ON PISIER’S INEQUALITY FOR UMD TARGETS

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Abstract. We prove an extension of Pisier’s inequality (1986) with a dimension independent constant for vector valued functions whose target spaces satisfy a relaxation of the UMD property.

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1. Introduction

Let \((X, \| \cdot \|_X)\) be a Banach space. For \(p \in [1, \infty)\), the vector valued \(L_p\) norm of a function \(f : \Omega \to X\) defined on a measure space \((\Omega, \mathcal{F}, \mu)\) is given by \(\|f\|_{L_p(\Omega; \mu; X)} = \int_\Omega \|f(\omega)\|_X^p \, d\mu(\omega)\). When \(\Omega\) is a finite set and \(\mu\) is the normalized counting measure, we will simply write \(\|f\|_{L_p(\Omega)}\).

Let \(\mathcal{C}_n = \{-1, 1\}^n\) be the discrete hypercube. For \(i \in \{1, \ldots, n\}\), the \(i\)-th partial derivative of a function \(f : \mathcal{C}_n \to X\) is defined by

\[
\forall \varepsilon \in \mathcal{C}_n, \quad \partial_i f(\varepsilon) \overset{\text{def}}{=} \frac{f(\varepsilon) - f(\varepsilon_1, \ldots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_n)}{2}.
\]

(1)

In [Pis86], Pisier showed that for every \(n \in \mathbb{N}\) and \(p \in [1, \infty)\), every \(f : \mathcal{C}_n \to X\) satisfies

\[
\left\| f - \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} f(\delta) \right\|_{L_p(\mathcal{C}_n; X)} \leq \mathfrak{P}^n_p(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^n \delta_i \partial_i f \right\|_{L_p(\mathcal{C}_n; X)}^p \right)^{1/p},
\]

(2)

with \(\mathfrak{P}^n_p(X) = 2e \log n\). Showing that \(\mathfrak{P}^n_p(X)\) is bounded by a constant depending only on \(p\) and the geometry of the given Banach space \(X\), is of fundamental importance in the theory of nonlinear type (see [Pis86, NS02]). The first positive and negative results in this direction were obtained by Talagrand in [Tal93], who showed that \(\mathfrak{P}^p_p(\mathbb{R}) = \Theta(1)\) and \(\mathfrak{P}^p_p(\ell_\infty) = \Theta(\log n)\) for every \(p \in [1, \infty)\).

Talagrand’s dimension independent scalar valued inequality (2) was greatly generalized in the range \(p \in (1, \infty)\) by Naor and Schechtman [NS02]. Recall that a Banach space \((X, \| \cdot \|_X)\) is called a UMD space if for every \(p \in (1, \infty)\), there exists a constant \(\beta_p \in (0, \infty)\) such that for every \(n \in \mathbb{N}\), every probability space \((\Omega, \mathcal{F}, \mu)\) and every filtration \(\{\mathcal{F}_i\}_{i=0}^n\) of sub-\(\sigma\)-algebras of \(\mathcal{F}\), every martingale \(\{\mathcal{M}_i : \Omega \to X\}_{i=0}^n\) adapted to \(\{\mathcal{F}_i\}_{i=0}^n\) satisfies

\[
\max_{\delta = (\delta_1, \ldots, \delta_n) \in \mathcal{C}_n} \left\| \sum_{i=1}^n \delta_i (\mathcal{M}_i - \mathcal{M}_{i-1}) \right\|_{L_p(\Omega; \mu; X)} \leq \beta_p \|\mathcal{M}_n - \mathcal{M}_0\|_{L_p(\Omega; \mu; X)}.
\]

(3)

The least constant \(\beta_p \in (0, \infty)\) for which (3) holds is called the UMD\(_p\) constant of \(X\) and is denoted by \(\beta_p(X)\). In [NS02], Naor and Schechtman proved that for every UMD Banach space \(X\) and \(p \in (1, \infty)\),

\[
\sup_{n \in \mathbb{N}} \mathfrak{P}^n_p(X) \leq \beta_p(X).
\]

(4)

Their result was later strengthened by Hytönen and Naor [HN13] in terms of the random martingale transform inequalities of Garling, see [Gar90]. Recall that a Banach space \((X, \| \cdot \|_X)\) is a UMD\(_+$

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space if for every \( p \in (1, \infty) \) there exists a constant \( \beta_p^+ \in (0, \infty) \) such that for every martingale \( \{M_i : \Omega \to X\}_{i=0}^n \) as before, we have
\[
\left( \frac{1}{2^n} \sum_{\delta \in \mathcal{E}_n} \left\| \sum_{i=1}^n \delta_i (M_i - M_{i-1}) \right\|_{L_p(\Omega, \mu; X)}^p \right)^{1/p} \leq \beta_p^+ \| M_n - M_0 \|_{L_p(\Omega, \mu; X)}.
\] (5)

Similarly, \( X \) is a \( \text{UMD}^- \) Banach space if for every \( p \in (1, \infty) \) there exists a constant \( \beta_p^- \in (0, \infty) \) such that for every martingale \( \{M_i : \Omega \to X\}_{i=0}^n \) as before, we have
\[
\| M_n - M_0 \|_{L_p(\Omega, \mu; X)} \leq \beta_p^- \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{E}_n} \left\| \sum_{i=1}^n \delta_i (M_i - M_{i-1}) \right\|_{L_p(\Omega, \mu; X)}^p \right)^{1/p}.
\] (6)

The least positive constants \( \beta_p^+, \beta_p^- \) for which (5) and (6) hold are respectively called the \( \text{UMD}^+ \) and \( \text{UMD}^- \) constants of \( X \) and denoted by \( \beta_p^+(X) \) and \( \beta_p^-(X) \). In [HN13], Hytönen and Naor showed that for every Banach space \( X \) whose dual \( X^* \) is a \( \text{UMD}^+ \) space and \( p \in (1, \infty) \),
\[
\sup_{n \in \mathbb{N}} \mathcal{Q}_p^n(X) \leq \beta_{p/(p-1)}^+(X^*).
\] (7)

In fact, in [HN13, Theorem 1.4], the authors proved a generalization (see (28)) of inequality (2) for a family of \( n \) functions \( \{f_i : \mathcal{E}_n \to X\}_{i=1}^n \) under the assumption that the dual of \( X \) is \( \text{UMD}^+ \).

The main result of the present note is a different inequality of this nature with respect to a Fourier analytic parameter of \( X \). For a Banach space \( (X, \| \cdot \|_X) \) and \( p \in (1, \infty) \), let \( \mathfrak{s}_p(X) \in (0, \infty) \) be the least constant \( s \in (0, \infty) \) such that the following holds. For every probability space \( (\Omega, \mathcal{F}, \mu) \), \( n \in \mathbb{N} \) and filtration \( \{\mathcal{F}_i\}_{i=1}^n \) of sub-\( \sigma \)-algebras of \( \mathcal{F} \) with corresponding vector valued conditional expectations \( \{E_i\}_{i=1}^n \), every sequence of functions \( \{f_i : \Omega \to X\}_{i=1}^n \) satisfies
\[
\left( \frac{1}{2^n} \sum_{\delta \in \mathcal{E}_n} \left\| \sum_{i=1}^n \delta_i E_i f_i \right\|_{L_p(\Omega, \mu; X)}^p \right)^{1/p} \leq s \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{E}_n} \left\| \sum_{i=1}^n \delta_i f_i \right\|_{L_p(\Omega, \mu; X)}^p \right)^{1/p}.
\] (8)

The square function inequality (8) originates in Stein’s classical work [Ste70], where he showed that \( \mathfrak{s}_p(\mathbb{R}) = \Theta(1) \) for every \( p \in (1, \infty) \). In the vector valued setting which is of interest here, it has been proven by Bourgain in [Bou86] that for every \( \text{UMD}^+ \) Banach space and \( p \in (1, \infty) \),
\[
\mathfrak{s}_p(X) \leq \beta_p^+(X).
\] (9)

For a function \( f : \mathcal{E}_n \to X \) and \( i \in \{0, 1, \ldots, n\} \) denote by
\[
\forall \varepsilon \in \mathcal{E}_n, \quad E_i f(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{2^{n-i}} \sum_{\delta_{i+1}, \ldots, \delta_n \in \{-1,1\}} f(\varepsilon_1, \ldots, \varepsilon_i, \delta_{i+1}, \ldots, \delta_n),
\] (10)
so that \( E_n f = f \) and \( E_0 f = \frac{1}{2^n} \sum_{\delta \in \mathcal{E}_n} f(\delta) \). The main result of this note is the following theorem.

**Theorem 1.** Fix \( p \in (1, \infty) \) and let \( (X, \| \cdot \|_X) \) be a Banach space with \( \mathfrak{s}_p(X) < \infty \). If, additionally, \( X \) is a \( \text{UMD}^- \) space, then for every \( n \in \mathbb{N} \) and functions \( f_1, \ldots, f_n : \mathcal{E}_n \to X \), we have
\[
\left\| \sum_{i=1}^n (E_i f_i - E_{i-1} f_i) \right\|_{L_p(\mathcal{E}_n; X)} \leq \mathfrak{s}_p(X) \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{E}_n} \left\| \sum_{i=1}^n \delta_i \partial_i f_i \right\|_{L_p(\mathcal{E}_n; X)}^p \right)^{1/p}.
\] (11)
Choosing \( f_1 = \cdots = f_n = f \), we deduce that the constants in Pisier’s inequality (2) satisfy
\[
\sup_{n \in \mathbb{N}} \mathcal{Q}_p^n(X) \leq \mathfrak{s}_p(X) \beta_p^-(X).
\] (12)
Combining (12) with Bourgain’s inequality (9), we deduce that sup_{n \in \mathbb{N}} \| \mathbf{F}_n^p \| \leq \beta_p^+(X) \beta_p^-(X)$, which is weaker than Naor and Schechtman’s bound (4). Nevertheless, it appears to be unknown (see [Pis16, p. 197]) whether every Banach space $X$ with $\mathbf{s}_p(X) < \infty$ is necessarily a UMD$^+$ space. Therefore, it is conceivable that there exist Banach spaces $X$ for which inequality (12) does not follow from the previously known results of [NS02, HN13]. We will see in Proposition 5 below that if the dual $X^*$ of a Banach space $X$ is UMD$^+$, then $X$ satisfies the assumptions of Theorem 1. Therefore, Theorem 1 also contains the aforementioned result of [HN13].

Moreover, Theorem 1 implies an inequality similar to [HN13, Theorem 1.4] (see also Remark 3 below for comparison), under different assumptions. We will need some standard terminology from discrete Fourier analysis. Recall that every function $f : \mathcal{C}_n \to X$ can be expanded in a Walsh series as

$$f = \sum_{A \subseteq \{1, \ldots, n\}} \hat{f}(A) w_A,$$

where the Walsh function $w_A : \mathcal{C}_n \to \{-1, 1\}$ is given by $w_A(\varepsilon) = \prod_{i \in A} \varepsilon_i$ for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{C}_n$, and $\hat{f}(A) \in X$. Moreover, the fractional hypercube Laplacian of a function $f : \mathcal{C}_n \to X$ is given by

$$\forall \alpha \in \mathbb{R}, \quad \Delta^\alpha \left( \sum_{A \subseteq \{1, \ldots, n\}} \hat{f}(A) w_A \right) \overset{\text{def}}{=} \sum_{A \subseteq \{1, \ldots, n\}, A \neq \emptyset} |A|^\alpha \hat{f}(A) w_A.$$

### Corollary 2.
Fix $p \in (1, \infty)$ and let $(X, \| \cdot \|_X)$ be a Banach space with $\mathbf{s}_p(X) < \infty$. If, additionally, $X$ is a UMD$^-$ space, then for every $n \in \mathbb{N}$ and functions $f_1, \ldots, f_n : \mathcal{C}_n \to X$, we have

$$\left\| \sum_{i=1}^n \Delta^{-1} \partial_i f_i \right\|_{L_p(\mathcal{C}_n; X)} \leq \mathbf{s}_p(X) \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^n \delta_i \partial_i f_i \right\|_{L_p(\mathcal{C}_n; X)}^p \right)^{\frac{1}{p}}.$$  

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## 2. Proofs

We first present the proof of Theorem 1.

**Proof of Theorem 1.** For a function $h : \mathcal{C}_n \to X$ and $i \in \{1, \ldots, n\}$ consider the averaging operator

$$\forall \varepsilon \in \mathcal{C}_n, \quad \mathcal{E}_i h(\varepsilon) \overset{\text{def}}{=} \frac{h(\varepsilon) + h(\varepsilon_1, \ldots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_n)}{2} = (\text{id} - \partial_i) h(\varepsilon),$$

where id is the identity operator. Then, for every $i \in \{0, 1, \ldots, n\}$ we have the identities

$$\mathcal{E}_i h = \mathcal{E}_{i+1} \circ \cdots \circ \mathcal{E}_n h = \mathbb{E}[h|\mathcal{F}_i],$$

where $\mathcal{F}_i = \sigma(\varepsilon_1, \ldots, \varepsilon_i)$. Since for every $i \in \{1, \ldots, n\}$,

$$\mathbb{E}[\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i | \mathcal{F}_{i-1}] = 0,$$

the sequence $\{\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i\}_{i=1}^n$ is a martingale difference sequence and thus the UMD$^-$ condition and (8) imply that

$$\left\| \sum_{i=1}^n (\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i) \right\|_{L_p(\mathcal{C}_n; X)} \overset{\text{6}}{\leq} \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^n \delta_i (\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i) \right\|_{L_p(\mathcal{C}_n; X)}^p \right)^{\frac{1}{p}} \overset{\text{16}}{=} \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^n \delta_i \mathcal{E}_i \partial_i f_i \right\|_{L_p(\mathcal{C}_n; X)}^p \right)^{\frac{1}{p}},$$

$$\overset{\text{8}}{=} \mathbf{s}_p(X) \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^n \delta_i \partial_i f_i \right\|_{L_p(\mathcal{C}_n; X)}^p \right)^{\frac{1}{p}}.$$
which completes the proof. □

We will now derive Corollary 2 from Theorem 1. The proof follows a symmetrization argument of [HN13].

**Proof of Corollary 2.** As noticed in (19) above, (11) can be equivalently written as

\[ \left\| \sum_{i=1}^{n} \mathcal{E}_i \partial_i f_i \right\|_{L_p(\mathcal{C}_n;X)} \leq \mathfrak{s}_p(X) \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i \partial_i f_i \right\|_{L_p(\mathcal{C}_n;X)}^p \right)^{1/p}. \]

Fix a permutation \( \pi \in S_n \) and consider the filtration \( \{ \mathcal{F}_i \}_{i=0}^{n} \) given by \( \mathcal{F}_i = \sigma(\mathcal{C}_1, \ldots, \mathcal{C}_i) \) with corresponding conditional expectations \( \mathcal{E}_\pi \). Repeating the argument of the proof of Theorem 1 for this filtration and the martingale difference sequence \( \{ \mathcal{E}_\pi \mathcal{F}_\pi(n(i)) - \mathcal{E}_\pi \mathcal{F}_\pi(n(-i)) \}_{i=1}^{n} \), we see that for every \( \pi \in S_n \),

\[ \left\| \sum_{i=1}^{n} \mathcal{E}_\pi^\pi \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) \right\|_{L_p(\mathcal{C}_n;X)} \leq \mathfrak{s}_p(X) \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) \right\|_{L_p(\mathcal{C}_n;X)}^p \right)^{1/p}, \]

(20)

since \( (\delta_1, \ldots, \delta_n) \) has the same distribution as \( (\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}) \). An obvious adaptation of (10) along with (13) shows that for every \( h : \mathcal{C}_n \to X \),

\[ \mathcal{E}_\pi^\pi h = \sum_{A \subseteq \{\pi(1), \ldots, \pi(i)\}} \hat{h}(A) w_A \]

(22)

where \( \hat{h}(A) \) are the Walsh coefficients of \( h \). Therefore, expanding each \( \mathcal{F}_\pi(n(i)) \) as a Walsh series (13) we have

\[ \forall \ i \in \{1, \ldots, n\}, \quad \mathcal{E}_\pi^\pi \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) = \sum_{A \subseteq \{1, \ldots, n\} \text{ max } \pi^{-1}(A) = i} \hat{f}_\pi(n(i))(A) w_A \]

(23)

and therefore

\[ \sum_{i=1}^{n} \mathcal{E}_\pi^\pi \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) = \sum_{A \subseteq \{1, \ldots, n\} \text{ A} \neq \emptyset} \mathcal{F}_\pi(\text{max } \pi^{-1}(A))(A) w_A. \]

(24)

Averaging (24) over all permutations \( \pi \in S_n \) and using the fact that \( \pi(\text{max } \pi^{-1}(A)) \) is uniformly distributed in \( A \), we get

\[ \frac{1}{n!} \sum_{\pi \in S_n} \sum_{i=1}^{n} \mathcal{E}_i^\pi \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) = \sum_{A \subseteq \{1, \ldots, n\} \text{ } A \neq \emptyset} \frac{1}{|A|} \sum_{i \in A} \hat{f}_i(A) w_A = \sum_{i=1}^{n} \frac{1}{|A|} \hat{f}_i(A) w_A = \sum_{i=1}^{n} \Delta^{-1} \partial_i f_i. \]

Hence, by convexity we finally deduce that

\[ \left\| \sum_{i=1}^{n} \Delta^{-1} \partial_i f_i \right\|_{L_p(\mathcal{C}_n;X)} \leq \frac{1}{n!} \sum_{\pi \in S_n} \left\| \sum_{i=1}^{n} \mathcal{E}_i^\pi \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) \right\|_{L_p(\mathcal{C}_n;X)} \]

(25)

\[ \leq \mathfrak{s}_p(X) \beta_p^-(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i \partial_\pi(n(i)) \mathcal{F}_\pi(n(i)) \right\|_{L_p(\mathcal{C}_n;X)}^p \right)^{1/p}, \]

which completes the proof. □
Lemma 4. If a space \( X \) is UMD\(^+ \), we will need the following lemma.

In [HN13, Theorem 1.4], it was shown that for every \( p \in (1, \infty) \) and every function \( F : \mathcal{C}_n \times \mathcal{C}_n \to X \),

\[
\left\| \sum_{i=1}^{n} \Delta^{-1} \partial_i F_i \right\|_{L_p(\mathcal{C}_n; X)} \leq \beta_p^+ (X^*) \left\| F \right\|_{L_p(\mathcal{C}_n \times \mathcal{C}_n; X)},
\]

(27)

In fact, since every Banach space whose dual is UMD\(^+ \) is K-convex (see [Pis16] and Section 3 below) the validity of inequality (27) is equivalent to its validity for functions of the form \( F(\varepsilon, \delta) = \sum_{i=1}^{n} \delta_i F_i(\varepsilon) \), where \( F_1, \ldots, F_n : \mathcal{C}_n \to X \). In other words, [HN13, Theorem 1.4] is equivalent to the fact that if \( X^* \) is UMD\(^+ \), then for every \( F_1, \ldots, F_n : \mathcal{C}_n \to X \) and \( p \in (1, \infty) \),

\[
\left\| \sum_{i=1}^{n} \Delta^{-1} \partial_i F_i \right\|_{L_p(\mathcal{C}_n; X)} \leq A_p(X) \left( \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i F_i \right\|_{L_p(\mathcal{C}_n; X)}^p \right)^{1/p},
\]

(28)

up to the value of the constant \( A_p(X) \). In particular, applying (28) to \( F_i = \partial^i f_i \), one recovers Corollary 2, so inequality (28) of [HN13] is formally stronger than (15) in the class of spaces whose dual is UMD\(^+ \).

3. Concluding remarks

In this section we will compare our result with existing theorems in the literature. Recall that a Banach \( X \) space is K-convex if \( X \) does not contain the family \( \{\ell^n_1\}_{n=1}^\infty \) with uniformly bounded distortion. We will need the following lemma.

Lemma 4. If a space \( (X, \| \cdot \|_X) \) satisfies \( s_p(X) < \infty \) for some \( p \in (1, \infty) \), then \( X \) is K-convex.

Proof. It is well known since Stein’s work [Ste70] that inequality (8) does not hold for \( p \in \{1, \infty\} \) even for scalar valued functions. In fact, an inspection of the argument in [Ste70, p. 105] shows that for every \( n \in \mathbb{N} \) there exists \( n \) functions \( g_1, \ldots, g_n : \mathcal{C}_n \to \{0, 1\} \) such that for every \( q \in (2, \infty) \),

\[
\left\| \left( \sum_{i=1}^{n} (E_i g_i)^2 \right)^{1/2} \right\|_{L_q(\mathcal{C}_n; \mathbb{R})} \gtrsim \left( \int_0^n y^{n/2} e^{-y} \, dy \right)^{1/q} \left\| \left( \sum_{i=1}^{n} g_i^2 \right)^{1/2} \right\|_{L_q(\mathcal{C}_n; \mathbb{R})},
\]

(29)

where \( \{E_i\}_{i=0}^{n} \) are the conditional expectations (10). Using the fact that \( L_\infty(\mathcal{C}^*_n; \mathbb{R}) \) is isomorphic to \( L_n(\mathcal{C}^*_n; \mathbb{R}) \), we thus deduce that

\[
\left\| \left( \sum_{i=1}^{n} (E_i g_i)^2 \right)^{1/2} \right\|_{L_\infty(\mathcal{C}_n; \mathbb{R})} \gtrsim \left( \int_0^n y^{n/2} e^{-y} \, dy \right)^{1/n} \left\| \left( \sum_{i=1}^{n} g_i^2 \right)^{1/2} \right\|_{L_\infty(\mathcal{C}_n; \mathbb{R})}.
\]

(30)

Therefore, by duality in \( L_\infty(\mathcal{C}^*_n; \ell^2_n) \) and Khintchine’s inequality [Khi23], we deduce that there exists \( n \) functions \( h_1, \ldots, h_n : \mathcal{C}_n \to \mathbb{R} \) such that

\[
\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i E_i h_i \right\|_{L_1(\mathcal{C}_n; \mathbb{R})} \gtrsim \frac{\sqrt{n}}{2^n} \sum_{\delta \in \mathcal{C}_n} \left\| \sum_{i=1}^{n} \delta_i h_i \right\|_{L_1(\mathcal{C}_n; \mathbb{R})}.
\]

(31)
Suppose that a Banach space $X$ with $s_p(X) < \infty$ is not $K$-convex, so that there exists a constant $K \in [1, \infty)$ such that for every $n \in \mathbb{N}$, there exists a linear operator $J_n : L_1(\mathbb{C}_n; \mathbb{R}) \to X$ satisfying
\[
\forall h \in L_1(\mathbb{C}_n; \mathbb{R}), \quad \|h\|_{L_1(\mathbb{C}_n; \mathbb{R})} \leq \|J_n h\|_X \leq K \|h\|_{L_1(\mathbb{C}_n; \mathbb{R})}.
\] (32)

Consider the functions $H_1, \ldots, H_n : \mathbb{C}_n \to L_1(\mathbb{C}_n; \mathbb{R})$ given by
\[
\forall \varepsilon, \varepsilon' \in \mathbb{C}_n, \quad [H_i(\varepsilon)](\varepsilon') = h_i(\varepsilon \varepsilon_1' \ldots \varepsilon_n \varepsilon'_n),
\] (33)
where $h_i \in L_1(\mathbb{C}_n; \mathbb{R})$ are the functions satisfying (31). Then, for every $i \in \{1, \ldots, n\}$, we have $[\varepsilon_i H_i(\varepsilon)](\varepsilon') = \varepsilon_i h_i(\varepsilon \varepsilon_1' \ldots \varepsilon_n \varepsilon'_n)$ and, by translation invariance, for every $\varepsilon, \delta \in \mathbb{C}_n$ we have
\[
\left\| \sum_{i=1}^n \delta_i \varepsilon_i H_i(\varepsilon) \right\|_{L_1(\mathbb{C}_n; \mathbb{R})} = \left\| \sum_{i=1}^n \delta_i \varepsilon_i h_i \right\|_{L_1(\mathbb{C}_n; \mathbb{R})} \quad \text{and} \quad \left\| \sum_{i=1}^n \delta_i H_i(\varepsilon) \right\|_{L_1(\mathbb{C}_n; \mathbb{R})} = \left\| \sum_{i=1}^n \delta_i h_i \right\|_{L_1(\mathbb{C}_n; \mathbb{R})}
\]

Therefore, considering the mappings $f_1, \ldots, f_n : \mathbb{C}_n \to X$ given by $f_i = J_n \circ H_i$, we see that
\[
\left( \frac{1}{2^n} \sum_{\delta \in \mathbb{C}_n} \left\| \sum_{i=1}^n \delta_i \varepsilon_i f_i \right\|_{L_p(\mathbb{C}_n; X)}^p \right)^{1/p} \geq K^{-1} \sqrt{n} \left( \frac{1}{2^n} \sum_{\delta \in \mathbb{C}_n} \left\| \sum_{i=1}^n \delta_i f_i \right\|_{L_p(\mathbb{C}_n; X)}^p \right)^{1/p},
\] (34)
thus showing that $s_p(X) \geq K^{-1} \sqrt{n}$, which is a contradiction. □

Recall that the $X$-valued Rademacher projection is defined to be
\[
\Rad(A) = \left\{ \sum_{i \in \{1, \ldots, n\}} f(A) w_i \right\} = \sum_{i=1}^n \hat{f} \{\{i\}\} w_{\{i\}}.
\] (35)

A deep theorem of Pisier [Pis82] asserts that a Banach space is $K$-convex if and only if
\[
\forall r \in (1, \infty), \quad K_r(X) \overset{\text{def}}{=} \sup_{n \in \mathbb{N}} \left\| \Rad \right\|_{L_r(\mathbb{C}_n; X) \to L_r(\mathbb{C}_n; X)} < \infty.
\] (36)

In particular, it follows from Lemma 4 that $s_p(X) < \infty$ for some $p \in (1, \infty)$ implies that $K_r(X) < \infty$ for every $r \in (1, \infty)$. We proceed by showing that Banach spaces belonging to the class considered in [HN13, Theorem 1.4] satisfy the assumptions of Theorem 1.

**Proposition 5.** Let $(X, \| \cdot \|_X)$ be a Banach space. If $X^*$ is a UMD$^+$ space, then $X$ is a UMD$^-$ space and $s_p(X) < \infty$ for every $p \in (1, \infty)$.

**Proof.** The fact that if $X^*$ is UMD$^+$, then $X$ is UMD$^-$ has been proven by Garling in [Gar90, Theorem 1], so we only have to prove that $s_p(X) < \infty$. Let $f_1, \ldots, f_n : \mathbb{C}_n \to X$ and $G^* : \mathbb{C}_n \times \mathbb{C}_n \to X^*$ be such that
\[
\left( \frac{1}{2^n} \sum_{\delta \in \mathbb{C}_n} \left\| \sum_{i=1}^n \delta_i \varepsilon_i f_i \right\|_{L_p(\mathbb{C}_n; X)}^p \right)^{1/p} = \frac{1}{4^n} \sum_{\varepsilon, \delta \in \mathbb{C}_n} \langle G^*(\varepsilon, \delta), \sum_{i=1}^n \delta_i \varepsilon_i f_i(\varepsilon) \rangle
\] (37)
and $\|G^*\|_{L_q(\mathbb{C}_n \times \mathbb{C}_n; X^*)} = 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $G^*_i : \mathbb{C}_n \to X^*$ be given by
\[
\forall \varepsilon \in \mathbb{C}_n, \quad G^*_i(\varepsilon) = \frac{1}{2^n} \sum_{\delta \in \mathbb{C}_n} \delta_i G^*_i(\varepsilon, \delta).
\] (38)

Then, since $X^*$ is UMD$^+$, we deduce that $X^*$ is also $K$-convex (this is proven in [Gar90] but it also follows by combining Bourgain’s inequality (8) with Lemma 4) and thus
\[
\left( \frac{1}{2^n} \sum_{\delta \in \mathbb{C}_n} \left\| \sum_{i=1}^n \delta_i G^*_i \right\|_{L_q(\mathbb{C}_n; X^*)}^q \right)^{1/q} \overset{(38)}{=} \left( \frac{1}{4^n} \sum_{\varepsilon, \delta \in \mathbb{C}_n} \| \Rad \delta G^*(\varepsilon, \delta) \|_X^q \right)^{1/q} \leq K_q(X^*).
\] (39)
Hence, we have
\[
\left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i E_i f_i \right\|_{L^p(\varepsilon_n;X)} \right)^{1/p} \leq \frac{1}{4^n} \sum_{\varepsilon, \delta \in \varepsilon_n} \left( \sum_{i=1}^{n} \delta_i G_i^s(\varepsilon), \sum_{i=1}^{n} \delta_i E_i f_i(\varepsilon) \right) = \frac{1}{2^n} \sum_{\varepsilon \in \varepsilon_n} \langle \varepsilon_i G_i^s, f_i \rangle = \frac{1}{4^n} \sum_{\varepsilon, \delta \in \varepsilon_n} \left( \sum_{i=1}^{n} \delta_i E_i G_i^s(\varepsilon), \sum_{i=1}^{n} \delta_i f_i(\varepsilon) \right)
\]
\[
\leq \left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i E_i G_i^s \right\|_{L^q(\varepsilon_n;X^s)} \right)^{1/q} \cdot \left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i f_i \right\|_{L^p(\varepsilon_n;X)} \right)^{1/p}.
\]
Therefore, combining (40) with (8) and (39), we deduce that
\[
\left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i E_i f_i \right\|_{L^p(\varepsilon_n;X)} \right)^{1/p} \leq s_q(X^s)\left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i E_i G_i^s \right\|_{L^q(\varepsilon_n;X^s)} \right)^{1/q} \cdot \left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i f_i \right\|_{L^p(\varepsilon_n;X)} \right)^{1/p}.
\]
\[
\leq s_q(X^s)K_q(X^s) \cdot \left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i f_i \right\|_{L^p(\varepsilon_n;X)} \right)^{1/p},
\]
which shows that \( s_p(X) \leq K_q(X^s)s_q(X^s). \)

We conclude by observing that spaces satisfying the assumptions of Theorem 1 are necessarily superreflexive (see [Pis16] for the relevant terminology).

**Lemma 6.** If a UMD\(^{-}\) Banach space \((X, \| \cdot \|_X)\) satisfies \(s_p(X) < \infty\), then \(X\) is superreflexive.

**Proof.** A theorem of Pisier [Pis73] asserts that a Banach space \(X\) is \(K\)-convex if and only if \(X\) has nontrivial Rademacher type. Therefore, we deduce from Lemma 4 that if \(s_p(X) < \infty\) for some \(p \in (1, \infty)\), then there exists \(s \in (1, 2)\) and \(T_s(X) \in (0, \infty)\) such that
\[
\forall \ x_1, \ldots, x_n \in X, \quad \left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i x_i \right\|_X^s \right)^{1/s} \leq T_s(X) \left( \sum_{i=1}^{n} \| x_i \|_X \right)^{1/s}.
\]

Therefore, if \(X\) also satisfies the UMD\(^{-}\) property, we deduce that for every \(X\)-valued martingale \(\{\mathcal{M}_i : \Omega \to X\}_{i=0}^n\),
\[
\| \mathcal{M}_n - \mathcal{M}_0 \|_{L_s(\Omega, \mu;X)} \leq \beta_s(X) \left( \frac{1}{2^n} \sum_{\delta \in \varepsilon_n} \left\| \sum_{i=1}^{n} \delta_i (\mathcal{M}_i - \mathcal{M}_{i-1}) \right\|_X^s \right)^{1/s} \leq \beta_s(X) T_s(X) \left( \sum_{i=1}^{n} \| \mathcal{M}_i - \mathcal{M}_{i-1} \|_X^s \right)^{1/s},
\]
which means that \(X\) has martingale type \(s\). Combining this with well known results linking martingale type and superreflexivity (see [Pis16]), we reach the desired conclusion. \(\square\)

Therefore, Theorem 1 establishes that \(\mathcal{P}_p^n(X) = \Theta(1)\) for \(X\) in a (strict, see [Gar90, Qiu12]) subclass of all superreflexive spaces. In the forthcoming manuscript [EN20], the bound \(\mathcal{P}_p^n(X) = o(\log n)\) is shown to hold for every superreflexive Banach space \(X\) and \(p \in (1, \infty)\).
References


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