ON PISIER'S INEQUALITY FOR UMD TARGETS

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Abstract. We prove an extension of Pisier's inequality (1986) with a dimension independent constant for vector valued functions whose target spaces satisfy a relaxation of the UMD property.

2010 Mathematics Subject Classification. Primary: 46B07; Secondary: 46B85, 42C10, 60G46.
Key words. Pisier's inequality, Banach space valued martingales, UMD Banach spaces.

1. Introduction

Let $(X, \|\cdot\|_X)$ be a Banach space. For $p \in [1,\infty)$, the vector valued L_p norm of a function $f : \Omega \to X$ defined on a measure space $(\Omega, \mathcal{F}, \mu)$ is given by $\|f\|^p_{L_p(\Omega,\mu;X)} = \int_\Omega \|f(\omega)\|_X^p \,\mathrm{d}\mu(\omega)$. When Ω is a finite set and μ is the normalized counting measure, we will simply write $\|f\|_{L_p(\Omega;X)}$.

Let $\mathcal{C}_n = \{-1,1\}^n$ be the discrete hypercube. For $i \in \{1,\ldots,n\}$, the i-th partial derivative of a function $f : \mathcal{C}_n \to X$ is defined by

$$\forall\, \varepsilon \in \mathcal{C}_n, \quad \partial_i f(\varepsilon) \stackrel{\mathrm{def}}{=} \frac{f(\varepsilon) - f(\varepsilon_1,\ldots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\ldots,\varepsilon_n)}{2}. \tag{1}$$

In [Pis86], Pisier showed that for every $n \in \mathbb{N}$ and $p \in [1,\infty)$, every $f : \mathcal{C}_n \to X$ satisfies

$$\Big\| f - \frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n} f(\delta)\Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{P}_p^n(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\partial_i f\Big\|^p_{L_p(\mathcal{C}_n;X)}\Big)^{1/p}, \tag{2}$$

with $\mathfrak{P}_p^n(X) \leqslant 2e\log n$. Showing that $\mathfrak{P}_p^n(X)$ is bounded by a constant depending only on p and the geometry of the given Banach space X, is of fundamental importance in the theory of nonlinear type (see [Pis86, NS02]). The first positive and negative results in this direction were obtained by Talagrand in [Tal93], who showed that $\mathfrak{P}_p^n(\mathbb{R}) = \Theta(1)$ and $\mathfrak{P}_p^n(\ell_\infty) = \Theta(\log n)$ for every $p \in [1,\infty)$.

Talagrand's dimension independent scalar valued inequality (2) was greatly generalized in the range $p \in (1,\infty)$ by Naor and Schechtman [NS02]. Recall that a Banach space $(X,\|\cdot\|_X)$ is called a UMD space if for every $p \in (1,\infty)$, there exists a constant $\beta_p \in (0,\infty)$ such that for every $n \in \mathbb{N}$, every probability space $(\Omega,\mathcal{F},\mu)$ and every filtration $\{\mathcal{F}_i\}_{i=0}^n$ of sub-σ-algebras of $\mathcal{F}$, every martingale $\{\mathcal{M}_i : \Omega \to X\}_{i=0}^n$ adapted to $\{\mathcal{F}_i\}_{i=0}^n$ satisfies

$$\max_{\delta=(\delta_1,\ldots,\delta_n)\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i(\mathcal{M}_i - \mathcal{M}_{i-1})\Big\|_{L_p(\Omega,\mu;X)} \leqslant \beta_p\|\mathcal{M}_n - \mathcal{M}_0\|_{L_p(\Omega,\mu;X)}. \tag{3}$$

The least constant $\beta_p \in (0,\infty)$ for which (3) holds is called the UMD$_p$ constant of X and is denoted by $\beta_p(X)$. In [NS02], Naor and Schechtman proved that for every UMD Banach space X and $p \in (1,\infty)$,

$$\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) \leqslant \beta_p(X). \tag{4}$$

Their result was later strengthened by Hytönen and Naor [HN13] in terms of the random martingale transform inequalities of Garling, see [Gar90]. Recall that a Banach space $(X,\|\cdot\|_X)$ is a UMD$^+$

The author was supported by a postdoctoral fellowship of the Fondation Sciences Mathématiques de Paris.

space if for every $p \in (1,\infty)$ there exists a constant $\beta_p^+ \in (0,\infty)$ such that for every martingale $\{\mathcal{M}_i : \Omega \to X\}_{i=0}^n$ as before, we have

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|_{L_p(\Omega,\mu;X)}^p\Big)^{1/p} \leqslant \beta_p^+\|\mathcal{M}_n-\mathcal{M}_0\|_{L_p(\Omega,\mu;X)}. \tag{5}$$

Similarly, X is a UMD$^-$ Banach space if for every $p\in(1,\infty)$ there exists a constant $\beta_p^-\in(0,\infty)$ such that for every martingale $\{\mathcal{M}_i : \Omega \to X\}_{i=0}^n$ as before, we have

$$\|\mathcal{M}_n-\mathcal{M}_0\|_{L_p(\Omega,\mu;X)} \leqslant \beta_p^-\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|_{L_p(\Omega,\mu;X)}^p\Big)^{1/p}. \tag{6}$$

The least positive constants β_p^+,β_p^- for which (5) and (6) hold are respectively called the UMD$_p^+$ and UMD$_p^-$ constants of X and denoted by $\beta_p^+(X)$ and $\beta_p^-(X)$. In [HN13], Hytönen and Naor showed that for every Banach space X whose dual X^* is a UMD$^+$ space and $p\in(1,\infty)$,

$$\sup_{n\in\mathbb{N}}\mathfrak{P}_p^n(X) \leqslant \beta_{p/(p-1)}^+(X^*). \tag{7}$$

In fact, in [HN13, Theorem 1.4], the authors proved a generalization (see (28)) of inequality (2) for a family of n functions $\{f_i : \mathcal{C}_n \to X\}_{i=1}^n$ under the assumption that the dual of X is UMD$^+$.

The main result of the present note is a different inequality of this nature with respect to a Fourier analytic parameter of X. For a Banach space $(X,\|\cdot\|_X)$ and $p\in(1,\infty)$, let $\mathfrak{s}_p(X)\in(0,\infty]$ be the least constant $\mathfrak{s}\in(0,\infty]$ such that the following holds. For every probability space $(\Omega,\mathcal{F},\mu)$, $n\in\mathbb{N}$ and filtration $\{\mathcal{F}_i\}_{i=1}^n$ of sub-σ-algebras of $\mathcal{F}$ with corresponding vector valued conditional expectations $\{\mathcal{E}_i\}_{i=1}^n$, every sequence of functions $\{f_i : \Omega\to X\}_{i=1}^n$ satisfies

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\mathcal{E}_i f_i\Big\|_{L_p(\Omega,\mu;X)}^p\Big)^{1/p} \leqslant \mathfrak{s}\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i f_i\Big\|_{L_p(\Omega,\mu;X)}^p\Big)^{1/p}. \tag{8}$$

The square function inequality (8) originates in Stein's classical work [Ste70], where he showed that $\mathfrak{s}_p(\mathbb{R})=\Theta(1)$ for every $p\in(1,\infty)$. In the vector valued setting which is of interest here, it has been proven by Bourgain in [Bou86] that for every UMD$^+$ Banach space and $p\in(1,\infty)$,

$$\mathfrak{s}_p(X)\leqslant\beta_p^+(X). \tag{9}$$

For a function $f:\mathcal{C}_n\to X$ and $i\in\{0,1,\ldots,n\}$ denote by

$$\forall\, \varepsilon\in\mathcal{C}_n,\quad \mathcal{E}_i f(\varepsilon) \stackrel{\mathrm{def}}{=} \frac{1}{2^{n-i}}\sum_{\delta_{i+1},\ldots,\delta_n\in\{-1,1\}} f(\varepsilon_1,\ldots,\varepsilon_i,\delta_{i+1},\ldots,\delta_n), \tag{10}$$

so that $\mathcal{E}_n f = f$ and $\mathcal{E}_0 f = \frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n} f(\delta)$. The main result of this note is the following theorem.

Theorem 1. *Fix $p\in(1,\infty)$ and let $(X,\|\cdot\|_X)$ be a Banach space with $\mathfrak{s}_p(X)<\infty$. If, additionally, X is a UMD$^-$ space, then for every $n\in\mathbb{N}$ and functions $f_1,\ldots,f_n:\mathcal{C}_n\to X$, we have*

$$\Big\|\sum_{i=1}^n(\mathcal{E}_i f_i-\mathcal{E}_{i-1}f_i)\Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}. \tag{11}$$

Choosing $f_1=\cdots=f_n=f$, we deduce that the constants in Pisier's inequality (2) satisfy

$$\sup_{n\in\mathbb{N}}\mathfrak{P}_p^n(X)\leqslant\mathfrak{s}_p(X)\beta_p^-(X). \tag{12}$$

Combining (12) with Bourgain's inequality (9), we deduce that $\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) \leqslant \beta_p^+(X)\beta_p^-(X)$, which is weaker than Naor and Schechtman's bound (4). Nevertheless, it appears to be unknown (see [Pis16, p. 197]) whether every Banach space X with $\mathfrak{s}_p(X) < \infty$ is necessarily a UMD$^+$ space. Therefore, it is conceivable that there exist Banach spaces X for which inequality (12) does not follow from the previously known results of [NS02, HN13]. We will see in Proposition 5 below that if the dual X^* of a Banach space X is UMD$^+$, then X satisfies the assumptions of Theorem 1. Therefore, Theorem 1 also contains the aforementioned result of [HN13].

Moreover, Theorem 1 implies an inequality similar to [HN13, Theorem 1.4] (see also Remark 3 below for comparison), under different assumptions. We will need some standard terminology from discrete Fourier analysis. Recall that every function $f : \mathcal{C}_n \to X$ can be expanded in a Walsh series as

$$f = \sum_{A\subseteq\{1,\ldots,n\}} \widehat{f}(A) w_A, \tag{13}$$

where the Walsh function $w_A : \mathcal{C}_n \to \{-1,1\}$ is given by $w_A(\varepsilon) = \prod_{i\in A} \varepsilon_i$ for $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \in \mathcal{C}_n$ and $\widehat{f}(A) \in X$. Moreover, the fractional hypercube Laplacian of a function $f : \mathcal{C}_n \to X$ is given by

$$\forall\ \alpha \in \mathbb{R}, \qquad \Delta^\alpha\Big(\sum_{A\subseteq\{1,\ldots,n\}} \widehat{f}(A)w_A\Big) \stackrel{\mathrm{def}}{=} \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\varnothing}} |A|^\alpha \widehat{f}(A) w_A. \tag{14}$$

Corollary 2. *Fix $p \in (1,\infty)$ and let $(X, \|\cdot\|_X)$ be a Banach space with $\mathfrak{s}_p(X) < \infty$. If, additionally, X is a UMD$^-$ space, then for every $n \in \mathbb{N}$ and functions $f_1,\ldots,f_n : \mathcal{C}_n \to X$, we have*

$$\Big\| \sum_{i=1}^n \Delta^{-1}\partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n} \Big\|\sum_{i=1}^n \delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}. \tag{15}$$

Acknowledgements. I would like to thank Assaf Naor for helpful discussions.

2. Proofs

We first present the proof of Theorem 1.

Proof of Theorem 1. For a function $h : \mathcal{C}_n \to X$ and $i \in \{1,\ldots,n\}$ consider the averaging operator

$$\forall\ \varepsilon \in \mathcal{C}_n, \qquad \mathsf{E}_i h(\varepsilon) \stackrel{\mathrm{def}}{=} \frac{h(\varepsilon) + h(\varepsilon_1,\ldots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\ldots,\varepsilon_n)}{2} = (\mathsf{id} - \partial_i)h(\varepsilon), \tag{16}$$

where id is the identity operator. Then, for every $i \in \{0,1,\ldots,n\}$ we have the identities

$$\mathcal{E}_i h = \mathsf{E}_{i+1}\circ\cdots\circ\mathsf{E}_n h = \mathbb{E}[h|\mathcal{F}_i], \tag{17}$$

where $\mathcal{F}_i = \sigma(\varepsilon_1,\ldots,\varepsilon_i)$. Since for every $i \in \{1,\ldots,n\}$,

$$\mathbb{E}\big[\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i \big| \mathcal{F}_{i-1}\big] = 0, \tag{18}$$

the sequence $\{\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i\}_{i=1}^n$ is a martingale difference sequence and thus the UMD$^-$ condition and (8) imply that

$$\begin{aligned}\Big\|\sum_{i=1}^n (\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i)\Big\|_{L_p(\mathcal{C}_n;X)} &\stackrel{(6)}{\leqslant} \beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i(\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i)\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p} \\ &\stackrel{(16)}{=} \beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\mathcal{E}_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p} \\ &\stackrel{(8)}{\leqslant} \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\end{aligned}, \tag{19}$$

which completes the proof. ◻

We will now derive Corollary 2 from Theorem 1. The proof follows a symmetrization argument of [HN13].

Proof of Corollary 2. As noticed in (19) above, (11) can be equivalently written as

$$\Big\|\sum_{i=1}^n \mathcal{E}_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}. \tag{20}$$

Fix a permutation $\pi\in S_n$ and consider the filtration $\{\mathcal{F}_i^\pi\}_{i=0}^n$ given by $\mathcal{F}_i^\pi=\sigma(\varepsilon_{\pi(1)},\ldots,\varepsilon_{\pi(i)})$ with corresponding conditional expectations $\{\mathcal{E}_i^\pi\}_{i=0}^n$. Repeating the argument of the proof of Theorem 1 for this filtration and the martingale difference sequence $\{\mathcal{E}_i^\pi f_{\pi(i)}-\mathcal{E}_{i-1}^\pi f_{\pi(i)}\}_{i=1}^n$, we see that for every $\pi\in S_n$,

$$\begin{aligned}\Big\|\sum_{i=1}^n \mathcal{E}_i^\pi\partial_{\pi(i)} f_{\pi(i)}\Big\|_{L_p(\mathcal{C}_n;X)} &\leqslant \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\partial_{\pi(i)} f_{\pi(i)}\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\\ &= \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},\end{aligned} \tag{21}$$

since $(\delta_1,\ldots,\delta_n)$ has the same distribution as $(\delta_{\pi(1)},\ldots,\delta_{\pi(n)})$. An obvious adaptation of (10) along with (13) shows that for every $h:\mathcal{C}_n\to X$,

$$\mathcal{E}_i^\pi h=\sum_{A\subseteq\{\pi(1),\ldots,\pi(i)\}}\widehat{h}(A)w_A \tag{22}$$

where $\widehat{h}(A)$ are the Walsh coefficients of h. Therefore, expanding each $f_{\pi(i)}$ as a Walsh series (13) we have

$$\forall\ i\in\{1,\ldots,n\},\qquad \mathcal{E}_i^\pi\partial_{\pi(i)}f_{\pi(i)}=\sum_{\substack{A\subseteq\{1\ldots,n\}\\ \max\pi^{-1}(A)=i}}\widehat{f_{\pi(i)}}(A)w_A \tag{23}$$

and therefore

$$\sum_{i=1}^n \mathcal{E}_i^\pi\partial_{\pi(i)}f_{\pi(i)}=\sum_{A\subseteq\{1,\ldots,n\}}\widehat{f_{\pi(\max\pi^{-1}(A))}}(A)w_A. \tag{24}$$

Averaging (24) over all permutations $\pi\in S_n$ and using the fact that $\pi(\max\pi^{-1}(A))$ is uniformly distributed in A, we get

$$\frac{1}{n!}\sum_{\pi\in S_n}\sum_{i=1}^n \mathcal{E}_i^\pi\partial_{\pi(i)}f_{\pi(i)}=\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\varnothing}}\frac{1}{|A|}\sum_{i\in A}\widehat{f}_i(A)w_A=\sum_{i=1}^n\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ i\in A}}\frac{1}{|A|}\widehat{f}_i(A)w_A=\sum_{i=1}^n\Delta^{-1}\partial_i f_i.$$

Hence, by convexity we finally deduce that

$$\begin{aligned}\Big\|\sum_{i=1}^n\Delta^{-1}\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)} &\leqslant \frac{1}{n!}\sum_{\pi\in S_n}\Big\|\sum_{i=1}^n \mathcal{E}_i^\pi\partial_{\pi(i)}f_{\pi(i)}\Big\|_{L_p(\mathcal{C}_n;X)}\\ &\overset{(21)}{\leqslant} \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},\end{aligned} \tag{25}$$

which completes the proof. ◻

Remark 3. In [HN13], Hytönen and Naor obtained a different extension of Pisier's inequality (2) for Banach spaces whose dual is UMD$^+$. For a function $F : \mathcal{C}_n \times \mathcal{C}_n \to X$ and $i \in \{1, \ldots, n\}$, let $F_i : \mathcal{C}_n \to X$ be given by

$$\forall\, \varepsilon \in \mathcal{C}_n, \qquad F_i(\varepsilon) \stackrel{\mathrm{def}}{=} \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \delta_i F(\varepsilon, \delta). \tag{26}$$

In [HN13, Theorem 1.4], it was shown that for every $p \in (1, \infty)$ and every function $F : \mathcal{C}_n \times \mathcal{C}_n \to X$,

$$\Big\| \sum_{i=1}^n \Delta^{-1} \partial_i F_i \Big\|_{L_p(\mathcal{C}_n; X)} \leqslant \beta^+_{p/(p-1)}(X^*) \|F\|_{L_p(\mathcal{C}_n \times \mathcal{C}_n; X)}. \tag{27}$$

In fact, since every Banach space whose dual is UMD$^+$ is K-convex (see [Pis16] and Section 3 below) the validity of inequality (27) is equivalent to its validity for functions of the form $F(\varepsilon, \delta) = \sum_{i=1}^n \delta_i F_i(\varepsilon)$, where $F_1, \ldots, F_n : \mathcal{C}_n \to X$. In other words, [HN13, Theorem 1.4] is equivalent to the fact that if X^* is UMD$^+$, then for every $F_1, \ldots, F_n : \mathcal{C}_n \to X$ and $p \in (1, \infty)$,

$$\Big\| \sum_{i=1}^n \Delta^{-1} \partial_i F_i \Big\|_{L_p(\mathcal{C}_n; X)} \leqslant A_p(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i F_i \Big\|^p_{L_p(\mathcal{C}_n; X)} \Big)^{1/p}, \tag{28}$$

up to the value of the constant $A_p(X)$. In particular, applying (28) to $F_i = \partial_i f_i$, one recovers Corollary 2, so inequality (28) of [HN13] is formally stronger than (15) in the class of spaces whose dual is UMD$^+$.

3. Concluding remarks

In this section we will compare our result with existing theorems in the literature. Recall that a Banach X space is K-convex if X does not contain the family $\{\ell_1^n\}_{n=1}^\infty$ with uniformly bounded distortion. We will need the following lemma.

Lemma 4. *If a space $(X, \|\cdot\|_X)$ satisfies $\mathfrak{s}_p(X) < \infty$ for some $p \in (1, \infty)$, then X is K-convex.*

Proof. It is well known since Stein's work [Ste70] that inequality (8) does not hold for $p \in \{1, \infty\}$ even for scalar valued functions. In fact, an inspection of the argument in [Ste70, p. 105] shows that for every $n \in \mathbb{N}$ there exists n functions $g_1, \ldots, g_n : \mathcal{C}_n \to \{0, 1\}$ such that for every $q \in (2, \infty)$,

$$\Big\| \Big(\sum_{i=1}^n \big(\mathcal{E}_i g_i\big)^2 \Big)^{1/2} \Big\|_{L_q(\mathcal{C}_n; \mathbb{R})} \gtrsim \Big(\int_0^n y^{q/2} e^{-y}\, \mathrm{d}y \Big)^{1/q} \Big\| \Big(\sum_{i=1}^n g_i^2 \Big)^{1/2} \Big\|_{L_q(\mathcal{C}_n; \mathbb{R})}, \tag{29}$$

where $\{\mathcal{E}_i\}_{i=0}^n$ are the conditional expectations (10). Using the fact that $L_\infty(\mathcal{C}_n; \mathbb{R})$ is 2-isomorphic to $L_n(\mathcal{C}_n; \mathbb{R})$, we thus deduce that

$$\begin{aligned} \Big\| \Big(\sum_{i=1}^n \big(\mathcal{E}_i g_i\big)^2 \Big)^{1/2} \Big\|_{L_\infty(\mathcal{C}_n; \mathbb{R})} &\gtrsim \Big(\int_0^n y^{n/2} e^{-y}\, \mathrm{d}y \Big)^{1/n} \Big\| \Big(\sum_{i=1}^n g_i^2 \Big)^{1/2} \Big\|_{L_\infty(\mathcal{C}_n; \mathbb{R})} \\ &\asymp \sqrt{n} \Big\| \Big(\sum_{i=1}^n g_i^2 \Big)^{1/2} \Big\|_{L_\infty(\mathcal{C}_n; \mathbb{R})} \end{aligned}. \tag{30}$$

Therefore, by duality in $L_\infty(\mathcal{C}_n; \ell_2^n)$ and Khintchine's inequality [Khi23], we deduce that there exists n functions $h_1, \ldots, h_n : \mathcal{C}_n \to \mathbb{R}$ such that

$$\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \mathcal{E}_i h_i \Big\|_{L_1(\mathcal{C}_n; \mathbb{R})} \gtrsim \frac{\sqrt{n}}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i h_i \Big\|_{L_1(\mathcal{C}_n; \mathbb{R})}. \tag{31}$$

Suppose that a Banach space X with $\mathfrak{s}_p(X)<\infty$ is not K-convex, so that there exists a constant $K\in[1,\infty)$ such that for every $n\in\mathbb{N}$, there exists a linear operator $\mathsf{J}_n:L_1(\mathcal{C}_n;\mathbb{R})\to X$ satisfying

$$\forall\, h\in L_1(\mathcal{C}_n;\mathbb{R}),\qquad \|h\|_{L_1(\mathcal{C}_n;\mathbb{R})}\leqslant \|\mathsf{J}_n h\|_X\leqslant K\|h\|_{L_1(\mathcal{C}_n;\mathbb{R})}. \tag{32}$$

Consider the functions $H_1,\ldots,H_n:\mathcal{C}_n\to L_1(\mathcal{C}_n;\mathbb{R})$ given by

$$\forall\, \varepsilon,\varepsilon'\in\mathcal{C}_n,\quad \big[H_i(\varepsilon)\big](\varepsilon')=h_i(\varepsilon_1\varepsilon_1',\ldots,\varepsilon_n\varepsilon_n'), \tag{33}$$

where $h_i\in L_1(\mathcal{C}_n;\mathbb{R})$ are the functions satisfying (31). Then, for every $i\in\{1,\ldots,n\}$, we have $[\mathcal{E}_iH_i(\varepsilon)](\varepsilon')=\mathcal{E}_ih_i(\varepsilon_1\varepsilon_1',\ldots,\varepsilon_n\varepsilon_n')$ and, by translation invariance, for every $\varepsilon,\delta\in\mathcal{C}_n$ we have

$$\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_iH_i(\varepsilon)\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})}=\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_ih_i\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})}\ \text{ and }\ \Big\|\sum_{i=1}^n\delta_iH_i(\varepsilon)\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})}=\Big\|\sum_{i=1}^n\delta_ih_i\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})}$$

Therefore, considering the mappings $f_1,\ldots,f_n:\mathcal{C}_n\to X$ given by $f_i=\mathsf{J}_n\circ H_i$, we see that

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\gtrsim K^{-1}\sqrt{n}\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}, \tag{34}$$

thus showing that $\mathfrak{s}_p(X)\gtrsim K^{-1}\sqrt{n}$, which is a contradiction. □

Recall that the X-valued Rademacher projection is defined to be

$$\mathsf{Rad}\Big(\sum_{A\subseteq\{1,\ldots,n\}}\widehat{f}(A)w_A\Big)\stackrel{\mathrm{def}}{=}\sum_{i=1}^n\widehat{f}(\{i\})w_{\{i\}}. \tag{35}$$

A deep theorem of Pisier [Pis82] asserts that a Banach space is K-convex if and only if

$$\forall\, r\in(1,\infty),\qquad \mathsf{K}_r(X)\stackrel{\mathrm{def}}{=}\sup_{n\in\mathbb{N}}\big\|\mathsf{Rad}\big\|_{L_r(\mathcal{C}_n;X)\to L_r(\mathcal{C}_n;X)}<\infty. \tag{36}$$

In particular, it follows from Lemma 4 that $\mathfrak{s}_p(X)<\infty$ for some $p\in(1,\infty)$ implies that $\mathsf{K}_r(X)<\infty$ for every $r\in(1,\infty)$. We proceed by showing that Banach spaces belonging to the class considered in [HN13, Theorem 1.4] satisfy the assumptions of Theorem 1.

Proposition 5. *Let $(X,\|\cdot\|_X)$ be a Banach space. If X^* is a UMD$^+$ space, then X is a UMD$^-$ space and $\mathfrak{s}_p(X)<\infty$ for every $p\in(1,\infty)$.*

Proof. The fact that if X^* is UMD$^+$, then X is UMD$^-$ has been proven by Garling in [Gar90, Theorem 1], so we only have to prove that $\mathfrak{s}_p(X)<\infty$. Let $f_1,\ldots,f_n:\mathcal{C}_n\to X$ and $G^*:\mathcal{C}_n\times\mathcal{C}_n\to X^*$ be such that

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}=\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\Big\langle G^*(\varepsilon,\delta),\sum_{i=1}^n\delta_i\mathcal{E}_if_i(\varepsilon)\Big\rangle \tag{37}$$

and $\|G^*\|_{L_q(\mathcal{C}_n\times\mathcal{C}_n;X^*)}=1$, where $\frac1p+\frac1q=1$. Let $G_i^*:\mathcal{C}_n\to X^*$ be given by

$$\forall\,\varepsilon\in\mathcal{C}_n,\qquad G_i^*(\varepsilon)=\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\delta_iG^*(\varepsilon,\delta). \tag{38}$$

Then, since X^* is UMD$^+$, we deduce that X^* is also K-convex (this is proven in [Gar90] but it also follows by combining Bourgain's inequality (9) with Lemma 4) and thus

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_iG_i^*\Big\|_{L_q(\mathcal{C}_n;X^*)}^q\Big)^{1/q}\stackrel{(38)}{=}\Big(\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\big\|\mathsf{Rad}_\delta G^*(\varepsilon,\delta)\big\|_X^q\Big)^{1/q}\leqslant\mathsf{K}_q(X^*). \tag{39}$$

Hence, we have

$$\begin{aligned}&\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\stackrel{(37)\wedge(38)}{=}\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\Big\langle\sum_{i=1}^n\delta_iG_i^*(\varepsilon),\sum_{i=1}^n\delta_i\mathcal{E}_if_i(\varepsilon)\Big\rangle\\&=\frac{1}{2^n}\sum_{\varepsilon\in\mathcal{C}_n}\langle G_i^*(\varepsilon),\mathcal{E}_if_i(\varepsilon)\rangle=\frac{1}{2^n}\sum_{\varepsilon\in\mathcal{C}_n}\langle\mathcal{E}_iG_i^*(\varepsilon),f_i(\varepsilon)\rangle=\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\Big\langle\sum_{i=1}^n\delta_i\mathcal{E}_iG_i^*(\varepsilon),\sum_{i=1}^n\delta_if_i(\varepsilon)\Big\rangle\\&\leqslant\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_iG_i^*\Big\|_{L_q(\mathcal{C}_n;X^*)}^q\Big)^{1/q}\cdot\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}.\end{aligned}\tag{40}$$

Therefore, combining (40) with (8) and (39), we deduce that

$$\begin{aligned}&\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\\&\qquad\stackrel{(8)}{\leqslant}\mathfrak{s}_q(X^*)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_iG_i^*\Big\|_{L_q(\mathcal{C}_n;X^*)}^q\Big)^{1/q}\cdot\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\\&\qquad\stackrel{(39)}{\leqslant}\mathfrak{s}_q(X^*)\mathsf{K}_q(X^*)\cdot\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},\end{aligned}\tag{41}$$

which shows that $\mathfrak{s}_p(X)\leqslant\mathsf{K}_q(X^*)\mathfrak{s}_q(X^*)$. □

We conclude by observing that spaces satisfying the assumptions of Theorem 1 are necessarily superreflexive (see [Pis16] for the relevant terminology).

Lemma 6. *If a UMD$^-$ Banach space $(X,\|\cdot\|_X)$ satisfies $\mathfrak{s}_p(X)<\infty$, then X is superreflexive.*

Proof. A theorem of Pisier [Pis73] asserts that a Banach space X is K-convex if and only if X has nontrivial Rademacher type. Therefore, we deduce from Lemma 4 that if $\mathfrak{s}_p(X)<\infty$ for some $p\in(1,\infty)$, then there exists $s\in(1,2]$ and $T_s(X)\in(0,\infty)$ such that

$$\forall\, x_1,\ldots,x_n\in X,\qquad\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_ix_i\Big\|_X^s\Big)^{1/s}\leqslant T_s(X)\Big(\sum_{i=1}^n\|x_i\|_X^s\Big)^{1/s}.\tag{42}$$

Therefore, if X also satisfies the UMD$^-$ property, we deduce that for every X-valued martingale $\{\mathcal{M}_i:\Omega\to X\}_{i=0}^n$,

$$\begin{aligned}\|\mathcal{M}_n-\mathcal{M}_0\|_{L_s(\Omega,\mu;X)}&\leqslant\beta_s^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|_{L_s(\Omega,\mu;X)}^s\Big)^{1/s}\\&\stackrel{(42)}{\leqslant}\beta_s^-(X)T_s(X)\Big(\sum_{i=1}^n\|\mathcal{M}_i-\mathcal{M}_{i-1}\|_{L_s(\Omega,\mu;X)}^s\Big)^{1/s},\end{aligned}\tag{43}$$

which means that X has martingale type s. Combining this with well known results linking martingale type and superreflexivity (see [Pis16]), we reach the desired conclusion. □

Therefore, Theorem 1 establishes that $\mathfrak{P}_p^n(X)=\Theta(1)$ for X in a (strict, see [Gar90, Qiu12]) subclass of all superreflexive spaces. In the forthcoming manuscript [EN20], the bound $\mathfrak{P}_p^n(X)=o(\log n)$ is shown to hold for every superreflexive Banach space X and $p\in(1,\infty)$.

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