$. The non-negativity of the determinant can be rewritten as

$$A(s) + \frac{1}{A(s)} \leq \frac{(x^2 - 1)^2}{y^2} + y^2 + 2x^2,$$

for every $s > 0$, which is equivalent to

$$c_B - 2 \leq \frac{(1 - x^2 - y^2)^2}{y^2}$$

and also

$$|y|\sqrt{c_B - 2} \leq 1 - x^2 - y^2.$$

The latter inequality can be rewritten as (33). This finishes the proof of the proposition. \square

3.2. Step 2. Szegő Theorem. In what follows we will be assuming that $B \in C([0, \infty)) \cap C^2((0, \infty))$ is such that $B', B'' > 0$, and $x \mapsto B(\sqrt{x + \varepsilon})$ satisfies (24) for each fixed $\varepsilon > 0$. Next, let us consider the lens domain in \mathbb{C} associated to B ,

$$(34) \quad \Omega_B := \left\{ z \in \mathbb{C} : \left| z \pm i\frac{\sqrt{c_B - 2}}{2} \right| \leq \frac{\sqrt{c_B + 2}}{2} \right\}.$$

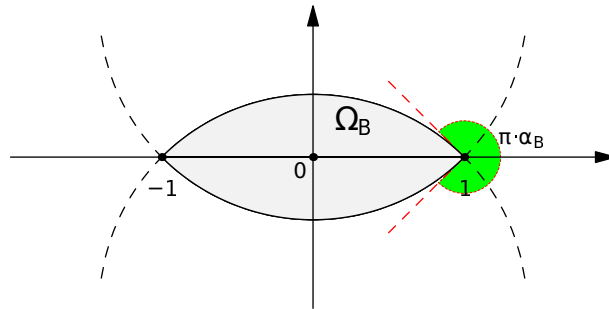


FIGURE 1. The domain Ω_B and the angle $\alpha_B = 1 + \frac{2}{\pi} \arctan\left(\frac{\sqrt{c_B - 2}}{2}\right)$

The domain Ω_B has an exterior angle at the point $(1, 0)$ which we are going to denote by $\pi \cdot \alpha_B$. A direct calculation reveals that

$$(35) \quad \alpha_B = 1 + \frac{2}{\pi} \arctan \left(\frac{\sqrt{c_B - 2}}{2} \right) \in [1, 2].$$

We will need the following Markov-type inequality in the complex domain Ω_B . For a compact set $K \subset \mathbb{C}$ and a polynomial P , we denote $\|P\|_{C(K)} = \sup_{z \in K} |P(z)|$ its supremum norm in K .

Proposition 15. *For any polynomial $P(w) = \sum_{j=0}^n a_j w^j$ with coefficients $a_j \in \mathbb{C}$, we have*

$$(36) \quad |P'(1)| \leq 10n^{\alpha_B} \|P\|_{C(\Omega_B)}.$$

Szegö was the first who investigated how the geometry of a domain in the complex plane affects the growth rate of the constant in Markov's inequality. The reader can find the bound $\|P'\|_{C(\Omega_B)} \leq C(B)n^{\alpha_B} \|P\|_{C(\Omega_B)}$ in [26], where the constant $C(B)$ depends on the domain Ω_B . We claim here that (36) holds with a universal constant, say $C(B) = 10$, which is independent of B . We could not locate the proof of this claim in the literature, so we include it here for the readers' convenience.

Proof of Proposition 15. Without loss of generality assume that $n > 10$, otherwise we can use the Markov inequality $|P'(1)| \leq n^2 \|P\|_{C([-1,1])} \leq 10n^{\alpha_B} \|P\|_{C(\Omega_B)}$.

We map conformally Ω_B^c , the complement of Ω_B , onto \mathbb{D}^c , the complement of the unit disk, using the map

$$\varphi(z) = \varphi_3 \circ \varphi_2 \circ \varphi_1(z),$$

where

$$\varphi_1(z) = \frac{z+1}{z-1}; \quad \varphi_2(z) = z^{1/\alpha_B}; \quad \varphi_3(z) = \frac{z+1}{z-1}.$$

Notice that the Möbius transformation $\varphi_1(z)$ maps $\varphi_1(-1) = 0$, $\varphi_1(1) = \infty$ and $\varphi_1(\infty) = 1$, thus $\varphi_1(\Omega_B^c)$ is the sector centered at $z = 0$ with angle $\pi\alpha_B$ and symmetric with respect to the positive x -semiaxis. Next, $\varphi_2(z)$ maps the sector to the right half plane $\Re z \geq 0$. Finally, $\varphi_3(z)$ maps the right half plane to the complement of the unit disk. It follows that

$$(37) \quad \varphi(z) = \frac{\left(\frac{z+1}{z-1}\right)^{1/\alpha_B} + 1}{\left(\frac{z+1}{z-1}\right)^{1/\alpha_B} - 1} = \alpha_B z + O\left(\frac{1}{z}\right), \quad \text{as } z \rightarrow \infty.$$

Next, let $P(z)$ be any polynomial of degree at most n on Ω_B such that $\|P\|_{C(\Omega_B)} \leq 1$ and consider the analytic function

$$\Omega_B^c \ni z \mapsto \psi(z) := \frac{P(z)}{\varphi^n(z)}.$$

The function ψ is regular at $z = \infty$ and bounded in absolute value by 1 on $\partial\Omega_B$. Thus, by the maximum principle, we get

$$|P(z)| \leq |\varphi(z)|^n \quad \text{for all } z \in \Omega_B^c.$$

Next, we will estimate $|P'(1)|$. Fix $0 < \delta < \frac{1}{10}$ to be determined later and let $C_\delta(1)$ be the circle of radius δ centered at the point $O(1, 0)$. Consider the arc $C_\delta(1) \setminus \Omega_B$, and let A, B be its endpoints. Let $\pi \cdot \alpha'_B = 2\pi - \angle AOB$ where the angle $\angle AOB$ is measured in radians. It follows from the alternate segment theorem and the law of cosines that

$$(38) \quad \alpha'_B = \alpha_B + \frac{1}{\pi} \arccos \left(1 - \frac{\delta^2}{2R^2} \right) \in (1, 2],$$

where R is the radius of the circles defining Ω_B . By Cauchy's integral formula, we have

$$\begin{aligned} |P'(1)| &= \left| \frac{1}{2\pi i} \oint_{C_\delta(1)} \frac{P(\zeta)}{(\zeta-1)^2} d\zeta \right| = \left| \frac{1}{2\delta} \int_0^{2\pi} \frac{P(1 + \delta e^{i\pi\theta})}{e^{i\pi\theta}} d\theta \right| \\ &\leq \frac{2 - \alpha'_B}{2\delta} + \frac{\alpha'_B}{2\delta} \max_{\theta \in [-\alpha'_B/2, \alpha'_B/2]} |\varphi(1 + \delta e^{i\pi\theta})|^n \leq \frac{1 + \max_{\theta \in [-\alpha'_B/2, \alpha'_B/2]} |\varphi(1 + \delta e^{i\pi\theta})|^n}{\delta}. \end{aligned}$$

We will need the following technical lemma.

Lemma 16. *For all $\gamma \in [1, 2]$ and $\delta, 0 < \delta < \frac{1}{10}$, we have*

$$(39) \quad \max_{\theta \in [-\gamma/2, \gamma/2]} |\varphi(1 + \delta e^{i\pi\theta})| \leq 1 + 2\delta^{1/\gamma},$$

where φ is given by (37).

Proof. Notice that

$$\varphi(1 + \delta e^{i\pi\theta}) \stackrel{(37)}{=} \frac{\left(\frac{2 + \delta e^{i\pi\theta}}{\delta e^{i\pi\theta}}\right)^{1/\gamma} + 1}{\left(\frac{2 + \delta e^{i\pi\theta}}{\delta e^{i\pi\theta}}\right)^{1/\gamma} - 1} = 1 + \frac{2\delta^{1/\gamma}}{(2e^{-i\pi\theta} + \delta)^{1/\gamma} - \delta^{1/\gamma}}.$$

Now, since $\gamma \in [1, 2]$, we see that for every $\delta < 1/10$ we have

$$|(2e^{-i\pi\theta} + \delta)^{1/\gamma} - \delta^{1/\gamma}| \geq (2 - \delta)^{1/\gamma} - \delta^{1/\gamma} \geq \left(2 - \frac{1}{10}\right)^{1/2} - \left(\frac{1}{10}\right)^{1/2} > 1$$

Thus we obtain that

$$\max_{\theta \in [-\gamma/2, \gamma/2]} |\varphi(1 + \delta e^{i\pi\theta})| \leq 1 + 2\delta^{1/\gamma},$$

which completes the proof of the lemma. \square

Proof of Proposition 15 (continued). It follows from (38) that since $R \geq 1$, for any $0 < \delta < 1/10$,

$$\alpha'_B \leq \alpha_B + \frac{1}{\pi} \arccos\left(1 - \frac{\delta^2}{2}\right) \leq \alpha_B + \frac{11\delta}{10\pi},$$

where the last inequality follows from the fact that $f(x) := \cos\left(\frac{11x}{10}\right) - 1 + \frac{x^2}{2} \leq 0$ for $x \in [0, 1/10]$. Next, choosing $\delta = n^{-\alpha_B}$ we thus get that

$$\frac{\alpha'_B}{\alpha_B} \leq 1 + \frac{1}{\pi} \arccos\left(1 - \frac{1}{2n^2}\right) \leq 1 + \frac{11}{10\pi n}$$

Therefore, applying Lemma 16, the inequality $\ln(1+x) \leq x$, $x > 0$, we get

$$\begin{aligned} |P'(1)| &\leq n^{\alpha_B} \left(1 + \exp\left(n \ln\left(1 + \frac{2}{n^{\alpha_B/\alpha'_B}}\right)\right)\right) \leq n^{\alpha_B} \left(1 + \exp\left(2n^{1 - \frac{\alpha_B}{\alpha'_B}}\right)\right) \\ &\leq n^{\alpha_B} \left(1 + \exp\left(2n^{\frac{1}{1 + \frac{10\pi n}{11}}}\right)\right) \leq n^{\alpha_B} \left(1 + \exp\left(2 \cdot 11^{\frac{1}{1 + 10\pi}}\right)\right) < 10n^{\alpha_B}. \end{aligned}$$

Here we have used the fact that $n \mapsto \frac{\ln n}{1 + cn}$ is decreasing for $n \geq 11$ provided that $c > \frac{1}{11(\ln(11) - 1)}$. This completes the proof. \square

3.3. Step 3. A duality argument and the proof of Theorem 10. To prove Theorem 10, we will use a duality argument which is inspired by a similar argument of Figiel [21, Theorem 14.6].

Lemma 17. *Fix a function $B \in C([0, \infty)) \cap C^2((0, \infty))$. For every positive integer n there exists a complex Radon measure $d\mu$ on Ω_B such that*

$$(40) \quad \int_{\Omega_B} z^\ell d\mu(z) = \ell, \quad \text{for all } \ell = 0, \dots, n,$$

and $\int_{\Omega_B} d|\mu| \leq 10n^{\alpha_B}$.

Proof. Fix a positive integer n , and consider the functional ψ on the space of polynomials of degree at most n on Ω_B , that is, $\mathcal{P}_n := \text{span}_{\mathbb{C}}\{z^\ell, \ell = 0, \dots, n\} \subset C(\Omega_B)$ given by

$$(41) \quad \psi \left(\sum_{\ell=0}^n a_\ell z^\ell \right) = \sum_{\ell=0}^n \ell a_\ell \quad \text{for all } a_j \in \mathbb{C}.$$

In other words, if P is a polynomial of degree at most n then, $\psi(P) = P'(1)$. It follows from Proposition 15 that for every such P , we have

$$(42) \quad |\psi(P)| \leq 10n^{\alpha_B} \|P\|_{C(\Omega_B)}.$$

Therefore, by the Hahn–Banach theorem, the functional $\psi \in (\mathcal{P}_n)^*$ can be extended to a functional $\Psi \in C(\Omega_B)^*$ with $\|\Psi\|_{(C(\Omega_B))^*} \leq 10n^{\alpha_B}$. However, by the Riesz representation theorem, the space $C(\Omega_B)^*$ can be identified with the Banach space of Radon measures on Ω_B equipped with the total variation norm and this completes the proof of the lemma. \square

Proof of Theorem 10. Take any complex-valued polynomial P of degree at most n on \mathbb{R}^k and $z \in \Omega_B$ and consider the measure μ supported on Ω_B given by Lemma 17. Then, we have

$$\begin{aligned} & \int_{\mathbb{R}^k} B \left(|LP(x)| \frac{1}{|\mu|(\Omega_B)} \right) d\gamma_k(x) \stackrel{(40)}{=} \int_{\mathbb{R}^k} B \left(\left| \int_{\Omega_B} T_z P(x) \frac{d\mu(z)}{|\mu|(\Omega_B)} \right| \right) d\gamma_k(x) \\ & \leq \int_{\mathbb{R}^k} B \left(\int_{\Omega_B} |T_z P(x)| \frac{d|\mu|(z)}{|\mu|(\Omega_B)} \right) d\gamma_k(x) \leq \int_{\Omega_B} \int_{\mathbb{R}^k} B(|T_z P(x)|) d\gamma_k(x) \frac{d|\mu|(z)}{|\mu|(\Omega_B)} \\ & \stackrel{(32)}{\leq} \int_{\Omega_B} \int_{\mathbb{R}^k} B(|P(x)|) d\gamma_k(x) \frac{d|\mu|(z)}{|\mu|(\Omega_B)} = \int_{\mathbb{R}^k} B(|P(x)|) d\gamma_k(x), \end{aligned}$$

where the second inequality follows from Jensen's inequality. After rescaling the coefficients of P and using Lemma 17, we deduce that the inequality

$$\int_{\mathbb{R}^k} B(|LP(x)|) d\gamma_k(x) \leq \int_{\mathbb{R}^k} B(10(\deg P)^{\alpha_B} |P(x)|) d\gamma_k(x)$$

holds true for all polynomials P on \mathbb{R}^k . \square

We can now easily deduce Theorem 8.

Proof of Theorem 8. Indeed, for $p > 1$ we can choose $B(s) = s^p$ in Theorem 10. Then

$$\frac{sB''(s)}{B'(s)} + \frac{B'(s)}{sB''(s)} = \frac{(p-1)^2 + 1}{p-1}$$

and therefore

$$\alpha_B = 1 + \frac{2}{\pi} \arctan \left(\frac{|p-2|}{2\sqrt{p-1}} \right).$$

Then, Theorem 10 implies that for every $p > 1$,

$$\|LP\|_{L^p(d\gamma_k)} \leq 10(\deg P)^{1 + \frac{2}{\pi} \arctan \left(\frac{|p-2|}{\sqrt{p-1}} \right)} \|P\|_{L^p(d\gamma_k)}$$

and letting $p \rightarrow 1^+$ we also deduce the endpoint case

$$\|LP\|_{L^1(d\gamma_k)} \leq 10(\deg P)^2 \|P\|_{L^1(d\gamma_k)},$$

which completes the proof of the theorem. \square

Remark 18. To directly prove Theorem 8, one could refer to the classical complex hypercontractivity [12] instead of invoking Proposition 14 in its full generality.

3.4. **The necessity of (25).** In addition to (24) let us require that for each point $t_0 > 0$ there exists $\delta = \delta(t_0)$ such that

$$(43) \quad R(t) = R(t_0) + R'(t_0)(t - t_0) + \frac{R''(t_0)}{2}(t - t_0)^2 + O(|t - t_0|^3)$$

holds for all $t > 0$ with $|t - t_0| < \delta(t_0)$. For example if $R \in C^3((0, \infty))$ then (43) holds. In particular, this means that for fixed complex numbers $a, b \in \mathbb{C}$ with $a \neq 0$ the function $t \mapsto R(|a + bt|^2)$ has the property (43).

Fix two complex numbers $a, b \in \mathbb{C}$ with $a \neq 0$, and consider a linear function $Q(x) = a + b\varepsilon x$ on \mathbb{R} , where $\varepsilon > 0$. Clearly $T_z Q(x) = a + b\varepsilon zx$. Since for any fixed $N > 0$, any polynomial P , and any constant $C > 0$ we have $\int_{|\varepsilon x| > C} |P(x)| d\gamma_1(x) = O(\varepsilon^N)$ as $\varepsilon \rightarrow 0$ we obtain that there exists a number $\delta = \delta(a, b, z)$ such that

$$\begin{aligned} & \int_{\mathbb{R}} R(|a + \varepsilon bzx|^2) d\gamma_1(x) \\ &= \int_{|\varepsilon x| \leq \delta} R(|a|^2) + R'(|a|^2)2\Re(\bar{a}bz)\varepsilon x + (R''(|a|^2)2(\Re(\bar{a}bz))^2 + R'(|a|^2)|bz|^2)|\varepsilon x|^2 d\gamma_1(x) + O(\varepsilon^3) \\ &= R(|a|^2) + (R''(|a|^2)2(\Re(\bar{a}bz))^2 + R'(|a|^2)|bz|^2)\varepsilon^2 + O(\varepsilon^3), \end{aligned}$$

as ε goes to zero. Similarly, there exists $\tilde{\delta} = \tilde{\delta}(a, b)$ such that

$$\begin{aligned} & \int_{\mathbb{R}} R(|a + \varepsilon bx|^2) d\gamma_1(x) \\ &= \int_{|\varepsilon x| \leq \tilde{\delta}} R(|a|^2) + R'(|a|^2)2\Re(\bar{a}b)\varepsilon x + (R''(|a|^2)2(\Re(\bar{a}b))^2 + R'(|a|^2)|b|^2)|\varepsilon x|^2 d\gamma_1(x) + O(\varepsilon^3) \\ &= R(|a|^2) + (R''(|a|^2)2(\Re(\bar{a}b))^2 + R'(|a|^2)|b|^2)\varepsilon^2 + O(\varepsilon^3), \end{aligned}$$

as $\varepsilon \rightarrow 0^+$. Using (31) we thus obtain

$$R''(|a|^2)2(\Re(\bar{a}bz))^2 + R'(|a|^2)|bz|^2 \leq R''(|a|^2)2(\Re(\bar{a}b))^2 + R'(|a|^2)|b|^2.$$

Denoting $w = \bar{a}b$ and $|a|^2 = x > 0$ the latter inequality, after multiplying the both sides by $|a|^2$, takes the form

$$(44) \quad 2xR''(x)(\Re(wz))^2 + R'(x)|wz|^2 \leq 2xR''(x)(\Re w)^2 + R'(x)|w|^2.$$

Since by changing $a \neq 0$ we can make x to be an arbitrary positive number, and by changing b we can make w to be an arbitrary complex number, we see that (44) coincides with (25). By continuity (44) holds also for $x = 0$. This proves the equivalence between (25) and (26). \square

4. PROOF OF THEOREM 6

Recall that $L = \Delta - x \cdot \nabla$ satisfies $LH_\alpha = -|\alpha|H_\alpha$. Define $(-L)^{1/2}H_\alpha = |\alpha|^{1/2}H_\alpha$ and extend it linearly to all polynomials P on \mathbb{R}^k . First we need the following lemma from [3, Lemma 5.6]. Since the argument is simple, we include the proof for the readers' convenience.

Lemma 19. *For any $p \geq 1$, any $k \geq 1$, and all polynomials P on \mathbb{R}^k we have*

$$(45) \quad \|(-L)^{1/2}P\|_{L^p(d\gamma_k)} \leq 2\|P\|_{L^p(d\gamma_k)}^{1/2}\|LP\|_{L^p(d\gamma_k)}^{1/2}.$$

Proof. Let $C := \int_0^\infty \frac{1-e^{-t}}{t^{3/2}} dt = 2\sqrt{\pi}$. Then for any $\lambda > 0$ we have

$$C\sqrt{\lambda} = \int_0^\infty \frac{1 - e^{-\lambda t}}{t^{3/2}} dt.$$

Therefore for any polynomial $P(x) = \sum_{|\alpha| \leq n} c_\alpha H_\alpha(x)$ and any number $M > 0$ we have

$$\begin{aligned}
\|(-L)^{1/2}P\|_{L^p(d\gamma_k)} &= \frac{1}{C} \left\| \sum_{|\alpha| \leq n} c_\alpha \left(\int_0^\infty \frac{1 - e^{-|\alpha|t}}{t^{3/2}} dt \right) H_\alpha \right\|_{L^p(d\gamma_k)} \\
&= \frac{1}{C} \left\| \int_0^M \int_0^t \left(\sum_{|\alpha| \leq n} c_\alpha e^{-|\alpha|s} |\alpha| H_\alpha \right) \frac{ds dt}{t^{3/2}} + \int_M^\infty \sum_{|\alpha| \leq n} c_\alpha (1 - e^{-|\alpha|t}) H_\alpha \frac{dt}{t^{3/2}} \right\|_{L^p(d\gamma_k)} \\
&\leq \frac{1}{C} \int_0^M \int_0^t \|T_{e^{-s}}LP\|_{L^p(d\gamma_k)} \frac{ds dt}{t^{3/2}} + \frac{1}{C} \int_M^\infty \|P - T_{e^{-t}}P\|_{L^p(d\gamma_k)} \frac{dt}{t^{3/2}} \\
&\leq \frac{2}{C} M^{1/2} \|LP\|_{L^p(d\gamma_k)} + \frac{4}{C} M^{-1/2} \|P\|_{L^p(d\gamma_k)},
\end{aligned}$$

where we used twice the fact that the operator $T_{e^{-t}}$ is the contraction in $L^p(d\gamma_k)$ for every $t \geq 0$. Finally choosing $M = 2\|P\|_{L^p(d\gamma_k)}\|LP\|_{L^p(d\gamma_k)}^{-1}$, one arrives at (45). \square

To deduce Theorem 6 from Theorem 8, we will need Meyer's Riesz transform inequalities in Gauss space [20] (see also [24] for a simpler proof and [8] for a stochastic calculus approach).

Theorem 20 (Meyer, [20]). *For each $p \in (1, \infty)$ there exist finite constants $C_p, c_p > 0$ such that, for any $k \geq 1$, and all polynomials P on \mathbb{R}^k we have*

$$(46) \quad c_p \|(-L)^{1/2}P\|_{L^p(d\gamma_k)} \leq \|\nabla P\|_{L^p(d\gamma_k)} \leq C_p \|(-L)^{1/2}P\|_{L^p(d\gamma_k)}.$$

We can now prove Theorem 6.

Proof of Theorem 6. Let P be any polynomial on \mathbb{R}^k . Then, we have

$$\begin{aligned}
\|\nabla P\|_{L^p(d\gamma_k)} &\stackrel{(46)}{\leq} C_p \|(-L)^{1/2}P\|_{L^p(d\gamma_k)} \stackrel{(45)}{\leq} 2C_p \|P\|_{L^p(d\gamma_k)}^{1/2} \|LP\|_{L^p(d\gamma_k)}^{1/2} \\
&\stackrel{(14)}{\leq} 2\sqrt{10}C_p (\deg P)^{\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{|p-2|}{\sqrt{p-1}}\right)} \|P\|_{L^p(d\gamma_k)},
\end{aligned}$$

and the proof is complete. \square

Remark 21. *When $p \rightarrow \infty$ the constant C_p , which comes from the boundedness of the Riesz transforms (46), goes to infinity. Therefore, for large enough values of p and polynomials P of small enough degree, the bound (6) is better than (7).*

5. PROOF OF PROPOSITION 2

In this section, we prove the sharp high dimensional L^∞ version of Freud's inequality, Proposition 2. The proof is an adaptation of the argument that Freud and Nevai [7] have used for the real line (see also further refinements of this technique in [14, 15]).

Proof of Proposition 2. Let us denote $\|f\|_{L^\infty(\Omega)} = \text{esssup}_{x \in \Omega} |f(x)|$ and $W_k(x) := e^{-|x|^2}$, the rescaled density of the Gaussian measure on \mathbb{R}^k . Take any polynomial P on \mathbb{R}^k of degree n and write $a_n = \sqrt{n/2}$. It was shown in [7] (see also [18, Section 8.2]) that there exists a universal constant $C > 0$ and a polynomial S_n on the real line of degree at most Cn such that

$$C^{-1}W_1(x) \leq S_n(x) \leq CW_1(x), \quad \text{for } |x| \leq 2a_n$$

and

$$|S'_n(x)| \leq C\sqrt{n}W_1(x), \quad \text{for } |x| \leq a_n.$$

In fact, one can take S_n to be the partial sums of the Taylor's series for $W_1(x)$ of order Cn . Clearly this polynomial is even because W_1 is so, therefore, the function $\rho_n(x) = S_n(|x|)$ is also a polynomial on \mathbb{R}^k . Taking into account that $W_k(x) = W_1(|x|)$ and that $|\nabla \rho_n(x)| = |S'_n(|x|)|$, we conclude that the estimates

$$(47) \quad C^{-1}W_k(x) \leq \rho_n(x) \leq CW_k(x) \quad \text{for } x \in B(2a_n)$$

and

$$(48) \quad |\nabla \rho_n(x)| \leq C\sqrt{n}W_k(x) \quad \text{on } x \in B(a_n),$$

also hold true, where $B(r)$ denotes the closed ball of radius r centered at the origin in \mathbb{R}^k .

Next, we will need the following well-known restricted range inequality, which follows, e.g., from [17, Theorem 1.8]. For any polynomial P on \mathbb{R}^k of degree at most n , we have

$$(49) \quad \|PW_k\|_{L^\infty(\mathbb{R}^k \setminus B(a_n))} \leq \|PW_k\|_{L^\infty(B(a_n))}.$$

In [17, Theorem 1.8] (see also [18, Theorem 6.2]), inequality (49) is stated for $k = 1$, i.e., for any polynomial G of degree at most n on \mathbb{R} we have

$$(50) \quad \|GW_1\|_{L^\infty(\mathbb{R} \setminus [-a_n, a_n])} \leq \|GW_1\|_{L^\infty([-a_n, a_n])}.$$

To deduce (49), it suffices to take an arbitrary unit vector v in \mathbb{R}^k , and apply (50) to $G(t) = P(vt)$. Thus, it follows that for any polynomial P on \mathbb{R}^k of degree at most n ,

$$(51) \quad \|PW_k^2\|_{L^\infty(\mathbb{R}^k \setminus B(a_n/\sqrt{2}))} \leq \|PW_k^2\|_{L^\infty(B(a_n/\sqrt{2}))}.$$

Since $|\nabla P|^2$ is a polynomial of degree at most $2n$ and $a_{2n} = \sqrt{2}a_n$, we have

$$\begin{aligned} \|\nabla P\|_{L^\infty(\mathbb{R}^k)} &\stackrel{(51)}{\leq} \|\nabla P\|_{L^\infty(B(a_{2n}/\sqrt{2}))} = \|\nabla P\|_{L^\infty(B(a_n))} \\ &\stackrel{(47)}{\leq} C\|\nabla P\|_{L^\infty(B(a_n))} \leq C(\|\nabla(P\rho_n)\|_{L^\infty(B(a_n))} + \|P|\nabla\rho_n|\|_{L^\infty(B(a_n))}) \\ &\stackrel{(52)\wedge(48)}{\leq} C \left[\frac{Bn}{\sqrt{n}}\|P\rho_n\|_{L^\infty(B(2a_n))} + C\sqrt{n}\|PW_k\|_{L^\infty(B(a_n))} \right] \leq A\sqrt{n}\|PW_k\|_{L^\infty(\mathbb{R}^k)}, \end{aligned}$$

for some universal constants $A, B > 0$. Here, we also used multidimensional Bernstein inequality

$$(52) \quad \|\nabla P\|_{L^\infty(B(R))} \leq \frac{Bd}{R}\|P\|_{L^\infty(B(2R))},$$

of Harris [9] (see also [25]), where $B > 0$ is a universal constant. Finally making the change of variables $x = y/\sqrt{2}$, and dividing of both sides of the one but last inequality by $\sqrt{(2\pi)^k}$ we obtain the estimate

$$\left\| |\nabla P(x)| \frac{e^{-|x|^2/2}}{\sqrt{(2\pi)^k}} \right\|_{L^\infty(\mathbb{R}^k)} \leq C\sqrt{n} \left\| P(x) \frac{e^{-|x|^2/2}}{\sqrt{(2\pi)^k}} \right\|_{L^\infty(\mathbb{R}^k)}$$

for a universal constant $C > 0$ and all polynomials P on \mathbb{R}^k of degree at most n . This finishes the proof of Proposition 2. \square

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