Geom Dedicata (2018) 192:57–86
https://doi.org/10.1007/s10711-017-0305-2

ORIGINAL PAPER

Simple root flows for Hitchin representations

Martin Bridgeman[1] · **Richard Canary**[2] ·
François Labourie[3] · **Andres Sambarino**[4]

Received: 14 July 2017 / Accepted: 27 November 2017 / Published online: 9 December 2017

Abstract We study special flows associated to Hitchin representations of a surface group, namely the simple root flows. We introduce and discuss the Liouvlle geodesic current which plays a singular role amongst all the natural invariant currents associated to such a representation. Finally, we discuss a rigidity result and a pressure metric associated to the first simple root flow.

Keywords Hitchin representations · Geodesic flows on surfaces · Geodesic currents · Pressure and thermodynamics

Mathematics Subject Classification (2000) 53C24 · 53C35 · 37D350 · 37D40

1 Introduction

Anosov representations [15,20] from a hyperbolic group to a semi-simple Lie group are characterized by their dynamical nature. In the context of projective Anosov representations,

---

Dedicated to a great mathematician on the occasion of his sixtieth birthday: our friend Bill Goldman

Bridgeman was partially suppported by Grants DMS-1500545 and DMS-1564410 and Canary was partially supported by Grants DMS-1306992 and DMS-1564362, from the National Science Foundation. Labourie and Sambarino were partially supported by the European Research Council under the *European Community*'s seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement No. FP7-246918 and by ANR project DynGéo ANR-16-CE40-0025.

✉ François Labourie
francois.labourie@gmail.com

1 Boston College, Chestnut Hill, MA 02467, USA

2 University of Michigan, Ann Arbor, MI 41809, USA

3 Laboratoire Jean-Alexandre Dieudonné, Université Côte d'Azur, Nice 06000, France

4 Université Pierre et Marie Curie (Paris VI), 4 Place Jussieu, Paris 75005, France

we [7] previously associated a metric Anosov flow to such a representation and showed that the (thermo)dynamical properties of this flow yield in turn new structures on the deformation space of these representations: entropy functions, (pressure) intersections and a pressure metric.

In this paper, we focus on Hitchin representations of a surface group into $\mathsf{PSL}_d(\mathbb{R})$. We associate a wealth of flows to a Hitchin representation, and hence geodesic currents, entropies, pressure forms etc., depending essentially on an element in the Weyl chamber.

Let us be more specific. If E is a real vector space of dimension d and S is a closed surface, a representation $\rho : \pi_1(S) \to \mathsf{PSL}(\mathsf{E})$ is *d-Fuchsian* if it is the composition of a Fuchsian representation into $\mathsf{PSL}(2,\mathbb{R})$ and an irreducible representation of $\mathsf{PSL}(2,\mathbb{R})$ into $\mathsf{PSL}(\mathsf{E})$. A representation $\rho : \pi_1(S) \to \mathsf{PSL}(\mathsf{E})$ is a *Hitchin representation* if it may be continuously deformed to a d-Fuchsian representation. Hitchin [17] showed that the *Hitchin component* $\mathcal{H}_d(S)$ of ($\mathsf{PGL}(E)$-conjugacy classes of) Hitchin representations into $\mathsf{PSL}(\mathsf{E})$ is an analytic manifold diffeomorphic to $\mathbb{R}^{(d^2-1)|\chi(S)|}$. Labourie [20] showed that a Hitchin representation is a discrete, faithful quasi-isometric embedding and that the image of every non-trivial element γ is diagonalizable over $\mathbb{R}$ with eigenvalues of distinct modulus:

$$\lambda_1(\rho(\gamma)) > \lambda_2(\rho(\gamma)) > \cdots > \lambda_d(\rho(\gamma)) > 0.$$

Moreover, there are Hölder-continuous, ρ-equivariant *limit curves* $\xi_\rho : \partial_\infty\pi_1(S) \to \mathbf{P}(\mathsf{E})$ and $\xi^*_\rho : \partial_\infty\pi_1(S) \to \mathbf{P}(\mathsf{E}^*)$ whose images are $\mathrm{C}^{1+\alpha}$-submanifolds. This last feature is very specific to Hitchin representations—see Sect. 3.2 (Theorem 3.2) and Guichard [14] for details.

Let $\mathcal{G}(S) = \partial_\infty\pi_1(S)^2\setminus\Delta$ be the space of distinct points in the Gromov boundary $\partial_\infty\pi_1(S)$ of $\pi_1(S)$. We say that a *flow over* $\mathcal{G}(S)$ is an $\mathbb{R}$-principal bundle L over $\mathcal{G}(S)$ equipped with a properly discontinuous and co-compact action of $\pi_1(S)$ by bundle automorphisms. The $\mathbb{R}$-action on the quotient space $\mathsf{U}_L = \mathsf{L}/\pi_1(S)$ is a flow, which justifies the terminology. Given a geodesic current ω, i.e. a $\pi_1(S)$-invariant locally finite measure on $\mathcal{G}(S)$, we define a pairing

$$\langle\omega \mid \mathsf{L}\rangle := \int_{\mathsf{U}_\mathsf{L}} \omega \otimes \mathrm{d}t$$

where $\mathrm{d}t$ is the element of arc length given by the $\mathbb{R}$ action.

We focus on the *simple root flows* associated to a Hitchin representation ρ (see Sect. 3.3). For each $i \in \{1,\ldots,d-1\}$ there is a flow $\mathsf{L}^{\alpha_i}_\rho$ over $\mathcal{G}(S)$ such that if δ_γ is the geodesic current with Dirac measure one on every (oriented) axis of an element conjugate to γ, then

$$\left\langle\delta_\gamma \mid \mathsf{L}^{\alpha_i}_\rho\right\rangle = L_{\alpha_i}(\rho(\gamma)) := \log\left(\frac{\lambda_i(\rho(\gamma))}{\lambda_{i+1}(\rho(\gamma))}\right).$$

Equivalently, if we let $\mathsf{U}_{\alpha_i}(\rho)$ be the quotient flow with associated element of arc length $\mathrm{d}s^{\alpha_i}_\rho$, then the period of $\mathsf{U}_{\alpha_i}(\rho)$ associated to $\gamma \in \pi_1(S)$ is given by $L_{\alpha_i}(\rho(\gamma))$, the *$L_{\alpha_i}$-length function*.

We also consider the *Hilbert flow* $\mathsf{L}^\mathsf{H}(\rho)$ associated to ρ which is determined, up to Hölder conjugacy, by

$$\left\langle\delta_\gamma \mid \mathsf{L}^\mathsf{H}_\rho\right\rangle = L_H(\rho(\gamma)) := \log\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_d(\rho(\gamma))}\right),$$

for any non-trivial $\gamma \in \pi_1(S)$.

Potrie and Sambarino show that the entropy of simple root flows is constant and characterize Fuchsian representations in terms of the entropy of the Hilbert flow.

Theorem 1.1 (Potrie–Sambarino [32]) *The topological entropy of a simple root flow is* 1 *for all Hitchin representations. Moreover, a Hitchin representation* $\rho \in \mathcal{H}_d(S)$ *is* d*-Fuchsian if and only if the topological entropy of the Hilbert flow is* $\frac{2}{d-1}$.

One of the main constructions of our paper is to single out, amongst all geodesic currents associated to a Hitchin representation, a specific asymmetric current called the *Liouville current* ω_ρ. This Liouville current was introduced in [21] and characterized by the cross ratio b_ρ of ρ as discussed in Sect. 4.1. If (t, x, y, z) are four points in cyclic order in $\partial_\infty \pi_1(S)$, then

$$\omega_\rho\left([t, x] \times [y, z]\right) = \frac{1}{2} \log \left(\frac{\langle u \mid \Phi \rangle \langle v \mid \Psi \rangle}{\langle u \mid \Psi \rangle \langle v \mid \Phi \rangle} \right).$$

where u, v, Φ and Ψ are non zero elements in $\xi_\rho(t)$, $\xi_\rho(x)$, $\xi^*_\rho(y)$ and $\xi^*_\rho(z)$ respectively.

As a consequence of Labourie's work on cross ratios for Hitchin representations [21], this gives an embedding of the space of all Hitchin representations into the space of geodesic currents.

Theorem 1.2 *If* ρ *and* σ *are two Hitchin representations—of possibly different dimensions—with the same Liouville current, then* $\rho = \sigma$.

The Liouville current enjoys the following properties.

Theorem 1.3 *If* ρ *is a Hitchin representation, then*

(1) *The current* ω_ρ *is the unique current—up to scalar multiplication—in the class of the Lebesgue measure for the* C^1 *structure on* $\mathcal{G}(S)$ *associated to the embedding* (ξ, ξ^*).
(2) *The measure* $\omega_\rho \otimes \mathrm{d}s^{\alpha_1}_\rho$ *is—up to scalar multiplication—the unique measure maximizing entropy for the flow* $\mathsf{U}_{\alpha_1}(\rho)$.
(3) *If* μ *is a geodesic current, then*

$$i\left(\mu, \omega_\rho\right) = \left\langle \mu \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle.$$

Our Liouville current is closely related to the symmetric Liouville currents defined by Bonahon [1], when $d = 2$, and Martone–Zhang [26]. In fact, one may view their Liouville currents as symmetrizations of our Liouville current.

We define the *Liouville volume* of a representation, by

$$\mathrm{vol}_{\mathsf{L}}(\rho) = i(\omega_\rho, \omega_\rho),$$

and establish the following volume rigidity result, which is motivated by work of Croke and Dairbekov [12].

Theorem 1.4 *If* $\rho, \eta \in \mathcal{H}_d(S)$, *then*

$$\left(\inf_{\gamma \in \pi_1(S)\setminus\{1\}} \frac{\left\langle \delta_\gamma \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle}{\left\langle \delta_\gamma \mid \mathsf{L}^{\mathsf{H}}_\eta \right\rangle} \right)^2 \leqslant \frac{\mathrm{vol}_{\mathsf{L}}(\rho)}{\mathrm{vol}_{\mathsf{L}}(\eta)} \leqslant \left(\sup_{\gamma \in \pi_1(S)\setminus\{1\}} \frac{\left\langle \delta_\gamma \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle}{\left\langle \delta_\gamma \mid \mathsf{L}^{\mathsf{H}}_\eta \right\rangle} \right)^2$$

and equality holds in either inequality if and only if either $\rho = \eta$ *or* $\rho = \eta^*$ *where* η^* *is the contragredient of* η.

When $d = 3$, we apply work of Tholozan [36, Theorem 3] to obtain a simpler volume rigidity result.

Corollary 1.5 *If $\rho \in \mathcal{H}_3(S)$, then*

$$\mathrm{vol}_{\mathsf{L}}(\rho) \geqslant 4\pi^2|\chi(S)|.$$

Moreover, equality holds if and only if ρ is 3-Fuchsian.

We return to the themes explored in [7], by constructing a new, hopefully more tractable, pressure metric on a Hitchin component. If $\rho, \eta \in \mathcal{H}_d(S)$, we define their *Liouville pressure intersection* to be

$$\mathbf{I}_{\alpha_1}(\rho, \eta) := \frac{1}{\left\langle \omega_\rho \mid \mathsf{L}_\rho^{\alpha_1} \right\rangle} \left\langle \omega_\rho \mid \mathsf{L}_\eta^{\alpha_1} \right\rangle.$$

If $\rho \in \mathcal{H}_d(S)$, we define a function $(\mathbf{I}_{\alpha_1})_\rho : \mathcal{H}_d(S) \to \mathbb{R}$ by letting $(\mathbf{I}_{\alpha_1})_\rho(\eta) = \mathbf{I}_{\alpha_1}(\rho, \eta)$.

Using the thermodynamic formalism developed by Bowen [3], Ruelle [33] and Parry–Pollicott [30] we show that $(\mathbf{I}_{\alpha_1})_\rho$ has a minimum at ρ, and its Hessian $\mathbf{P}_{\alpha_1}$ at ρ is positive semi-definite. We call $\mathbf{P}_{\alpha_1}$ the *Liouville pressure quadratic form*. This construction is motivated by Thurston's version of the Weil–Petersson metric on Teichmüller space (see Wolpert [37]) as re-interpreted by Bonahon [1], McMullen [27] and Bridgeman [5].

We show that $\mathbf{P}_{\alpha_1}$ is non-degenerate, hence gives rise to a Riemannian metric, and apply work of Wolpert [37] to see that it restricts to a multiple of the Weil–Petersson metric on the Fuchsian locus.

Theorem 1.6 *The Liouville pressure quadratic form $\mathbf{P}_{\alpha_1}$ is a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(S)$, that restricts to a scalar multiple of the the Weil–Petersson metric on the Fuchsian locus.*

The main tool in the proof of the non-degeneracy of $\mathbf{P}_{\alpha_1}$ is that the L_{α_1}-length functions of elements of $\pi_1(S)$ generate the cotangent space of the Hitchin component. More precisely, if $\gamma \in \pi_1(S)$, let $L^\gamma_{\alpha_1} : \mathcal{H}_d(S) \to \mathbb{R}$ be given by

$$L^\gamma_{\alpha_1}(\rho) = L_{\alpha_1}(\rho(\gamma)) = \left\langle \delta_\gamma \mid \mathsf{L}_\rho^{\alpha_1} \right\rangle.$$

Theorem 1.7 *If $\rho \in \mathcal{H}_d(S)$, then the set*

$$\left\{ \mathrm{D}_\rho L^\gamma_{\alpha_1} \right\}_{\gamma \in \pi_1(S)}$$

generates, as a vector space, the cotangent space $\mathsf{T}^*_\rho \mathcal{H}_d(S)$.

We can also give an interpretation of $\mathbf{I}_{\alpha_1}$ in terms more reminiscent of the construction in [7]. This interpretation generalizes to give pressure quadratic forms associated to other simple roots. If $T > 0$ and $i \in \{1, \ldots, d-1\}$, let

$$R_{\alpha_i}(\rho, T) = \left\{ [\gamma] \in [\pi_1(S)] \backslash \{[1]\} \mid L_{\alpha_i}(\rho(\gamma)) \leqslant T \right\}.$$

We then define an associated pressure intersection

$$\mathbf{I}_{\alpha_i}(\rho, \eta) = \lim_{T \to \infty} \frac{1}{\# R_{\alpha_i}(\rho, T)} \sum_{\gamma \in R_{\alpha_i}(\rho, T)} \frac{L_{\alpha_i}(\eta(\gamma))}{L_{\alpha_i}(\rho(\gamma))}.$$

The associated function $(\mathbf{I}_{\alpha_i})_\rho$ has a minimum at ρ, and we again obtain, by considering the Hessian of $(\mathbf{I}_{\alpha_1})_\rho$, a positive semi-definite quadratic pressure form $\mathbf{P}_{\alpha_i}$. It is natural to ask when $\mathbf{P}_{\alpha_i}$ is non-degenerate. In a final section, we observe that $\mathbf{P}_{\alpha_n}$ is degenerate on $\mathcal{H}_{2n}(S)$ at any Hitchin representation with image (conjugate into) $\mathsf{PSp}(2n)$, see Proposition 8.1.

We recall that our original pressure metric from [7] was obtained as the Hessian of a renormalized pressure intersection

$$\mathbf{J}(\rho,\eta)=\frac{h(\rho)}{h(\eta)}\lim_{T\to\infty}\frac{1}{\#R_1(\rho,T)}\sum_{\gamma\in R_1(\rho,T)}\frac{L_1\left(\eta(\gamma)\right)}{L_1\left(\rho(\gamma)\right)}$$

where $L_1\left(\rho(\gamma)\right)=\log\lambda_1\left(\rho(\gamma)\right)$, $R_1(\rho,T)=\{[\gamma]\in[\pi_1(S)]\backslash\{[1]\}\mid L_1\left(\rho(\gamma)\right)\leqslant T\}$ and the spectral radius entropy $h(\rho)$ is the exponential growth rate of $R_1(\rho,T)$.

There are two main advantages of the Liouville pressure metric with respect to the pressure metric defined in [7]. First, due to work of Potrie and Sambarino [32], we do not have to renormalize the pressure intersection by an entropy. Second, the Bowen–Margulis measure associated to the first simple root is directly related to the cross ratio of the representation. We hope that these two facts will make the Liouville pressure metric more accessible to computation. It follows from work of Zhang [38] and Theorem 1.4 that the Liouville volume is non-constant on $\mathcal{H}_d(S)$ when $d\geqslant 3$, so one cannot directly use the Hessian of intersection to construct a metric, as Bonahon [1, Theorem 19] does to reconstruct the Weil–Petersson metric when $d=2$.

2 Dynamical background

In Sects. 2.1 and 2.2 we recall the thermodynamic formalism of Bowen and Ruelle [3,4,33], which was further developed by Parry and Pollicott [30]. We then discuss geodesic currents (in Sect. 2.3) and describe the relationship between contracting line bundles and flows (in Sect. 2.4).

2.1 Basic definitions

Let X be a compact metric space and $\phi=\{\phi_t:X\to X\}_{t\in\mathbb{R}}$ be a topologically transitive, metric Anosov flow on X. (Metric Anosov flows were first defined by Pollicott [31] who called them Smale flows.) Let O_ϕ be the collection of periodic orbits of the flow ϕ and and define

$$R_\phi(T)=\left\{a\in O_\phi\mid \ell(a)\leqslant T\right\}$$

where $\ell(a)$ is the period of a. The *topological entropy* of the flow ϕ is given by

$$h(\phi)=\lim_{T\to\infty}\frac{\log\#R_\phi(T)}{T}.$$

If $\alpha>0$, let $\mathrm{Hol}^\alpha(X,\mathbb{R})$ be the space of α-Hölder continuous functions on X. If $f\in\mathrm{Hol}^\alpha(X,\mathbb{R})$, let

$$\ell_f(a)=\int_X f\,\mathrm{d}\hat{\delta}_a$$

where $\hat{\delta}_a$ is a ϕ-invariant measure supported on a with total mass $\ell(a)$. Let

$$R_\phi(f,T)=\left\{a\in O_\phi\mid \ell_f(a)\leqslant T\right\}$$

and define

$$h_\phi(f)=\lim_{T\to\infty}\frac{\log\#R_\phi(f,T)}{T}.$$

If f is positive, we obtain a new flow ϕ^f on X by reparametrizing ϕ by f. Concretely, ϕ^f is determined by the formula

$$\phi^f_{k_f(x,t)}(x) = \phi^f_t(x)$$

where $k_f(x,t) = \int_0^t f(\phi_s x)\mathrm{d}s$ for all $x \in X$ and $t \in \mathbb{R}$. Notice that if $\mathrm{d}s$ is an element of arc length for the flow lines of ϕ, then $f\mathrm{d}s$ is an element of arc length for the flow lines of ϕ^f.

The flow ϕ^f is Hölder orbit equivalent to ϕ and if $a \in O_\phi = O_{\phi^f}$, then $\ell_f(a)$ is the period of a in the flow ϕ^f. In this case, $h_\phi(f)$ is the topological entropy $h(\phi^f)$ of the flow ϕ^f.

We will say that $f, g \in \mathrm{Hol}^\alpha(X,\mathbb{R})$ are *Livšic cohomologuous* if there exists $U : X \to \mathbb{R}$ such that for all $x \in X$ one has

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x).$$

Recall that f and g are Livšic cohomologous if and only if $\ell_f(a) = \ell_g(a)$ for all $a \in O_\phi$. Moreover, if f and g are positive, then ϕ^f and ϕ^g are Hölder conjugate if and only if f and g are Livšic cohomologuous (see Livšic [25]).

If $\mathcal{M}_\phi$ is the space of ϕ-invariant probability measures on X and $m \in \mathcal{M}_\phi$, let $h(\phi, m)$ be the metric entropy of m. Then, for $f \in \mathrm{Hol}^\alpha(X,\mathbb{R})$, the *topological pressure* is

$$\mathbf{P}_\phi(f) = \sup_{m\in\mathcal{M}_\phi} \left\{ h(\phi, m) + \int_X f\mathrm{d}m \right\}.$$

A measure that attains this supremum is called an *equilibrium state* for f and an equilibrium state for the zero function is called a *measure of maximal entropy*.

If $f \in \mathrm{Hol}^\alpha(X,\mathbb{R})$ is positive, Bowen [2, Theorem 5.11] (see also Pollicott [31, Theorem 9]) showed that the measure of maximal entropy for ϕ^f is given by the *Bowen–Margulis measure* for ϕ^f

$$\lim_{T\to\infty} \frac{1}{\#R_\phi(f,T)} \sum_{a\in O_X} \frac{\hat{\delta}_a}{\ell_f(a)}$$

where $\hat{\delta}_a$ is the product of Dirac measure on the orbit a and the element of arc length on a in ϕ^f.

We make use of the following result of Sambarino [34, Lemma 2.4].

Lemma 2.1 *Suppose that $f \in \mathrm{Hol}^\alpha(X,\mathbb{R})$ is positive. If $m_{-h_\phi(f)f}$ is the equilibrium state of $-h_\phi(f)f$, then*

$$\mathrm{d}m^\# = \frac{f\mathrm{d}m_{-h_\phi(f)f}}{\int f\mathrm{d}m_{-h_\phi(f)f}}$$

is the measure of maximal entropy of ϕ^f.

If $f, g \in \mathrm{Hol}^\alpha(X,\mathbb{R})$ are positive, we define their *pressure intersection*[1] by

$$\mathbf{I}(f,g) = \lim_{T\to\infty} \frac{1}{\#R_\phi(f,T)} \sum_{a\in R_\phi(f,T)} \frac{\ell_g(a)}{\ell_f(a)} = \frac{\int g\mathrm{d}m_{-h(f)f}}{\int f\mathrm{d}m_{-h(f)f}}. \tag{1}$$

[1] We emphasize the terminology pressure intersection which is meant to distinguish pressure intersection from the intersection defined by Bonahon [1].

The last equation follows from [7, Section 3.4]. We define the *renormalized pressure intersection* by

$$\mathbf{J}(f,g) = \frac{h_\phi(g)}{h_\phi(f)}\mathbf{I}(f,g).$$

In [7, Cor. 2.5, Propositions 3.11 and 3.12], we used results of Parry–Pollicott [30] and Ruelle [33] to prove the following.

Proposition 2.2 *If ϕ is a topologically transitive metric Anosov flow on a compact metric space X, then*

(1) *If $f \in \mathrm{Hol}^\alpha(X,\mathbb{R})$ is positive, then the function $\mathbf{J}_f$ defined by $\mathbf{J}_f(g) = \mathbf{J}(f,g)$ has a global minimum at f. Therefore,* Hess $\mathbf{J}_f$ *is positive semi-definite.*
(2) *If $\{f_t\}_{t\in(-\epsilon,\epsilon)} \subset \mathrm{Hol}^\alpha(X,\mathbb{R})$ is a smooth one-parameter family of positive functions, then*

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\mathbf{J}(f_0, f_t) = 0$$

if and only if, for every $a \in O_\phi$, one has

$$\frac{\partial}{\partial t}\bigg|_{t=0} h_\phi(f_t)\ell_{f_t}(a) = 0.$$

(3) *If $\{f_u\}_{u\in M}$ and $\{g_v\}_{v\in M'}$ are analytic families of positive α-Hölder functions parametrized by analytic manifolds M and M', then $\mathbf{J}(f_u, g_v)$ is an analytic function on $M \times M'$.*

2.2 Expansion on periodic orbits

Assume now that X is a manifold and that ϕ is a $\mathrm{C}^{1+\alpha}$ Anosov flow with unstable bundle E^u. Denote by $\lambda^u_\phi : X \to (0,\infty)$ the *infinitesimal expansion rate* on the unstable direction, defined by

$$\lambda^u_\phi(x) = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{1}{\kappa}\int_0^\kappa \log\det\left(d_x\phi_{t+s}|E^u\right) ds$$

for some $\kappa > 0$.

We record the following observations (see [32, Section 2.2] for further discussion):

(1) If $a \in O_\phi$, then

$$\ell_{\lambda^u_\phi}(a) = \int_a \lambda^u_\phi = \log\det\left(d_x\phi_{\ell(a)}|E^u\right)$$

is the total expansion of ϕ along a.
(2) The Livšic-cohomology class of λ^u_ϕ does not depend on κ.
(3) If ϕ^{-1} is the inverse flow $\phi^{-1}_t = \phi_{-t}$, it follows from Livšic's Theorem [25] that ϕ preserves a measure in the class of Lebesgue if and only if λ^u_ϕ is Livšic cohomologuous to $\lambda^u_{\phi^{-1}}$.

We make crucial use of the following classical result of Sinai, Ruelle and Bowen.

Theorem 2.3 (Sinai–Ruelle–Bowen [4]) *Let ϕ be a $\mathrm{C}^{1+\alpha}$ Anosov flow on a compact manifold X, then $\mathbf{P}(-\lambda^u_\phi) = 0$. Moreover, if ϕ preserves a measure in the class of Lebesgue, then this measure is the equilibrium state of $-\lambda^u_\phi$.*

Bowen and Ruelle state their result in the C^2 setting, but the proof may be extended to the $C^{1+\alpha}$ setting by applying [16, Propositions 19.16 and 20.4.2].

2.3 Geodesic currents

Let Γ be a hyperbolic group which is not virtually cyclic. Let $\mathcal{G}(\Gamma)$ be the space of pair of distinct points, which we think of as the space of *oriented geodesics*, on the Gromov boundary $\partial_\infty\Gamma$ of Γ:

$$\mathcal{G}(\Gamma) := \{(x, y) \in \partial_\infty\Gamma \mid x \neq y\}.$$

A *geodesic current* for Γ is a Γ-invariant locally finite measure on $\mathcal{G}(\Gamma)$. If γ is a primitive infinite order element of Γ with attracting fixed point $\gamma_+ \in \partial_\infty\Gamma$ and repelling fixed point γ_- and $\delta_{(x,y)}$ is the Dirac measure supported at $(x, y) \in \mathcal{G}(\Gamma)$, we define the geodesic current

$$\delta_\gamma := \sum_{\hat{\gamma}\in[\gamma]} \delta_{(\hat{\gamma}_-,\hat{\gamma}_+)}.$$

where $[\gamma]$ is the conjugacy class of γ in Γ. If $\alpha = \gamma^n$ where α is primitive and $n > 0$, we let $\delta_\alpha = n\delta_\gamma$.

We let $\mathcal{C}(\Gamma)$ denote the space of geodesic currents on Γ and endow it with the weak-* topology. When $\Gamma = \pi_1(S)$, for a closed surface S, we write $\mathcal{G}(S)$ and $\mathcal{C}(S)$ for $\mathcal{G}(\pi_1(S))$ and $\mathcal{C}(\pi_1(S))$.

Following Bonahon [1, Section 4.2], we define a continuous, symmetric, bilinear pairing, called the *intersection*

$$i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}$$

so that if $\alpha, \beta \in \Gamma$, then $i(\delta_\alpha, \delta_\beta)$ is the geometric intersection number of the curves on S representing α and β. Let $\mathcal{DG}(S) \subset \mathcal{G}(S) \times \mathcal{G}(S)$ denote the space of pairs (x, y) and (u, v) of oriented geodesics which intersect, i.e. so that x and y lie in distinct components of $\partial_\infty\pi_1(S) - \{u, v\}$. We then define

$$i(\mu, \nu) = \int_{\mathcal{DG}(S)/\pi_1(S)} \mathrm{d}\mu \otimes \mathrm{d}\nu.$$

A geodesic current is *symmetric* if it is invariant by the involution $\iota : (x, y) \mapsto (y, x)$. Bonahon [1] works entirely in the setting of symmetric geodesic currents. In fact, he defines a geodesic current as a measure on the space $\widehat{\mathcal{G}}(\Gamma) = \mathcal{G}(\Gamma)/\iota$ of unordered pairs of distinct points in $\partial_\infty\Gamma$. A geodesic current μ in our sense naturally pushes forward to a geodesic current $\hat{\mu}$ in the sense of Bonahon. Moreover, if $\mu, \nu \in \mathcal{C}(S)$, then $i(\mu, \nu)$ agrees with the intersection, in the sense of Bonahon, of $\hat{\mu}$ and $\hat{\nu}$.

2.4 Contracting line bundles and flows

Gromov [13] defined a geodesic flow $\mathsf{U}(\Gamma)$ for a hyperbolic group Γ, which is well-defined up to Hölder orbit equivalence, see Champetier [9] and Mineyev [28] for detailed constructions. The closed orbits of $\mathsf{U}(\Gamma)$ are in one-to-one correspondence with conjugacy classes of infinite order elements of Γ. There is a trivial Hölder $\mathbb{R}$ principal bundle $\mathsf{L}_\Gamma = \widetilde{\mathsf{U}}(\Gamma)$ over $\mathcal{G}(\Gamma)$ equipped with a properly discontinuous action of Γ by bundle automorphisms, so that L_Γ/Γ equipped with the flow coming from the action of $\mathbb{R}$ is Hölder orbit equivalent to $\mathsf{U}(\Gamma)$. Moreover, $\widetilde{\mathsf{U}}(\Gamma)$ may be parametrized as $\mathcal{G}(\Gamma) \times \mathbb{R}$ where the action of $\mathbb{R}$ is by translation in the second factor. We will mostly be interested in the situation where $\mathsf{U}(\Gamma)$ is metric Anosov.

In [7, Section 5], we showed that, whenever a Γ admits an Anosov representation, $\mathsf{U}(\Gamma)$ is indeed metric Anosov. In this paper, we will focus on the case where $\Gamma = \pi_1(S)$, in which case $\mathsf{U}(\Gamma)$ may be taken to be the geodesic flow on the unit tangent bundle of a hyperbolic surface Y homeomorphic to S, and will be denoted $\mathsf{U}(S)$, and L_Γ may be identified with the geodesic flow on the unit tangent bundle of the universal cover of Y, and will be denoted $\mathsf{U}(\tilde{S})$.

A *flow over* $\mathcal{G}(\Gamma)$ is a Hölder $\mathbb{R}$-principal line bundle L over $\mathcal{G}(\Gamma)$ equipped with a properly discontinuous action of Γ by Hölder bundle automorphisms, so that the quotient flow on $\mathsf{U}_\mathsf{L} := \mathsf{L}/\Gamma$ is Hölder orbit equivalent to the geodesic flow of Γ. In other words, one may think of a flow over $\mathcal{G}(\Gamma)$ as a parametrization of the geodesic flow of Γ.

Given a geodesic current ω and a flow L over $\mathcal{G}(\Gamma)$, we define a pairing

$$\langle \omega \mid \mathsf{L}\rangle := \int_{\mathsf{U}_\mathsf{L}} \omega \otimes \mathrm{d}t$$

where $\mathrm{d}t$ is the element of arc length on U_L given by the $\mathbb{R}$-action. Given a flow L the function $\omega \mapsto \langle \omega \mid \mathsf{L}\rangle$ from $\mathcal{C}(\Gamma)$ to $\mathbb{R}$ is continuous.

We observe that, for every non trivial element γ in Γ, $\langle \delta_\gamma \mid \mathsf{L}\rangle$ is the length of the periodic orbit associated to γ in U_L, or, equivalently, the translation distance of the action of γ on the fiber $\mathsf{L}_{(\gamma^-,\gamma^+)}$. The map $\gamma \mapsto \langle \delta_\gamma \mid \mathsf{L}\rangle$ is the *length spectrum* of L. If $\mathsf{U}(\Gamma)$ is metric Anosov, then, by Livšic's Theorem, the length spectrum determines the quotient flow U_L up to Hölder conjugacy.

Let M be a Hölder line bundle over $\mathsf{U}(\Gamma)$, equipped with a lift of the geodesic flow $\{\psi_t\}_{t\in\mathbb{R}}$ on $\mathsf{U}(\Gamma)$ to a Hölder flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on M by bundle automorphisms (i.e. the restriction of Ψ_t is a linear automorphism from M_z to $\mathsf{M}_{\psi_t(z)}$ for all $z \in \mathsf{U}(\Gamma)$ and all $t \in \mathbb{R}$). We say that M is *contracting* if there exist a metric $\|\cdot\|$ on M and $t_0 > 0$ so that

$$\|\Psi_{t_0}(u)\| \leqslant \frac{1}{2}\|u\|,$$

for all u in M. Every such line bundle has a *contraction spectrum* $\gamma \mapsto c(\gamma)$, where if the periodic orbit of $\mathsf{U}(\Gamma)$ associated to $\gamma \in \pi_1(S)$ has period t_γ, then

$$\|\Psi_{t_\gamma}(v)\| = e^{-c(\gamma)}\|v\|$$

for any vector v in a fiber over the periodic orbit. Again Livšic's Theorem guarantees that two line bundle with the same contracting spectrum are isomorphic.

The notions of contracting line bundles and flow are equivalent.

Proposition 2.4 *Let Γ be a hyperbolic group whose geodesic flow is metric Anosov. Then*

(1) *Given a contracting line bundle M over $\mathsf{U}(\Gamma)$, there exists a flow over $\mathcal{G}(\Gamma)$ whose length spectrum coincides with the contracting spectrum of M.*
(2) *Conversely, given a flow L over $\mathcal{G}(\Gamma)$, there exists a contracting line bundle over $\mathsf{U}(\Gamma)$ whose contracting spectrum is the length spectrum of L.*

Proof Given a contracting line bundle M over $\mathsf{U}(\Gamma)$, we construct a flow L_M over $\mathcal{G}(\Gamma)$ by the following procedure

(1) First, lift M to a line bundle $\widetilde{\mathsf{M}}$ over $\widetilde{\mathsf{U}}(\Gamma)$ and let $\{\widetilde{\Psi}_t\}_{t\in\mathbb{R}}$ be the lift of the flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on M.
(2) We consider the corresponding $\mathbb{R}$-principal line bundle $\widehat{\mathsf{L}}_\mathsf{M}$ over $\widetilde{\mathsf{U}}(\Gamma)$ equipped with an action of Γ by bundle automorphisms; concretely the fiber of $\widehat{\mathsf{L}}_\mathsf{M}$ over $(x, y, s) \in \widetilde{\mathsf{U}}(\Gamma)$

is $(\widetilde{\mathsf{M}}_{(x,y,s)} - \{0\})/\pm 1$, i.e. non-zero vectors up to sign, and the action of $t \in \mathbb{R}$ takes $[v] \in (\widehat{\mathsf{L}_\mathsf{M}})_{(x,y,s)}$ to $[e^t v]$.

(3) Let $\pi : \tilde{\mathsf{U}}(\Gamma) \to \mathcal{G}(\Gamma)$. We define $\mathsf{L}_\mathsf{M} := \pi_* \widehat{\mathsf{L}_\mathsf{M}}$, that is the bundle whose sheaf of sections are the sections of $\widehat{\mathsf{L}_\mathsf{M}}$ invariant by the flow: More explicitly, for all $t \in \mathbb{R}$, $(x, y, s) \in \tilde{\mathsf{U}}(\Gamma)$ and $[v] \in (\widehat{\mathsf{L}_\mathsf{M}})_{(x,y,s)}$, we identify $[v]$ with $[\widetilde{\Psi}_t(v)]$ and notice that the quotient is a principal $\mathbb{R}$-bundle over $\mathcal{G}(\Gamma)$.

The proof of [7, Proposition 4.2] generalizes immediately to yield the first part of our proposition.

We now establish our second claim. Let L be a flow over $\mathcal{G}(\Gamma)$. Consider the trivial bundle $\widetilde{\mathsf{M}} = \mathsf{L} \times \mathbb{R}$ over L equipped with the trivial lift of the action of Γ given by $\gamma(x, v) = (\gamma x, v)$. Lift the flow $\{\tilde{\phi}_t\}_{t\in\mathbb{R}}$ on L to the flow

$$\widetilde{\Psi_t}(x, v) = \left(\tilde{\phi}_t(x), e^{-t} v\right)$$

on $\widetilde{\mathsf{M}}$. These two actions commute and we obtain a contracting line bundle $\mathsf{M} := \widetilde{\mathsf{M}}/\Gamma$ over U_L equipped with the quotient flow $\{\Psi_t\}_{t\in\mathbb{R}}$ whose contracting spectrum agrees with the length spectrum of L. □

As an immediate consequence, the tensor product on principal $\mathbb{R}$ bundles gives rise to an inner product, also called the tensor product,

$$(\mathsf{L}_0, \mathsf{L}_1) \mapsto \mathsf{L}_0 \otimes \mathsf{L}_1,$$

on geodesic flows, which is equivalent to the tensor product of the corresponding contracting line bundles. The length spectrum of the tensor product is then the sum of the two length spectra and thus for any geodesic current $\mu \in \mathcal{C}(\Gamma)$

$$\langle \mu \mid \mathsf{L}_0 \otimes \mathsf{L}_1 \rangle = \langle \mu \mid \mathsf{L}_0 \rangle + \langle \mu \mid \mathsf{L}_1 \rangle$$

since any current may be approximated by linear combinations of currents associated to group elements. Given a positive number t, which we may view as an element of $\mathrm{Aut}(\mathbb{R})$, we can renormalise the action of $\mathbb{R}$ on the $\mathbb{R}$-bundle L to obtain a new bundle L^t so that

$$\left\langle \mu \mid \mathsf{L}^t \right\rangle = t \left\langle \mu \mid \mathsf{L} \right\rangle .$$

One can check then that for a positive integer n,

$$\mathsf{L}^n = \overbrace{\mathsf{L} \otimes \cdots \otimes \mathsf{L}}^{n}.$$

3 Hitchin representations and their associated flows

In Sects. 3.1 and 3.2 we recall the definitions and basic properties of projective Anosov and Hitchin representations. In Sects. 3.3 and 3.4 we use the techniques of Sect. 2.4 to construct families of flows associated to such representations.

3.1 Projective Anosov representations

It will occasionally be useful to work in the more general class of *projective Anosov* representations. A representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ with domain a hyperbolic group Γ has *transverse projective limit maps* if there exist continuous, ρ-equivariant functions

$$\xi_\rho : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$$

and

$$\xi^*_\rho : \partial_\infty \Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$$

so that if x and y are distinct points in $\partial_\infty \Gamma$, then

$$\xi_\rho(x) \oplus \ker \xi^*_\rho(y) = \mathbb{R}^d.$$

Recall that a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$, with domain a hyperbolic group Γ, gives rise to a flat $\mathbb{R}^d$-bundle E_ρ over $\mathsf{U}(\Gamma)$ and that the geodesic flow ϕ on $\mathsf{U}(\Gamma)$ lifts to a flow ψ_ρ parallel to the flat connection on E_ρ. Explicitly, let $\tilde{E}_\rho = \widetilde{\mathsf{U}}(\Gamma) \times \mathbb{R}^d$ and let $(\tilde{\psi}_\rho)_t(z, v) = (\tilde{\phi}_t(z), v)$ where $\tilde{\phi}_t$ is the lift of the geodesic flow ϕ_t on $\mathsf{U}(\Gamma)$ to $\widetilde{\mathsf{U}}(\Gamma)$. The group Γ acts on $\tilde{E}_\rho$ by the action of Γ on the first factor and $\rho(\Gamma)$ on the second factor, and the quotient is the flat bundle E_ρ and the flow $\{(\tilde{\psi}_\rho)_t\}_{t\in\mathbb{R}}$ descends to a flow ψ_ρ on E_ρ.

A representation ρ with transverse projective limit maps determines a ψ_ρ-invariant splitting $\Xi_\rho \oplus \Theta_\rho$ of the flat bundle E_ρ over $\mathsf{U}(\Gamma)$. Concretely, the lift $\tilde{\Xi}_\rho$ of Ξ_ρ has fiber $\xi_\rho(x)$ and the lift $\tilde{\Theta}_\rho$ of Θ_ρ has fiber $\ker \xi^*_\rho(y)$ over the point $(x, y, t) \in \widetilde{\mathsf{U}}(\Gamma)$. One says that ρ is *projective Anosov* if the resulting flow on the associated bundle

$$\mathrm{Hom}(\Theta_\rho, \Xi_\rho) = \Xi_\rho \otimes \Theta^*_\rho$$

is contracting.

Projective Anosov representations are quasi-isometric embeddings with finite kernel, see [15, Theorem 5.3] and [22, Theorem 1.0.1] for Hitchin representations. The following result is also a standard consequence of the definitions, see, for example, [7, Proposition 2.6]).

Lemma 3.1 *If $\rho : \pi_1(S) \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov and $\gamma \in \pi_1(S)$ is non-trivial, then $\rho(\gamma)$ is proximal with attracting line $\xi(\gamma_+)$ and repelling hyperplane $\theta(\gamma_-)$. Moreover, there exist positive constants B and C such that*

$$\log \frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))} \geqslant B\ell(\gamma) - C$$

where $\ell(\gamma)$ is the reduced word length of γ.

3.2 Hitchin representations

If $\rho : \pi_1(S) \to \mathsf{PSL}_d(\mathbb{R})$ is a Hitchin representation, it admits a lift $\tilde{\rho} : \pi_1(S) \to \mathsf{SL}(d, \mathbb{R})$. We will abuse notation and denote the flat bundle $E_{\tilde{\rho}}$ associated to this lift by E_ρ. (The flat bundle depends on the choice of lift, but this choice will not matter for our purposes).

Labourie [20] showed that every Hitchin representation ρ admits a continuous ρ-equivariant limit map $\hat{\xi}_\rho : \partial_\infty \pi_1(S) \to \mathcal{F}_d$ where $\mathcal{F}_d$ is the space of complete flags in $\mathbb{R}^d$. We summarize its crucial properties below.

Theorem 3.2 (Labourie [20]) *If $\rho \in \mathcal{H}_d(S)$, there exists a unique ρ-equivariant Hölder continuous map $\hat{\xi}_\rho : \partial_\infty \pi_1(S) \to \mathcal{F}_d$ such that*

(1) *If $d = n_1 + \cdots + n_k$, where each $n_k \in \mathbb{N}$, and $\{z_1, \ldots, z_k\} \subset \partial_\infty \pi_1(S)$ are pairwise distinct, then*

$$\hat{\xi}^{(n_1)}_\rho(z_1) \oplus \cdots \oplus \hat{\xi}^{(n_k)}_\rho(z_k) = \mathbb{R}^d.$$

(2) *The image* $\hat{\xi}^{(1)}_\rho(\partial_\infty\pi_1(S))$ *is a* $C^{1+\alpha}$ *manifold for some* $\alpha > 0$.
(3) *The splitting* $\bigoplus_{i=1}^d \tilde{\mathsf{M}}^i_\rho$ *of* $\widetilde{E}_\rho$ *into line bundles so that* $(\tilde{\mathsf{M}}^i_\rho)_{(x,y,t)} = \hat{\xi}^{(i)}_\rho(x) \cap \hat{\xi}^{(n-i+1)}_\rho(y)$ *descends to a splitting* $\bigoplus_{i=1}^d \mathsf{M}^i_\rho$ *of* E_ρ *into line bundles, so that* $\mathsf{M}^i_\rho \otimes (\mathsf{M}^j_\rho)^*$ *is contracting if* $i < j$.

It is well-known that any exterior power of a (lift of a) Hitchin representation is projective Anosov (see for example Guichard–Wienhard [15, Pop. 4.4]). Guichard has shown conversely in [14] that the existence of such limit maps characterize Hitchin representations.

Proposition 3.3 *If* $\rho \in \mathcal{H}_d(S)$, $\tilde{\rho} : \pi_1(S) \to \mathsf{SL}(d, \mathbb{R})$ *is a lift of* ρ, $k \in \{1, \ldots, d-1\}$, *and* $\mathsf{E}^k\tilde{\rho} : \pi_1(S) \to \mathsf{SL}(\Lambda^k\mathbb{R}^d)$ *is the* k^{th} *exterior power of* $\tilde{\rho}$, *then* $\mathsf{E}^k\tilde{\rho}$ *is projective Anosov.*

If $\rho \in \mathcal{H}_d(S)$, then

$$\xi_\rho = \xi_{\tilde{\rho}} = \hat{\xi}^{(1)}_\rho \quad \text{and} \quad \ker \xi^*_\rho(x) = \ker \xi^*_{\tilde{\rho}}(x) = \hat{\xi}^{(d-1)}_\rho(x),$$

i.e. $\xi^*_\rho(x)$ is the projective class of linear functionals with kernel $\hat{\xi}^{(d-1)}_\rho(x)$. More generally, if $\rho \in \mathcal{H}_d(S)$, we may choose for each $x \in \partial_\infty\pi_1(S)$ a basis $\{e_i(\rho, x)\}$ for $\mathbb{R}^d$ so that $\hat{\xi}^{(j)}_\rho(x)$ is spanned by $\{e_1(\rho, x), \ldots, e_j(\rho, x)\}$. The limit maps for $\mathsf{E}^k\tilde{\rho}$ are given by

$$\xi_{\mathsf{E}^k\tilde{\rho}}(x) = \langle e_1(\rho, x) \wedge \cdots \wedge e_k(\rho, x)\rangle$$

and

$$\ker \xi^*_{\mathsf{E}^k\tilde{\rho}}(x) = \left\langle e_{j_1}(\rho, x) \wedge \cdots \wedge e_{j_k}(\rho, x) \mid 1 \leqslant j_1 < j_2 < \cdots < j_k, \ j_k > k\right\rangle.$$

One may check directly that $\mathrm{Hom}(\Theta_{\mathsf{E}^k\tilde{\rho}}, \Xi_{\mathsf{E}^k\tilde{\rho}})$ is contracting and hence that $\mathsf{E}^k\tilde{\rho}$ is projective Anosov, by applying part (3) of Theorem 3.2.

If we apply Lemma 3.1 to the exterior product $\mathsf{E}^i\tilde{\rho}$ of a (lift of a) Hitchin representation we obtain:

Lemma 3.4 *If* $\rho \in \mathcal{H}_d(S)$ *and* $i \in \{1, \ldots, d-1\}$, *then there exist* $B_i > 0$ *and* C_i *so that*

$$\log \frac{\lambda_i(\rho(\gamma))}{\lambda_{i+1}(\rho(\gamma))} \geqslant B_i \ell(\gamma) - C_i$$

where $\ell(\gamma)$ *is the reduced word length of* γ.

Lemma 3.4 can also be derived directly from part (3) of Theorem 3.2.

3.3 Flows for Hitchin representations

Theorem 3.2 provides several contracting line bundles over $\mathsf{U}(S)$ associated to a Hitchin representation $\rho \in \mathcal{H}_d(S)$.

(1) The spectral radius line bundle M^1_ρ.
(2) The simple root line bundles $\mathsf{M}^{\alpha_i}_\rho := \mathsf{M}^i_\rho \otimes (\mathsf{M}^{i+1}_\rho)^*$.
(3) The Hilbert line bundle $\mathsf{M}^{\mathsf{H}}_\rho := \mathsf{M}^1_\rho \otimes (\mathsf{M}^d_\rho)^*$.

Proposition 2.4 shows that the associated flows

(1) The spectral radius flow $\mathsf{U}_1(\rho) := \mathsf{L}^1_\rho/\pi_1(S)$.
(2) The simple root flows $\mathsf{U}_{\alpha_i}(\rho) := \mathsf{L}^{\alpha_i}_\rho/\pi_1(S)$.
(3) The Hilbert flow $\mathsf{U}_{\mathsf{H}}(\rho) := \mathsf{L}^{\mathsf{H}}_\rho/\pi_1(S)$.

are all Hölder orbit equivalent to $\mathsf{U}(S)$. The corresponding length spectra are

(1) The spectral radius length $L_1(\rho(\gamma)) := \log(\lambda_1(\rho(\gamma)))$.
(2) The simple root length $L_{\alpha_i}(\rho(\gamma)) := \log\left(\frac{\lambda_i(\rho(\gamma))}{\lambda_{i+1}(\rho(\gamma))}\right)$.
(3) The Hilbert length $L_{\mathsf{H}}(\rho(\gamma)) := \log\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_d(\rho(\gamma))}\right)$.

More generally, given any positive linear combination $L_\phi = a_1 L_{\alpha_1} + \cdots + a_{d-1} L_{\alpha_{d-1}}$ of the simple root length functions, we can find a flow $\mathsf{U}_\phi(\rho)$ so that the period of $\gamma \in \pi_1(S)$ is given by

$$L_\phi(\rho(\gamma)) = a_1 L_{\alpha_1}(\rho(\gamma)) + \cdots + a_{d-1} L_{\alpha_{d-1}}(\rho(\gamma)).$$

(See the discussion in Sect. 2.4.)

Finally, we observe that, by Theorem 3.2 and [32, Proposition 6.2], the flow M^1_ρ is obtained as a pullback of a smooth line bundle over the $\mathrm{C}^{1+\alpha}$-submanifold $(\xi_\rho \times \xi^*_\rho)(\mathcal{G}(S))$ of $\mathbb{P}(\mathbb{R}^d) \times \mathbb{P}^*(\mathbb{R}^d)$, so L^1_ρ inherits the structure of a $\mathrm{C}^{1+\alpha}$-flow.

Potrie and Sambarino [32, Proposition 6.2] show that the unstable manifold E^u_ρ for L^1_ρ at a point above $(x, y) \in \mathcal{G}(S)$ may be identified with $\mathrm{Hom}\left(\hat{\xi}^{(2)}_\rho(x) \cap \hat{\xi}^{(d-1)}_\rho(y), \hat{\xi}^{(1)}_\rho(x)\right)$ and so the *infinitesmal expansion rate* λ^u_ρ of $\mathsf{U}_1(\rho)$ has the property that

$$\int_\gamma \lambda^u_\rho \mathrm{d}s^1_\rho = L_{\alpha_1}(\rho(\gamma))$$

for all $\rho \in \mathcal{H}_d(S)$, where $\mathrm{d}s^1_\rho$ is the element of arc length of $\mathsf{U}_1(\rho)$.

It follows that the reparametrization of $\mathsf{U}_1(\rho)$ by λ^u_ρ is Hölder conjugate to $\mathsf{U}_{\alpha_1}(\rho)$. They then apply results of Sinai, Ruelle and Bowen [4], to conclude that the entropy of $\mathsf{U}_{\alpha_1}(\rho)$ is 1. They further show, with a more sophisticated argument in the general case, that all the simple root flows have entropy 1.

Theorem 3.5 (Potrie–Sambarino [32, Theorem B]) *If $\rho \in \mathcal{H}_d(S)$ and $i \in \{1, \ldots, d-1\}$, then $\mathsf{U}_{\alpha_i}(\rho)$ has topological entropy 1.*

Remark One may also construct a flow Hölder conjugate to $\mathsf{U}_{\alpha_i}(\rho)$ by constructing, as is done in Sambarino [34], a positive Hölder function on $\mathsf{U}(S)$ whose periods are given by $L_{\alpha_i}(\rho(\gamma))$, see also Potrie–Sambarino [32].

3.4 The spectral radius flow of a projective Anosov representation

Proposition 2.4 implies that if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov, then the contracting line bundle Ξ_ρ over $\mathsf{U}(\Gamma)$ gives rise to a spectral radius flow L^ρ_1 over $\mathcal{G}(\Gamma)$ with quotient $\mathsf{U}_1(\rho)$ so that the closed orbit associated to $\gamma \in \Gamma$ has period $L_1(\rho(\gamma)) = \log(\lambda_1(\rho(\gamma)))$.

The spectral radius flow $\mathsf{U}_1(\rho)$ is Hölder orbit equivalent to $\mathsf{U}(\Gamma)$. In [7], we prove that, up to Hölder conjugacy, the reparametrization function can be chosen to vary analytically in a neighborhood of ρ.

Proposition 3.6 [7, Proposition 6.2] *Let $\{\rho_u : \pi_1(S) \to \mathsf{SL}(d, \mathbb{R})\}_{u \in D}$ be a real analytic family of projective Anosov homomorphisms parameterized by a disk D about the origin 0. Then, there exists a sub-disk D_0 about 0, $\alpha > 0$ and a real analytic family $\{f_u : \mathsf{U}(\Gamma) \to \mathbb{R}\}_{u \in D_0}$ of positive α-Hölder functions such that if $\gamma \in \Gamma$, then $\ell_{f_u}(\gamma) = \log \lambda_1(\rho_u(\gamma))$.*

4 Liouville currents for Hitchin representations

In Sects. 4.1 and 4.2 we recall Labourie's cross ratio, define our Liouville current and prove that it determines the Hitchin representation. In Sect. 4.3 we establish relationships between the Liouville current, Hilbert length L_{H}, the Bowen–Margulis current μ_ρ for $\mathsf{U}_{\alpha_1}(\rho)$, and the equilibrium state $m_{-\lambda^u_\rho}$ for the (negative of the) infinitesimal expansion rate on $\mathsf{U}_1(\rho)$.

4.1 Labourie's cross ratio

If V is a finite dimensional real vector space, let

$$\mathbb{P}^{(2)} = \mathbb{P}(V) \times \mathbb{P}(V^*) - \{(L, \Phi) : L \in \ker \Phi\}$$

and

$$\mathbb{P}^{(4)} = \{(L, \Phi, D, \Psi) : L \notin \ker \Psi \text{ and } D \notin \ker \Phi\}.$$

Consider the cross ratio on $\mathbb{P}^{(4)}$ defined by

$$\mathbb{B}(L, \Phi, D, \Psi) = \frac{\varphi(u)}{\psi(u)} \frac{\psi(v)}{\varphi(v)},$$

where $\varphi \in \Phi$, $\psi \in \Psi$, $u \in L$ and $v \in D$ are all non-zero. Notice that the result does not depend on the choices of φ, ψ, u and v. Labourie observes that $\mathbb{B}$ is the polarized cross-ratio associated to a symplectic form on $\mathbb{P}^{(2)}$.

Proposition 4.1 (Labourie [21, Propositions 4.7, 5.4]) *There exists a symplectic form Ω on $\mathbb{P}^{(2)}$ so that if $(L, \Phi, D, \Psi) \in \mathbb{P}^{(4)}$, then*

$$\mathbb{B}(L, \Phi, D, \Psi) = e^{\int G^*\Omega}$$

where $G : [0, 1]^2 \to \mathbb{P}^{(2)}$ is a map such that the images of the vertices of $[0, 1]^2$ are (L, Φ), (L, Ψ), (D, Φ) and (D, Ψ) and the image of every boundary segment is contained in either $\mathbb{P}(V) \times \{\cdot\}$ or $\{\cdot\} \times \mathbb{P}(V^*)$.

Moreover, if ρ is Hitchin, the restriction of the symplectic form Ω to the $\mathrm{C}^{1+\alpha}$-submanifold $(\xi_\rho \times \xi^*_\rho)(\mathcal{G}(S))$ is non-degenerate.

Given $\rho \in \mathcal{H}_d(S)$, Labourie defined a cross ratio b_ρ on

$$\partial_\infty\pi_1(S)^{(4)} = \{(x, y, z, t) \in \partial_\infty\pi_1(S)^4 \mid x \neq t,\ y \neq z\}$$

by setting

$$\mathsf{b}_\rho(x, y, z, t) = \mathbb{B}\left(\xi_\rho(x), \xi^*_\rho(y), \xi_\rho(z), \xi^*_\rho(t)\right).$$

Labourie and McShane [23, Theorem 9.0.3] show that $\mathsf{b}_\rho(x, z, t, y) > 1$ if (t, x, y, z) is cyclically ordered in $\partial_\infty\pi_1(S)$.

Labourie [21, Theorem 1.1] proves that this cross ratio determines the representation and has rank d. For any pair of $(p + 1)$-tuples of pairwise distinct points $X = (x_0, \ldots, x_p)$ and $Y = (y_0, \ldots, y_p)$ in $\partial_\infty\pi_1(S)$, we define

$$\chi_p(\mathsf{b}_\rho)(X, Y) = \det\Big(\mathsf{b}_\rho(x_i, y_j, x_0, y_0)\Big)_{i,j\in\{1,\ldots,p\}}.$$

Theorem 4.2 (Labourie [21, Theorem 1.1]) *If $\rho, \sigma \in \mathcal{H}_d(S)$, then $\mathsf{b}_\rho = \mathsf{b}_\sigma$ if and only if $\rho = \sigma$. Moreover, $\chi_d(\mathsf{b}_\rho) \equiv 0$ and $\chi_{d-1}(\mathsf{b}_\rho)$ never vanishes.*

Labourie [21, Theorem 1.1] also shows that the facts that $\chi_d(\mathsf{b}_\rho) \equiv 0$ and $\chi_{d-1}(\mathsf{b}_\rho)$ never vanishes characterize cross ratios of Hitchin representations into $\mathsf{PSL}_d(\mathbb{R})$ among all $\pi_1(S)$-invariant functions on $\partial_\infty\pi_1(S)^{(4)}$ satisfying the basic properties of a cross ratio.

4.2 Liouville currents: basic definitions

Let ω_ρ be the geodesic current defined by

$$\omega_\rho\left([t,x]\times[y,z]\right) = \frac{1}{2}\log \mathsf{b}_\rho(x,z,t,y) > 0$$

when (x,y,z,t) is a cyclically ordered 4-tuple in the circle $\partial_\infty\pi_1(S)$ and $[x,y]$ denotes the points between x and y in this cyclic ordering.

Proposition 4.1 implies that

$$\omega_\rho\left([t,x]\times[y,z]\right) = \frac{1}{2}\int_{\xi_\rho([t,x])\times\xi^*_\rho([y,z])}\Omega,$$

so ω_ρ is a measure on $\mathcal{G}(S)$ which is absolutely continuous with respect to the Lebesgue measure obtained by identifying $\mathcal{G}(S)$ with the $\mathrm{C}^{1+\alpha}$-manifold $(\xi_\rho\times\xi^*_\rho)\,(\mathcal{G}(S))$. We call ω_ρ the *Liouville current*.

We observe that the Liouville current also determines the Hitchin representation.

Theorem 1.2 *If $\rho\in\mathcal{H}_d(S)$ and $\eta\in\mathcal{H}_m(S)$, then $\omega_\rho=\omega_\eta$ if and only if $\rho=\eta$.*

Proof of Theorem 1.2 Suppose that $\omega_\rho=\omega_\eta$. By definition,

$$\mathsf{b}_\rho(x,y,z,t) = \omega_\rho([z,x]\times[t,y]) = \omega_\eta([z,x]\times[t,y]) = \mathsf{b}_\eta(x,y,z,t)$$

whenever (z,x,t,y) is cyclically ordered. Similarly, if (z,x,y,t) is cyclically ordered, then

$$\mathsf{b}_\rho(x,y,z,t) = \frac{1}{\omega_\rho([z,x]\times[y,t])} = \frac{1}{\omega_\eta([z,x]\times[y,t])} = \mathsf{b}_\eta(x,y,z,t)$$

(One may summarize these two observations, by saying that $\mathsf{b}_\rho(x,y,z,t)=\mathsf{b}_\eta(x,y,z,t)$ whenever the pairs (x,z) and (y,t) have non-intersecting axes, i.e. y and t lie in the same component of $\partial_\infty\pi_1(S)-\{x,z\}$.)

Suppose that $m>d$. Let $X=(x_0,x_1,\ldots,x_{m-1})$ and $Y=(y_0,\ldots,y_{m-1})$ be two m-tuples in $\partial_\infty\pi_1(S)$ so that $(x_0,x_1,\ldots,x_{m-1},y_0,y_1,\ldots,y_{m-1})$ is cyclically ordered. It follows from the previous paragraph that $\mathsf{b}_\rho(x_i,y_j,x_0,y_0)=\mathsf{b}_\eta(x_i,y_j,x_0,y_0)$ for all $i,j>0$. Theorem 4.2 then implies that every $(d+1)\times(d+1)$ minor of $\left(\mathsf{b}_\rho(x_i,y_j,x_0,y_0)\right)_{i,j\in\{1,\ldots,m-1\}}$ is zero, yet

$$\det\left(\mathsf{b}_\rho(x_i,y_j,x_0,y_0)\right)_{i,j\in\{1,\ldots,m-1\}} = \det\left(\mathsf{b}_\eta(x_i,y_j,x_0,y_0)\right)_{i,j\in\{1,\ldots,m-1\}} \neq 0$$

which is impossible. Therefore, we may assume that $m=d$.

By Theorem 4.2, it suffices to prove that ω_ρ determines the cross-ratio $\mathsf{b}_\rho(x,y,z,t)$ of any 4-tuple $(x,y,z,t)\in\partial_\infty\pi_1(S)^{(4)}$. By the observations in the first paragraph, and symmetry, it suffices to also consider the case where (x,y,z,t) is cyclically ordered.

Fix a cyclically ordered configuration $(x_d,y_d,x_0,y_0)\in\partial_\infty\pi_1(S)^{(4)}$. Choose pairwise distinct points $\{x_1,\ldots,x_{d-1}\}$ and $\{y_1,\ldots,y_{d-1}\}$ in $\partial_\infty\pi_1(S)$ so that $(x_0,x_1,\ldots,x_{d-1},y_0,$

$\dots, y_{d-1}, x_d, y_d)$ is cyclically ordered. Let $X = (x_0, \dots, x_d)$ and $Y = (y_0, \dots, y_d)$. Theorem 4.2 implies that

$$\chi_d\left((\mathsf{b}_\rho)(X,Y)\right) = \det\left(\mathsf{b}_\rho(x_i, y_j, x_0, y_0)\right)_{i,j\in\{1,\dots,d\}}$$
$$= \det\left(\mathsf{b}_\sigma(x_i, y_j, x_0, y_0)\right)_{i,j\in\{1,\dots,d\}} = \chi_d(\mathsf{b}_\eta)(X,Y) = 0.$$

If i and j are not both d, then either (x_0, x_i, y_0, y_j) or (x_0, x_i, y_j, y_0) is cyclically ordered, so $\mathsf{b}_\rho(x_i, y_j, x_0, y_0) = \mathsf{b}_\eta(x_i, y_j, x_0, y_0)$. One sees that all the coefficients in the matrices above agree except for the term where $i = j = d$, moreover, again applying Theorem 4.2, we see that the minors

$$\det\left(\mathsf{b}_\rho(x_i, y_j, x_0, y_0)\right)_{i,j\in\{1,\dots,d-1\}} = \det\left(\mathsf{b}_\sigma(x_i, y_j, x_0, y_0)\right)_{i,j\in\{1,\dots,d-1\}} \neq 0$$

agree and are non-zero. It follows that, $\mathsf{b}_\rho(x_d, y_d, x_0, y_0) = \mathsf{b}_\eta(x_d, y_d, x_0, y_0)$. This completes the proof. □

Corollary 4.3 *The Liouville current is symmetric if and only if* $\rho = \rho^*$.

Proof There is a natural identification of $\mathbb{P}(\mathbb{R}^d)$ with $\mathbb{P}((\mathbb{R}^d)^*)$, given by identifying $v \in \mathbb{R}^d$ to the linear functional $w \to v \cdot w$. So, given a representation $\rho \in \mathcal{H}_d(S)$, $\xi_{\rho_*} = \xi^*_\rho$ and $\xi^*_{\rho^*} = \xi_\rho$. Therefore,

$$w_{\rho^*}([t,x]\times[y,z]) = w_\rho([y,z]\times[t,x]) = w_\rho\big(\iota([t,x]\times[y,z])\big)$$

whenever (x, y, z, t) is cyclically ordered. It follows that ω_ρ is symmetric if and only if $\omega_\rho = \omega_{\rho_*}$. Theorem 1.2 then completes the proof. □

4.3 Liouville currents, equilibrium states and Bowen–Margulis measures

We define the current

$$\mu_\rho = \lim_{T\to\infty} \frac{1}{\#R_{\alpha_1}(\rho,T)} \sum_{[\gamma]\in R_{\alpha_1}(\rho,T)} \frac{1}{\left\langle \gamma \mid \mathsf{L}^{\alpha_1}_\rho \right\rangle} \delta_\gamma, \tag{2}$$

where $R_{\alpha_1}(\rho, T)$ is the set of closed orbits of $\mathsf{U}_{\alpha_1}(\rho)$ of period at most T. As was discussed in Sect. 2.1, the measure of maximal entropy for $\mathsf{U}_{\alpha_1}(\rho)$ is the Bowen–Margulis measure for $\mathsf{U}_{\alpha_1}(\rho)$, given by

$$\mu_\rho \otimes \mathrm{d}s^{\alpha_1}_\rho = \lim_{T\to\infty} \frac{1}{\#R_{\alpha_1}(\rho,T)} \sum_{[\gamma]\in R_{\alpha_1}(\rho,T)} \frac{1}{\left\langle \gamma \mid \mathsf{L}^{\alpha_1}_\rho \right\rangle} \hat{\delta}_\gamma$$

where $\mathrm{d}s^{\alpha_1}_\rho$ is the element of arc length on $\mathsf{U}_{\alpha_1}(\rho)$. We will refer to μ_ρ as the *Bowen–Margulis current* for $\mathsf{U}_{\alpha_1}(\rho)$.

The following result is an enlarged version of Theorem 1.3 from the introduction.

Theorem 4.4 *Suppose that* $\rho \in \mathcal{H}_d(S)$*,* ω_ρ *is its Liouville current,* λ^u_ρ *is the infinitesmal expansion rate of* $\mathsf{U}_1(\rho)$ *and* μ_ρ *is the Bowen–Margulis current for* $\mathsf{U}_{\alpha_1}(\rho)$.

(1) *If* $\gamma \in \pi_1(S)$*, then* $i(\delta_\gamma, \omega_\rho) = L_{\mathsf{H}}(\rho(\gamma)) = \left\langle \delta_\gamma \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle$.
(2) *If* $\mu \in \mathcal{C}(S)$*, then* $i(\mu, \omega_\rho) = \left\langle \mu \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle$.
(3) *The equilibrium state* $m_{-\lambda^u_\rho}$ *for the Hölder potential* $-\lambda^u_\rho$ *on* $\mathsf{U}_1(\rho)$ *is a scalar multiple of* $\omega_\rho \otimes \mathrm{d}s^1_\rho$ *where* $\mathrm{d}s^1_\rho$ *is the element of arc length on* $\mathsf{U}_1(\rho)$.

(4) *The equilibrium state* $m_{-\lambda^u_\rho}$ *is a scalar multiple of* $\mu_\rho \otimes \mathrm{d}s^1_\rho$.
(5) *The measure of maximal entropy for* $\mathsf{U}_{\alpha_1}(\rho)$ *is a scalar multiple of* $\omega_\rho \otimes \mathrm{d}s^{\alpha_1}_\rho$.
(6) *The Liouville current* ω_ρ *is a scalar multiple of the Bowen–Margulis current* μ_ρ.

Proof A standard computation, see for example [21, Proposition 5.8], shows that, for all $\gamma \in \pi_1(S)$,

$$\begin{aligned} i(\delta_\gamma, \omega_\rho) &= \omega_\rho([\gamma_+, \gamma_-] \times [x, \gamma(x)]) + \omega_\rho([\gamma_+, \gamma_-] \times [y, \gamma(y)]) \\ &= \frac{1}{2}\big(\log \mathsf{b}_\rho(\gamma_-, \gamma(x), \gamma_+, x) + \log \mathsf{b}_\rho(\gamma_-, \gamma(y), \gamma_+, y)\big) \\ &= \log \frac{\lambda_1(\rho(\gamma))}{\lambda_d(\rho(\gamma))} \\ &= L_H(\rho(\gamma)) \\ &= \left\langle \delta_\gamma \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle \end{aligned}$$

where x and y are in distinct components of $\partial_\infty \pi_1(S) - \{\gamma_-, \gamma_+\}$.

Since every current is a limit of positive linear combinations of currents associated to elements of $\pi_1(S)$ and the intersection function is continuous in the weak-* topology, we see that

$$i(\mu, \omega_\rho) = \left\langle \mu \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle$$

whenever $\mu \in \mathcal{C}(S)$.

Since ω_ρ is a measure on $\mathcal{G}(S)$ which is absolutely continuous with respect to the pullback of the Lebesgue measure on the $\mathrm{C}^{1+\alpha}$-submanifold $(\xi_\rho \times \xi^*_\rho)(\mathcal{G}(S))$, $\omega_\rho \otimes \mathrm{d}s^1_\rho$ is in the class of the Lebesgue measure on the $\mathrm{C}^{1+\alpha}$ manifold $\mathsf{U}_1(\rho)$. Theorem 2.3 implies that $\omega_\rho \otimes \mathrm{d}s^1_\rho$ is a scalar multiple of the equilibrium state $m_{-\lambda^u_\rho}$ for $-\lambda^u_\rho$ on $\mathsf{U}_1(\rho)$, i.e.

$$m_{-\lambda^u_\rho} = \frac{\omega_\rho \otimes \mathrm{d}s^1_\rho}{\langle \omega_\rho \mid \mathsf{L}^1_\rho \rangle}. \tag{3}$$

Since $\mathsf{U}_{\alpha_1}(\rho)$ is Hölder conjugate to the reparametrization of $\mathsf{U}_1(\rho)$ by λ^u_ρ and $\mathsf{U}_{\alpha_1}(\rho)$ has topological entropy 1, the equilibrium measure $m_{-\lambda^u_\rho}$ is a scalar multiple of the pullback of the measure of maximal entropy $\mu_\rho \otimes \mathrm{d}s^{\alpha_1}_\rho$ for $\mathsf{U}_{\alpha_1}(\rho)$ to $\mathsf{U}_1(\rho)$, see Lemma 2.1, i.e.

$$m_{-\lambda^u_\rho} = \frac{\mu_\rho \otimes \mathrm{d}s^1_\rho}{\langle \mu_\rho \mid \mathsf{L}^1_\rho \rangle}. \tag{4}$$

Since, by Eqs. (3) and (4), $\mu_\rho \otimes \mathrm{d}s^1_\rho$ is a scalar multiple of $\omega_\rho \otimes \mathrm{d}s^1_\rho$, we see that μ_ρ is a scalar multiple of ω_ρ. Therefore, the measure of maximal entropy $\mu_\rho \otimes \mathrm{d}s^{\alpha_1}_\rho$ for $\mathsf{U}_{\alpha_1}(\rho)$ is a scalar multiple of $\omega_\rho \otimes \mathrm{d}s^{\alpha_1}_\rho$. □

As an immediate corollary, we obtain an expression for the intersection of two Liouville currents.

Corollary 4.5 *If* $\rho \in \mathcal{H}_m(S)$ *and* $\eta \in \mathcal{H}_d(S)$, *then*

$$i(\omega_\rho, \omega_\eta) = \left\langle \omega_\rho \mid \mathsf{L}^{\mathsf{H}}_\eta \right\rangle.$$

Remark Since symmetric geodesic currents are determined by their periods [29, Theorem 2], our Liouville current ω_ρ pushes forward to Bonahon's Liouville current on $\widehat{\mathcal{G}}(S)$, if $d = 2$, and to the symmetric Liouville current defined by Martone and Zhang [26], if $d > 2$.

5 Liouville volume rigidity

Recall that we define the *Liouville volume* of $\rho \in \mathcal{H}_d(S)$ by

$$\mathrm{vol}_{\mathsf{L}}(\rho) = \left\langle \omega_\rho \mid \mathsf{L}^{\mathsf{H}}_\rho \right\rangle$$

so Corollary 4.5 implies that

$$\mathrm{vol}_{\mathsf{L}}(\rho) = i(\omega_\rho, \omega_\rho).$$

In this section, we apply Corollary 4.5, an argument of Labourie [19, Lemma 5.1] and a length spectrum rigidity result [7, Theorem 11.2] to obtain a Liouville volume rigidity result.

Theorem 5.1 *If $\rho, \eta \in \mathcal{H}_d(S)$, then*

$$\frac{\mathrm{vol}_{\mathsf{L}}(\rho)}{\mathrm{vol}_{\mathsf{L}}(\eta)} \geqslant \left(\inf_{\gamma \in \pi_1(S)-\{1\}} \frac{L_{\mathsf{H}}(\rho(\gamma))}{L_{\mathsf{H}}(\eta(\gamma))} \right)^2 .$$

Moreover, equality holds if and only if either $\rho = \eta$ or $\rho = \eta^*$ where η^* is the contragredient of η.

Notice that, $\inf_{\gamma \in \pi_1(S)-\{1\}} \frac{L_{\mathsf{H}}(\rho(\gamma))}{L_{\mathsf{H}}(\eta(\gamma))}$ is finite and non-zero, since Hitchin representations are well-displacing (see [22, Theorem 6.1.3]). However, if $d > 2$, it can be arbitrarily close to 0 or ∞ (see Zhang [38]).

Proof Let $K = \inf_{\gamma \in \pi_1(S)-\{1\}} \frac{L_{\mathsf{H}}(\rho(\gamma))}{L_{\mathsf{H}}(\eta(\gamma))}$ so that if $\gamma \in \pi_1(S) - \{1\}$, then

$$i(\delta_\gamma, \omega_\rho) = L_{\mathsf{H}}(\rho(\gamma)) \geqslant K L_{\mathsf{H}}(\eta(\gamma)) = K\, i(\delta_\gamma, \omega_\eta).$$

Since ω_ρ and ω_η are both limits of positive linear combinations of currents associated to elements of $\pi_1(S)$, this implies that

$$i(\omega_\rho, \omega_\rho) \geqslant K\, i(\omega_\rho, \omega_\eta) \quad \text{and} \quad i(\omega_\eta, \omega_\rho) \geqslant K\, i(\omega_\eta, \omega_\eta).$$

Therefore, using the fact that i is symmetric,

$$\mathrm{vol}_{\mathsf{L}}(\rho) = i(\omega_\rho, \omega_\rho) \geqslant K\, i(\omega_\rho, \omega_\eta) = K\, i(\omega_\eta, \omega_\rho) \geqslant K^2\, i(\omega_\eta, \omega_\eta) = K^2\, \mathrm{vol}_{\mathsf{L}}(\eta).$$

Now assume that, in addition, $\mathrm{vol}_{\mathsf{L}}(\rho) = K^2\, \mathrm{vol}_{\mathsf{L}}(\eta)$, so

$$i(\omega_\rho, \omega_\rho) = K\, i(\omega_\rho, \omega_\eta) \quad \text{and} \quad i(\omega_\rho, \omega_\eta) = K\, i(\omega_\eta, \omega_\eta).$$

Since $\mathsf{U}_{\mathsf{H}}(\rho)$, $\mathsf{U}_{\mathsf{H}}(\eta)$ and $\mathsf{U}_{\alpha_1}(\eta)$ are all Hölder orbit equivalent to $\mathsf{U}(S)$, we may assume that, up to Hölder conjugacy, there exist positive Hölder functions $g : \mathsf{U}(S) \to \mathbb{R}$ and $j : \mathsf{U}(S) \to \mathbb{R}$ so that

$$\mathrm{d}s^{\mathsf{H}}_\rho = g \mathrm{d}s^{\mathsf{H}}_\eta \quad \text{and} \quad j \mathrm{d}s^{\alpha_1}_\eta = \mathrm{d}s^{\mathsf{H}}_\eta.$$

So, applying Corollary 4.5,

$$\int g \mathrm{d}\omega_\eta \otimes \mathrm{d}s^{\mathsf{H}}_\eta = \int \mathrm{d}\omega_\eta \otimes \mathrm{d}s^{\mathsf{H}}_\rho = i(\omega_\eta, \omega_\rho) = K\, \mathrm{vol}_{\mathsf{L}}(\eta)$$

and

$$\int (g-K)\, \mathrm{d}\omega_\eta \otimes \mathrm{d}s^{\mathsf{H}}_\eta = \int g\, \mathrm{d}\omega_\eta \otimes \mathrm{d}s^{\mathsf{H}}_\eta - K \int \mathrm{d}\omega_\eta \otimes \mathrm{d}s^{\mathsf{H}}_\eta = K\, \mathrm{vol}_{\mathsf{L}}(\eta) - K\, \mathrm{vol}_{\mathsf{L}}(\eta) = 0. \tag{5}$$

On the other hand, since $L_{\mathsf{H}}(\rho(\gamma)) \geqslant K L_{\mathsf{H}}(\eta(\gamma))$,

$$\int_\gamma (g-K)\,\mathrm{d}s_\eta^{\mathsf{H}} = L_{\mathsf{H}}(\rho(\gamma)) - K L_{\mathsf{H}}(\eta(\gamma)) \geqslant 0$$

for all $\gamma \in \pi_1(S) - \{1\}$.

Let $f = (g-K)j$. We will apply the argument of [19, Lemma 5.1] to establish our rigidity claim. If $\gamma \in \pi_1(S)$, then

$$\int_\gamma f \mathrm{d}s_\eta^{\alpha_1} = \int_\gamma (g-K)\,\mathrm{d}s_\eta^{\mathsf{H}} \geqslant 0.$$

Since measures supported on periodic orbits are dense in the space $\mathcal{M}_{\mathsf{U}_{\alpha_1}(\eta)}$ of all flow invariant probability measures on $\mathsf{U}_{\alpha_1}(\eta)$ (see Sigmund [35]), we see that

$$\int f \mathrm{d}\mu \geqslant 0, \tag{6}$$

for all $\mu \in \mathcal{M}_{\mathsf{U}_{\alpha_1}(\eta)}$.

Since μ_η is a multiple of ω_η, Eq. (5) implies that

$$\int f\ \mathrm{d}\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1} = \int (g-K)j\ \mathrm{d}\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1} = \int (g-K)\ \mathrm{d}\mu_\eta \otimes \mathrm{d}s_\eta^{\mathsf{H}} = 0, \tag{7}$$

so

$$\begin{aligned}\sup_{\mu\in\mathcal{M}_{\mathsf{U}_{\alpha_1}(\eta)}} \left(h(\mu) - \int f \mathrm{d}\mu\right) &\leqslant \sup_{\mu\in\mathcal{M}_{\mathsf{U}_{\alpha_1}(\eta)}} h(\mu)\\ &= h\left(\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}\right)\\ &= h\left(\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}\right) - \int f\ \mathrm{d}\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}\\ &\leqslant \sup_{\mu\in\mathcal{M}_{\mathsf{U}_{\alpha_1}(\eta)}} \left(h(\mu) - \int f \mathrm{d}\mu\right)\end{aligned}$$

where the first inequality follows from inequality (6), the equality in the second line holds because $\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}$ is the measure of maximal entropy for $\mathsf{U}_{\alpha_1}(\eta)$, the equality in the third line follows from Eq. (7) and the final inequality holds by definition. Therefore,

$$\mathbf{P}(-f) = \sup_{\mu\in\mathcal{M}_{\mathsf{U}_{\alpha_1}(\eta)}} \left(h(\mu) - \int f \mathrm{d}\mu\right) = h\left(\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}\right) - \int f\ \mathrm{d}\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1},$$

so $\mu_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}$ is the equilibrium state for $-f$. Since $\omega_\eta \otimes \mathrm{d}s_\eta^{\alpha_1}$ is also the equilibrium state for the zero function, [16, Proposition 20.3.10] implies that $-f$ is Livšic cohomologuous to a constant function A. However, $A = 0$ since

$$\int f \mathrm{d}\omega_\eta \otimes \mathrm{d}s_\eta^{\alpha_1} = 0.$$

It follows that for all $\gamma \in \pi_1(S)$,

$$L_{\mathsf{H}}(\rho(\gamma)) - K L_{\mathsf{H}}(\eta(\gamma)) = \int_\gamma f \mathrm{d}s_\eta^{\alpha_1} = 0.$$

Therefore, $L_{\mathsf{H}}(\rho(\gamma)) = K L_{\mathsf{H}}(\eta(\gamma))$ for all $\gamma \in \pi_1(S)$.

We recall that since ρ and σ are projective Anosov, Adρ and Ad(σ) are also projective Anosov (see [15, Section 10.2]). Since $\lambda_1(\mathrm{Ad}\rho(\gamma)) = L_{\mathsf{H}}(\rho(\gamma))$ for all $\gamma \in \pi_1(S)$, [7, Theorem 11.2] implies that $K = 1$ and either Adρ = Adη or Adρ = Adη^*. Therefore, either $\rho = \eta$ or $\rho = \eta^*$. (When $d = 3$, we could apply earlier results of Cooper–Delp [11] or Kim [18]). □

We obtain the following corollary, stated in the introduction as Theorem 1.4, by symmetry.

Corollary 5.2 *If $\rho, \eta \in \mathcal{H}_d(S)$, then*

$$\left(\inf_{\gamma\in\pi_1(S)\setminus\{1\}} \frac{L_{\mathsf{H}}(\rho(\gamma))}{L_{\mathsf{H}}(\eta(\gamma))}\right)^2 \leqslant \frac{\mathrm{vol}_{\mathsf{L}}(\rho)}{\mathrm{vol}_{\mathsf{L}}(\eta)} \leqslant \left(\sup_{\gamma\in\pi_1(S)\setminus\{1\}} \frac{L_{\mathsf{H}}(\rho(\gamma))}{L_{\mathsf{H}}(\eta(\gamma))}\right)^2$$

and equality holds in either inequality if and only if either $\rho = \eta$ or $\rho = \eta^*$.

If $\rho \in \mathcal{H}_3(S)$, Tholozan [36, Theorem 3] showed that there exists a 3-Fuchsian representation $\sigma = \tau_3 \circ \sigma_0$, where $\sigma_0 : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ is Fuchsian and $\tau_3 : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$ is the irreducible representation, so that ρ dominates σ, i.e. $L_{\mathsf{H}}(\rho(\gamma)) \geqslant L_{\mathsf{H}}(\sigma(\gamma))$ for all $\gamma \in \pi_1(S)$. Since $\omega_\sigma = 2\omega_{\sigma_0}$ and $i(\omega_{\sigma_0}, \omega_{\sigma_0}) = \pi^2|\chi(S)|$ (see Bonahon [1, Proposition 15]), Corollary 5.2 implies that $\mathrm{vol}_{\mathsf{L}}(\rho) \geqslant \mathrm{vol}_{\mathsf{L}}(\sigma) = 4\pi^2|\chi(S)|$.

Corollary 5.3 *If $\rho \in \mathcal{H}_3(S)$, then*

$$\mathrm{vol}_{\mathsf{L}}(\rho) \geqslant 4\pi^2|\chi(S)|.$$

Moreover, equality holds if and only if ρ is 3*-Fuchsian.*

If $\rho \in \mathcal{H}_3(S)$, then, see Choi–Goldman [10], there exists a strictly convex open domain Ω_ρ in $\mathbb{RP}^2$ so that $\rho(\pi_1(S))$ acts properly discontinuously and cocompactly on Ω_ρ. It would be interesting to explore the relationship between $\mathrm{vol}_{\mathsf{L}}(\rho)$ and other notions of volume for $\Omega_\rho/\rho(\pi_1(S))$.

If $\sigma = \tau_d \circ \sigma_0 \in \mathcal{H}_d(S)$ is d-Fuchsian, then $\omega_\sigma = (d-1)\omega_{\sigma_0}$, so $\mathrm{vol}_{\mathsf{L}}(\sigma) = (d-1)^2\pi^2|\chi(S)|$. It is known that not every $\rho \in \mathcal{H}_d(S)$ dominates a Fuchsian representation, but one might still ask the following question.

Question *Is it true that, for all $d > 3$,*

$$\mathrm{vol}_{\mathsf{L}}(\rho) \geqslant (d-1)^2\pi^2|\chi(S)|$$

for all $\rho \in \mathcal{H}_d(S)$? If so, does equality hold if and only if ρ is d-Fuchsian?

6 Pressure quadratic forms associated to simple roots

In [8, Section 3], we describe a general procedure for producing pressure metrics on deformation spaces of representations based on the constructions in McMullen [27], Bridgeman [5] and [7]. The first step in the process is to associate a flow to each representation. One then defines an associated pressure intersection and renormalized pressure intersection. Fundamental properties from the thermodynamic formalism, as summarized in Proposition 2.2, then guarantee that the Hessian of the renormalized intersection gives rise to a non-negative quadratic form on the tangent space to the deformation space. The resulting quadratic form may or may not be positive definite and the analysis of its degeneracy is typically the most difficult step in this procedure.

Recall that, in Sect. 3.3, we associated a family $\mathsf{U}_{\alpha_i}(\rho)$ of simple root flows to a Hitchin representation. We interpret the next result to say that this family of flows varies analytically over the Hitchin component.

Proposition 6.1 *For all $i \in \{1, \dots, d-1\}$ and $\rho \in \mathcal{H}_d(S)$, there exists a neighborhood V_i of ρ in $\mathcal{H}_d(S)$, $\nu_i > 0$ and an analytic map $T_i : V_i \to \mathrm{Hol}^{\nu_i}(\mathsf{U}(S))$ such that if $\sigma \in V_i$, then $T_i(\sigma)$ is positive and $\ell_{T_i(\sigma)}(\gamma) = L_{\alpha_i}(\sigma(\gamma))$ for all $\gamma \in \pi_1(S)$.*

Notice that the conclusion of Proposition 6.1 implies that the reparametrization of $\mathsf{U}(S)$ by $T_i(\sigma)$ is Hölder conjugate to $\mathsf{U}_{\alpha_i}(\sigma)$.

Proof Let $\rho \in \mathcal{H}_d(S)$. Proposition 3.6 implies that there exists a neighborhood W_1 of ρ, $\beta_1 > 0$ and an analytic map $S_1 : W_1 \to \mathrm{Hol}^{\beta_1}(\mathsf{U}(S))$, so that

$$\ell_{S_1(\sigma)}(\gamma) = \log \lambda_1(\sigma(\gamma))$$

for all $\gamma \in \pi_1(S)$ and $\sigma \in W_1$. Similarly, since for all $i \in \{2, \dots, d-1\}$, the exterior power $\mathsf{E}^i\tilde{\rho}$ of a lift of ρ is projective Anosov, by Proposition 3.3, Proposition 3.6 implies that there exists a neighborhood W_i of ρ in $\mathcal{H}_d(S)$, $\beta_i > 0$ and an analytic map $S_i : W_i \to \mathrm{Hol}^{\beta_i}(\mathsf{U}(S), \mathbb{R})$ so that if $\sigma \in W_i$, then

$$\ell_{S_i(\sigma)}(\gamma) = \log \lambda_1\left(E^i\tilde{\sigma}(\gamma)\right) = \log\left(\lambda_1(\sigma(\gamma))\,\lambda_2(\sigma(\gamma))\cdots\lambda_i(\sigma(\gamma))\right)$$

for all $\gamma \in \pi_1(S)$.

Let $\widehat{V}_1 = W_1 \cap W_2$ and $\hat{\nu}_1 = \min\{\beta_1, \beta_2\}$ and define an analytic map $\widehat{T}_1 : V_1 \to \mathrm{Hol}^{\hat{\nu}_1}(\mathsf{U}(S))$ by setting $\widehat{T}_1(\sigma) = 2S_1(\sigma) - S_2(\sigma)$. Then

$$\ell_{\widehat{T}_1(\sigma)}(\gamma) = 2\log\lambda_1(\sigma(\gamma)) - \log\lambda_1\left(E^2\sigma(\gamma)\right) = \log\left(\frac{\lambda_1(\sigma(\gamma))}{\lambda_2(\sigma(\gamma))}\right) = L_{\alpha_1}(\sigma(\gamma))$$

for all $\gamma \in \pi_1(S)$ and $\sigma \in \widehat{V}_1$.

More generally, if $i \in \{2, \dots, d-2\}$, let $\widehat{V}_i = W_1 \cap W_2 \cap \cdots \cap W_{i+1}$ and $\hat{\nu}_i = \min\{\beta_1, \dots, \beta_{i+1}\}$, and define $\widehat{T}_i : \widehat{V}_i \to \mathrm{Hol}^{\hat{\nu}_i}(\mathsf{U}(S))$ by setting

$$\widehat{T}_i(\sigma) = 2S_i(\sigma) - S_{i+1}(\sigma) - S_{i-1}(\sigma).$$

One easily checks that $\ell_{\widehat{T}_i(\sigma)}(\gamma) = L_{\alpha_i}(\sigma(\gamma))$ for all $\gamma \in \pi_1(S)$ and $\sigma \in \widehat{V}_i$. Finally, we define $\widehat{T}_{d-1} : \widehat{V}_1 \to \mathrm{Hol}^{\hat{\nu}_{d-1}}(\mathsf{U}(S))$, where $\hat{\nu}_{d-1} = \hat{\nu}_1$, by $\widehat{T}_{d-1}(\sigma) = \widehat{T}_1(\sigma) \circ F$ where $F : \mathsf{U}(S) \to \mathsf{U}(S)$ is given by $F(v) = -v$, and check that $\ell_{\widehat{T}_{d-1}(\sigma)}(\gamma) = L_{\alpha_1}\left(\sigma(\gamma^{-1})\right) = L_{\alpha_{d-1}}(\sigma(\gamma))$ for all $\gamma \in \pi_1(S)$ and $\sigma \in \widehat{V}_{d-1} = \widehat{V}_1$.

It remains to alter each $\widehat{T}_i$ so that, after restricting to a sub-neighborhood of $\widehat{V}_i$, the image consists of positive functions. Since $\widehat{T}_i(\rho)$ has positive periods, it is Livšic cohomologous to a positive τ_i-Hölder function f_i, for some $\tau_i > 0$ (see [34, Lemma 3.8]). Define $T_i : \widehat{V}_i \to \mathrm{Hol}^{\nu_i}(\mathsf{U}(S))$, where $\nu_i = \min\{\hat{\nu}_i, \tau_i\}$, by setting

$$T_i(\sigma) = \widehat{T}_i(\sigma) + \left(f_i - \widehat{T}_i(\rho)\right).$$

We now check that T_i has the properties we claimed.

(1) Since $\widehat{T}_i$ is analytic, and T_i is a translate of $\widehat{T}_i$, T_i is also analytic.

(2) Since $f_i - \widehat{T}_i(\rho)$ is Livšic cohomologous to 0, $T_i(\sigma)$ is Livšic cohomologous to $\widehat{T}_i(\sigma)$. In particular, they have the same periods, so $\ell_{T_i(\sigma)} = \ell_{\widehat{T}_i(\sigma)}(\gamma) = L_{\alpha_i}(\sigma(\gamma))$ for all $\gamma \in \pi_1(S)$ and $\sigma \in \widehat{V}_1$.

(3) Since $\mathsf{U}(S)$ is compact, the set of positive functions is an open subset of $\mathrm{Hol}^{\nu_i}(\mathsf{U}(S))$. Since $T_i(\rho)$ is a positive function and T_i is analytic, hence continuous, there is a neighbourhood $V_i \subset \widehat{V}_i$ of ρ so that $T_i(\sigma)$ is a positive function for all $\sigma \in V_i$. □

We then define the pressure intersection

$$\mathbf{I}_{\alpha_i}(\rho, \eta) = \lim_{T\to\infty} \frac{1}{\#R_{\alpha_i}(\rho, T)} \sum_{\gamma \in R_{\alpha_i}(\rho,T)} \frac{L_{\alpha_i}(\eta(\gamma))}{L_{\alpha_i}(\rho(\gamma))} = \mathbf{I}\left(f^i_\rho, f^i_\eta\right)$$

for all $\rho, \eta \in \mathcal{H}_d(S)$, where

$$R_{\alpha_i}(\rho, T) = \{[\gamma] \in [\pi_1(S)]\backslash\{[1]\} \mid L_{\alpha_i}(\rho(\gamma)) \leqslant T\}$$

and the reparametrizations of $\mathsf{U}(S)$ by f^i_ρ and f^i_η are Hölder conjugate to $\mathsf{U}_{\alpha_i}(\rho)$ and $\mathsf{U}_{\alpha_i}(\eta)$. For fixed $\rho \in \mathcal{H}_d(S)$, we further define $(\mathbf{I}_{\alpha_i})_\rho : \mathcal{H}_d(S) \to \mathbb{R}$ by

$$(\mathbf{I}_{\alpha_i})_\rho(\sigma) = \mathbf{I}_{\alpha_i}(\rho, \sigma)$$

for all $\sigma \in \mathcal{H}_d(S)$. If V_i is the neighorhood of ρ and T_i is the map provided by Proposition 6.1, then

$$\mathbf{I}_{\alpha_i}(\sigma, \eta) = \mathbf{I}(T_i(\sigma), T_i(\eta))$$

for all $\sigma, \eta \in V_i$. By Theorem 3.5, $\mathsf{U}_{\alpha_i}(\sigma)$ has entropy 1, for all $\sigma \in \mathcal{H}_d(S)$ and all i, so

$$\mathbf{I}_{\alpha_i}(\sigma, \eta) = \mathbf{I}(T_i(\sigma), T_i(\eta)) = \mathbf{J}(T_i(\sigma), T_i(\eta))$$

for all $\sigma, \eta \in V_i$. Proposition 2.2 then implies that

$$\mathbf{P}_{\alpha_i}|_{\mathsf{T}_\rho\mathcal{H}_d(S)} = \mathrm{Hess}_\rho(\mathbf{I}_{\alpha_i})_\rho$$

is positive semi-definite and varies analytically over $\mathcal{H}_d(S)$.

By construction, the extended mapping class group and the contragredient preserve each $\mathbf{P}_{\alpha_i}$. It follows immediately from work of Wolpert [37], that the restriction of each $\mathbf{P}_{\alpha_i}$ to the Fuchsian locus is a positive multiple of the Weil–Petersson metric. Since $L_{\alpha_i}(\rho(\gamma)) = L_{\alpha_{d-i}}\left(\rho(\gamma^{-1})\right)$ for all $\rho \in \mathcal{H}_d(S)$ and $\gamma \in \pi_1(S)$, we see that $\mathbf{I}_{\alpha_i}(\rho, \sigma) = \mathbf{I}_{\alpha_{d-i}}(\rho, \sigma)$ for all $\rho, \sigma \in \mathcal{H}_d(S)$, so $\mathbf{P}_{\alpha_i} = \mathbf{P}_{\alpha_{d-i}}$ for all i.

We combine these observations with the non-degeneracy criterion provided by Proposition 2.2 to obtain:

Proposition 6.2 *For each $i \in \{1, \ldots, d-1\}$, there exists a positive semi-definite, analytic, quadratic form $\mathbf{P}_{\alpha_i}$ on $\mathsf{T}\mathcal{H}_d(S)$, which is invariant under the action of the mapping class group and restricts to a multiple of the Weil–Petersson metric on the Fuchsian locus. Moreover, if $\{\rho_t\}_t \in (-\epsilon, \epsilon)$ is a smooth one-parameter family in $\mathcal{H}_d(S)$, then $\|\dot\rho_0\|^2_{\mathbf{P}_{\alpha_i}} = 0$ if and only if*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \langle \gamma \mid \mathsf{L}^{\alpha_i}_{\rho_t}\rangle = \left.\frac{\partial}{\partial t}\right|_{t=0} L_{\alpha_i}(\rho_t(\gamma)) = 0$$

for all $\gamma \in \pi_1(S)$.

Remark Labourie and Wentworth [24] evaluate the original pressure metric at the Fuchsian locus. They remark [24, Section 6.6] that their analysis should extend to the pressure quadratic forms $\mathbf{P}_{\alpha_i}$.

Finally, we observe that, as was claimed in the introduction, we may rewrite the Liouville pressure intersection $\mathbf{I}_{\alpha_1}$ as

$$\mathbf{I}_{\alpha_1}(\rho,\eta)=\frac{1}{\left\langle \omega_\rho \mid \mathsf{L}_\rho^{\alpha_1}\right\rangle}\left\langle \omega_\rho \mid \mathsf{L}_\eta^{\alpha_1}\right\rangle.$$

Notice that, by Theorem 4.4, ω_ρ is a scalar multiple of μ_ρ so $\omega_\rho = c_\rho\mu_\rho$ for some $c_\rho \in \mathbb{R}$. Since $\mu_\rho = \frac{1}{\#R_{\alpha_1}(\rho,T)}\sum_{R_{\alpha_1}(\rho,T)}\frac{\delta_\gamma}{L_{\alpha_1}(\rho(\gamma))}$, we see that $\left\langle \omega_\rho \mid \mathsf{L}_\rho^{\alpha_1}\right\rangle = c_\rho$ and

$$\left\langle \omega_\rho \mid \mathsf{L}_\eta^{\alpha_1}\right\rangle = c_\rho \lim_{T\to\infty}\frac{1}{\#R_{\alpha_1}(\rho,T)}\sum_{R_{\alpha_1}(\rho,T)}\frac{\left\langle \delta_\gamma \mid \mathsf{L}_\eta^{\alpha_1}\right\rangle}{L_{\alpha_1}(\rho(\gamma))} = c_\rho \lim_{T\to\infty}\frac{1}{\#R_{\alpha_1}(\rho,T)}\sum_{R_{\alpha_1}(\rho,T)}\frac{L_{\alpha_1}(\eta(\gamma))}{L_{\alpha_1}(\rho(\gamma))}$$

Therefore,

$$\frac{1}{\left\langle \omega_\rho \mid \mathsf{L}_\rho^{\alpha_1}\right\rangle}\left\langle \omega_\rho \mid \mathsf{L}_\eta^{\alpha_1}\right\rangle = \lim_{T\to\infty}\frac{1}{\#R_{\alpha_1}(\rho,T)}\sum_{R_{\alpha_1}(\rho,T)}\frac{L_{\alpha_1}(\eta(\gamma))}{L_{\alpha_1}(\rho(\gamma))} = \mathbf{I}_{\alpha_1}(\rho,\eta).$$

7 The Liouville pressure quadratic form is a Riemannian metric

The main work of this section is to show that the derivatives of the L_{α_1}-length functions generate the cotangent space of the Hitchin component.

Theorem 1.7 *If $\rho \in \mathcal{H}_d(S)$, then the set*

$$\left\{\mathrm{D}_\rho L_{\alpha_1}^\gamma\right\}_{\gamma\in\pi_1(S)}$$

generates the cotangent space $\mathsf{T}_\rho^*\mathcal{H}_d(S)$.

Theorem 1.7 and Proposition 6.2 together imply that the Liouville pressure quadratic form is a Riemannian metric.

Theorem 1.6 *The Liouville pressure quadratic form $\mathbf{P}_{\alpha_1}$ is a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(S)$, that restricts to a scalar multiple of the the Weil–Petersson metric on the Fuchsian locus.*

Proof of Theorem 1.6 Suppose that $v \in \mathsf{T}_\rho\mathcal{H}_d(S)$ and $\|v\|_{\mathbf{P}_{\alpha_1}} = 0$. Proposition 6.2 implies that $\mathrm{D}_\rho L_{\alpha_1}^\gamma(v) = 0$ for all $\gamma \in \pi_1(S)$. Theorem 1.7 then implies that $v = 0$. Therefore, since we already know it is positive semi-definite, $\mathbf{P}_{\alpha_1}$ is positive definite. The remainder of the theorem follows from Proposition 6.2. □

The remainder of the section will be taken up with the proof of Theorem 1.7. Theorem 1.7 generalizes [7, Proposition 10.1], which asserts that derivatives of the spectral radius functions generate the cotangent space, and its proof follows a similar outline. We use an analysis of the asymptotic behavior of the L_{α_1}-length functions to show that if $\mathrm{D}_\rho L_{\alpha_1}^\gamma(v) = 0$ for all γ, then the derivatives of functions which record the eigenvalues are also trivial in the direction v. We then apply [7, Proposition 10.1] itself to finish the proof, but we could also have observed that the derivatives of all trace functions are trivial in the direction v and applied standard facts about character varieties.

7.1 Transversality results

Let $\widehat{\rho(\gamma)}$ be the lift of $\rho(\gamma)$ to $\mathsf{SL}(d,\mathbb{R})$ so that all of its eigenvalues are positive. Suppose that $\{e_1(\rho(\gamma)),\ldots,e_d(\rho(\gamma))\}$ is a basis of $\mathbb{R}^d$ consisting of eigenvectors for $\rho(\gamma)$ so that

$$\widehat{\rho(\gamma)}\,(e_i(\gamma)) = \lambda_i(\rho(\gamma))\, e_i(\rho(\gamma))$$

for all i. Then we may write

$$\widehat{\rho(\gamma)} = \sum_{i=1}^{d} \lambda_i(\rho(\gamma))\, \mathbf{p}_i(\rho(\gamma))$$

where $\mathbf{p}_i(\rho(\gamma))$ is the projection onto the eigenline spanned by $e_i(\rho(\gamma))$ parallel to the hyperplane spanned by the other basis elements.

In [6], we prove that if α and β have non-intersecting axes and $\rho \in \mathcal{H}_d(S)$, then the bases $\{e_i(\rho(\alpha))\}$ and $\{e_i(\rho(\beta))\}$ have strong transversality properties, which generalize the transversality properties established by Labourie in [20].

Theorem 7.1 [6, Cor. 4.1] *If $\rho \in \mathcal{H}_d(S)$, $\alpha,\beta \in \pi_1(S) - \{1\}$ and α and β have non-intersecting axes, then any d elements of*

$$\{e_1(\rho(\alpha)),\ldots,e_d(\rho(\alpha)), e_1(\rho(\beta)),\ldots,e_d(\rho(\beta))\}$$

span $\mathbb{R}^d$. In particular,

$$\mathbf{p}_i(\rho(\alpha))\big(e_j(\rho(\beta))\big) \neq 0$$

for any $i, j \in \{1,\ldots,d\}$.

If $\rho \in \mathcal{H}_d(S)$ and $\mathsf{S}^2\rho : \pi_1(S) \to \mathsf{SL}\left(\mathsf{S}^2(\mathbb{R}^d)\right)$ is the second symmetric product of a lift of ρ to a representation into $\mathsf{SL}(d,\mathbb{R})$, then

$$\mathsf{S}^2\rho(\gamma) = \sum_{i\leqslant j}^{d} \lambda_i(\rho(\gamma))\,\lambda_j(\rho(\gamma))\,\mathbf{p}_{ij}(\rho(\gamma))$$

and if $\mathsf{E}^2\rho : \pi_1(S) \to \mathsf{SL}\left(\mathsf{E}^2(\mathbb{R}^d)\right)$ is the second exterior product of a lift of ρ to a representation into $\mathsf{SL}(d,\mathbb{R})$, then

$$\mathsf{E}^2\rho(\gamma) = \sum_{i<j}^{d} \lambda_i(\rho(\gamma))\,\lambda_j(\rho(\gamma))\,\mathbf{q}_{ij}(\rho(\gamma))$$

where $\mathbf{p}_{ij}(\rho(\gamma))$ is the projection onto the eigenline spanned by $e_i(\rho(\gamma))\cdot e_j(\rho(\gamma))$ and $\mathbf{q}_{ij}(\rho(\gamma))$ is the projection onto the eigenline $e_i(\rho(\gamma))\wedge e_j(\rho(\gamma))$ parallel to the hyperplane spanned by the other products of basis elements. (Notice that $\mathsf{E}^2\rho$ and $\mathsf{S}^2\rho$ are independent of the choice of lift of ρ to a representation into $\mathsf{SL}(d,\mathbb{R})$.) Then

$$\begin{aligned}
\mathbf{p}_{ii}(\rho(\gamma))(v\cdot w) &= \mathbf{p}_i(\rho(\gamma))(v)\cdot \mathbf{p}_i(\rho(\gamma))(w),\\
\mathbf{p}_{ij}(\rho(\gamma))(v\cdot w) &= \mathbf{p}_i(\rho(\gamma))(v)\cdot \mathbf{p}_j(\rho(\gamma))(w) + \mathbf{p}_j(\rho(\gamma))(v)\cdot \mathbf{p}_i(\rho(\gamma))(w) \quad \text{for } i\neq j, \text{ and}\\
\mathbf{q}_{ij}(\rho(\gamma))(v\wedge w) &= \mathbf{p}_i(\rho(\gamma))(v)\wedge \mathbf{p}_j(\rho(\gamma))(w) - \mathbf{p}_j(\rho(\gamma))(v)\wedge \mathbf{p}_i(\rho(\gamma))(w).
\end{aligned}$$

We use Theorem 7.1 to prove that various terms arising in our asymptotic analysis are non-zero.

Lemma 7.2 *If $\alpha, \beta \in \pi_1(S)$ have non-intersecting axes and $\rho \in \mathcal{H}_d(S)$, then*

(1) $\mathrm{Tr}\left(\mathbf{p}_{ii}(\rho(\alpha))\,\mathbf{p}_{kk}(\rho(\beta))\right) \neq 0,$ *for all $i, k \in \{1, \ldots, d\}$,*
(2) $\mathrm{Tr}\left(\mathbf{q}_{ij}(\rho(\alpha))\,\mathbf{q}_{kl}(\rho(\beta))\right) \neq 0$ *if $i, j, k, l \in \{1, \ldots, d\}$, $i \neq j$ and $k \neq l$,*
(3) $\mathrm{Tr}\left(\mathbf{p}_{ii}(\rho(\alpha))\,\mathsf{S}^2\rho(\beta)\right) \neq 0$ *if $i \in \{1, \ldots, d\}$, and*
(4) $\mathrm{Tr}\left(\mathbf{q}_{ij}(\rho(\alpha))\,\mathsf{E}^2\rho(\beta)\right) \neq 0$ *if $i, j \in \{1, \ldots, d\}$ and $i \neq j$.*

Proof We fix $\rho \in \mathcal{H}_d(S)$ and identify $\rho(\gamma)$ with γ, for all $\gamma \in \pi_1(S)$, throughout the proof for notational simplicity. Choose bases $\{e_1(\alpha), \ldots, e_d(\alpha)\}$ and $\{e_1(\beta), \ldots, e_d(\beta)\}$ and define $\mathbf{t}_{ij}(\alpha, \beta)$ so that

$$\mathbf{p}_i(\alpha)\left(e_j(\beta)\right) = \mathbf{t}_{ij}(\alpha, \beta)e_i(\alpha)$$

for all $i, j \in \{1, \ldots, d\}$. Theorem 7.1 implies that $\mathbf{t}_{ij}(\alpha, \beta) \neq 0$ for all i and j, so

$$\mathrm{Tr}\left(\mathbf{p}_{ii}(\alpha)\mathbf{p}_{kk}(\beta)\right) = \mathbf{t}_{ik}(\alpha, \beta)^2\mathbf{t}_{ki}(\beta, \alpha)^2 \neq 0.$$

If $i < j$ and $k < l$, we define $\mathbf{s}_{ijkl}(\alpha, \beta)$ by the equation

$$e_k(\beta) \wedge e_l(\beta) \bigwedge_{r \neq i,j} e_r(\alpha) = \mathbf{s}_{ijkl}(\alpha, \beta)\left(e_i(\alpha) \wedge e_j(\alpha) \bigwedge_{r \neq i,j} e_r(\alpha)\right).$$

Theorem 7.1 implies that $\mathbf{s}_{ijkl}(\alpha, \beta) \neq 0$, so

$$\mathrm{Tr}\left(\mathbf{q}_{ij}(\alpha)\mathbf{q}_{kl}(\beta)\right) = \mathbf{s}_{ijkl}(\alpha, \beta)\mathbf{s}_{klij}(\beta, \alpha) \neq 0.$$

Notice that we may choose the basis $\{e_i(\beta\alpha\beta^{-1})\}_{i=1}^d = \{\beta\left(e_i(\alpha)\right)\}_{i=1}^d$, in which case

$$\mathsf{S}^2\rho(\beta)\left(e_i(\alpha) \cdot e_i(\alpha)\right) = e_i(\beta\alpha\beta^{-1}) \cdot e_i(\beta\alpha\beta^{-1}).$$

One then computes that

$$\mathrm{Tr}\left(\mathbf{p}_{ii}(\alpha)\mathsf{S}^2\rho(\beta)\right) = \mathbf{t}_{ii}(\alpha, \beta\alpha\beta^{-1})^2 \neq 0.$$

(Notice that if α and β have non-intersecting axes, then so do α and $\beta\alpha\beta^{-1}$.)
Similarly,

$$\mathrm{Tr}\left(\mathbf{q}_{ij}(\alpha)\mathsf{E}^2\rho(\beta)\right) = \mathbf{s}_{ijij}(\alpha, \beta\alpha\beta^{-1}) \neq 0.$$

□

7.2 Trace asymptotics

If $\gamma \in \pi_1(S)$, let $\Lambda^\gamma_{\alpha_1} : \mathcal{H}_d(S) \to \mathbb{R}$ be given by

$$\Lambda^\gamma_{\alpha_1}(\rho) = \frac{\lambda_1\left(\rho(\gamma)\right)}{\lambda_2\left(\rho(\gamma)\right)}$$

and notice that $L^\gamma_{\alpha_1} = \log \Lambda^\gamma_{\alpha_1}$. An asymptotic analysis of traces yields:

Lemma 7.3 *If $\alpha, \beta \in \pi_1(S)$ have non-intersecting axes and $\rho \in \mathcal{H}_d(S)$, then*

$$\lim_{n\to\infty} \frac{\Lambda^{\alpha^n\beta^n}_{\alpha_1}(\rho)}{\Lambda^{\alpha^n}_{\alpha_1}(\rho)\Lambda^{\beta^n}_{\alpha_1}(\rho)} = \frac{\mathrm{Tr}\left(\mathbf{p}_{11}\left(\rho(\alpha)\right)\mathbf{p}_{11}\left(\rho(\beta)\right)\right)}{\mathrm{Tr}\left(\mathbf{q}_{12}\left(\rho(\alpha)\right)\mathbf{q}_{12}\left(\rho(\beta)\right)\right)} \neq 0$$

and

$$\lim_{n\to\infty} \frac{\Lambda_{\alpha_1}^{\alpha^n\beta}(\rho)}{\Lambda_{\alpha_1}^{\alpha^n}(\rho)} = \frac{\mathrm{Tr}\left(\mathbf{p}_{11}\left(\rho(\alpha)\right) \mathsf{S}^2\rho(\beta)\right)}{\mathrm{Tr}\left(\mathbf{q}_{12}(\rho(\alpha)\mathsf{E}^2\rho(\beta)\right)} \neq 0$$

Proof We again fix $\rho \in \mathcal{H}_d(S)$ and identify $\rho(\gamma)$ with γ throughout the proof for notational simplicity. One can compute that

$$\frac{\mathrm{Tr}\left(\mathsf{S}^2\rho(\alpha^n\beta^n)\right)}{\mathrm{Tr}\left(\mathsf{E}^2\rho(\alpha^n\beta^n)\right)} = \frac{\sum_{1\leqslant i\leqslant j\leqslant d}\lambda_i(\alpha^n\beta^n)\lambda_j(\alpha^n\beta^n)}{\sum_{1\leqslant i<j\leqslant d}\lambda_i(\alpha^n\beta^n)\lambda_j(\alpha^n\beta)} = \frac{\lambda_1(\alpha^n\beta^n)^2(1+a_n)}{\lambda_1(\alpha^n\beta^n)\lambda_2(\alpha^n\beta^n)(1+b_n)}$$

Since $\lim_{n\to\infty}\frac{\lambda_j(\alpha^n\beta^n)}{\lambda_i(\alpha^n\beta^n)} = 0$ if $i > j$, by Lemma 3.4, $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

Similarly,

$$\begin{aligned}\frac{\mathrm{Tr}\left(\mathsf{S}^2\rho(\alpha^n)\mathsf{S}^2\rho(\beta^n)\right)}{\mathrm{Tr}\left(\mathsf{E}^2\rho(\alpha^n)\mathsf{E}^2\rho(\beta^n)\right)} &= \frac{\mathrm{Tr}\left(\left(\sum_{i\leqslant j}\lambda_i(\alpha^n)\lambda_j(\alpha^n)\mathbf{p}_{ij}(\alpha)\right)\left(\sum_{i\leqslant j}\lambda_i(\beta^n)\lambda_j(\beta^n)\mathbf{p}_{ij}(\beta)\right)\right)}{\mathrm{Tr}\left(\left(\sum_{i<j}\lambda_i(\alpha^n)\lambda_j(\alpha^n)\mathbf{q}_{ij}(\alpha)\right)\left(\sum_{i<j}\lambda_i(\beta^n)\lambda_j(\beta^n)\mathbf{q}_{ij}(\beta)\right)\right)}\\ &= \Lambda_{\alpha_1}^{\alpha^n}(\rho)\Lambda_{\alpha_1}^{\beta^n}(\rho)\frac{\mathrm{Tr}\left(\mathbf{p}_{11}(\alpha)\mathbf{p}_{11}(\beta)\right)(1+c_n)}{\mathrm{Tr}\left(\mathbf{q}_{12}(\alpha)\mathbf{q}_{12}(\beta)\right)(1+d_n)}\end{aligned}$$

where $c_n \to 0$ and $d_n \to 0$. Since the two expression are equal, we may take limits to obtain the first equality in the statement. Notice that Lemma 7.2 is being used to guarantee that $\mathrm{Tr}\,(\mathbf{p}_{11}(\alpha)\mathbf{p}_{11}(\beta))$ and $\mathrm{Tr}\,(\mathbf{q}_{12}(\alpha)\mathbf{q}_{12}(\beta))$ are non-zero so that the right-hand expression makes sense and is non-zero.

To establish the second equality, we compute that

$$\frac{\mathrm{Tr}\left(\mathsf{S}^2\rho(\alpha^n\beta)\right)}{\mathrm{Tr}\left(\mathsf{E}^2\rho(\alpha^n\beta)\right)} = \frac{\lambda_1(\alpha^n\beta)^2(1+a'_n)}{\lambda_1(\alpha^n\beta)\lambda_2(\alpha^n\beta)(1+b'_n)},$$

where $a'_n \to 0$ and $b'_n \to 0$, and that

$$\begin{aligned}\frac{\mathrm{Tr}\left(\mathsf{S}^2\rho(\alpha^n)\mathsf{S}^2\rho(\beta)\right)}{\mathrm{Tr}\left(\mathsf{E}^2\rho(\alpha^n)\mathsf{E}^2\rho(\beta)\right)} &= \frac{\mathrm{Tr}\left(\left(\sum_{i\leqslant j}\lambda_i(\alpha^n)\lambda_j(\alpha^n)\mathbf{p}_{ij}(\alpha)\right)\mathsf{S}^2\rho(\beta)\right)}{\mathrm{Tr}\left(\left(\sum_{i<j}\lambda_i(\alpha^n)\lambda_j(\alpha^n)\mathbf{q}_{ij}(\alpha)\right)\mathsf{E}^2\rho(\beta)\right)}\\ &= \Lambda_{\alpha_1}^{\alpha^n}(\rho)\frac{\mathrm{Tr}\left(\mathbf{p}_{11}(\alpha)\mathsf{S}^2(\rho(\beta)\right)(1+c'_n)}{\mathrm{Tr}\left(\mathbf{q}_{12}(\alpha)\mathsf{E}^2\rho(\beta)\right)(1+d'_n)}\end{aligned}$$

where $c'_n \to 0$ and $d'_n \to 0$. We obtain the second equation by setting the two expressions above equal, taking limits and applying Lemma 7.2 to guarantee that the right-hand expression makes sense and is non-zero. □

7.3 Derivatives of eigenvalue functions

Let $\lambda_i^\gamma : \mathcal{H}_d(S) \to \mathbb{R}$ be given by $\lambda_i^\gamma(\rho) = \lambda_i\,(\rho(\gamma))$.

Proposition 7.4 *If $v \in T_\rho\mathcal{H}_d(S)$ and $\mathrm{D}_\rho L_{\alpha_1}^\gamma(v) = 0$ for all $\gamma \in \pi_1(S)$, then $\mathrm{D}_\rho\lambda_i^\gamma(v) = 0$, for all $i = 1, \ldots, d$ and all $\gamma \in \pi_1(S)$.*

Notice that the assumptions of Proposition 7.4 are equivalent to the assumption that $\mathrm{D}_\rho\Lambda_{\alpha_1}^\gamma(v) = 0$ for all $\gamma \in \pi_1(S)$. The proof of Proposition 7.4 makes use of the following elementary lemma:

Lemma 7.5 *Let $a_i, b_i, c_i, d_i, w_i \in \mathbb{R}$, for $i = 1, \ldots k$, with $w_1 > w_2 > \cdots > w_k > 0$. If, for every $n \in \mathbb{N}$,*

$$\sum_{i=1}^{k}(a_i + nb_i)w_i^n = \sum_{i=1}^{k}(c_i + nd_i)w_i^n,$$

then $a_i = c_i$ and $b_i = d_i$ for all i.

Proof We first divide by nw_1^n and take the limit to see that

$$b_1 = \lim_{n\to\infty}\frac{1}{nw_1^n}\left(\sum_{i=1}^{k}(a_i + nb_i)w_i^n\right) = \lim_{n\to\infty}\frac{1}{nw_1^n}\left(\sum_{i=1}^{k}(c_i + nd_i)w_i^n\right) = d_1.$$

We then subtract $nb_1w_1^n$ from each side, divide by w_1^n, and pass to a limit to conclude that $a_1 = c_1$.

We may then remove the first order terms and proceed iteratively. □

Proof of Proposition 7.4 We will show that, if $\gamma \in \pi_1(S)$, then $\mathrm{D}_\rho(\log\lambda_i^\gamma)(v) = \mathrm{D}_\rho(\log\lambda_1^\gamma)(v)$ for all i. Since $\lambda_1^\gamma \cdots \lambda_d^\gamma = 1$,

$$\mathrm{D}_\rho(0)(v) = \mathrm{D}_\rho\left(\log\lambda_1^\gamma\right)(v) + \cdots + \mathrm{D}_\rho\left(\log\lambda_d^\gamma\right)(v) = d\,\mathrm{D}_\rho\left(\log\lambda_1^\gamma\right)(v) = 0,$$

which in turn implies that $\mathrm{D}_\rho\lambda_i^\gamma(v) = 0$ for all i.

We first notice that, since $\mathrm{D}_\rho L_{\alpha_1}^\gamma(v) = 0$, $\mathrm{D}_\rho(\log\lambda_2^\gamma)(v) = \mathrm{D}_\rho(\log\lambda_1^\gamma)(v)$ for all $\gamma \in \pi_1(S)$. We proceed iteratively. Assume that $\mathrm{D}_\rho(\log\lambda_i^\gamma)(v) = \mathrm{D}_\rho(\log\lambda_1^\gamma)(v)$ for all $i < m$ and $\gamma \in \pi_1(S)$. Notice that this is equivalent to the claim that $\mathrm{D}_\rho\lambda_i^\gamma(v) = \mathrm{D}_\rho\lambda_1^\gamma(v)$ for all $i < m$ and $\gamma \in \pi_1(S)$.

Fix $\alpha \in \pi_1(S) - \{1\}$ and let β be an element of $\pi_1(S)$, so that α and β have non-intersecting axes and consider the family of analytic functions $\{F_n : \mathcal{H}_d(S) \to \mathbb{R}\}_{n\in\mathbb{N}}$ defined by

$$F_n(\rho) = \frac{\left(\frac{\mathrm{Tr}(\mathbf{p}_{11}(\rho(\alpha))\mathsf{S}^2\rho(\beta^n))}{\mathrm{Tr}(\mathbf{q}_{12}(\rho(\alpha))\mathsf{E}^2\rho(\beta^n))}\right)}{\left(\Lambda_1^\beta(\rho)\right)^n\left(\frac{\mathrm{Tr}\left(\mathbf{p}_{11}(\rho(\alpha))\mathbf{p}_{11}(\rho(\beta))\right)}{\mathrm{Tr}\left(\mathbf{q}_{12}(\rho(\alpha))\mathbf{q}_{12}(\rho(\beta))\right)}\right)}.$$

Notice that, by Lemma 7.3, the numerator of F_n is an analytic function which is a limit of analytic functions which, by assumption, have derivative zero in the direction v, so the numerator has derivative zero in direction v. We may similarly use our assumptions and Lemma 7.3 to show that the denominator of F_n has derivative zero in direction v. Therefore, $\mathrm{D}_\rho F_n(v) = 0$ for all $n \in \mathbb{N}$.

We adopt the shorthand $\lambda_i = \lambda_i^\rho(\beta)$ and expand the above equation to see that

$$F_n(\rho) = \frac{\sum_{i\leqslant j} a_{ij}(\rho)\left(\frac{\lambda_i}{\lambda_1}\right)^n\left(\frac{\lambda_j}{\lambda_1}\right)^n}{\sum_{i<j} b_{ij}(\rho)\left(\frac{\lambda_i}{\lambda_1}\right)^n\left(\frac{\lambda_j}{\lambda_2}\right)^n} = \frac{\sum_{i\leqslant j} a_{ij}(\rho)u_i^n u_j^n}{\sum_{i<j} b_{ij}(\rho)u_i^n v_j^n}$$

where

$$a_{ij}(\rho) = \frac{\mathrm{Tr}\left(\mathbf{p}_{11}\left(\rho(\alpha)\right)\mathbf{p}_{ij}\left(\rho(\beta)\right)\right)}{\mathrm{Tr}\left(\mathbf{p}_{11}\left(\rho(\alpha)\right)\mathbf{p}_{11}\left(\rho(\beta)\right)\right)}, \quad b_{ij}(\rho) = \frac{\mathrm{Tr}\left(\mathbf{q}_{12}\left(\rho(\alpha)\right)\mathbf{q}_{ij}\left(\rho(\beta)\right)\right)}{\mathrm{Tr}\left(\mathbf{q}_{12}\left(\rho(\alpha)\right)\mathbf{q}_{12}\left(\rho(\beta)\right)\right)},$$
$$u_i = \frac{\lambda_i}{\lambda_1}, \text{ and } v_i = \frac{\lambda_i}{\lambda_2}.$$

In particular, $a_{11} = b_{12} = 1$ and, by Lemma 7.2, $b_{1m} \neq 0$ for all m. Since $\mathrm{D}_\rho F_n(v) = 0$ for all $n \in \mathbb{N}$,

$$\left(\mathrm{D}_\rho \left(\sum_{i \leqslant j} a_{ij} u_i^n u_j^n\right)(v)\right)\left(\left(\sum_{k<l} b_{kl} u_k^n v_l^n\right)(\rho)\right) = \left(\left(\sum_{i \leqslant j} a_{ij} u_i^n u_j^n\right)(\rho)\right)\left(\mathrm{D}_\rho \left(\sum_{k<l} b_{kl} u_k^n v_l^n\right)(v)\right)$$

Letting $U_{ijkl} = u_i u_j u_k v_l$, this becomes

$$\sum_{i \leqslant j, k<l} \left(\dot{a}_{ij} b_{kl} + n a_{ij} b_{kl} \left(\frac{\dot{u}_i}{u_i} + \frac{\dot{u}_j}{u_j}\right)\right) U_{ijkl}^n = \sum_{i \leqslant j, k<l} \left(a_{ij} \dot{b}_{kl} + n a_{ij} b_{kl} \left(\frac{\dot{u}_k}{u_k} + \frac{\dot{v}_l}{v_l}\right)\right) U_{ijkl}^n.$$

We group terms where U_{ijkl} agree and order so that, as sets, $\{w_s\}_{s=1}^M = \{U_{ijkl}\}_{i \leqslant j, k<l}$ and $w_i > w_{i+1} > 0$ for all i. We may rewrite the expression above as

$$\sum_{s=1}^M (A_s + n B_s) w_s^n = \sum_{s=1}^M (C_s + n D_s) w_s^n$$

where A_s, B_s, C_s and D_s are constants depending only on s and not on n. Lemma 7.5 implies that $A_s = C_s$ and $B_s = D_s$ for all s.

By our iterative hypothesis, $\mathrm{D}_\rho(\log \lambda_i)(v) = \mathrm{D}_\rho(\log \lambda_1)$ for all $i < m$, and $m > 2$. Therefore, $\dot{u}_i = \dot{v}_i = 0$ for all $i < m$. Since $m > 2$, the iterative step of the proof will be completed if we show that either $\dot{u}_m = 0$ or $\dot{v}_m = 0$.

Consider s_1 such that $w_{s_1} = U_{111m} = v_m$ and notice that

$$B_{s_1} = \sum_{\{i \leqslant j,\ k<l \ \mid \ U_{ijkl} = v_m\}} \left(a_{ij} b_{kl} \left(\frac{\dot{u}_i}{u_i} + \frac{\dot{u}_j}{u_j}\right)\right).$$

If $w_{s_1} = U_{ijkl}$, then $U_{ijkl} = u_i u_j u_k v_l = v_m$. Since $1 = u_1 > u_2 > \cdots > u_d > 0$ and $1 \geqslant v_i > u_i$ for all $i \geqslant 2$, we see that $u_j \geqslant u_i u_j u_k v_l = v_m > u_m$, so $i \leqslant j < m$. Since $\dot{u}_i = 0$ if $i < m$, we see that $B_{s_1} = 0$.

A similar analysis yields that if $U_{ijkl} = u_i u_j u_k v_l = v_m$, then $k < m$ and $l \leqslant m$. Therefore, $\dot{u}_k = 0$ and $\dot{v}_l = 0$ if $l \neq m$. However, if $l = m$, then $i = j = k = 1$, so

$$D_{s_1} = \sum_{\{i \leqslant j, k<l \ \mid \ U_{ijkl} = v_m\}} \left(a_{ij} b_{kl} \left(\frac{\dot{u}_k}{u_k} + \frac{\dot{v}_l}{v_l}\right)\right) = a_{11} b_{1m} \left(\frac{\dot{v}_m}{v_m}\right) = b_{1m} \left(\frac{\dot{v}_m}{v_m}\right),$$

so, since $D_{s_1} = B_{s_1} = 0$, we conclude that $b_{1m} \dot{v}_m = 0$. Since we have previously observed that $b_{1m} \neq 0$, it must be that $\dot{v}_m = 0$ which completes the proof. □

7.4 Proof of Theorem 1.7

If $v \in T_\rho \mathcal{H}_d(S)$ and $\mathrm{D}_\rho L_{\alpha_1}^\gamma(v) = 0$ for all $\gamma \in \pi_1(S)$, then, by Proposition 7.4, $\mathrm{D}_\rho \lambda_i^\gamma(v) = 0$ for all i and all $\gamma \in \pi_1(S)$. However, Proposition 10.1 in [7] guarantees that $\{\mathrm{D}_\rho \lambda_1^\gamma\}_{\gamma \in \pi_1(S)}$ generates the cotangent space to $\mathcal{H}_d(S)$ at ρ, so our proof is complete. □

8 Degeneracy of $\mathbf{P}_{\alpha_n}$ on $\mathcal{H}_{2n}(S)$

Bridgeman [5] showed that the pressure metric on quasifuchsian space is degenerate on the Fuchsian locus. In [8, Section 5.8], we construct a pressure metric on $\mathcal{H}_d(S)$ which is associated to the Hilbert length of elements of the image and similarly prove that this metric is degenerate on the fixed point locus of the contragredient involution. A very similar argument yields that $\mathbf{P}_{\alpha_n}$ is degenerate on $\mathcal{H}_{2n}(S)$.

Recall that the contragredient involution $\tau : \mathcal{H}_{2n}(S) \to \mathcal{H}_{2n}(S)$ fixes the submanifold $\mathcal{H}(S, \mathsf{PSp}(2n))$ of Hitchin representations with image in $\mathsf{PSp}(2n)$.

Proposition 8.1 *The pressure quadratic form* $\mathbf{P}_{\alpha_n}$ *on* $\mathcal{H}_{2n}(S)$ *is degenerate on* $\mathcal{H}(S, \mathsf{PSp}(2n))$. *In particular, if* $\rho \in \mathcal{H}(S, \mathsf{PSp}(2n))$, $v \in \mathsf{T}_\rho\mathcal{H}_d(S)$ *and* $D\tau_\rho(v) = -v$, *then* $||v||_{\mathbf{P}_{\alpha_n}} = 0$.

Proof Suppose that $\rho \in \mathcal{H}(S, \mathsf{PSp}(2n))$, $v \in T_\rho\mathcal{H}_d(S)$ and $D\tau_\rho(v) = -v$. We choose a path $\{\rho_t\}_{t\in(\epsilon,\epsilon)}$ in $\mathcal{H}_{2n}(S)$ such that $\dot{\rho}_0 = v$ and $\tau(\rho_t) = \rho_{-t}$ for all $t \in (-\epsilon, \epsilon)$. Since $\lambda_i\left(\sigma(\gamma^{-1})\right) = (\lambda_{2n-i}(\tau(\sigma)(\gamma)))^{-1}$ for all i and all $\sigma \in \mathcal{H}_{2n}$,

$$L_{\alpha_n}(\rho_t(\gamma)) = \log\left(\frac{\lambda_n(\rho_t(\gamma))}{\lambda_{n+1}(\rho_t(\gamma))}\right) = L_{\alpha_n}(\rho_{-t}(\gamma))$$

for all $t \in (-\epsilon, \epsilon)$ and $\gamma \in \pi_1(S)$. Therefore,

$$\left.\frac{d}{dt}\right|_{t=0} L_{\alpha_n}(\rho_t(\gamma)) = 0$$

for all $\gamma \in \pi_1(S)$, so Proposition 6.2 implies that $\|\dot{\rho}_0\|_{\mathbf{P}_{\alpha_n}} = \|v\|_{\mathbf{P}_{\alpha_n}} = 0$. □

It is natural to wonder whether a similar symmetry is responsible for all degeneracies of pressure metrics constructed in this fashion.

Acknowledgements The authors would like to thank Harrison Bray, Francois Ledrappier, Ralf Spatzier and Tengren Zhang for helpful conversations.

References

1. Bonahon, F.: The geometry of Teichmüller space via geodesic currents. Invent. Math. **92**, 139–162 (1988)
2. Bowen, R.: Periodic orbits for hyperbolic flows. Am. J. Math. **94**, 1–30 (1972)
3. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Springer-Verlag Lecture Notes in Mathematics, vol. 470. Springer, Berlin, Heidelberg (1975)
4. Bowen, R., Ruelle, D.: The ergodic theory of axiom A flows. Invent. Math. **29**, 181–202 (1975)
5. Bridgeman, M.: Hausdorff dimension and the Weil–Petersson extension to quasifuchsian space. Geom. Top. **14**, 799–831 (2010)
6. Bridgeman, M., Canary, R., Labourie, F.: Simple length rigidity for Hitchin representations. http://front.math.ucdavis.edu/1703.07336 (2017)
7. Bridgeman, M., Canary, R., Labourie, F., Sambarino, A.: The pressure metric for Anosov representations. Geom. Funct. Anal. **25**, 1089–1179 (2015)
8. Bridgeman, M., Canary, R., Sambarino, A.: An introduction to pressure metrics on higher Teichmüller spaces. Ergod. Theory Dyn. Syst. arXiv:1507.03894 **(to appear)**
9. Champetier, C.: Petite simplification dans les groupes hyperboliques. Ann. Fac. Sci. Toulouse Math. **3**, 161–221 (1994)
10. Choi, S., Goldman, W.: The classification of real projective structures on compact surfaces. Bull. AMS **34**, 161–171 (1997)
11. Cooper, D., Delp, K.: The marked length spectrum of a projective manifold or orbifold. Proc. AMS **138**, 3361–3376 (2010)

12. Croke, C., Dairbekov, N.: Lengths and volumes in Riemannian manifolds. Duke Math. J. **125**, 1–14 (2004)
13. Gromov, M.: Hyperbolic groups. In: Gersten, S.M. (ed) Essays in Group Theory. MSRI Publications, vol. 8, Springer, New York, NY (1987)
14. Guichard, O.: Composantes de Hitchin et représentations hyperconvexes de groupes de surface. J. Diff. Geom **80**, 391–431 (2008)
15. Guichard, O., Wienhard, A.: Anosov representations: domains of discontinuity and applications. Invent. Math. **190**, 357–438 (2012)
16. Hasselblatt, B., Katok, A.: Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, Cambridge (1997)
17. Hitchin, N.: Lie groups and Teichmüller space. Topology **31**, 449–473 (1992)
18. Kim, I.: Rigidity and deformation space of strictly convex real projective structures. J. Diff. Geom. **58**, 189–218 (2001)
19. Labourie, F.: Fuchsian affine actions of surface groups. J. Diff. Geom. **59**, 15–31 (2001)
20. Labourie, F.: Anosov flows, surface groups and curves in projective space. Invent. Math. **165**, 51–114 (2006)
21. Labourie, F.: Cross ratios, surface groups, $\mathsf{S}L(n, \mathbb{R})$ and diffeomorphisms of the circle. Publ. Math. IHES **106**, 139–213 (2007)
22. Labourie, F.: Cross ratios, Anosov representations and the energy functional on Teichmüller space. Ann. Sci. ENS **41**, 437–469 (2008)
23. Labourie, F., McShane, G.: Cross ratios and identities for higher Thurston theory. Duke Math. J. **148**, 279–345 (2009)
24. Labourie, F., Wentworth, R.: Variations along the Fuchsian locus. Ann. Sci. Éc. Norm. Supér. **41**(3), 437–469
25. Livšic, A.: Cohomology of dynamical systems. Math. USSR Izvestija **6**, 1296–1320 (1972)
26. Martone, G., Zhang, T.: Positively ratioed representations. Comm. Math. Helv. arXiv:1609.01245 **(to appear)**
27. McMullen, C.: Thermodynamics, dimension and the Weil–Petersson metric. Invent. Math. **173**, 365–425 (2008)
28. Mineyev, I.: Flows and joins of metric spaces. Geom. Top. **9**, 403–482 (2005)
29. Otal, J.P.: Le spectre marqué des longueurs des surfaces à courbure négative. Ann. Math. **131**, 151–162 (1990)
30. Parry, W., Pollicott, M.: Zeta functions and the periodic orbit structure of hyperbolic dynamics. Astérisque **187–188**, 1–268 (1990)
31. Pollicott, M.: Symbolic dynamics for Smale flows. Am. J. Math. **109**, 183–200 (1987)
32. Potrie, R., Sambarino, A.: Eigenvalues and entropy of a Hitchin representation. Invent. Math. **209**, 885–925
33. Ruelle, D.: Thermodynamic Formalism. Addison-Wesley, London (1978)
34. Sambarino, A.: Quantitative properties of convex representations. Comm. Math. Helv. **89**, 443–488 (2014)
35. Sigmund, K.: On the space of invariant measures for hyperbolic flows. Am. J. Math. **94**, 31–37 (1972)
36. Tholozan, N.: Volume entropy of Hilbert metrics and length spectrum of Hitchin representations into $\mathsf{P}SL(3, \mathbb{R})$. Duke Math. J. **166**, 1377–1403 (2017)
37. Wolpert, S.: Thurston's Riemannian metric for Teichmüller space. J. Diff. Geom. **23**, 143–174 (1986)
38. Zhang, T.: Degeneration of Hitchin representations along internal sequences. Geom. Funct. Anal. **25**, 1588–1645 (2015)