CORRIGENDUM: AN EXACT SEQUENCE FOR CONTACT- AND SYMPLECTIC HOMOLOGY

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The goal of this note is to make two corrections to the authors' paper "An exact sequence for contact- and symplectic homology", *Invent. Math.* 175 (2009), no. 3, 611–680, hereafter referred to as [3]. The first correction concerns the transversality assumptions for the definition of linearized contact homology. The second correction concerns the description of the connecting map D in terms of the contact complex. These corrections are described respectively in Sections 1 and 2 below.

Referencing convention. The equations in [3] are numbered by $1, 2, 3, \dots$. The equations in the current note are numbered by $x.1, x.2, x.3, \dots$ where x is the number of the section. Thus no confusion should arise.

1. Transversality assumptions

For the reader's convenience we recall the notation from [3]: (M, ξ) is a contact manifold with symplectic filling (W, ω), such that ω admits a primitive near the boundary which restricts to a positive contact form λ on the boundary; the symplectic completion of (W, ω) is denoted $(\widehat{W}, \widehat{\omega})$; J is a compatible almost complex structure on $\widehat{W}$ which is cylindrical on the symplectization part of $\widehat{W}$; J_∞ is the corresponding compatible almost complex structure on the symplectization $\mathbb{R} \times M$; a denotes a free homotopy class of loops in W, $i : M \hookrightarrow W$ is the inclusion, and $i^{-1}(a)$ denotes the set of free homotopy classes of loops in M which are mapped to a via the inclusion; $\mathcal{P}_\lambda^{i^{-1}(a)}$ is the set of periodic Reeb orbits of λ in $i^{-1}(a)$.

In [3, Remark 9, p. 633] we formulated conditions (A) and (B_a) under which linearized contact homology $HC_*^{i^{-1}(a)}(\lambda, J)$ was supposed to be well-defined. It was brought to our attention by M. Abouzaid, J. Latschev and J. Nelson that condition (B_a) is not strong enough as shown by the following example. Consider the ellipsoid $E := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2/a_1^2 + |z_2|^2/a_2^2 = 1\}$ with $0 < a_1 < 2a_2$ and a_1, a_2 rationally independent. The contact form is induced by $\frac{1}{2}(\sum x_i dy_i - y_i dx_i)$ on $\mathbb{R}^4 \equiv \mathbb{C}^2$, and the closed Reeb orbits are iterates γ_k^i, $i = 1, 2$, $k \geq 1$ of the simple orbits $\gamma^1 := E \cap \{z_2 = 0\}$, $\gamma^2 := E \cap \{z_1 = 0\}$. Since a_1, a_2 are rationally independent these orbits are nondegenerate, and since $a_1 < a_2$ the Conley-Zehnder indices of γ_1^1, γ_2^1 are $\mu(\gamma_1^1) = 3$, $\mu(\gamma_2^1) = 5$ and their grading is $|\gamma_1^1| = 2$, $|\gamma_2^1| = 4$. Let J be some generic almost complex structure as above. The moduli space $\mathcal{M}(\gamma_1^1, \emptyset; J)$ (cf. [3, p. 630]) is nonempty and has dimension 2 (the asymptote γ_1^1 is simple and transversality can be achieved for generic J). On the other hand, the moduli space $\mathcal{M}(\gamma_2^1, \gamma_1^1, \gamma_1^1; J_\infty)$ (cf. [3, p. 629]) has virtual dimension 0 and is always nonempty, since it contains the double branched covers of the trivial cylinder over

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γ_1^1, which form a 2-dimensional family. Thus transversality can never be achieved for this moduli space. The point now is that pairs $[u,v] \in \mathcal{M}_c(\gamma_2^1, \gamma_1^1; J)$ with $u \in \mathcal{M}(\gamma_2^1, \gamma_1^1, \gamma_1^1; J_\infty)$ and $v \in \mathcal{M}(\gamma_1^1, \emptyset; J)$ fall outside the scope of assumption (B_a) in [3, Remark 9, p. 633] since dim $\mathcal{M}(\gamma_1^1, \emptyset; J) > 0$. On the other hand, nothing prevents *a priori* such a pair to appear on the boundary of $\mathcal{M}(\gamma_2^1, \gamma_1^1; J_\infty)$, which would destroy the relation $\partial^2 = 0$ for the differential in linearized contact homology.

Condition (A) is not sufficient either as shown by the following idealized example: an index 2 cylinder in the symplectization could break into an index 1 pair of pants and an index 1 plane in the symplectization, which can be viewed in $\widehat{W}$. In order to analyze the contribution of this configuration one needs to ensure regularity of index 1 planes in $\widehat{W}$.

Correction for conditions (A) *and* (B_a) *of* [3, Remark 9, p. 633]

To define linearized contact homology $HC_*^{i^{-1}(a)}(\lambda, J)$ we must strengthen assumptions (A) and (B_a) in [3] to assumptions $(\widetilde{A})$ and $(\widetilde{B}_a)$ below. Thus, we assume the existence of an almost complex structure J that satisfies the following two regularity conditions:

- $(\widetilde{A})$ J is regular for holomorphic planes in $\widehat{W}$ which belong to moduli spaces $\mathcal{M}(\gamma', \emptyset; J)$ of virtual dimension ≤ 1;
- $(\widetilde{B}_a)$ J_∞ is regular for punctured holomorphic cylinders asymptotic at $\pm\infty$ to closed Reeb orbits in $i^{-1}(a)$, belonging to moduli spaces of virtual dimension ≤ 2, and asymptotic at the punctures to elements $\gamma' \in \mathcal{P}^{i^{-1}(0)}(\lambda)$ such that there exists a J-holomorphic building of type $0|1|k_+$, $k_+ \geq 0$ in the sense of [2, §8.1] with exactly one positive puncture and asymptote γ'. (By definition, a building of type $0|1|k_+$ has 1 level in $\widehat{W}$ and k_+ levels in $\mathbb{R} \times M$.)

The main point of assumption $(\widetilde{B}_a)$ is that we allow the virtual dimension of the J-holomorphic building with asymptote γ' to be arbitrary, whereas in (B_a) of [3] we had considered only buildings of virtual dimension 0. Condition $(\widetilde{A})$ strengthens condition (A) of [3] in that we also assume regularity for holomorphic planes that belong to moduli spaces of virtual dimension 1.

We wish to stress that assumptions $(\widetilde{A})$ and $(\widetilde{B}_a)$ depend on λ and J. In no way do they suffice to prove invariance of $HC_*^{i^{-1}(a)}(\lambda, J)$ with respect to deformations of λ or J. As explicitly stated in [3, Remark 7, p. 633] the invariance with respect to λ and J needs the polyfold formalism currently being developed by Hofer, Wysocki and Zehnder [5]. Alternatively, invariance follows from the isomorphism between $HC_*^{i^{-1}(a)}(\lambda, J)$ and positive S^1-equivariant symplectic homology $SH_*^{S^1,+}(W, \omega)$ with $\mathbb{Q}$-coefficients [4]. The proofs of [3] are written for a specific choice of λ and J that obey the assumptions above.

We give below the proof that $HC_*^{i^{-1}(a)}(\lambda, J)$ is defined under assumptions $(\widetilde{A})$ and $(\widetilde{B}_a)$. Along the way, we need to correct the equations (36) and (77) in [3].

Correction for equation (36)

We denote $\mathcal{M}^B(\gamma', \emptyset; J_\infty)$ the moduli space of J_∞-holomorphic planes in the symplectization $\mathbb{R} \times M$. Its virtual dimension is $\bar{\mu}(\gamma') + 2\langle c_1(\xi), B\rangle$. We define

$$\partial_0 : C_*^{i^{-1}(a)}(\lambda) \to \Lambda_\omega$$

by

$$\partial_0(\gamma') := \sum_{\substack{B \in H_2(M;\mathbb{Z}) \\ |\gamma'| = |e^B| + 1}} \Big(\sum_{[F] \in \mathcal{M}^B(\gamma', \emptyset; J_\infty)/\mathbb{R}} \epsilon(F) \Big) e^B.$$

The correct form of equation (36) is

$$\partial \circ \partial = 0, \qquad \partial_0 + e \circ \partial = 0.$$

Proof of the fact that assumptions $(\widetilde{A})$ *and* $(\widetilde{B}_a)$ *imply equation (36), namely*

$$\partial \circ \partial = 0, \qquad \partial_0 + e \circ \partial = 0.$$

We first prove the identity $\partial_0 + e \circ \partial = 0$. To prove that $\partial_0(\gamma_1') + e \circ \partial(\gamma_1') = 0$ for $\gamma_1' \in \mathcal{P}_\lambda^{i^{-1}(a)}$ we examine the boundary of the 1-dimensional moduli spaces $\mathcal{M}^{A_1}(\gamma_1', \emptyset; J)$ for $A_1 \in H_2(W;\mathbb{Z})$ such that $|\gamma_1'| - |e^{A_1}| = 1$. By assumption $(\widetilde{A})$ the almost complex structure J is regular for this moduli space, so that its boundary points come in pairs with opposite signs (this is where we use the stronger assumption $(\widetilde{A})$ instead of (A)). We claim that the boundary is given by

$$\mathcal{M}^{A_1}(\gamma_1', \emptyset; J_\infty)/\mathbb{R} \ \cup \bigcup_{\gamma_2' \in \mathcal{P}_\lambda} \Big[\mathcal{M}_c^{A_2}(\gamma_1', \gamma_2'; J)/\mathbb{R} \times \mathcal{M}^{A_1 - A_2}(\gamma_2', \emptyset; J) \Big]. \tag{1.1}$$

This implies the desired identity since the two terms in the above union correspond to $\partial_0(\gamma_1')$ and $e \circ \partial(\gamma_1')$ respectively (note that the sets under the second union sign in (1.1) are not necessarily disjoint). To prove that the boundary has this form we appeal to the SFT compactness theorem [2], by which the boundary elements of the above moduli space correspond to holomorphic buildings of type $0|1|k_+$ with $k_+ \geq 1$. It is enough to prove that $k_+ = 1$, in which case the two terms in the union correspond to the level in $\widehat{W}$ being empty, respectively non-empty. We analyze first the case in which the level in $\widehat{W}$ is non-empty. Assumption $(\widetilde{B}_a)$ implies that all levels in the symplectization have index ≥ 1. Assumption $(\widetilde{A})$ implies that the components of the level in $\widehat{W}$ have index ≥ 0 (this would be still true assuming only (A)). Since the total index of the building is 1 we infer that there can be a single level in the symplectization, i.e. $k_+ = 1$, and moreover that the holomorphic planes in $\widehat{W}$ by which it is capped have index 0. This configuration corresponds to the second term in the above union. We now analyze the case in which the level in $\widehat{W}$ is empty. The same argument as before shows that $k_+ = 1$, so that the level contains a single connected component. This corresponds to the term $\mathcal{M}^{A_1}(\gamma_1', \emptyset; J_\infty)/\mathbb{R}$ in the above union.

We now prove the identity $\partial \circ \partial = 0$. To prove that $\partial \circ \partial(\overline{\gamma}') = 0$ for $\overline{\gamma}' \in \mathcal{P}_\lambda^{i^{-1}(a)}$ we examine the boundary of the 1-dimensional moduli spaces $\mathcal{M}_c^A(\overline{\gamma}', \underline{\gamma}'; J)/\mathbb{R}$ for $A \in H_2(W;\mathbb{Z})$ such that $|\overline{\gamma}'| - |\underline{\gamma}'| - |e^A| = 2$. Assumptions $(\widetilde{A})$ and $(\widetilde{B}_a)$ ensure that the almost complex structure J is regular for these moduli spaces, so that their boundary points come in pairs with opposite signs. We claim that the boundary is given by

$$\bigcup_{\gamma_1'\in\mathcal{P}_\lambda}\Big[\mathcal{M}_c^{A-A_1}(\overline{\gamma}',\gamma_1';J)/\mathbb{R}\times\mathcal{M}_c^{A_1}(\gamma_1',\underline{\gamma}';J)/\mathbb{R}\Big] \tag{1.2}$$
$$\cup\bigcup_{\gamma_1'\in\mathcal{P}_\lambda}\mathcal{M}_{c,1}^{A-A_1}(\overline{\gamma}',\underline{\gamma}',\gamma_1';J)/\mathbb{R}\ \times$$
$$\left(\mathcal{M}^{A_1}(\gamma_1',\emptyset;J_\infty)/\mathbb{R}\ \cup\bigcup_{\gamma_2'\in\mathcal{P}_\lambda}\Big[\mathcal{M}_c^{A_2}(\gamma_1',\gamma_2';J)/\mathbb{R}\times\mathcal{M}^{A_1-A_2}(\gamma_2',\emptyset;J)\Big]\right).$$

Here $\mathcal{M}_{c,1}^{A-A_1}(\overline{\gamma}',\underline{\gamma}',\gamma_1';J)$ denotes the moduli space of punctured cylinders in the symplectization with asymptotes $\overline{\gamma}',\underline{\gamma}'$ capped at all but one of the punctures with rigid holomorphic planes in $\widehat{W}$, asymptotic at the special puncture to γ_1', and representing the class $A-A_1$ (compare with a similar moduli space in equation (77)). The claim implies the desired identity as follows. The signed count of the elements of the set described by the first line is the coefficient of $e^A\underline{\gamma}'$ in $\partial\circ\partial(\overline{\gamma}')$. The signed count of the elements of the set described on the second and third line is $\sum_{\gamma_1',A_1}\#\mathcal{M}_{c,1}^{A-A_1}(\overline{\gamma}',\underline{\gamma}',\gamma_1';J)/\mathbb{R}\times n_{\gamma_1',A_1}$, where n_{γ_1',A_1} describes the signed count of elements of the set described on the third line. Comparing with formula (1.1) we see that n_{γ_1',A_1} is the coefficient of e^{A_1} in $\partial_0+e\circ\partial(\gamma_1')$, hence 0.

To prove the claim we again appeal to the SFT compactness theorem [2], by which the boundary of the 1-dimensional moduli space $\mathcal{M}_c^A(\overline{\gamma}',\underline{\gamma}';J)/\mathbb{R}$ containing a 1-parameter family of punctured cylinders of index 2 in $\mathbb{R}\times M$ capped by rigid holomorphic planes in $\widehat{W}$ consists of holomorphic buildings of type $0|1|k_+$ with $k_+\geq 2$. We prove that we have $k_+=2$. Indeed, assumption $(\widetilde{B}_a)$ implies that the index of all levels in the symplectization is ≥ 1, and assumption $(\widetilde{A})$ implies that the index of all levels in $\widehat{W}$ is ≥ 0. Since the total index is 2 we obtain that $k_+=2$. The first line of (1.2) describes the situation in which $\overline{\gamma}'$, $\underline{\gamma}'$ are asymptotes of punctured cylinders which lie on different levels in the symplectization. The second and third lines of (1.2) describe the situation in which $\overline{\gamma}'$, $\underline{\gamma}'$ are the two asymptotes at $\pm\infty$ of a punctured cylinder of index 1 in the symplectization. This punctured cylinder is capped by rigid holomorphic planes in $\widehat{W}$ at all punctures but one, where it is capped by an index 1 holomorphic building of type $0|1|1$. The latter is described by the third line in (1.2). □

Correction for equation (77)

We must add in equation (77) the term

$$\cup\bigcup_{\gamma_1'\in\mathcal{P}_\lambda^{\leq\alpha}}\Big[\mathcal{M}_{c,1}^{A_1}(p',q,\gamma_1';H_\infty^\rho,\{f_\gamma,f_\gamma'\},J)\times\ \mathcal{M}^{A-A_1}(\gamma_1',\emptyset;J_\infty)/\mathbb{R}\Big].$$

Accordingly, the first displayed equation on p. 661 must be corrected to

$$\mathcal{M}^{A-A_1}(\gamma_1',\emptyset;J_\infty)/\mathbb{R}\ \cup\bigcup_{\gamma_2'\in\mathcal{P}_\lambda^{\leq\alpha}}\Big[\mathcal{M}_c^{A_2}(\gamma_1',\gamma_2';J)/\mathbb{R}\times\mathcal{M}^{A-A_1-A_2}(\gamma_2',\emptyset;J)\Big]$$

and the equation $e\circ\partial(\gamma_1')=0$ on the following line has to be corrected to $\partial_0(\gamma_1')+e\circ\partial(\gamma_1')=0$.

Correction for the discussion of the examples in [3, Remark 9, p. 634]

In Examples (i) and (ii) the stronger assumption $(\widetilde{B}_0)$ is violated. Nevertheless, linearized contact homology is defined for *ad hoc*, geometric reasons, and the long exact sequence of Theorem 1 still holds. In Example (iii) we only need to strengthen the condition dim $L \geq 4$ to dim $L \geq 5$, so that $(\widetilde{A})$ and $(\widetilde{B}_a)$ hold.

Example (i). As in [3, p. 633] one sees that condition $(\widetilde{A})$ is satisfied. However, assumption $(\widetilde{B}_0)$ is never satisfied. To see this, one can refer to the example of the ellipsoid $E = \{(z_1, z_2) \in \mathbb{C}^2 \, : \, |z_1|^2/a_1^2 + |z_2|^2/a_2^2 = 1\}$ that we presented at the beginning of Section 1. In that example we exhibited an index 0 pair of pants in the symplectization, capped at one of the punctures by a plane of index 2. Similar examples can be constructed in any dimension. Subcritical Stein manifolds are obtained by attaching subcritical Weinstein handles; these always contain such contact ellipsoids which violate condition $(\widetilde{B}_0)$.

Although assumption $(\widetilde{B}_0)$ is violated, the linearized contact homology group $HC_*^{i^{-1}(0),\leq\alpha}(\lambda_\alpha, J_\alpha)$ is well-defined for geometric reasons [10] for special choices of λ_α and J_α. The contact form corresponds to a choice of sufficiently thin handles. The differential for linearized contact homology counts index 1 punctured cylinders in the symplectization, capped at the punctures with index 0 planes. Since the latter do not exist, we are left with cylinders. Mei-Lin Yau proved the following: (i) there exists a regular almost complex structure J_α for these cylinders; (ii) each such cylinder corresponds to a gradient trajectory. The contact differential consists therefore of a certain number of copies of the Morse differential (depending on the maximal multiplicity allowed by the bound on the action), so that $\partial \circ \partial = 0$.

Proposition 9 shows that the long exact sequence of Theorem 1 holds for dimensional reasons.

Example (ii). Given an upper bound α on the action, we choose a Morse-Bott perturbation $(\widehat{\omega}_\epsilon, J_\epsilon)$ with $\epsilon > 0$ small enough and depending on α. In this setup, we proved in [3] that (A) and (B_0) were satisfied for curves having asymptotes of action $\leq \alpha$. While assumption $(\widetilde{A})$ is still satisfied, assumption $(\widetilde{B}_0)$ is *not* satisfied anymore. Indeed, the curves involved in condition $(\widetilde{A})$ are holomorphic planes with simple asymptote in the fibers over minima of f (these have index 0) and holomorphic planes with simple asymptote in the fibers over the unstable manifolds of critical points of f having index 1 (these have index 1). Since these holomorphic planes are regular before perturbation, they remain regular for J_ϵ; thus $(\widetilde{A})$ is satisfied. To see that $(\widetilde{B}_0)$ is not satisfied, let us consider a pair of pants which doubly covers a trivial cylinder over a simple orbit in the fiber, capped at one puncture by a rigid plane in the same fiber. If the fiber lies over a critical point of f of index k, the index of such a pair of pants is $2 - k$, which is ≤ 0 as soon as $k \geq 2$. Such a curve always exists and cannot therefore be regular.

However, although assumption $(\widetilde{B}_0)$ is violated, the linearized contact homology group $HC_*^{i^{-1}(0),\leq\alpha}(\lambda_\epsilon, J_\epsilon)$ is well-defined for geometric reasons. Indeed, the linearized contact differential counts index 1 generalized holomorphic curves in the sense of [1] (or Morse-Bott broken holomorphic curves in the terminology of [3]). Each holomorphic curve in such a configuration is a punctured cylinder, capped at the punctures with index 0 planes lying in the fibers. Such index 0 planes arise only when the respective fiber lies over a minimum of f and the asymptote is simple. Denoting by $m-1 \geq 0$ the number of punctures, the index of such a curve is $2(m-1) = (\dim B + 2(m-1)) - \dim B$, which is ≥ 2 if $m-1 \geq 1$. For the total index of the configuration to be 1, the holomorphic curve part can thus only be a trivial cylinder with no punctures. Thus the linearized contact differential coincides with the Morse differential for f and its square is zero.

As shown by Proposition 10 and Proposition 11, this example is an instance where the long exact sequence of Theorem 1 holds for geometric reasons.

Example (iii). The condition $\dim L \geq 4$ must be strengthened to $\dim L \geq 5$, so that $(\widetilde{A})$ becomes vacuous. The rest of the discussion holds verbatim.

Remark 1.1. J. Nelson has recently exhibited a large class of 3-dimensional contact manifolds for which transversality can be achieved for cylindrical contact homology [8].

2. The map D

It was pointed out to us by T. Ekholm that Proposition 8 is true as stated only in the absence of bad orbits. In case there are bad orbits, we omitted a term in the definition of the chain map which induces the map D. We describe now this additional term and complete the proof of Proposition 8.

Denote by $\mathcal{P}_\lambda^{bad}$ the set of unparametrized closed Reeb orbits which are bad (see p. 627 for the definition). Given $\overline{\gamma}', \underline{\gamma}' \in \mathcal{P}_\lambda$, $A \in H_2(W;\mathbb{Z})$ we denote by

$$\mathcal{M}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J) := \bigcup_{\gamma' \in \mathcal{P}_\lambda^{bad},\, A_1 \in H_2(W;\mathbb{Z})} \left[\mathcal{M}_c^{A_1}(\overline{\gamma}', \gamma'; J)\right] \times \left[\mathcal{M}_c^{A-A_1}(\gamma', \underline{\gamma}'; J)\right]$$

the set of pairs of equivalence classes $([u'], [u''])$ for the equivalence relation given by ignoring the asymptotic markers $\underline{L}', \overline{L}''$ corresponding to the common asymptote γ'. The decorations "2" and "*bad*" for the moduli space are motivated by the fact that it consists of curves with two sublevels such that the intermediate asymptote is a bad orbit. In the situation $\mu(\overline{\gamma}') - \mu(\underline{\gamma}') + 2\langle c_1(TW), A\rangle = 2$ and for a generic choice of the points $P_{\gamma'}$ the moduli spaces $\mathcal{M}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J)$ are rigid and one can associate a sign $\epsilon(u)$ to each of their elements via coherent orientations and fibered products. Given an element $u \in \left[\mathcal{M}_c^{A_1}(\overline{\gamma}', \gamma'; J)\right] \times \left[\mathcal{M}_c^{A-A_1}(\gamma', \underline{\gamma}'; J)\right] \subset \mathcal{M}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J)$ we define a sign

$$\bar{\epsilon}(u) := \epsilon(u)\epsilon(\gamma'),$$

where $\epsilon(\gamma') \in \{\pm 1\}$ is uniquely determined by the relation $\delta^0(\gamma'_m) = \epsilon(\gamma')2\gamma'_M$ (compare [3, Proposition 1, p. 641]).

Given the free homotopy class a in W we define a map

$$\Delta^{bad} : C_*^{i^{-1}(a)}(\lambda) \to C_{*-2}^{i^{-1}(a)}(\lambda),$$

$$\Delta^{bad}(\overline{\gamma}') = -\frac{1}{2} \cdot \sum_{\substack{\underline{\gamma}', A \\ |\underline{\gamma}' e^A| = |\overline{\gamma}'| - 2}} \frac{1}{\kappa_{\underline{\gamma}'}} \sum_{u \in \mathcal{M}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J)} \bar{\epsilon}(u) e^A \underline{\gamma}'. \tag{2.1}$$

The correct statement of Proposition 8 is the following.

Proposition 2.1. *The map $\Delta + \Delta^{bad}$ defined by (85) and (2.1) is a chain map, and induces in homology the map D in the long exact sequence of Theorem 1.*

As a preparation, let us denote $CM_*^{good/bad}$, respectively $Cm_*^{good/bad}$ the Λ_ω-submodules of the complex $BC_*^{i^{-1}(a)}(\lambda)$ (see [3, p. 637]) defined by

$$CM_*^{good/bad} := \bigoplus_{\gamma' \ good/bad} \Lambda_\omega\langle\gamma'_M\rangle, \qquad Cm_*^{good/bad} := \bigoplus_{\gamma' \ good/bad} \Lambda_\omega\langle\gamma'_m\rangle.$$

We denote $CM_* := CM_*^{good} \oplus CM_*^{bad}$ and $Cm_* := Cm_*^{good} \oplus Cm_*^{bad}$. The components δ^0, δ^1, and δ^2 of the differential δ behave as follows with respect to the splitting

$$BC_*^{i^{-1}(a)}(\lambda) = (CM_*^{good} \oplus Cm_*^{good}) \oplus (CM_*^{bad} \oplus Cm_*^{bad}).$$

- δ^0 vanishes on $CM_*^{good} \oplus Cm_*^{good} \oplus CM_*^{bad}$ and $\delta^0 : Cm_*^{bad} \to CM_{*-1}^{bad}$ is an isomorphism over $\mathbb{Q}$.
- $\delta^1(CM_*^{bad}) = 0$, $\delta^1(CM_*^{good}) \subset CM_{*-1}$, $\delta^1(Cm_*) \subset Cm_{*-1}^{good}$.
- $\delta^2(Cm_*) = 0$, $\delta^2(CM_*) \subset Cm_{*-1}$ and $\delta^2(CM_*^{bad}) \subset Cm_{*-1}^{good}$.

The assertions concerning δ^0 follow from Proposition 1. That $\delta^1(CM_*^{bad}) = 0$ follows from the fact that there is an even number of choices with alternating signs for the asymptotic marker at the bad orbit with the point constraint. The same argument shows that $\delta^1(Cm_*) \subset Cm_{*-1}^{good}$. The last assertion on δ^2 follows from the following computation:

$$\begin{aligned} \delta^2(CM_*^{bad}) \ = \ \delta^2\delta^0(Cm_{*+1}^{bad}) \ &= \ -\delta^0\delta^2(Cm_{*+1}^{bad}) - (\delta^1)^2(Cm_{*+1}^{bad}) \\ &= \ -(\delta^1)^2(Cm_{*+1}^{bad}) \ \subset \ Cm_{*-1}^{good}. \end{aligned}$$

Proof of Proposition 2.1. We first reinterpret Δ^{bad} by expressing the moduli spaces $\mathcal{M}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J)$ in terms of moduli spaces of capped punctured S^1-parametrized holomorphic cylinders. Given $\overline{\gamma}', \underline{\gamma}' \in \mathcal{P}_\lambda$, $A \in H_2(W;\mathbb{Z})$ we denote

$$\widetilde{\mathcal{M}}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J) = \bigcup_{\gamma' \in \mathcal{P}_\lambda^{bad},\, A_1 \in H_2(W;\mathbb{Z})} \widetilde{\mathcal{M}}_c^{A_1}(P_{\overline{\gamma}'}, S'_\gamma; J) \times \widetilde{\mathcal{M}}_c^{A-A_1}(S'_\gamma, P_{\underline{\gamma}'}; J).$$

(The moduli spaces $\widetilde{\mathcal{M}}_c^{A}(P_{\overline{\gamma}'}, S'_{\underline{\gamma}}; J)$ and $\widetilde{\mathcal{M}}_c^{A}(S'_{\overline{\gamma}}, P_{\underline{\gamma}'}; J)$ are defined on p. 670.) It follows from the definition that there is a bijective correspondence

$$\mathcal{M}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J) \sim \widetilde{\mathcal{M}}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J).$$

Hence $\widetilde{\Delta}^{bad} := \Theta^{-1} \circ \Delta^{bad} \circ \Theta : C_*^{i^{-1}(a)}(\lambda) \otimes H_0(S^1) \to C_{*-2}^{i^{-1}(a)}(\lambda) \otimes H_1(S^1)$, where Θ is defined in (48), acts by

$$\widetilde{\Delta}^{bad}(\overline{\gamma}'_M) = -\frac{1}{2} \cdot \sum_{\substack{\underline{\gamma}', A \\ |\underline{\gamma}' e^A| = |\overline{\gamma}'| - 2}} \ \sum_{u \in \widetilde{\mathcal{M}}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J)} \bar{\epsilon}(u) e^A \underline{\gamma}'_m.$$

Here we use the notation $\bar{\epsilon}(u) := \epsilon(u)\epsilon(\gamma')$, where $\epsilon(u)$ is the coherent orientation sign for $u \in \widetilde{\mathcal{M}}_c^{A_1}(P_{\overline{\gamma}'}, S'_\gamma; J) \times \widetilde{\mathcal{M}}_c^{A-A_1}(S'_\gamma, P_{\underline{\gamma}'}; J) \subset \widetilde{\mathcal{M}}_{2,c}^{A,bad}(P_{\overline{\gamma}'}, P_{\underline{\gamma}'}; J)$.

We denote $C_*^{good} = CM_*^{good} \oplus Cm_*^{good}$, $C_*^{bad} = CM_*^{bad} \oplus Cm_*^{bad}$ and consider the splitting $BC_*^{i^{-1}(a)}(\lambda) = C_*^{good} \oplus C_*^{bad}$. With respect to this splitting, the differential δ has the form

$$\delta = \begin{pmatrix} \tilde\alpha & \tilde\beta \\ \tilde\gamma & \tilde\delta \end{pmatrix}.$$

The map $\tilde\delta : CM_*^{bad} \oplus Cm_*^{bad} \to CM_{*-1}^{bad} \oplus Cm_{*-1}^{bad}$ is given by

$$\tilde\delta = \begin{pmatrix} 0 & \delta^0 \\ 0 & 0 \end{pmatrix}.$$

In particular, the complex $(C_*^{bad}, \tilde\delta)$ is acyclic over $\mathbb{Q}$. In fact, it admits the contracting homotopy $h : C_*^{bad} \to C_{*+1}^{bad}$ given by

$$h = \begin{pmatrix} 0 & 0 \\ (\delta^0)^{-1} & 0 \end{pmatrix}.$$

By [6, Lemma 2.1.6] (Killing Contractible Subcomplexes Lemma), the map

$$(Id, -h\circ\tilde\gamma) : (C_*^{good}, \tilde\alpha - \tilde\beta\circ h\circ\tilde\gamma) \to (BC_*^{i^{-1}(a)}(\lambda), \delta)$$

is a quasi-isomorphism.

Now, the maps $\tilde\alpha : CM_*^{good} \oplus Cm_*^{good} \to CM_{*-1}^{good} \oplus Cm_{*-1}^{good}$, $\tilde\beta : CM_*^{bad} \oplus Cm_*^{bad} \to CM_{*-1}^{good} \oplus Cm_{*-1}^{good}$ and $\tilde\gamma : CM_*^{good} \oplus Cm_*^{good} \to CM_{*-1}^{bad} \oplus Cm_{*-1}^{bad}$ are given by

$$\tilde\alpha = \begin{pmatrix} \pi^{good}\circ\delta^1 & 0 \\ \pi^{good}\circ\delta^2 & \delta^1 \end{pmatrix}, \qquad \tilde\beta = \begin{pmatrix} 0 & 0 \\ \delta^2 & \delta^1 \end{pmatrix}, \qquad \tilde\gamma = \begin{pmatrix} \pi^{bad}\circ\delta^1 & 0 \\ \pi^{bad}\circ\delta^2 & 0 \end{pmatrix},$$

where $\pi^{good} : C_*^{good} \oplus C_*^{bad} \to C_*^{good}$ and $\pi^{bad} : C_*^{good} \oplus C_*^{bad} \to C_*^{bad}$ are the obvious projections. Note that the map $(Id, -h\circ\tilde\gamma)$ preserves the filtration (43) and therefore induces a map of spectral sequences. This map is an isomorphism on the first page, and therefore an isomorphism on the second page as well. Since

$$\tilde\alpha - \tilde\beta\circ h\circ\tilde\gamma = \begin{pmatrix} \pi^{good}\circ\delta^1 & 0 \\ \pi^{good}\circ\delta^2 - \delta^1\circ(\delta^0)^{-1}\circ\pi^{bad}\circ\delta^1 & \delta^1 \end{pmatrix}$$

we infer that the differential D on the second page of the spectral sequence is induced by the chain map $\pi^{good}\circ\delta^2 - \delta^1\circ(\delta^0)^{-1}\circ\pi^{bad}\circ\delta^1$. We have already seen in the proof of Proposition 8 that the term $\pi^{good}\circ\delta^2$ coincides with $\tilde\Delta$. That the term $-\delta^1\circ(\delta^0)^{-1}\circ\pi^{bad}\circ\delta^1$ coincides with $\tilde\Delta^{bad}$ follows directly from the definitions. This finishes the proof. □

Remark 2.2. *For a general filtered complex with differential $\delta = \delta^0 + \delta^1 + \delta^2 + \dots$, the differential $\bar\delta^2$ on the second page of the spectral sequence is* not *induced by a chain map on the first page. In our situation it was possible to exhibit a chain map inducing $\bar\delta^2$ thanks to the fact that δ^0 could be absorbed in an acyclic complex.*

In the general case the definition of $\bar\delta^2$ is the following. An element $x \in E^0_{\delta;*,0}$ defines a class $[x]_1$ in $E^1_{\delta;*,0}$ if $\delta^0(x) = 0$ (this is automatic in our situation). This class is a cycle with respect to $\bar\delta^1$ if and only if $\delta^1(x) + \delta^0(y) = 0$ for some $y \in E^0_{\delta;*-1,1}$. By definition of the spectral sequence determined by a filtration (see *for example* [7, §2.2] *and* [9, Example 7.1]*), the image of the corresponding class* $[x]_2 \in E^2_{\delta;*,0}$ *through* $\bar{\delta}^2$ *is given by*

$$\bar{\delta}^2[x]_2 := [\delta^2(x) + \delta^1(y)]_2.$$

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