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Eigenvalues and entropy of a Hitchin representation

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Abstract We show that the critical exponent of a representation ρ in the Hitchin component of $\mathrm{PSL}(d, \mathbb{R})$ is bounded above, the least upper bound being attained only in the Fuchsian locus. This provides a rigid inequality for the area of a minimal surface on $\rho\backslash X$, where X is the symmetric space of $\mathrm{PSL}(d, \mathbb{R})$. The proof relies in a construction useful to prove a regularity statement: if the Frenet equivariant curve of ρ is smooth, then ρ is Fuchsian.

Contents

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1 Introduction

Let Σ be a closed orientable surface of genus ≥ 2. A representation $\pi_1\Sigma \to \mathrm{PSL}(d, \mathbb{R})$ is *Fuchsian* if it factors as

$$\pi_1\Sigma \to \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R}),$$

where the first arrow is a choice of a hyperbolic metric on Σ, and the second arrow is the (unique up to conjugation) irreducible linear action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^d$.[1]

A *Hitchin component* of $\mathrm{PSL}(d, \mathbb{R})$ is a connected component of

$$\mathfrak{X}(\pi_1\Sigma, \mathrm{PSL}(d, \mathbb{R})) = \hom(\pi_1\Sigma, \mathrm{PSL}(d, \mathbb{R}))/\,\mathrm{PSL}(d, \mathbb{R})$$

that contains a Fuchsian representation. Hitchin [23] proved that there are either one, or two Hitchin components (according to d odd or even respectively), and that each of these components is diffeomorphic to an open $|\chi(\Sigma)| \cdot \dim \mathrm{PSL}(d, \mathbb{R})$-dimensional Euclidean ball. When $d = 2$ these two components correspond to the Teichmüller space of Σ with a fixed orientation. A Hitchin component appears then as a higher rank generalization of Teichmüller space. Denote by $\mathrm{Hitchin}(\Sigma, d)$ this (these) component(s).

The analogy with Teichmüller space is carried on. Labourie [27] shows that a representation in $\mathrm{Hitchin}(\Sigma, d)$ (from now on a *Hitchin representation*) is discrete, irreducible and faithful, and consists of purely loxodromic elements. Guichard–Wienhard [21] proved that Hitchin components are deformation spaces of geometric structures on closed manifolds. Bridgeman–Canary–Labourie–Sambarino [12] provide a Weil–Petersson-type Riemannian metric on $\mathrm{Hitchin}(\Sigma, d)$, invariant under the mapping class group of Σ.

Denote by X the symmetric space of $\mathrm{PSL}(d, \mathbb{R})$, and by d_X a distance on X induced by a $\mathrm{PSL}(d, \mathbb{R})$-invariant Riemannian metric on X. If Δ is a discrete subgroup of $\mathrm{PSL}(d, \mathbb{R})$, the *critical exponent* of Δ is defined by

[1] This is standard, see Guichard [20] for an explicit construction.

$$h_X(\Delta) = \lim_{s\to\infty} \frac{\log \#\{g \in \Delta : d_X(o, g\cdot o) \leq s\}}{s},$$

for some (any) $o \in X$.

Introduced by Margulis [31] in the negatively curved setting, this invariant associated to a discrete group of isometries has been object of numerous deep results. Recall for example the Patterson–Sullivan theory used for precise orbital counting, or its rigid structure due to Besson–Courtois–Gallot [7], Bowen [10] and Bourdon [9], just to name a few.

This paper is concerned on the rigidity problem for Hitchin representations (the orbital counting problem has already been treated in [37]). Normalize d_X so that the totally geodesic embedding of $\mathbb{H}^2$ in X, induced by the morphism $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$ has curvature -1. The main result of this work is the following theorem.

Theorem A *For all* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *one has* $h_X(\rho(\pi_1\Sigma)) \leq 1$ *and equality only holds if* ρ *is Fuchsian.*

Theorem A confirms the current philosophy that deformations in higher rank spaces should decrease the critical exponent, as opposed to deformations on rank 1 spaces (i.e. pinched negative curvature) where the critical exponent increases (see Bowen's fundamental paper [10] on quasi-Fuchsian representations). It would be interesting to find a global explanation for these two different phenomena, today understood independently: in rank 1 the critical exponent is the Hausdorff dimension of the limit set, bounded below by the topological dimension; in higher rank (as we shall see below) it is the possibility of growing in different directions that forces h_X to decrease.

This philosophy probably originated in Bishop–Steger's work [8], where they show that if $\rho, \eta \in \mathrm{Hitchin}(\Sigma, 2)$ then

$$h^{(1,1)}(\rho, \eta) = \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\pi_1\Sigma] : |\rho\gamma| + |\eta\gamma| \leq s\}}{s} \leq 1/2,$$

where $|g|$ is the translation distance of g in $\mathbb{H}^2$ and $[\pi_1\Sigma]$ denotes the set of conjugacy classes of $\pi_1\Sigma$. Moreover, equality implies $\rho = \eta$. As noticed by Burger [13], this is a rank-2 problem, associated to the product representation $\rho \times \eta : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

An analogous result holds for *Benoist representations*.[2] These are homomorphisms $\rho : \Gamma \to \mathrm{PGL}(n+1,\mathbb{R})$ where Γ is a word-hyperbolic group, such that $\rho(\Gamma)$ preserves an open convex set $\Omega \subset \mathbb{P}(\mathbb{R}^{n+1})$ properly contained on an affine chart, and such that the quotient $\rho(\Gamma)\backslash\Omega$ is compact. The

[2] These are also called *divisible convex sets with strictly convex boundary,* or *strictly convex projective structures on closed manifolds.*

Hilbert metric on Ω induces a $\rho(\Gamma)$-invariant Finsler metric on Ω. Crampon [15] proved that the topological entropy of the geodesic flow on $\mathsf{T}^1\rho(\Gamma)\backslash\Omega$ associated to this metric, is bounded above by $n-1$ and equality only holds if Ω is an ellipsoid. We provide a new proof of Crampon's result in Sect. 7.

It is consequence of Choi–Goldman's work [14] that the space of Benoist representations of $\pi_1\Sigma$ coincides with $\mathrm{Hitchin}(\Sigma, 3)$.

Before explaining the main ideas of the proof let us remark that, as explained by Labourie [26, Section 1.4], the inequality in Theorem A implies a (rigid) inequality concerning the area of a minimal surface on $\rho(\pi_1\Sigma)\backslash X$. Recall from Labourie [28] that the *minimal area* of ρ is defined by

$$\mathrm{MinArea}(\rho) = \inf\{e_\rho(J) : J \in \mathrm{Hitchin}(\Sigma, 2)\},$$

where $e_\rho(J)$ is the energy of the unique harmonic ρ-equivariant map from Σ equipped with J to $\rho(\pi_1\Sigma)\backslash X$. It follows from Hitchin's construction that such a harmonic map is an immersion (see Sanders [39] for details). Standard computations imply that the metric induced on this immersed surface is necessarily negatively curved and hence its topological entropy is bounded above by $h_X(\rho(\pi_1\Sigma))$. Applying a theorem of Katok [24, Theorem B] one has

$$\mathrm{MinArea}(\rho) \geq \frac{-2\pi\chi(\Sigma)}{h_X^2(\rho(\pi_1\Sigma))},$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Consequently, Theorem A implies the following:

Corollary 1.1 *Let* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *then*

$$\mathrm{MinArea}(\rho) \geq -2\pi\chi(\Sigma)$$

and equality only holds if ρ *is Fuchsian.*

This is a theorem of Labourie [26, Theorem 1.4.1] when the Zariski closure of ρ has rank 2, proved using Higgs bundles techniques.

Finally, let us note that Theorem A is still open for the Hitchin components of the real split simple groups $\mathrm{PSO}(n, n)$ $(n \geq 4)$ and the exceptional real split Lie groups (except G_2). This is due to the fact that the Frenet property of Labourie's equivariant flag curve (see below) is only known to hold for $\mathrm{Hitchin}(\Sigma, d)$ (and hence for the groups $\mathrm{PSp}(2k, \mathbb{R})$, $\mathrm{PSO}(k, k+1)$ and G_2, since their respective Hitchin components are canonically embeded in $\mathrm{Hitchin}(\Sigma, d)$ for $d = 2k$, $2k+1$ and 7 respectively).

1.1 Proof of Theorem A: The asymptotic location of eigenvalues

The general method is not specific to the Hitchin component. Indeed, our method applied in different situations gives an improvement of Crampon's result and a generalization of Bishop–Steger's theorem to arbitrary products such as

$$\mathrm{Hitchin}(\Sigma, d_1) \times \cdots \times \mathrm{Hitchin}(\Sigma, d_k),$$

replacing $1/2$ with a proper upper bound. We will explain here how the idea works in the Hitchin component, and leave to Sect. 7 the case of Benoist's representations.

The first step of the proof of Theorem A reposes on some previous results of Quint [34] and Sambarino [35] which relate the critical exponent with the (asymptotic) location of the eigenvalues of a Hitchin representation.

Let $\mathfrak{a} = \{a \in \mathbb{R}^d : a_1 + \cdots + a_d = 0\}$ be a Cartan subalgebra of $\mathfrak{sl}(d, \mathbb{R})$ and denote by $\varepsilon_i(a) = a_i$. Let

$$\mathfrak{a}^+ = \{a \in \mathfrak{a} : a_1 \geq \cdots \geq a_d\}$$

be a closed Weyl chamber and $\Pi = \{\sigma_i = \varepsilon_i - \varepsilon_{i+1} \in \mathfrak{a}^* : i \in \{1, \ldots, d-1\}\}$ the set of simple roots associated to the choice of $\mathfrak{a}^+$. Denote by $\lambda : \mathrm{PSL}(d, \mathbb{R}) \to \mathfrak{a}^+$ the *Jordan projection*:

$$\lambda(g) = (\lambda_1(g), \ldots, \lambda_d(g)),$$

consisting on the log of the modulus of the eigenvalues of g (possibly with repetition) and in decreasing order.

For $\rho \in \mathrm{Hitchin}(\Sigma, d)$ denote by $\mathscr{L}_\rho$ the closed cone of $\mathfrak{a}^+$ generated by $\{\lambda(\rho\gamma) : \gamma \in \pi_1\Sigma\}$. This cone contains all possible directions where $\lambda(\rho(\pi_{\mathbf{1}}\Sigma))$ is. A finer invariant is to understand *how many* eigenvalues of ρ are on a given direction inside $\mathscr{L}_\rho$. Denote by $\mathscr{L}_\rho^* = \{\varphi \in \mathfrak{a}^* : \varphi|\mathscr{L}_\rho \geq 0\}$ the *dual cone* of $\mathscr{L}_\rho$. For $\varphi \in \mathscr{L}_\rho^*$ define its *entropy* by

$$h_\rho^\varphi = \lim_{s \to \infty} \frac{\log \#\{[\gamma] \in [\pi_1\Sigma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}.$$

A linear form φ belongs to the interior of $\mathscr{L}_\rho^*$ if and only if h_ρ^φ is finite and positive (Lemma 2.7). The main object we are interested in is the set

$$\mathcal{D}_\rho = \{\varphi : h_\rho^\varphi \in (0, 1]\}.$$

Proposition 4.11 states that $\mathcal{D}_\rho$ is a convex subset of $\mathfrak{a}^*$, and the formula $h_\rho^{t\varphi} = h_\rho^\varphi / t$ implies that if $\varphi \in \mathcal{D}_\rho$ then $t\varphi \in \mathcal{D}_\rho$ for all $t \geq 1$. Moreover, its boundary $\partial\mathcal{D}_\rho = \{\varphi : h^\varphi_\rho = 1\}$ is a codimension 1 closed analytic submanifold of $\mathfrak{a}^*$. The shape of $\mathcal{D}_\rho$ will be crucial in the sequel.

Recall that d_X is a distance on X induced by a $\mathrm{PSL}(d, \mathbb{R})$-invariant Riemannian metric on X. Denote by $\| \ \|_\mathfrak{a}$ the Euclidean norm on $\mathfrak{a}$ (invariant under the Weyl group) induced by d_X, and by $\| \ \|_{\mathfrak{a}^*}$ the induced norm on $\mathfrak{a}^*$. One has the following result.[3]

Proposition 1.2 (Quint [34, Corollary 3.1.4] + [35, Corollary 4.4]) *Let* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *then*

$$h_X(\rho(\pi_1\Sigma)) = \min\{\|\varphi\|_{\mathfrak{a}^*} : \varphi \in \mathcal{D}_\rho\}.$$

Example 1.3 The irreducible linear action $\tau_d : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R})$ is given by the canonical action of $\mathrm{PSL}(2, \mathbb{R})$ on the $(d-1)$-symmetric power $\mathsf{S}^{d-1}(\mathbb{R}^2)$ of $\mathbb{R}^2$. If $g \in \mathrm{PSL}(2, \mathbb{R})$ one has $\lambda_{\mathrm{PSL}(2,\mathbb{R})}(g) = (|g|/2, -|g|/2)$, where $|g|$ denotes the translation distance of g, and hence

$$\lambda(\tau_d g) = \frac{|g|}{2}(d-1, d-3, \ldots, 3-d, 1-d).$$

Thus, for all $\sigma \in \Pi$ one has $\sigma(\lambda(\tau_d g)) = |g|$. Moreover if φ belongs to the affine hyperplane generated by Π,

$$V_\Pi = \left\{ \sum_{\sigma\in\Pi} t_\sigma \sigma : \sum t_\sigma = 1 \right\},$$

then $\varphi(\lambda(\tau_d g)) = |g|$. Consequently, if $\rho_0 \in \mathrm{Hitchin}(\Sigma, d)$ is Fuchsian then $\partial\mathcal{D}_{\rho_0} = V_\Pi$. Since d_X is normalized such that the totally geodesic embedding of $\mathbb{H}^2$ in X to have curvature -1, the Fuchsian representation ρ_0 has critical exponent equal to 1. One concludes, using Proposition 1.2, that

$$\min\{\|\varphi\|_{\mathfrak{a}^*} : \varphi \in V_\Pi\} = 1$$

and this minimum is realized in the dual space of the Cartan algebra

$$\{(d-1, d-3, \ldots, 3-d, 1-d)t : t \in \mathbb{R}\}$$

of $\tau_d(\mathrm{PSL}(2, \mathbb{R}))$.

The proof of Theorem A consists in a deeper understanding of the set $\mathcal{D}_\rho$ for a given $\rho \in \mathrm{Hitchin}(\Sigma, d)$, and its relative position with respect to V_Π.

[3] Proposition 1.2 actually holds on a much more general setting, see Sect. 1.4.

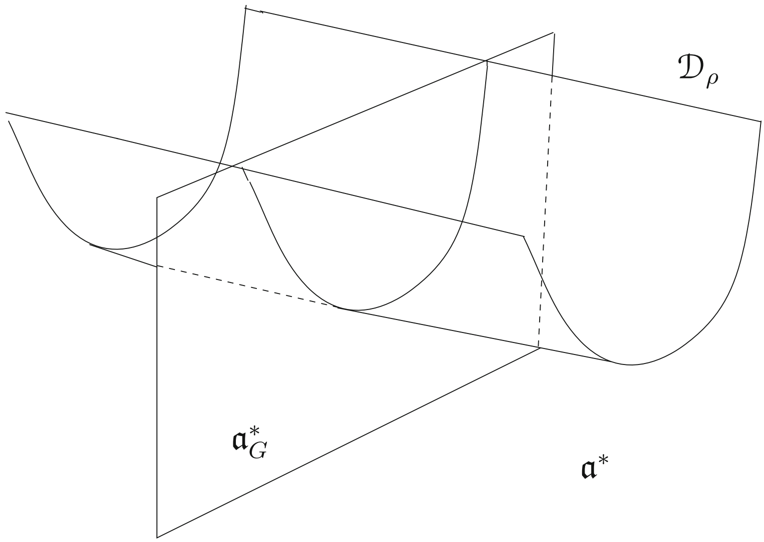

Fig. 1 The set $\mathcal{D}_\rho$ when $\mathfrak{a}_G^*$ is a strict subspace of $\mathfrak{a}^*$

Denote by $G = G_\rho$ the Zariski closure of $\rho(\pi_1\Sigma)$. The group G is necessarily semisimple.[4] Choose a Cartan subalgebra $\mathfrak{a}_G \subset \mathfrak{a}$ and a Weyl chamber $\mathfrak{a}_G^+ \subset \mathfrak{a}^+$. Consider the restriction map $\mathrm{rt} : \mathfrak{a}^* \to \mathfrak{a}_G^*$, defined by $\mathrm{rt}(\varphi) = \varphi|\mathfrak{a}_G$. Observe that, since the vector space spanned by $\{\lambda(\rho\gamma) : \gamma \in \pi_1\Sigma\}$ is $\mathfrak{a}_G$, the entropy of a given linear form φ, is the entropy of $\mathrm{rt}(\varphi)$.

Remark 4.10 and Proposition 4.11 below imply that $\mathrm{rt}(\mathcal{D}_\rho)$ is strictly convex. Since $\| \|_{\mathfrak{a}}$ is Euclidean one can (and will) identify the space $\mathfrak{a}_G^*$ with a subspace of $\mathfrak{a}^*$. Namely, denote by $p_G : \mathfrak{a} \to \mathfrak{a}_G$ the orthogonal projection, then

$$\mathfrak{a}_G^* = \{\varphi \in \mathfrak{a}^* : \varphi = \varphi \circ p_G.\}.$$

The set $\mathcal{D}_\rho$ is hence a convex set, whose intersection with $\mathfrak{a}_G^*$ is strictly convex (see Fig. 1).

The second important step in the proof of Theorem A is the following theorem, its statement arose from an insightful discussion between the second author with Bertrand Deroin and Nicolas Tholozan.

Theorem B *For every* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *and* $\sigma \in \Pi$ *one has* $h_\rho^\sigma = 1$.

Theorem B states that the simple roots σ always belong to $\partial\mathcal{D}_\rho$, regardless of $\rho \in \mathrm{Hitchin}(\Sigma, d)$. Let us explain how this implies Theorem A.

[4] It is reductive, since it acts irreducibly on $\mathbb{R}^d$ (Labourie [27, Lemma 10.1]) and has no center, since moreover $\forall\gamma \in \pi_1\Sigma$, $\rho(\gamma)$ is proximal (see Benoist [6]).

Proof of Theorem A Let Δ_Π be the convex hull of Π, denote by $\operatorname{int}\Delta_\Pi$ its relative interior and consider $\rho \in \operatorname{Hitchin}(\Sigma, d)$. Since $\mathcal{D}_\rho$ is convex and $\Pi \subset \partial\mathcal{D}_\rho$ one has $\Delta_\Pi \subset \mathcal{D}_\rho$. Hence, Proposition 1.2 and the computations in Example 1.3 give

$$h_X(\rho) = \min\{\|\varphi\|_{\mathfrak{a}^*} : \varphi \in \mathcal{D}_\rho\} \leq \min\{\|\varphi\|_{\mathfrak{a}^*} : \varphi \in \Delta_\Pi\} = 1.$$

If $h_X(\rho) = 1$, then the intersection $\partial\mathcal{D}_\rho \cap \operatorname{int}\Delta_\Pi$ is non-empty, thus $\operatorname{int}\Delta_\Pi \subset \partial\mathcal{D}_\rho$. Moreover, since $\partial\mathcal{D}_\rho$ is closed one has $\Delta_\Pi \subset \partial\mathcal{D}_\rho$.

Since $\mathcal{D}_\rho \cap \mathfrak{a}_G^*$ is strictly convex, the only possibility is for $\mathfrak{a}_G^*$ to be 1-dimensional, i.e. the Zariski closure of ρ has rank 1.[5] Moreover, $\mathfrak{a}_G = \{(d-1, d-3, \ldots, 1-d)t : t \in \mathbb{R}\}$. Since a purely loxodromic matrix does not commute with a one-parameter compact group, G_ρ is simple and actually its Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ (recall the classification of rank 1 real-algebraic simple Lie groups). Hence, the group G_ρ is a finite covering of $\operatorname{PSL}(2,\mathbb{R})$. Since G_ρ is linear the connected component of the identity $(G_\rho)_0$ is isomorphic to $\operatorname{PSL}(2,\mathbb{R})$. Since ρ can be connected to a Fuchsian representation, for every $\gamma \in \pi_1\Sigma$ there exists a path, through purely loxodromic matrices, from $\rho(\gamma)$ to a diagonalizable matrix with eigenvalues of the same sign. This implies that $\rho(\gamma)$ has all its eigenvalues of the same sign and hence belongs to $(G_\rho)_0$. This completes the proof. □

In fact, Theorem B and the last proof provide a rigid upper bound for the entropy of each linear form in the interior of the dual cone $(\mathfrak{a}^+)^*$. Indeed, if $\varphi \in \operatorname{int}(\mathfrak{a}^+)^*$ then it is a linear combination of elements in Π with (strictly) positive coefficients, i.e. the half line $\mathbb{R}_+ \cdot \varphi$ intersects $\operatorname{int}\Delta_\Pi$. Notice that h_ρ^φ is the only number such that

$$h_\rho^\varphi \varphi \in \partial\mathcal{D}_\rho.$$

The upper bound of $\rho \mapsto h_\rho^\varphi$ is hence the number $c(\varphi)$ such that $c(\varphi)\varphi \in \Delta_\Pi$ (see Fig. 2).

Corollary 1.4 *Consider* $\varphi \in \operatorname{int}(\mathfrak{a}^+)^*$, *then for all* $\rho \in \operatorname{Hitchin}(\Sigma, d)$ *one has* $h_\rho^\varphi \leq c(\varphi)$, *and equality only holds if* ρ *is Fuchsian.*

In particular, considering the linear form $\varphi_{1d}(a) = (a_1 - a_d)/2 = (\sum \sigma_i)/2$, one has $h_\rho^{\varphi_{1d}} \leq 2/(d-1)$. Also, notice that $\varphi_1(a) = a_1 = \frac{1}{d}\sum_{j=1}^{d-1}(d-j)\sigma_j$ therefore one also has $c(\varphi_1) = 2/(d-1)$.

[5] A recent classification of possible Zariski closures of a Hitchin representation, obtained by Guichard [19], implies directly that G_ρ is isomorphic to $\operatorname{PSL}(2,\mathbb{R})$. In our present situation a direct proof of this fact is possible and easy, so we include it for completeness.

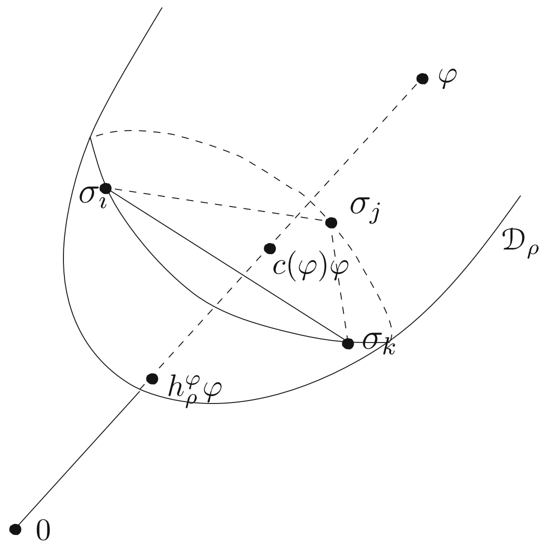

Fig. 2 The simple roots force the linear form in $\mathcal{D}_\rho$ closest to the origin, to be below a certain affine subspace

In [38, Corollary 3.4] a similar inequality is proved, namely $\alpha h_\rho^{\varphi_1} \leq 2/(d-1)$, where α is the Hölder exponent of Labourie's equivariant flag curve (see below) for a visual metric on $\partial\pi_1\Sigma$ (induced by a choice of a hyperbolic metric on Σ). These two rigid inequalities are different in nature: while equality in Corollary 1.4 implies that a totally geodesic copy of $\mathbb{H}^2$ is preserved, [38, Corollary 3.4] states that equality in $\alpha h_\rho^{\varphi_1} \leq 2/(d-1)$ recognizes a specific representation in $\tau_d(\mathrm{PSL}(2,\mathbb{R}))$.

It is interesting to remark that the same argument shows the existence of linear forms whose entropy is bounded from *below* (when defined). For example: $(1+\varepsilon_1)\sigma_1 - \sum_2^d \varepsilon_i\sigma_i$ for small enough $\varepsilon_i > 0$ works.

Furthermore, the special shape of $\partial\mathcal{D}_\rho$ actually provides a 'simple' criterion to determine the rank of the Zariski closure of a Hitchin representation. Observe that Δ_Π is a $(d-1)$-dimensional simplex. Let $F_k \subset \Delta_\Pi$ be a k-dimensional face and denote by int F_k its relative interior.

Corollary 1.5 *Consider* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *and assume that* $(\mathrm{int}\, F_k) \cap \partial\mathcal{D}_\rho \neq \emptyset$, *then* $\mathrm{rank}(G_\rho) \leq \dim \mathfrak{a} - k$.

Proof As in the proof of Theorem A, the fact that $(\mathrm{int}\, F_k) \cap \partial\mathcal{D}_\rho \neq \emptyset$ implies that $F_k \subset \partial\mathcal{D}_\rho$. Since $\partial\mathcal{D}_\rho$ is a closed analytic submanifold of $\mathfrak{a}$ (Proposition 4.11), one concludes that the affine space V_{F_k} spanned by F_k is contained in $\partial\mathcal{D}_\rho$.

Recall that $\mathcal{D}_\rho \cap \mathfrak{a}^*_{G_\rho}$ is strictly convex, thus $\mathfrak{a}^*_{G_\rho}$ is transverse to a k-dimensional affine space. Hence $\dim \mathfrak{a}_{G_\rho} + k \leq \dim \mathfrak{a}$. This finishes the proof.

1.2 Theorem B: Finding a suitable Anosov flow

The proof of Theorem B is based on the following (SRB)-principle (Corollary 2.13): If ϕ is a $\mathrm{C}^{1+\alpha}$ Anosov flow on a closed manifold X, and $\lambda^u : X \to \mathbb{R}_+$ denotes the infinitesimal expansion rate in the unstable direction, then the reparametrization of ϕ by λ^u has topological entropy equal to 1.

The proof of Theorem B goes by finding, for each $i \in \{2, \ldots, d-1\}$, an Anosov flow whose periodic orbits are indexed in $[\pi_1\Sigma]$, such that the total expansion rate along the periodic orbit $[\gamma] \in [\pi_1\Sigma]$ is given by

$$\int_{[\gamma]} \lambda^u = \sigma_{i-1}(\lambda(\rho\gamma)).$$

In $\mathrm{Hitchin}(\Sigma, d)$ our construction only works locally, i.e. on a neighborhood of the Fuchsian locus, nevertheless the construction is global in the Hitchin components of the groups G_2, $\mathrm{PSp}(2k, \mathbb{R})$ and $\mathrm{PSO}(k, k+1)$. Analyticity of the entropy function will allow us to conclude Theorem B in the whole component $\mathrm{Hitchin}(\Sigma, d)$.

A basic tool for understanding Hitchin representations is *Labourie's* [27] *equivariant flag curve*. Let $\mathscr{F}$ be the space of complete flags of $\mathbb{R}^d$, then given $\rho \in \mathrm{Hitchin}(\Sigma, d)$ there exists an equivariant Hölder-continuous map $\zeta = \zeta(\rho) : \partial\pi_1\Sigma \to \mathscr{F}$. One denotes by $\zeta_i(x)$ the i-dimensional subspace of $\mathbb{R}^d$ associated to $\zeta(x)$.

This equivariant map is a *Frenet curve*, i.e. for every decomposition $n = d_1 + \cdots + d_k \leq d$ ($d_i \in \mathbb{N}$), and $x_1, \ldots, x_k \in \partial\pi_1\Sigma$ pairwise distinct, the subspaces $\zeta_{d_i}(x_i)$ are in direct sum, and moreover

$$\lim_{(x_i)\to x} \bigoplus_1^k \zeta_{d_i}(x_i) = \zeta_n(x).$$

This condition implies that one can recover ζ from ζ_1 and we shall sometimes call ζ_1 the *Frenet equivariant curve* of ρ too.

The existence of this curve guarantees that each $\rho\gamma$ is diagonalizable, indeed, if γ_+ and γ_- are the attracting and repelling points of γ on $\partial\pi_1\Sigma$, then for $i \in \{1, \ldots, d\}$ one has that

$$\ell_i(\gamma_+, \gamma_-) = \zeta_i(\gamma_+) \cap \zeta_{d-i+1}(\gamma_-)$$

is a $\rho\gamma$-invariant line, and its associated eigenvalue has modulus $e^{\lambda_i(\rho\gamma)}$. The Frenet condition implies that the projective trace of ζ, i.e. $\zeta_1(\partial\pi_1\Sigma)$, is a C^1-submanifold of $\mathbb{P}(\mathbb{R}^d)$.

Denote by $\partial^2\pi_1\Sigma = \{(x, y) \in (\partial\pi_1\Sigma)^2 : x \neq y\}$. We prove in Proposition 5.4 that the function $\ell_i : \partial^2\pi_1\Sigma \to \mathbb{P}(\mathbb{R}^d)$ defined by

$$\ell_i(x, y) := \zeta_i(x) \cap \zeta_{d-i+1}(y),$$

provides a $C^{1+\alpha}$ submanifold of $\mathbb{P}(\mathbb{R}^d)$, namely

$$L^i_\rho := \{\ell_i(x, y) : (x, y) \in \partial^2\pi_1\Sigma\}.$$

Moreover when $i = 2, \ldots, d-1$, the tangent space $T_{\ell_i(x,y)} L^i_\rho$ splits as

$$\hom(\ell_i(x, y), \ell_{i-1}(x, y)) \oplus \hom(\ell_i(x, y), \ell_{i+1}(x, y)).$$

Consider now the bundle $\widetilde{\mathsf{F}}^i_\rho$ over L^i_ρ whose fiber $\mathsf{M}^i_\rho(x, y)$ at $\ell_i(x, y)$ consists on the elements of $\ell_i(x, y)$, i.e.

$$\mathsf{M}^i_\rho(x, y) = \{v \in \ell_i(x, y) - \{0\}\}/v \sim -v.$$

The fiber bundle $\widetilde{\mathsf{F}}^i_\rho$ is equipped with the action of $\rho(\pi_1\Sigma)$ and with a commuting $\mathbb{R}$-action, defined on each fiber by

$$\widetilde{\phi}^i_t(v) = e^{-t}v.$$

Theorem C *There exists a neighborhood U of the Fuchsian locus on* $\mathrm{Hitchin}(\Sigma, d)$, *such that if $\rho \in U$ then, for every $i \in \{2, \ldots, d-1\}$ with $i \neq (d+1)/2$, the action of $\rho(\pi_1\Sigma)$ on $\widetilde{\mathsf{F}}^i_\rho$ is properly discontinuous and cocompact. The flow ϕ^i induced on the quotient $\mathsf{F}^i_\rho = \rho(\pi_1\Sigma)\backslash\widetilde{\mathsf{F}}^i_\rho$ is a* $C^{1+\alpha}$ *Anosov flow, whose unstable distribution is given by $E^u_i =$* $\hom(\ell_i(x, y), \ell_{i-1}(x, y))$.

Theorem C is the statement of Corollary 6.3. Sections 5 and 6 are devoted to its proof.

Example 1.6 When $d = 3$ the representation ρ preserves a proper open convex set $\Omega \subset \mathbb{P}(\mathbb{R}^3)$ and the map ℓ_2 is a 2-fold covering from the annulus $\partial^2\Omega$ to the Möbis strip $\mathbb{P}(\mathbb{R}^3) - \overline{\Omega}$ (see Barbot [1]). If moreover $\rho \in \mathrm{Hitchin}(\Sigma, 3)$ is Fuchsian, then $\lambda_2(\rho\gamma) = 0$ for all $\gamma \in \pi_1\Sigma$, hence each $v \in \mathsf{M}^2_\rho(\gamma_+, \gamma_-)$ is fixed by $\rho\gamma$. Thus, the action of $\rho(\pi_1\Sigma)$ on $\widetilde{\mathsf{F}}^i_\rho$ is not proper. A similar situation occurs for $d = 2k-1$ and $i = k$.

Remark 1.7 The neighborhood U of Theorem C is explicit. For $i \in \{2, \ldots, d-1\}$ denote by

$$U_i = \{\rho \in \mathrm{Hitchin}(\Sigma, d) : \mathscr{L}_\rho \cap \ker \varepsilon_i = \{0\}\}$$

(U_1 and U_d are uninteresting since $\mathfrak{a}^+ \cap \ker \varepsilon_1 = \mathfrak{a}^+ \cap \ker \varepsilon_d = \{0\}$). This is an open set (Corollary 4.9) that contains the Fuchsian locus except when $d = 2k-1$ and $i = k$. Theorem C is proved for $U = \bigcap_{i \neq (d+1)/2} U_i$. Notice that the case $\mathrm{Hitchin}(\Sigma, 3)$ needs to be treated separately, we do so in Sect. 7.

Assume from now on that $d \neq 3$ and that $i \neq (d+1)/2$. Let U be the neighborhood provided by Theorem C and consider $\rho \in U$. Since ϕ^i is a $\mathrm{C}^{1+\alpha}$ Anosov flow, one can consider the expansion rate $\lambda^u : \mathsf{F}^i_\rho \to \mathbb{R}_+$ along the unstable distribution E^u_i defined by

$$\lambda^u(x) = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{1}{\kappa} \int_0^\kappa \log\det(d_x\phi^i_{t+s}|E^u_i)ds$$

(for any $\kappa > 0$, see Sect. 2.2). Corollary 6.3 states that if $\gamma \in \pi_1\Sigma$ then

$$\int_\gamma \lambda^u = \sigma_{i-1}(\lambda(\rho\gamma)),$$

i.e. if one reparametrizes ϕ^i with λ^u, then the period of the periodic orbit $[\gamma]$ is $\sigma_{i-1}(\lambda(\rho\gamma))$.

Corollary 2.13 states that the reparametrization of ϕ^i by λ^u has topological entropy 1. Since the topological entropy of an Anosov flow is the exponential growth rate of its periodic orbits, one concludes

$$1 = \lim_{s\to\infty} \frac{\log\#\{[\gamma] \in [\pi_1\Sigma] : \sigma_{i-1}(\lambda(\rho\gamma)) \leq s\}}{s} = h^{\sigma_{i-1}}_\rho.$$

The unstable distribution of the inverse flow $v \mapsto \phi^i_{-t}v$, is $\hom(\ell_i(x,y), \ell_{i+1}(x,y))$, so the same argument proves that $h^{\sigma_i}_\rho = 1$. Finally, observe that even though $i \neq (d+1)/2$, we have achieved all possible simple roots.

One concludes that for all $\sigma \in \Pi$, the function $\rho \mapsto h^\sigma_\rho$ is constant equal 1 on the open set U. Since $\mathrm{Hitchin}(\Sigma, d)$ is an analytic manifold (Hitchin [23]), Corollary 4.9 implies that this map is analytic on $\mathrm{Hitchin}(\Sigma, d)$, hence, it is globally constant. This finishes the proof of Theorem B.

1.3 Further consequences

Labourie [27] observes that if $\rho \in \mathrm{Hitchin}(\Sigma, d)$ and its equivariant Frenet curve $\zeta_1 : \partial\pi_1\Sigma \to \mathbb{P}(\mathbb{R}^d)$ is of class C^∞ then one can recover the flag curve by means of its derivatives, namely

$$\zeta_k = \zeta_1 \oplus \zeta_1' \oplus \cdots \oplus \zeta_1^{(k-1)},$$

where $\zeta_1^{(i)}$ is the i-th derivative of ζ_1 in an affine chart. He also remarks that there is no reason for ζ_1 to be of class C^∞, we prove in Sect. 8 the following theorem.

Theorem D *Let ρ be a Hitchin representation such that ζ_1 is of class* C^∞, *then ρ is Fuchsian.*

1.4 Historical comments

A slightly different version of the set $\mathcal{D}_\rho$ was introduced by Burger [13] for product representations $\rho = \rho_1 \times \rho_2 : \Gamma \to G_1 \times G_2$, where G_i is a simple rank 1 group, and $\rho_i : \Gamma \to G_i$ is convex cocompact. It is also dual to Quint's [34] *growth indicator function*, defined for a Zariski-dense subgroup of a real-algebraic semisimple Lie group. Quint's definition involves the Cartan projection (instead of the Jordan projection) and with his definition Proposition 1.2 holds for any such subgroup (Quint [34]). The relation between our definition and his, established in [35], (is only known to) holds for a Anosov representation of a hyperbolic group with respect to a minimal parabolic subgroup.

The statement of Theorem B arose from a discussion between the second author with Bertrand Deroin and Nicolas Tholozan. Using random walk techniques, they prove [16] that if $\rho, \eta \in \mathrm{Hitchin}(\Sigma, d)$ and $\sigma \in \Pi$ then

$$\sup_{\gamma \in \pi_1 \Sigma} \frac{\sigma(\rho\gamma)}{\sigma(\eta\gamma)} \geq 1.$$

Their theorem suggested that Theorem B should be true and it is quite possible that their method also provides a proof.

The construction of the flow $\phi^i = (\phi^i_t : \mathsf{F}^i_\rho \to \mathsf{F}^i_\rho)_{t\in\mathbb{R}}$ is analogous to the construction of the geodesic flow of a projective Anosov representation in [12], this construction is explained in Sect. 3. The advantage of considering this variation is that one can guarantee further regularity of the objects on consideration, which is needed to apply the Sinai–Ruelle–Bowen Theorem. The geodesic flow of a projective Anosov irreducible representation was introduced in [36] under the terminology of convex representations.

2 Reparametrizations and thermodynamic formalism

Let X be a compact metric space, $\phi = (\phi_t)_{t\in\mathbb{R}}$ a continuous flow on X without fixed points and V a finite dimensional real vector space. Consider a continuous map $f : X \to V$, and denote by $p(\tau)$ the period of a ϕ-periodic orbit τ. The *period* of τ for f is defined by

$$\int_\tau f = \int_0^{p(\tau)} f(\phi_s x)ds,$$

for any $x \in \tau$.

We say that a map $U : X \to V$ is C^1 *in the direction of the flow* ϕ, if for every $x \in X$, the map $t \mapsto U(\phi_t x)$ is of class C^1, and the map

$$x \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x)$$

is continuous. Two continuous maps, $f, g : X \to V$ are *Livšic-cohomologous* if there exists a map U, which is C^1 in the direction of the flow, such that for all $x \in X$ one has

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x).$$

Notice that if this is the case then $\int f dm = \int g dm$ for any ϕ-invariant measure m. In particular, f and g have the same periods.

If $f : X \to \mathbb{R}$ is positive, then f has a positive minimum and hence for every $x \in X$ the function $\kappa_f : X \times \mathbb{R} \to V$, defined by $\kappa_f(x,t) = \int_0^t f(\phi_s x) ds$, is an increasing homeomorphism of $\mathbb{R}$. Thus there is a continuous function $\alpha_f : X \times \mathbb{R} \to \mathbb{R}$ that verifies

$$\alpha_f(x, \kappa_f(x,t)) = \kappa_f(x, \alpha_f(x,t)) = t, \tag{1}$$

for every $(x,t) \in X \times \mathbb{R}$.

Definition 2.1 The *reparametrization* of ϕ by $f : X \to \mathbb{R}_{>0}$, is the flow $\psi = \psi^f = (\psi_t)_{t\in\mathbb{R}}$ on X defined by $\psi_t(x) = \phi_{\alpha_f(x,t)}(x)$, for all $t \in \mathbb{R}$ and $x \in X$. If f is Hölder-continuous, we say that ψ is a Hölder reparametrization of ϕ.

By definition, the period of a periodic orbit τ for ψ^f is the period of τ for f. Denote by $\mathcal{M}^\phi$ the space of ϕ-invariant probability measures on X. The *pressure* of a continuous function $f : X \to \mathbb{R}$, is defined by

$$P(f) = P(\phi, f) = \sup_{m \in \mathcal{M}^\phi} h(\phi, m) + \int_X f dm,$$

where $h(\phi, m)$ is the metric entropy of m for ϕ. A probability measure m, on which the least upper bound is attained, is called an *equilibrium state* of f. An equilibrium state for $f \equiv 0$ is called a *measure of maximal entropy*, and its entropy is called the *topological entropy* of ϕ, denoted by $h_{\mathrm{top}}(\phi)$.

Lemma 2.2 ([36, Lemma 2.4]) *Let $f : X \to \mathbb{R}_{>0}$ be a continuous function. Assume the equation*

$$P(\phi, -sf) = 0 \quad s \in \mathbb{R},$$

has a finite positive solution h, then h is the topological entropy of ψ^f. In particular the solution is unique. Conversely if $h_{\mathrm{top}}(\psi^f)$ is finite then it is a solution to the last equation.

2.1 (Metric) Anosov flows and vector valued potentials

We will now define *metric Anosov flows*. The transfer of classical results from axiom A flows to this more general setting is provided by Pollicott's work [33], and references therein.

As before ϕ denotes a continuous flow on the compact metric space X. For $\varepsilon > 0$ one defines the *local stable set* of x by

$$\begin{aligned} W^s_\varepsilon(x) = \{y \in X : d(\phi_t x, \phi_t y) \le \varepsilon \quad \forall t > 0 \\ \text{and} \quad d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\} \end{aligned}$$

and the *local unstable set* by

$$\begin{aligned} W^u_\varepsilon(x) = \{y \in X : d(\phi_{-t} x, \phi_{-t} y) \le \varepsilon \quad \forall t > 0 \\ \text{and} \quad d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Definition 2.3 We will say that ϕ is a *metric Anosov flow* if the following holds:

- There exist positive constants C, λ and ε such that for every $x \in X$, every $y \in W^s_\varepsilon(x)$ and every $t > 0$ one has

  $$d(\phi_t(x), \phi_t(y)) \le Ce^{-\lambda t}$$

  and such that for every $y \in W^u_\varepsilon(x)$ one has

  $$d(\phi_{-t}(x), \phi_{-t}(y)) \le Ce^{-\lambda t}.$$

- There exists $\delta > 0$ and a continuous map $\nu : \{(x, y) \in X \times X : d(x, y) < \delta\} \to \mathbb{R}$ such that $\nu(x, y)$ is the unique value such that $W^u_\varepsilon(\phi_\nu x) \cap W^s_\varepsilon(y)$ is non empty, and consists of exactly one point.

A flow is said to be *transitive* if it has a dense orbit. From now on we will assume that ϕ is a transitive metric Anosov flow.

Theorem 2.4 (Livšic [30]) *Consider a Hölder-continuous map $f : X \to V$, if $\int_\tau f = 0$ for every periodic orbit τ, then f is Livšic-cohomologous to* 0.

Consider a Hölder-continuous function $f : X \to \mathbb{R}$ with non-negative periods and define its *entropy* by

$$h_f = \limsup_{s\to\infty} \frac{1}{s} \log \# \left\{ \tau \text{ periodic} : \int_\tau f \leq s \right\} \in [0, \infty].$$

Clearly, the entropy of a function only depends on the periods of the function, therefore two Livšic cohomologous functions have the same entropy. One has the following lemma.

Lemma 2.5 (Ledrappier [29, Lemma 1]+[36, Lemma 3.8]) *Consider a Hölder-continuous function $f : X \to \mathbb{R}$ with non-negative periods. Then the following statements are equivalent:*

- *the function f is Livšic-cohomologous to a positive Hölder-continuous function,*
- *there exists $\kappa > 0$ such that $\int_\tau f > \kappa p(\tau)$ for every periodic orbit τ,*
- *the entropy $h_f \in (0, \infty)$.*

Denote by $\mathrm{Holder}^\alpha(X, V)$ the space of Hölder-continuous $V-$valued maps with exponent α. For $f \in \mathrm{Holder}^\alpha(X, V)$ denote by $\|f\|_\infty := \max |f|$ and

$$K_f = \sup \frac{\|f(p) - f(q)\|}{d(p, q)^\alpha},$$

one then defines the norm of f by $\|f\|_\alpha = \|f\|_\infty + K_f$.

The vector space $(\mathrm{Holder}^\alpha(X, V), \|\ \|_\alpha)$ is a Banach space and Livšic's theorem implies that the vector space of functions Livšic-cohomologous to 0 is a closed subspace. Denote by $\mathrm{Livsic}^\alpha(X, V)$ the quotient Banach space, and by $[\ ]_L$ the projection.

Denote by $\mathrm{Livsic}^\alpha_+(X, \mathbb{R})$ the subset of $\mathrm{Livsic}^\alpha(X, \mathbb{R})$ consisting of functions Livšic-cohomologous to a positive function.

Lemma 2.6 ([35, Lemma 2.13]) *The entropy function $h : \mathrm{Livsic}^\alpha_+(X) \to \mathbb{R}_{>0}$, defined by $f \mapsto h_f$, is analytic.*

Consider now a Hölder-continuous map $f : X \to V$, and denote by $\mathscr{L}_f$ the closed cone of V generated by the periods of f

$$\left\{ \int_\tau f : \tau \text{ periodic} \right\}.$$

Assume its *dual cone*, defined by $\mathscr{L}_f^* = \{\varphi \in V^* : \varphi|\mathscr{L}_f \geq 0\}$, is different from $\{0\}$. The *entropy* of $\varphi \in \mathscr{L}_f^*$ is defined by $h_f^\varphi = h_{\varphi\circ f}$. The following lemma is now direct using Lemma 2.5 (see also Sambarino [37, Lemma 3.2]).

Lemma 2.7 *If there exists $\varphi \in \mathcal{L}_f^*$ with finite entropy then it belongs to the interior of $\mathcal{L}_f^*$. If this is the case, any linear form $\varphi \in \mathcal{L}_f^*$ has finite and positive entropy if and only if it belongs to the interior of $\mathcal{L}_f^*$.*

We will assume from now on that there exists a linear form in $\mathcal{L}_f^*$ with finite entropy.

In view of the last lemma, one considers the open subset of $\mathrm{Livsic}^\alpha(X, V)$ defined by

$$\mathrm{Livsic}_+^\alpha(X, V) = \{[f]_L : \exists\varphi \in \mathcal{L}_f^* \text{ with } h_f^\varphi \in (0, \infty)\}.$$

Lemma 2.8 *The map* $\mathrm{Livsic}_+^\alpha(X, V) \to \{$*compact subsets of* $\mathbb{P}(V)\}$ *defined by*

$$f \mapsto \mathbb{P}(\mathcal{L}_f),$$

is continuous.

Proof Recall that the space $\mathcal{M}^\phi$ of ϕ-invariant probability measures is compact. Moreover, since ϕ is Anosov, periodic orbits viewed as invariant probability measures[6] are dense in $\mathcal{M}^\phi$ (c.f. Anosov's closing lemma, see Sigmund [40]). Consequently, the set

$$\mathcal{K}_f = \left\{\int f dm : m \in \mathcal{M}^\phi\right\}$$

is compact and generates the cone $\mathcal{L}_f$. Moreover, $f \mapsto \mathcal{K}_f$ is continuous.

In order to show that its projectivisation is also continuous, we need to show that $0 \notin \mathcal{K}_f$, but since $\varphi(f)$ is Livšic-cohomologous to a positive function, there exists $k > 0$ such that $\varphi(\int f dm) > k$ for all $m \in \mathcal{M}^\phi$. This finishes the proof. □

Summarizing one obtains the following:

Corollary 2.9 *Consider $f_0 \in \mathrm{Livsic}_+^\alpha(X, V)$ and $\varphi \in \mathrm{int}\,\mathcal{L}_{f_0}^*$, then the entropy function defined by $f \mapsto h_f^\varphi$ is analytic on a neighborhood U of f_0 such that $\varphi \in \mathrm{int}\,\mathcal{L}_f^*$ for all $f \in U$.*

We say that $f \in \mathrm{Livsic}_+^\alpha(X, V)$ is *non-arithmetic on V* if the additive group generated by its periods is dense in V. Consider the set

$$\mathcal{D}_f = \{\varphi \in V^* : P(-\varphi \circ f) \leq 0\}.$$

6 To a periodic orbit τ one associates the invariant probability measure $r \mapsto \dfrac{1}{p(\tau)} \displaystyle\int_\tau r$.

It follows from the definition of pressure that $\mathcal{D}_f$ is convex, and that if $\varphi \in \mathcal{D}_f$ then $t\varphi \in \mathcal{D}_f$ for all $t \geq 1$.

Proposition 2.10 ([35, Propositions 4.5 and 4.7]) *The set $\mathcal{D}_f$ coincides with the set $\{\varphi \in \mathscr{L}_f^* : h_f^\varphi \in (0, 1]\}$, its boundary $\partial\mathcal{D}_f$ coincides with the set*

$$\{\varphi \in \mathscr{L}_f^* : h_f^\varphi = 1\},$$

and is a codimension 1 closed analytic submanifold of V. If moreover f is non-arithmetic on V, then $\mathcal{D}_f$ is strictly convex.

2.2 SRB measures and reparametrizations

In this subsection we recall some classical results in the Sinai–Ruelle–Bowen theory and reinterpret them in the context of reparametrizations. It is common in the literature to state this type of results under a C^2-hypothesis. We shall explain how those results work in the $C^{1+\alpha}$-context.

Assume from now on that X is a compact manifold and that the flow ϕ is C^1. We say that ϕ is *Anosov* if the tangent bundle of X splits as a sum of three $d\phi_t$-invariant bundles

$$TX = E^s \oplus E^0 \oplus E^u,$$

and there exist positive constants C and c such that: E^0 is the direction of the flow and for every $t \geq 0$ one has: for every $v \in E^s$

$$\|d\phi_t v\| \leq Ce^{-ct}\|v\|,$$

and for every $v \in E^u$

$$\|d\phi_{-t} v\| \leq Ce^{-ct}\|v\|.$$

If ϕ is an Anosov flow let $\lambda^u : X \to \mathbb{R}_+$ be the *infinitesimal expansion rate* on the unstable direction, defined by

$$\lambda^u(x) = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{1}{\kappa}\int_0^\kappa \log\det(d_x\phi_{t+s}|E^u)ds$$

for some $\kappa > 0$.

Remark 2.11 Notice that by definition, if τ is a periodic orbit then

$$\int_\tau \lambda^u = \log\det d_x\phi_{p(\tau)}|E^u,$$

for any $x \in \tau$. Moreover, it is a direct consequence of Livšic's Theorem 2.4 that the Livšic-cohomology class of λ^u does not depend on κ, hence it will not appear in the notation.

Theorem 2.12 (Sinai–Ruelle–Bowen [11]) *Let ϕ be a* $\mathrm{C}^{1+\alpha}$ *Anosov flow on a compact manifold X, then $P(-\lambda^u) = 0$.*

This is statement is proved in Bowen–Ruelle [11, Proposition 4.4] assuming ϕ is C^2. Let us now give some hints on why the proof carries on in the $\mathrm{C}^{1+\alpha}$-setting. The C^2-hypothesis in [11] appears for three reasons:

– In order to guarantee that the function $x \mapsto E^u(x)$ is Hölder-continuous. This holds for $\mathrm{C}^{1+\alpha}$ Anosov flows too (see for example Katok–Hasselblatt [25, Proposition 19.1.6]).
– In order to show that $t \mapsto \log\det(d_x\phi_t|E^u)$ is C^1. By using our function λ^u this is no longer necessary as long as we show that the volume lemma holds for λ^u.
– To prove the *volume lemma* ([11, Lemma 4.2]) relating the function they define with the rate of decrease of the volume of Bowen balls. This can be proved in our context, for the function λ^u, by following the same scheme as [25, Proposition 20.4.2].

Theorem 2.12 together with Lemma 2.2 give immediately the following corollary.

Corollary 2.13 *Let ϕ be a* $\mathrm{C}^{1+\alpha}$ *Anosov flow, then the topological entropy of the reparametrization of ϕ by λ^u is* 1.

In Sect. 8 we make use of the following well known result. Denote by $\lambda^s : X \to \mathbb{R}$ the infinitesimal expansion rate of the inverse flow $(\phi_{-t})_{t\in\mathbb{R}}$.

Theorem 2.14 (Sinai–Ruelle–Bowen [11]) *Let ϕ be a* $\mathrm{C}^{1+\alpha}$ *Anosov flow on a compact manifold X, then ϕ preserves a measure in the class of Lebesgue if and only if λ^u and λ^s are Livšic-cohomologous.*

3 Projective Anosov representations

The main purpose of this section and Sect. 4 is to extend several results from [36] and [35] to the Anosov representations setting. We present here some general results from [12] on *projective Anosov representations*. These representations are a basic tool to study general Anosov representations (introduced by Labourie [27]), as we shall see in the next section. A more explanatory and detailed exposition on this class of representations is Labourie [27], Guichard–Wienhard [21], [36] and [12].

Let Γ be a word hyperbolic group.

Definition 3.1 A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ has *transverse maps* if there exist two continuous ρ-equivariant maps $(\xi, \xi^*) : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}((\mathbb{R}^d)^*)$ such that if $x \neq y$ then $\xi(y) \oplus \ker \xi^*(x) = \mathbb{R}^d$.

In order to define the Anosov property for a representation with transverse maps, we need to recall the *Gromov geodesic flow* of Γ. Gromov [18] (see also Mineyev [32]) defines a proper cocompact action of Γ on $\partial^2\Gamma \times \mathbb{R}$, which commutes with the action of $\mathbb{R}$ by translation on the final factor. The action of Γ restricted to $\partial^2\Gamma$ is the diagonal action.

There is a metric on $\partial^2\Gamma \times \mathbb{R}$, well-defined up to Hölder equivalence, so that Γ acts by isometries, every orbit of the $\mathbb{R}$ action gives a quasi-isometric embedding and the traslation flow on the $\mathbb{R}$-coordinate acts by bi-Lipschitz homeomorphisms. This flow on $\widetilde{\mathsf{U}\Gamma} = \partial^2\Gamma \times \mathbb{R}$ descends to a flow ϕ on the quotient $\mathsf{U}\Gamma = \partial^2\Gamma \times \mathbb{R}/\Gamma$. This flow is called *the geodesic flow* of Γ.

If ρ has transverse maps, the equivariant maps (ξ, ξ^*) provide two fiber bundles over $\widetilde{\mathsf{U}\Gamma}$, denoted by $\widetilde{\Xi}$ and $\widetilde{\Theta}$ respectively, whose fibers at $(x, y, t) \in \widetilde{\mathsf{U}\Gamma}$ are respectively $\widetilde{\Xi}(x, y, t) = \xi(x)$ and $\widetilde{\Theta}(x, y, t) = \ker \xi^*(y)$. The diagonal action of Γ on $\widetilde{\Xi}$ and $\widetilde{\Theta}$ is properly discontinuous (because it is on $\widetilde{\mathsf{U}\Gamma}$) and one obtains two vector bundles Ξ and Θ over $\mathsf{U}\Gamma$.

The geodesic flow of Γ on $\widetilde{\mathsf{U}\Gamma}$ extends to $\widetilde{\Xi}$ and $\widetilde{\Theta}$ by acting trivially on the fibers. This flow induces a flow on the respective quotients. Denote by $\psi = (\psi_t)_{t\in\mathbb{R}}$ the induced flow on the bundle $\Xi^* \otimes \Theta$.

The representation ρ is *projective Anosov* if it has transverse maps and the flow ψ is contracting to the past, i.e. there exist $C, c > 0$ such that for all $w \in \Xi^* \otimes \Theta$ and $t > 0$ one has

$$\|\psi_{-t} w\| \le C e^{-ct} \|w\|,$$

where $\| \ \|$ is a Euclidean metric on the bundle $\Xi^* \otimes \Theta$.

For $g \in \mathrm{PGL}(d, \mathbb{R})$, denote by $\lambda_1(g)$ the logarithm of the spectral radius of some lift $\widetilde{g} \in \mathrm{GL}(d, \mathbb{R})$ of g, with $\det \widetilde{g} \in \{-1, 1\}$. We say that g is *proximal* if the generalized eigenspace of $\widetilde{g}$ of eigenvalue with modulus $e^{\lambda_1(g)}$ has dimension 1. Such eigenline, denoted by g_+, is an *attractor* for g on $\mathbb{P}(\mathbb{R}^d)$, and its g-invariant complement g_- (i.e. $\mathbb{R}^d = g_+ \oplus g_-$) is its *repelling hyperplane*. The following lemma is standard (see Guichard–Wienhard [21, Lemma 3.1]).

Lemma 3.2 *Let ρ be a projective Anosov representation, then for every non-torsion $\gamma \in \Gamma$, the element $\rho(\gamma)$ is proximal on $\mathbb{P}(\mathbb{R}^d)$, its attractive line is $\xi(\gamma_+)$ and its repelling hyperplane is* $\ker \xi^*(\gamma_-)$.

The equivariant maps are unique, since they are continuous (in fact Hölder-continuous [12, Lemma 2.5]) and uniquely defined on a dense set of $\partial\Gamma$.

Denote by $\mathrm{L}_\rho = \xi(\partial\Gamma)$ and by $\mathrm{L}^*_\rho = \xi^*(\partial\Gamma)$. If ρ is irreducible, these are the *limit sets* (on $\mathbb{P}(\mathbb{R}^d)$ and $\mathbb{P}((\mathbb{R}^d)^*)$ respectively) of $\rho(\Gamma)$, introduced by Guivarc'h [22] and Benoist [2]. Denote by

$$\mathrm{L}^{(2)}_\rho = (\xi, \xi^*)(\partial^2\Gamma) = \{(x, y) \in \mathrm{L}_\rho \times \mathrm{L}^*_\rho : \mathbb{R}^d = \ker y \oplus x\}.$$

Consider the tautological bundle $\widetilde{\mathsf{U}\Gamma}_\rho$ over $\mathrm{L}^{(2)}_\rho$, whose fiber at (x, y) is defined by

$$\mathsf{M}_\rho(x, y) = \{(v, \varphi) : v \in x, \varphi \in y \quad \text{and} \quad \varphi(v) = 1\}/(v, \varphi) \sim -(v, \varphi).$$

The bundle $\widetilde{\mathsf{U}\Gamma}_\rho$ is equipped with a flow $\widetilde{\phi}^\rho = (\widetilde{\phi}^\rho_t)$ defined by

$$\widetilde{\phi}^\rho_t(x, y, (v, \varphi)) = (x, y, (e^t v, e^{-t}\varphi)),$$

that commutes with the natural action of $\rho(\Gamma)$. It is a consequence of the following theorem that the action of $\rho(\Gamma)$ on $\widetilde{\mathsf{U}\Gamma}_\rho$ is properly discontinuous and cocompact. The induced flow ϕ^ρ on the quotient $\mathsf{U}\Gamma_\rho = \rho(\Gamma)\backslash\widetilde{\mathsf{U}\Gamma}_\rho$ is called *the geodesic flow* of ρ.

Theorem 3.3 (Bridgeman–Canary–Labourie–Sambarino [12, Section 4]) *Let ρ be a projective Anosov representation, then there exists a ρ-equivariant Hölder-continuous homeomorphism $E : \widetilde{\mathsf{U}\Gamma}_\rho \to \widetilde{\mathsf{U}\Gamma}$, which is an orbit equivalence for the respective geodesic flows. The geodesic flow of ρ is a transitive metric Anosov flow and its stable and unstable laminations are given by (the induced on the quotient of)*

$$\widetilde{W}^s(x_0, y_0, (v_0, \varphi_0)) = \{(x_0, y, (v_0, \varphi)) : y \in \mathrm{L}^*_\rho - \{x_0\},\ \varphi \in y,\ \varphi(v_0) = 1\}$$

and

$$\widetilde{W}^u(x_0, y_0, (v_0, \varphi_0)) = \{(x, y_0, (v, \varphi_0)) : x \in \mathrm{L}_\rho - \{y_0\},\ v \in x,\ \varphi_0(v) = 1\}.$$

Periodic orbits of ϕ^ρ are in bijective correspondence with conjugacy classes of primitive elements of Γ (i.e. not a positive power of some other element in Γ), namely, if γ is such an element then its associated periodic orbit is the projection of $(\gamma_+, \gamma_-, (v, \varphi))$, for (any) $\varphi \in \xi^*(\gamma_-)$ and $v \in \xi(\gamma_+)$.

Since $\xi(\gamma_+)$ is the attracting line of $\rho(\gamma)$ (Lemma 3.2), one obtains

$$\gamma(\gamma_+, \gamma_-, (v, \varphi)) = (\gamma_+, \gamma_-, (e^{\lambda_1(\rho\gamma)}v, e^{-\lambda_1(\rho\gamma)}\varphi)).$$

Consequently, the period of such periodic orbit is $\lambda_1(\rho\gamma)$.

Hence, since the flows ϕ^ρ and ϕ are Hölder orbit equivalent, there exists a Hölder-continuous positive function $f_\rho : \mathsf{U}\Gamma \to \mathbb{R}_+$ such that for every non-torsion $\gamma \in \Gamma$, one has $\int_\gamma f_\rho = \lambda_1(\rho\gamma)$. Such f_ρ is unique up to Livšic-cohomology.

Theorem 3.4 (Bridgeman–Canary–Labourie–Sambarino [12, Proposition 6.2]) *Let $\{\rho_u : \Gamma \to \mathrm{PGL}(d, \mathbb{R})\}_{u\in D}$ be an analytic family*[7] *of projective Anosov representations. Then $u \mapsto [f_{\rho_u}]_L$ is analytic.*

The *entropy* of ρ is the topological entropy of the geodesic flow ϕ^ρ, and can be computed by

$$h_\rho = \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\} \in (0, \infty)}{s}.$$

4 General Anosov representations

The concept of Anosov representation originated in Labourie [27] and is further developed in Guichard–Wienhard [21].

Let G be a real-algebraic semisimple Lie group. Let K be a maximal compact subgroup of G and τ the Cartan involution on $\mathfrak{g}$ whose fixed point set is the Lie algebra of K. Consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and $\mathfrak{a}$ a maximal abelian subspace contained in $\mathfrak{p}$.

Let Σ be the set of roots of $\mathfrak{a}$ on $\mathfrak{g}$, consider $\mathfrak{a}^+$ a closed Weyl chamber, Σ^+ the set of positive roots associated to $\mathfrak{a}^+$ and Π the set of simple roots determined by Σ^+. To each subset θ of Π one associates a pair of opposite parabolic subgroups P_θ and $\overline{P_\theta}$ of G, whose Lie algebras are, by definition,[8]

$$\mathfrak{p}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\langle\Pi-\theta\rangle} \mathfrak{g}_{-\alpha}$$

and

$$\overline{\mathfrak{p}_\theta} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha\in\langle\Pi-\theta\rangle} \mathfrak{g}_{\alpha}$$

where $\langle\theta\rangle$ is the set of positive roots generated by θ and

$$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : [v, w] = \alpha(v)w\ \forall v \in \mathfrak{a}\}.$$

[7] We are assuming this family is far from the singular set of projective Anosov representations.

[8] Note that we use the opposite convention than Guichard–Wienhard [21], our P_θ is their P_{θ^c}.

Let W be the Weyl group of Σ and denote by $u_0 : \mathfrak{a} \to \mathfrak{a}$ the longest element in W : i.e. u_0 is the unique element in W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is the defined by $\mathrm{i} = -u_0$. Every parabolic subgroup is conjugated to a unique P_θ, in particular $\overline{P_\theta}$ is conjugated to $P_{\mathrm{i}(\theta)}$ where

$$\mathrm{i}(\theta) = \{\alpha \circ \mathrm{i} : \alpha \in \theta\}.$$

Denote by $\mathscr{F}_\theta = G/P_\theta$. The set $\mathscr{F}_{\mathrm{i}(\theta)} \times \mathscr{F}_\theta$ possesses a unique open G-orbit, which we will denote by $\mathscr{F}_\theta^{(2)}$.

Example 4.1 If $G = \mathrm{PGL}(d, \mathbb{R})$ then $\mathfrak{a} = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : a_1 + \cdots + a_d = 0\}$, a Weyl chamber is

$$\mathfrak{a}^+ = \{(a_1, \ldots, a_d) \in \mathfrak{a} : a_1 \geq \cdots \geq a_d\},$$

the set of positive roots associated to $\mathfrak{a}^+$ is $\Sigma^+ = \{a \mapsto a_i - a_j : 1 \leq i < j \leq d\}$ and the simple roots are $\Pi = \{\sigma_i : i \in \{1, \ldots, d-1\}\}$ where $\sigma_i(a) = a_i - a_{i+1}$. The opposition involution is $\mathrm{i}(a) = (-a_d, \ldots, -a_1)$. The parabolic group P_Π is the stabilizer of a complete flag, and $\mathscr{F}_\Pi^{(2)}$ is the space of pairs of flags in general position, i.e. $(\{V_i\}, \{W_i\}) \in \mathscr{F}_\Pi^{(2)}$ if $V_i \oplus W_{d-i} = \mathbb{R}^d$ for every i.

Let Γ be a word hyperbolic group and consider a representation $\rho : \Gamma \to G$. Consider the trivial bundle $\widetilde{\mathsf{U}\Gamma} \times \mathscr{F}_\theta^{(2)}$, and extend the geodesic flow of Γ to this bundle by acting trivially on the second coordinate. Passing to the quotient one obtains a flow ϕ on the bundle $\Gamma \backslash (\widetilde{\mathsf{U}\Gamma} \times \mathscr{F}_\theta^{(2)}) \to \mathsf{U}\Gamma$.

The representation ρ is (P_θ, G)*-Anosov* if there exists a ρ-equivariant section $(\xi_\theta, \xi_{\mathrm{i}(\theta)}) : \widetilde{\mathsf{U}\Gamma} \to \mathscr{F}_\theta^{(2)}$, invariant under the geodesic flow of Γ and such that its image is a hyperbolic set for ϕ whose stable distribution is the tangent space to $\{\cdot\} \times \mathscr{F}_{\mathrm{i}(\theta)}$.

Denote by $\mathrm{HA}_\theta(\Gamma, G)$ the space of (P_θ, G)-Anosov representations of Γ. Labourie [27] and Guichard–Wienhard [21] proved that this is an open subset of the space $\hom(\Gamma, G)$.

From the definitions one obtains that a representation is projective Anosov if and only if it is $(\mathsf{P}_1, \mathrm{PGL}(d, \mathbb{R}))$-Anosov, where P_1 is the stabilizer of a line in $\mathbb{R}^d$. This follows from the following remark (see [12, Proposition 2.11] for a proof).

Remark 4.2 Consider a decomposition $\mathbb{R}^d = \ell \oplus V$, where ℓ is a line and V a hyperplane, then the tangent space $T_\ell \mathbb{P}(\mathbb{R}^d)$ is canonically identified with $\hom(\ell, V)$.

Projective Anosov representations are useful to study general Anosov representations, as Theorem 4.4 below shows. Let $\{\omega_\alpha\}_{\alpha\in\Pi}$ be the set of fundamental weights of Π.

Proposition 4.3 (Tits [41]) *For each $\alpha \in \Pi$ there exists a finite dimensional proximal[9] irreducible representation $\Lambda_\alpha : G \to \mathrm{PGL}(V_\alpha)$, such that the highest weight χ_α of Λ_α is an integer multiple of the fundamental weight ω_α. All other weights of Λ_α are of the form*

$$\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta \beta$$

where $n_\beta \in \mathbb{N}$.

In other words, if $g \in G$ then $\lambda_1(\Lambda_\alpha(g)) = k_\alpha \omega_\alpha(\lambda(g))$, where $\lambda : G \to \mathfrak{a}^+$ is *the Jordan projection* of G.

Theorem 4.4 (Guichard–Wienhard [21, Lemma 3.18 + Theorem 4.10]) *Consider $\rho \in \mathrm{HA}_\theta(\Gamma, G)$, then for every $\alpha \in \theta$ the composition $\Lambda_\alpha \circ \rho : \Gamma \to \mathrm{PGL}(V_\alpha)$ is projective Anosov.*

Let

$$\mathfrak{a}_\theta = \bigcap_{\alpha\in\Pi-\theta} \ker \alpha$$

be the Lie algebra of the center of the reductive group $P_\theta \cap \overline{P_\theta}$, where $\overline{P_\theta}$ is the opposite parabolic group of P_θ. Consider also $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ the only projection invariant under the group $W_\theta = \{w \in W : w \text{ fixes pointwise } \mathfrak{a}_\theta\}$. Note that, if $\alpha \in \theta$ then $\omega_\alpha = \omega_\alpha \circ p_\theta$, (see for example Quint [34, Lemme 2.2.3]). Define $\lambda_\theta : G \to \mathfrak{a}_\theta$ by $\lambda_\theta = p_\theta \circ \lambda$.

Corollary 4.5 *Consider $\rho \in \mathrm{HA}_\theta(\Gamma, G)$, then there exists a Hölder-continuous map $f_\rho^\theta : \mathsf{U}\Gamma \to \mathfrak{a}_\theta$, such that for every non-torsion conjugacy class $[\gamma] \in [\Gamma]$ one has*

$$\int_{[\gamma]} f_\rho^\theta = \lambda_\theta(\rho\gamma).$$

Moreover, if $\{\rho_u\}_{u\in D}$ is an analytic family[10] on $\mathrm{HA}_\theta(\Gamma, G)$, then $u \mapsto [f_{\rho_u}^\theta]_L$ is analytic.

9 I.e. $\Lambda_\alpha(G)$ contains a proximal matrix.

10 We are assuming this family is far from the singular set of $\mathrm{HA}_\theta(\Gamma, G)$.

Proof For[11] each $\alpha\in\theta$ the representation $\Lambda_\alpha\circ\rho$ is projective Anosov (Theorem 4.4), hence Theorem 3.3 guarantees the existence of a Hölder-continuous function $f_\rho^\alpha:\mathsf{U}\Gamma\to\mathbb{R}_+$ such that for all non-torsion $\gamma\in\Gamma$ one has:

$$\int_{[\gamma]} f_\rho^\alpha=\lambda_1(\Lambda_\alpha\rho(\gamma))=k_\alpha\omega_\alpha(\lambda(\rho\gamma)).$$

Note that, since $\alpha\in\theta$ one has $\omega_\alpha(\lambda(\rho\gamma))=\omega_\alpha(\lambda_\theta(\rho\gamma))$ (recall $\omega_\alpha=\omega_\alpha\circ p_\theta$), and observe that the set of fundamental weights $\{\omega_\alpha\}_{\alpha\in\theta}$ is a basis of $\mathfrak{a}_\theta^*$. Hence, there exists $f_\rho^\theta:\mathsf{U}\Gamma\to\mathfrak{a}_\theta$ such that, for all $\alpha\in\theta$ one has

$$k_\alpha\omega_\alpha(f_\rho^\theta)=f_\rho^\alpha.$$

Theorem 3.4 finishes the proof. □

4.1 Limit cones

Let Δ a discrete subgroup of G. The *limit cone* of Δ (introduced by Benoist [2]) is the closed cone generated by $\{\lambda(g):g\in\Delta\}$ and is denoted by $\mathscr{L}_\Delta$.

Proposition 4.6 *Consider* $\rho\in\mathrm{HA}_\theta(\Gamma,G)$. *Then* $\mathscr{L}_{\rho(\Gamma)}$ *does not intersect the walls* $\ker\alpha$ *for every* $\alpha\in\theta\cup\mathrm{i}(\theta)$.

Example 4.7 The proposition is optimal in the following sense: If $\rho:\pi_1\Sigma\to\mathrm{PSO}(3,1)\subset\mathrm{PSL}(4,\mathbb{R})$ is a quasi-Fuchsian representation then it is projective Anosov. Its limit cone is the Weil chamber of the Cartan algebra of $\mathrm{PSO}(3,1)$, which does not intersect the walls $\ker\sigma_1$ and $\ker\sigma_3$ but is contained in the wall $\ker\sigma_2$.

Proof Assume first that $\rho:\Gamma\to\mathrm{PGL}(d,\mathbb{R})$ is projective Anosov. We have to show that its limit cone does not intersect the walls $\ker\sigma_1$ and $\ker\sigma_{d-1}$.

Consider a non-torsion element $\gamma\in\Gamma$. Recall that if $v\in\xi(\gamma_+)$ then $\rho(\gamma)v=\pm e^{\lambda_1(\rho\gamma)}v$, and that $e^{\lambda_2(\rho\gamma)}$ is the spectral radius of $\rho(\gamma)|\ker\xi^*(\gamma_-)$. Consider a Euclidean metric $\{\|\ \|_p\}_{p\in\mathsf{U}\Gamma}$ on the bundle $\Xi^*\otimes\Theta$. This metric lifts to a ρ-equivariant family of norms indexed on $\widetilde{\mathsf{U}\Gamma}$, still denoted by $\{\|\ \|_p\}_{p\in\widetilde{\mathsf{U}\Gamma}}$.

Consider $p=(\gamma_-,\gamma_+,t)\in\widetilde{\mathsf{U}\Gamma}$, $\varphi:\xi(\gamma_+)\to\mathbb{R}$ and $w\in\ker\xi^*(\gamma_-)$, then

$$\|\varphi\otimes w\|_{\phi_{-n|\gamma|}p}\le Ce^{-n|\gamma|c}\|\varphi\otimes w\|_p.$$

[11] The first statement is proved in [35], under the stronger hypothesis that $\rho(\Gamma)$ is Zariski-dense.

Since $\phi_{-n|\gamma|}p = \gamma^{-n}p$ and the norms are equivariant, one has $\|\varphi \otimes w\|_{\phi_{-n|\gamma|}p} = \|\rho(\gamma^n)\varphi \otimes w\|_p$, consequently

$$e^{n(\lambda_2(\rho\gamma)-\lambda_1(\rho\gamma))}\|\varphi \otimes w\|_p \leq Ce^{-n|\gamma|c}\|\varphi \otimes w\|_p.$$

Hence

$$\frac{\lambda_1(\rho\gamma)-\lambda_2(\rho\gamma)}{|\gamma|} > c,$$

for a $c > 0$ independent of γ. Finally, Theorem 3.3 implies the existence of $M > m > 0$ such that for every non-torsion $\gamma \in \Gamma$ one has

$$M > \frac{\lambda_1(\rho\gamma)}{|\gamma|} > m.$$

These two equations give $\mathscr{L}_{\rho(\Gamma)} \cap \ker \sigma_1 = \{0\}$. Since $\mathscr{L}_\rho$ is i-invariant and $\sigma_{d-1} = \sigma_1 \circ \mathrm{i}$, we obtain $\mathscr{L}_\rho \cap \ker \sigma_{d-1} = \{0\}$.

Assume now that ρ is P_θ-Anosov. Consider $\alpha \in \theta$ and recall that $\Lambda_\alpha \circ \rho$ is projective Anosov (Theorem 4.4). The proof finishes by applying the last paragraph to $\Lambda_\alpha \circ \rho$, and by recalling that for all $g \in G$ one has

$$\alpha(\lambda(g)) = \lambda_1(\Lambda_\alpha g) - \lambda_2(\Lambda_\alpha g).$$

$\square$

If $\rho \in \mathrm{HA}_\theta(\Gamma, G)$ more information is given on the closed cone of $\mathfrak{a}_\theta$ generated by $\{\lambda_\theta(\rho\gamma) : \gamma \in \Gamma\}$. Denote this cone by $\mathscr{L}_\rho^\theta = \mathscr{L}_{f_\rho^\theta}$ (where f_ρ^θ is given by Corollary 4.5), denote its dual cone by $\mathscr{L}_\rho^{\theta *} = \{\varphi \in \mathfrak{a}_\theta^* : \varphi|\mathscr{L}_\rho^{\theta *} \geq 0\}$. For $\varphi \in \mathscr{L}_\rho^{\theta *}$ define its *entropy* by

$$h_\rho^\varphi = \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda_\theta(\rho\gamma)) \leq s\}}{s}.$$

The following remark is direct from Lemma 2.7.

Remark 4.8 A linear form φ belongs to $\mathrm{int}\,\mathscr{L}_\rho^{\theta *}$ if and only if $h_\rho^\varphi \in (0, +\infty)$.

Corollary 4.9 *The function* $\mathrm{HA}_\theta(\Gamma, G) \to \{$*compact subsets of* $\mathbb{P}(\mathfrak{a}_\theta)\}$ *given by* $\rho \mapsto \mathbb{P}(\mathscr{L}_\rho^\theta)$ *is continuous. Consider* $\rho_0 \in \mathrm{HA}_\theta(\Gamma, G)$ *and* $\varphi \in \mathrm{int}\,\mathscr{L}_{\rho_0}^{\theta *}$*. Then the function*

$$\rho \mapsto h_\rho^\varphi$$

is analytic in a neighborhood U *of* ρ_0 *such that* $\varphi \in \mathrm{int}\,\mathscr{L}_\rho^{\theta *}$ *for every* $\rho \in U$.

Proof Follows from Corollary 4.5, Lemma 2.8 and Corollary 2.9. □

We say that $\rho \in \mathrm{HA}_\theta(\Gamma, G)$ is *non-arithmetic on* $\mathfrak{a}_\theta$ if the group generated by $\{\lambda_\theta(\rho\gamma) : \gamma \in \Gamma\}$ is dense in $\mathfrak{a}_\theta$. In the language of Sect. 2, this is to say that the function f_ρ^θ is non-arithmetic on $\mathfrak{a}_\theta$.

Remark 4.10 Benoist's theorem [4, Main theorem] asserts that if Δ is a Zariski-dense subgroup of G, then the group generated by $\{\lambda(g) : g \in \Delta\}$ is dense in $\mathfrak{a}$. Hence, if $\rho \in \mathrm{HA}_\theta(\Gamma, G)$ is Zariski-dense, then it is non-arithmetic on $\mathfrak{a}_\theta$.

If $\rho \in \mathrm{HA}_\theta(\Gamma, G)$ denote by $\mathcal{D}_\rho^\theta = \mathcal{D}_{f_\rho^\theta}$. The following is a direct consequence of Proposition 2.10.

Proposition 4.11 *Consider* $\rho \in \mathrm{HA}_\theta(\Gamma, G)$, *then the set*

$$\partial\mathcal{D}_\rho^\theta = \{\varphi \in \mathscr{L}_\rho^{\theta *} : h_\rho^\varphi = 1\},$$

is a codimension 1 closed analytic submanifold of $\mathfrak{a}_\theta^*$. *If moreover* ρ *is non-arithmetic on* $\mathfrak{a}_\theta$, *then the set* $\mathcal{D}_\rho^\theta = \{\varphi \in \mathscr{L}_\rho^{\theta *} : h_\rho^\varphi \leq 1\}$ *is strictly convex.*

5 The *i*-th eigenvalue

Let Σ be a closed orientable surface of genus ≥ 2 and denote by $\Gamma = \pi_1\Sigma$. Consider a P_Π-Anosov representation $\rho : \Gamma \to \mathrm{PSL}(d, \mathbb{R})$ and denote by $\zeta : \partial\Gamma \to \mathscr{F}$ its equivariant map. We will say that ζ is a *Frenet curve* if for every decomposition $n = d_1 + \cdots + d_k \leq d$ ($d_i \in \mathbb{N}$), and $x_1, \ldots, x_k \in \partial\Gamma$ pairwise distinct, one has that the spaces $\zeta_{d_i}(x_i)$ are in direct sum, and moreover

$$\lim_{(x_i)\to x} \bigoplus_1^k \zeta_{d_i}(x_i) = \zeta_n(x),$$

where $\zeta_i(x)$ is the i-dimensional space of the flag $\zeta(x)$.

Theorem 5.1 (Labourie [27, Theorems 4.1 and 4.2]) *Consider* $\rho \in \mathrm{Hitchin}(\Sigma, d)$, *then* ρ *is* P_Π*-Anosov and* ζ *is a Frenet curve.*

There is a nice converse to this statement due to Guichard [20].

Denote by $\mathrm{Gr}_k(\mathbb{R}^d)$ the Grassmanian of k-dimensional subspaces of $\mathbb{R}^d$. The Frenet condition implies that if $d_1 + d_2 \leq d$ where $d_1, d_2 \in \mathbb{N}$, then the function $\overline{\zeta} = \overline{\zeta}_{d_1,d_2} : (\partial\Gamma)^2 \to \mathrm{Gr}_{d_1+d_2}(\mathbb{R}^d)$ defined by

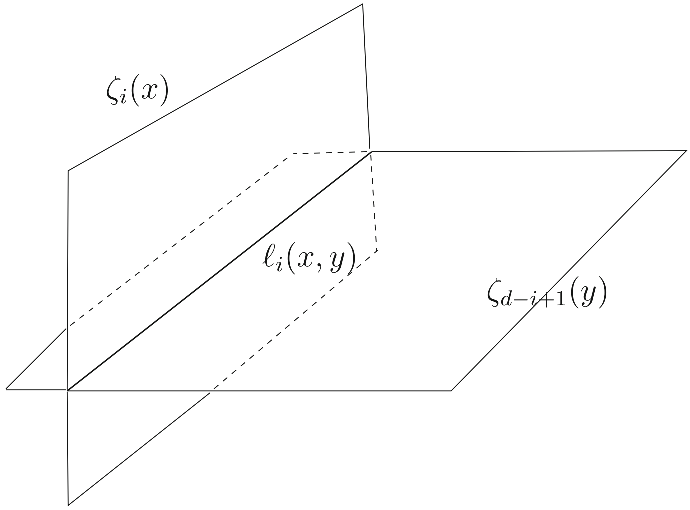

Fig. 3 The i-th eigenvalue

$$\overline{\zeta}(x, y) = \begin{cases} \zeta_{d_1}(x) \oplus \zeta_{d_2}(y) & \text{if } x \neq y \\ \zeta_{d_1+d_2}(x) & \text{if } x = y \end{cases} \tag{2}$$

is (uniformly) continuous.

Labourie [27] actually provides an even stronger transversality condition which he calls Property (H): given $x, y, z \in \partial\pi_1\Sigma$ pairwise distinct then for every $i \in \{1, \ldots, d\}$ one has

$$\zeta_{d-i+1}(y) \oplus (\zeta_{d-i+1}(z) \cap \zeta_i(x)) \oplus \zeta_{i-2}(x) = \mathbb{R}^d.$$

By combining [27, Proposition 8.2, Lemma 8.4, Lemma 9.1] one obtains:

Theorem 5.2 (Labourie [27]) *The Frenet curve of a Hitchin representation verifies* Property (H).

For each $i \in \{1, \ldots, d\}$ consider the map $\ell_i : \partial^2\Gamma \to \mathbb{P}(\mathbb{R}^d)$ defined by (Fig. 3)

$$\ell_i(x, y) = \zeta_i(x) \cap \zeta_{d-i+1}(y).$$

With this definition, Property (H) can be expressed as follows: For $x, z, t \in \partial\pi_1\Sigma$ pairwise distinct one has:

$$\zeta_{d-i+1}(t) \oplus \ell_i(x, z) \oplus \zeta_{i-2}(x) = \mathbb{R}^d$$

Remark 5.3 Note that each ℓ_i is Hölder-continuous and that for all non-torsion $\gamma \in \Gamma$, the line $\ell_i(\gamma_+, \gamma_-)$ is the eigenline of $\rho(\gamma)$ whose associated eigenvalue has modulus $e^{\lambda_i(\rho\gamma)}$. Observe also that $\ell_1(x, y) = \zeta_1(x)$ only depends on x.

For $i \in \{2, \ldots, d-1\}$ let

$$E_i^u(x, y) = \hom(\ell_i(x, y), \ell_{i-1}(x, y))$$

and

$$E_i^s(x, y) = \hom(\ell_i(x, y), \ell_{i+1}(x, y)).$$

Notice that these bundles are Hölder-continuous on both variables. The purpose of this section is to prove the following proposition.

Proposition 5.4 *Consider* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *and* $2 \leq i \leq d/2$, *then the space*

$$\mathrm{L}_\rho^i = \{\ell_i(x, y) : (x, y) \in \partial^2\Gamma\}$$

is a $\mathrm{C}^{1+\alpha}$ *submanifold of* $\mathbb{P}(\mathbb{R}^d)$. *The tangent space to* L_ρ^i *at* $\ell_i(x, y)$ *is canonically identified with* $E_i^u(x, y) \oplus E_i^s(x, y)$.

This proposition implies the same statement for all $i \in \{1, \ldots, d-1\}$ since $\ell_1(x, y) = \zeta_1(x)$ is C^1 by the Frenet property,[12] and for $i > d/2$ one has $\ell_i(x, y) = \ell_{d-i+1}(y, x)$.

5.1 Proof of Proposition 5.4

Since ρ is P_Π-Anosov, the map $\ell_i : \partial^2\Gamma \to \mathbb{P}(\mathbb{R}^d)$ is Hölder-continuous. Let us prove that, except on special cases, it is injective. Indeed, notice that if $i = 1$ (resp. $i = d$) one has that $\ell_1(x, y) = \zeta_1(x)$ (resp. $\ell_d(x, y) = \zeta_1(y)$) and if $d = 2k - 1$ then ℓ_k is not injective neither: $\ell_k(x, y) = \ell_k(y, x)$.

Lemma 5.5 *The map* $\ell_i : \partial^2\Gamma \to \mathbb{P}(\mathbb{R}^d)$ *is injective for every* $i \notin \{1, (d+1)/2, d\}$.

Proof Assume first that $2 \leq i < (d+1)/2$. Thus, $2 \leq i \leq d/2$. Observe that, since $i + i \leq d$, one has $\zeta_i(x) \cap \zeta_i(y) = \{0\}$ for every $(x, y) \in \partial^2\Gamma$. Thus, if $\ell_i(x, z) = \ell_i(y, t)$ then $x = y$.

[12] And indeed, the tangent space can be expressed in terms of the function ζ_2 and therefore it is $\mathrm{C}^{1+\alpha}$.

Hence, we need to show that if

$$\ell_i(x,z) = \ell_i(x,t)$$

then $z = t$. But if x, z, t are pairwise distinct then Property (H) (Theorem 5.2) implies

$$\zeta_{d-i+1}(t) \oplus \ell_i(x,z) \oplus \zeta_{i-2}(x) = \mathbb{R}^d,$$

this contradicts the fact that $\ell_i(x,z) = \ell_i(x,t) \subset \zeta_{d-i+1}(t)$. Finally, if $i > (d+1)/2$ then $d-i+1 < (d+1)/2$. The equality $\ell_i(x,y) = \ell_{d-i+1}(y,x)$ together with the last paragraph gives injectivity. This finishes the proof. □

We need the following technical lemma.

Lemma 5.6 *Consider a k-dimensional vector subspace W of $\mathbb{R}^d$, and consider an incomplete flag $\{V_{d-k+i} : i \in \{0,\dots,k\}\}$, such that $W \oplus V_{d-k} = \mathbb{R}^d$. Then* $\dim W \cap V_{d-k+i} = i$.

Proof When $i = 1$ the lemma follows easily. Assume now that the space $V_i' = W \cap V_{d-k+i}$ has dimension i. Applying the base step in the quotient space $\mathbb{R}^d / V_i'$ finishes the proof. □

We can now compute the 'partial derivatives' of ℓ_i. Define the maps $e_i^u, e_i^s : \partial^2\Gamma \to \mathrm{Gr}_2(\mathbb{R}^d)$ by

$$e_i^u(x,y) = \zeta_i(x) \cap \zeta_{d-i+2}(y)$$

and

$$e_i^s(x,y) = e_{d-i+1}^u(y,x) = \zeta_{i+1}(x) \cap \zeta_{d-i+1}(y).$$

Notice that injectivity implies that $\ell_i(x,y) + \ell_i(x,z)$ has dimension 2 (i.e. the sum is direct), we have the following:

Lemma 5.7 *For $i \notin \{1, (d+1)/2, d\}$ and x, y, z pairwise distinct, one has*

$$\lim_{z\to y} \ell_i(x,z) \oplus \ell_i(x,y) = e_i^u(x,y),$$

and $\lim_{z\to y} \ell_i(z,x) \oplus \ell_i(y,x) = e_i^s(y,x)$.

Proof The second statement follows from the first and the equalities $\ell_i(x,y) = \ell_{d-i+1}(y,x)$ and $e_i^s(x,y) = e_{d-i+1}^u(y,x)$. We will focus hence on the first convergence.

Since $\zeta_i(x) \cap \zeta_{d-i}(y) = \{0\}$, one has $\zeta_{d-i+1}(y) = \zeta_{d-i}(y) \oplus \ell_i(x, y)$. Since $i \geq 2$ one has $(d - i + 1) + 1 \leq d$, and therefore the Frenet condition implies

$$\zeta_1(z) \oplus \zeta_{d-i+1}(y) = \zeta_1(z) \oplus \zeta_{d-i}(y) \oplus \ell_i(x, y).$$

Intersecting with $\zeta_i(x)$ one has

$$(\zeta_1(z) \oplus \zeta_{d-i+1}(y)) \cap \zeta_i(x) = (\zeta_1(z) \oplus \zeta_{d-i}(y) \oplus \ell_i(x, y)) \cap \zeta_i(x).$$

Since ζ is a Frenet curve Lemma 5.6 implies that the left hand side of the equality has dimension 2 and also implies that $\dim(\zeta_1(z) \oplus \zeta_{d-i}(y)) \cap \zeta_i(x) = 1$. Since $\ell_i(x, y) \in \zeta_i(x)$ we conclude that

$$(\zeta_1(z) \oplus \zeta_{d-i+1}(y)) \cap \zeta_i(x) = ([\zeta_1(z) \oplus \zeta_{d-i}(y)] \cap \zeta_i(x)) \oplus \ell_i(x, y). \quad (3)$$

Given $\varepsilon > 0$, consider $\delta > 0$ from uniform continuity of $\overline{\zeta}$ (Eq. (2)). If $d(z, y) \leq \delta$ then $\zeta_1(z) \oplus \zeta_{d-i+1}(y)$ is ε-close to $\zeta_{d-i+2}(y)$, hence the left hand side of Eq. (3) is ε-close to $e_i^u(x, y)$.

Moreover, if $d(z, y) < \delta$ one has that $\zeta_1(z) \oplus \zeta_{d-i}(y)$ is ε-close to $\zeta_{d-i+1}(z)$. Thus $(\zeta_1(z) \oplus \zeta_{d-i}(y)) \cap \zeta_i(x)$ is ε-close to $\ell_i(x, z)$. Furthermore $\ell_i(x, z) \cap \ell_i(x, y) = \{0\}$ since $z \neq y$, hence the right hand side of Eq. (3) is ε-close to $\ell_i(x, z) \oplus \ell_i(x, y)$. Thus, Eq. (3) implies that

$$d_{\mathrm{Gr}_2(\mathbb{R}^d)}(e_i^u(x, y), \ell_i(x, z) \oplus \ell(x, y)) < 2\varepsilon.$$

□

Using Lemmas 5.5 and 5.7 we can finish the proof of Proposition 5.4

For $2 \leq i \leq d - 1$, denote by $\ell_i^*(x, y) = \zeta_{i-1}(x) \oplus \zeta_{d-i}(y)$ and note that $\ell_i(x, y) \oplus \ell_i^*(x, y) = \mathbb{R}^d$. Consider now the affine chart of $\mathbb{P}(\mathbb{R}^d)$ defined by this decomposition, i.e. fix $v \in \ell_i(x, y)$ and consider the map $\vartheta : \ell_i^*(x, y) \to \mathbb{P}(\mathbb{R}^d)$ defined by

$$w \mapsto \mathbb{R}(w + v).$$

This map identifies $\ell_i^*(x, y)$ with $\mathbb{P}(\mathbb{R}^d - \mathbb{P}(\ell_i^*(x, y)))$.

Denote by $w_i(a, b) \in \ell_i^*(x, y)$ the point defined by $\vartheta(w_i(a, b)) = \ell_i(a, b)$. This map may only be defined near (x, y), but this is not an issue. Observe that $\vartheta^{-1}(\ell_i(x, z) \oplus \ell_i(x, y))$ is the straight line defined by 0 and $w_i(x, z)$. The same holds for $\vartheta^{-1}(\ell_i(z, y) \oplus \ell_i(x, y))$. Lemma 5.7 implies that the set $\vartheta^{-1}\,\mathrm{L}_\rho^i$ has partial derivatives. Moreover, these partial derivatives are Hölder-continuous since they can be expressed in terms of the maps ζ_k.

This implies that $\vartheta^{-1}\,\mathrm{L}^i_\rho$ is $\mathrm{C}^{1+\alpha}$ (near 0), and that its tangent space at 0 is

$$\vartheta^{-1}(e^u_i(x,y)) \oplus \vartheta^{-1}(e^s_i(x,y)) = \ell_{i-1}(x,y) \oplus \ell_{i+1}(x,y).$$

We conclude that L^i_ρ is $\mathrm{C}^{1+\alpha}$ and that its tangent space at $\ell_i(x,y)$ is $E^u_i(x,y) \oplus E^s(x,y)$ (see Remark 4.2). This finishes the proof. □

6 Theorem C: The Anosov flow associated to ℓ_i

Let $\rho \in \mathrm{Hitchin}(\Sigma, d)$, denote by $\Gamma = \pi_1\Sigma$ and consider the manifold L^i_ρ provided by Proposition 5.4. Let $\widetilde{\mathsf{F}}^i_\rho$ be the tautological line bundle over L^i_ρ whose fiber $\mathsf{M}^i_\rho(x,y)$ at $\ell_i(x,y)$ consists on the elements of $\ell_i(x,y)$, i.e.

$$\mathsf{M}^i_\rho(x,y) = \{v \in \ell_i(x,y) - \{0\}\}/v \sim -v.$$

The fiber bundle $\widetilde{\mathsf{F}}^i_\rho$ is equipped with the action of $\rho(\Gamma)$ and with a commuting $\mathbb{R}$-action, defined on each fiber by

$$\widetilde{\phi}^i_t(v) = e^{-t}v.$$

Recall that $\mathfrak{a}$ is the Cartan algebra of $\mathfrak{sl}(d,\mathbb{R})$ and that $\varepsilon_i \in \mathfrak{a}$ is defined by $\varepsilon_i(a_1,\ldots,a_d) = a_i$. The purpose of this section is to prove the following theorem.

Theorem 6.1 *Assume $\mathscr{L}_\rho \cap \ker\varepsilon_i = \{0\}$, then there exists a ρ-equivariant Hölder-continuous homeomorphism $E : \widetilde{\mathsf{F}^i_\rho} \to \widetilde{\mathsf{U}\Gamma}$ that preserves the orbits of the respective flows.*

Consequently the action of $\rho(\Gamma)$ on $\widetilde{\mathsf{F}^i_\rho}$ is properly discontinuous and cocompact and the quotient flow ϕ^i on $\mathsf{F}^i_\rho = \rho(\Gamma)\backslash\widetilde{\mathsf{F}^i_\rho}$ is a change of speed of the geodesic flow of Γ. Moreover one has the following proposition.

Proposition 6.2 *Assume $\mathscr{L}_\rho \cap \ker\varepsilon_i = \{0\}$, then ϕ^i is a $\mathrm{C}^{1+\alpha}$ Anosov flow whose unstable distribution E^u_i is given by (the induced on the quotient by)* $\hom(\ell_i(x,y), \ell_{i-1}(x,y))$. *Consequently the expansion rate $\lambda^u : \mathsf{F}^i_\rho \to \mathbb{R}_+$ verifies that for every $\gamma \in \Gamma$ one has that:*

$$\int_{[\gamma]} \lambda^u = \sigma_{i-1}(\lambda(\rho\gamma)).$$

Lets prove Proposition 6.2 assuming Theorem 6.1.

Proof Since $\widetilde{\mathsf{F}}^i_\rho$ is a $\mathrm{C}^{1+\alpha}$ manifold and the action of $\rho(\pi_1\Sigma)$ on it is linear, we obtain that $\mathsf{F}^i_\rho = \rho(\pi_1\Sigma)\backslash\widetilde{\mathsf{F}}^i_\rho$ is $\mathrm{C}^{1+\alpha}$ and so is ϕ^i.

Theorem 6.1 implies that ϕ^i is Hölder conjugate to a reparametrization of an Anosov flow (i.e. the geodesic flow of Γ), hence it is metric Anosov with respect to the metric induced by the quotient: To prove this last assertion, the only thing to check is the existence of local (strong) stable and unstable manifolds since the uniform contraction and expansion follows from the fact that the reparametrizing function is positive. The existence of local (strong) stable and unstable manifolds follows from classical graph transform arguments.

The differential $d\phi^i_t$ of ϕ^i_t preserves the distribution E^u_i induced on the quotient by $\hom(\ell_i(x,y),\ell_{i+1}(x,y))$. Along the periodic orbits, the local unstable manifolds are tangent to E^u_i. Since the expansion of the local unstable manifolds is uniformly exponential, it follows that there exists T such that for all p in a periodic orbit one has

$$\|d\phi^T_i|E^u_i(p)\| \geq 2.$$

Since periodic orbits are dense and E^u_i is continuous one concludes that E^u_i is expanded uniformly in time. The symmetric argument gives uniform contraction of E^s_i.

Finally, if $\gamma\in\Gamma$ then recall that $\ell_i(\gamma_+,\gamma_-)$ is the eigenline of $\rho\gamma$ associated to the eigenvalue of modulus $\exp\lambda_i(\rho\gamma)$. Hence one has

$$\gamma\cdot(\ell_i(\gamma_+,\gamma_-),v) = (\ell_i(\gamma_+,\gamma_-),\rho\gamma(v)) = \widetilde{\phi^i}_{\lambda_i(\rho\gamma)}(\ell_i(\gamma_+,\gamma_-),v).$$

Thus, if one considers a Γ-invariant Riemannian metric $\|\ \|$ on $\widetilde{\mathsf{F}^i_\rho}$ and $\varphi\in\hom(\ell_i(\gamma_+,\gamma_-),\ell_{i-1}(\gamma_+,\gamma_-))$ one has that

$$\begin{aligned}\|d\widetilde{\phi^i}_{\lambda_i(\rho\gamma)}(\varphi)\| = \|\gamma\cdot\varphi\| &= \|\exp(\lambda_{i-1}(\rho\gamma)-\lambda_i(\rho\gamma))\varphi\|\\ &= \exp(\sigma_{i-1}(\lambda(\rho\gamma)))\|\varphi\|.\end{aligned}$$

Hence Remark 2.11 implies that, for x in the periodic orbit corresponding to γ one has

$$\int_{[\gamma]}\lambda^u = \log\det(d_x\phi^i_{\lambda_i(\rho\gamma)}|E^u_i) = \sigma_{i-1}(\lambda(\rho\gamma)).$$

This finishes the proof. □

Notice that Corollary 4.9 implies that the map $\rho\mapsto\mathbb{P}(\mathscr{L}_\rho)$ is continuous on $\mathrm{Hitchin}(\Sigma,d)$ and hence

$$U_i = \{\rho\in\mathrm{Hitchin}(\Sigma,d) : \mathscr{L}_\rho\cap\ker\varepsilon_i = \{0\}\}$$

is an open set. If ρ_0 is Fuchsian, then

$$\mathscr{L}_{\rho_0} = \mathfrak{a}^+_{\mathrm{PSL}(2,\mathbb{R})} = \{(d-1, d-3, \ldots, 3-d, 1-d)t : t \in \mathbb{R}_+\}.$$

Hence, if $i \in \{2, \ldots, d-1\}$ with $i \neq (d+1)/2$ then $\mathscr{L}_{\rho_0} \cap \ker \varepsilon_i = \{0\}$. This is to say, the Fuchsian locus is contained in the open set $U = \bigcap_{i \neq (d+1)/2} U_i$. One has the following corollary.

Corollary 6.3 (Theorem C) *If ρ belongs to the neighborhood U of the Fuchsian locus, then Proposition 6.2 holds for ρ.*

6.1 Hölder cocycles

In this subsection we recall a basic tool of [36]. Consider a CAT(-1) space X and denote by ∂X its visual boundary. For a discrete subgroup Γ of Isom X, denote by L_Γ its limit set on ∂X. Let $\widetilde{\mathsf{U}\Gamma}$ denote the space of parametrized complete geodesics,

$$\widetilde{\mathsf{U}\Gamma} = \{\theta : (-\infty, \infty) \to X : \theta \text{ is a complete geodesic with } \theta_{-\infty}, \theta_\infty \in \mathrm{L}_\Gamma\}.$$

The group Γ naturally acts on $\widetilde{\mathsf{U}\Gamma}$, and we denote by $\mathsf{U}\Gamma = \Gamma \backslash \widetilde{\mathsf{U}\Gamma}$ its quotient. We will say that Γ is *convex cocompact* if the space $\mathsf{U}\Gamma$ is compact. If this is the case we will naturally identify L_Γ with the Gromov boundary $\partial\Gamma$ of Γ.

We will now focus on cocycles for the action of Γ on $\partial^2\Gamma = (\partial\Gamma)^2 - \{(x, x) : x \in \partial\Gamma\}$. The main references for this subsection are Ledrappier [29] and [36, Section 5]. The usual setting is to consider cocycles on $\partial\Gamma$, however, it is convenient to use $\partial^2\Gamma$ since our cocycles are naturally defined in this space.

Definition 6.4 A *Hölder cocycle* is a function $c : \Gamma \times \partial^2\Gamma \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x, y) = c(\gamma_0, \gamma_1(x, y)) + c(\gamma_1, x, y)$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $(x, y) \in \partial^2\Gamma$, and where $c(\gamma, \cdot)$ is a Hölder map for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Given a Hölder cocycle c and a non-torsion $\gamma \in \Gamma$, the *period* of γ for c is defined by

$$p_c(\gamma) = c(\gamma, \gamma_+, \gamma_-),$$

where γ_+ is the attractive fixed point of γ on $\partial\Gamma$, and γ_- is the repelling one. The cocycle property implies that $p_c(\gamma)$ only depends on the conjugacy class $[\gamma] \in [\Gamma]$.

Two Hölder cocycles c, c' are *cohomologous*, if there exists a Hölder-continuous function $U : \partial^2\Gamma \to \mathbb{R}$, such that for all $\gamma \in \Gamma$ one has

$$c(\gamma, x, y) - c'(\gamma, x, y) = U(\gamma x, \gamma y) - U(x, y).$$

Theorem 6.5 (Ledrappier [29]) *Let c be a Hölder cocycle on $\partial^2\Gamma$, then there exists a Hölder-continuous function $f_c : \mathsf{U}\Gamma \to \mathbb{R}$, such that for every non-torsion $[\gamma]$, one has*

$$\int_{[\gamma]} f_c = p_c(\gamma).$$

Proof This is a slight variation from Ledrappier's theorem, but the proof follows verbatim. Indeed, one can find an explicit formula for such f_c as follows (Ledrappier [29], p. 105). Fix a point $o \in X$ and consider a C^∞ function $F : \mathbb{R} \to \mathbb{R}$ with compact support such that $F(0) = 1$, $F'(0) = F''(0) = 0$ and $F(t) > 1/2$ if $|t| \leq 2\sup\{d_X(p, \Gamma \cdot o) : p \in X\}$.

We can assume that $t \mapsto F(d_X(\theta(t), p))$ is differentiable on t for every $\theta \in \widetilde{\mathsf{U}\Gamma}$ and $p \in X$.

Let $A : \widetilde{\mathsf{U}\Gamma} \to \mathbb{R}$ be

$$A(\theta) = \sum_{\gamma\in\Gamma} F(d_X(\theta(0), \gamma o))e^{-c(\gamma^{-1},\theta_{-\infty},\theta_\infty)}. \tag{4}$$

The function $f_c : \widetilde{\mathsf{U}\Gamma} \to \mathbb{R}$ defined by

$$f_c(\theta) = -\left.\frac{d}{dt}\right|_{t=0} \log A(\widetilde{\phi}_t\theta), \tag{5}$$

where $\widetilde{\phi}_t\theta \in \widetilde{\mathsf{U}\Gamma}$ is the parametrized geodesic $s \mapsto \theta(s + t)$, is Γ-invariant and verifies $\int_{[\gamma]} f_c = c(\gamma, \gamma_-, \gamma_+)$. □

If c is a Hölder cocycle with non-negative periods, one defines the *entropy* of c by

$$h_c = \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma] : p_c(\gamma) \leq t\} \in [0, \infty].$$

As in [36] one has the following reparametrizing theorem:

Theorem 6.6 ([36, Theorem 3.2]) *Let c be a Hölder cocycle with non-negative periods and $h_c \in (0, \infty)$, then the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ defined by*

$$\gamma(x, y, t) = (\gamma x, \gamma y, t - c(\gamma, x, y))$$

is proper and cocompact. Moreover, the translation flow $\psi = (\psi_t)_{t\in\mathbb{R}}$ *on the quotient* $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$ *is Hölder conjugated to a reparametrization of the geodesic flow of* Γ. *The topological entropy of* ψ *is* h_c.

Proof The only difference between the actual statement of [36, Theorem 3.2] is that the cocycle c is defined on $\partial^2\Gamma$ (as opposed to $\partial\Gamma$), nevertheless the proof follows verbatim provided Ledrappier's Theorem 6.5. □

6.2 Proof of Theorem 6.1

Since $\mathscr{L}_\rho \cap \ker \varepsilon_i = \{0\}$ one has either $\varepsilon_i \in \operatorname{int} \mathscr{L}_\rho^*$, or $-\varepsilon_i \in \operatorname{int} \mathscr{L}_\rho^*$. In order to simplify notation assume $\varepsilon_i \in \operatorname{int} \mathscr{L}_\rho^*$. Remark 4.8 states that if this is the case then

$$h_\rho^{\varepsilon_i} = \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\pi_1\Sigma] : \lambda_i(\rho\gamma) \le s\}}{s} \in (0, +\infty).$$

Consider a norm $\| \ \|$ on $\mathbb{R}^d$. The Hölder cocycle $c : \pi_1\Sigma \times \partial^2\pi_1\Sigma \to \mathbb{R}$, defined by

$$c(\gamma, x, y) = \log \frac{\|\rho\gamma \cdot v\|}{\|v\|},$$

for any $v \in \ell_i(x, y)$, has periods $c(\gamma, \gamma_+, \gamma_-) = \lambda_i(\rho\gamma)$. Since $h_\rho^{\lambda_i} \in (0, \infty)$ the Reparametrizing Theorem 6.6 implies that the action of $\pi_1\Sigma$ on $\partial^2\pi_1\Sigma \times \mathbb{R}$ via c,

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t - c(\gamma, x, y))$$

is properly discontinuous and cocompact, moreover, the translation on the $\mathbb{R}$ coordinate is (conjugated to) a reparametrization of the geodesic flow of Σ (for a (any) hyperbolization on Σ fixed beforehand).

The proof of Theorem 6.1 is achieved by observing that the map $\widetilde{\mathsf{F}}^i_\rho \to \partial^2\pi_1\Sigma \times \mathbb{R}$ defined by

$$(\ell_i(x, y), v) \mapsto (x, y, \log \|v\|)$$

is $\pi_1\Sigma$-equivariant for the cocycle c (recall Lemma 5.5). This finishes the proof.

7 Benoist representations

Let Γ be a hyperbolic group. A *Benoist representation* is a homomorphism $\rho : \Gamma \to \mathrm{PGL}(n+1, \mathbb{R})$ such that $\rho(\Gamma)$ preserves an open convex set $\Omega = \Omega_\rho$

properly contained on an affine chart, and such that the quotient $\rho(\Gamma)\backslash\Omega$ is compact. Benoist [5] has proved that under these conditions, the set Ω is necessarily strictly convex and its boundary is a $\mathrm{C}^{1+\alpha}$ submanifold of $\mathbb{P}(\mathbb{R}^{n+1})$.

The geodesic flow $\phi = (\phi_t : \mathsf{T}^1(\rho(\Gamma)\backslash\Omega) \to \mathsf{T}^1(\rho(\Gamma)\backslash\Omega))_{t\in\mathbb{R}}$ for the Hilbert metric on $\rho(\Gamma)\backslash\Omega$ is a $\mathrm{C}^{1+\alpha}$ Anosov flow (Benoist [5]). Denote by $\overline{\varphi} \in \mathfrak{a}^*$ the functional $\overline{\varphi} = (\varepsilon_1 - \varepsilon_{n+1})/2$. The topological entropy of ϕ is

$$h_{\mathrm{top}}(\phi) = \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \overline{\varphi}(\lambda(\rho\gamma)) \leq s\}}{s}.$$

Crampon [15] has proved that $h_{\mathrm{top}}(\phi) \leq n-1$, and equality only holds if Ω is an ellipsoid, or equivalently, the Hilbert metric is Riemannian.

Benoist representations are projective Anosov representations, they are hence P_θ-Anosov where $\theta = \{\sigma_1, \sigma_n\} \subset \Pi$. Consider the vector space $\mathfrak{a}_\theta = \bigcap_{i=2}^{n-1} \ker \sigma_i$. Its dual space $\mathfrak{a}_\theta^* \subset \mathfrak{a}^*$ is spanned by the fundamental weights $\omega_1(a) = \omega_{\sigma_1}(a) = a_1$ and

$$\omega_n(a) = \omega_{\sigma_n}(a) = \sum_1^n a_i = -a_{n+1}.$$

Denote by $\varphi^u, \varphi^s \in \mathfrak{a}_\theta^*$ the linear forms defined by $\varphi^u = n\omega_1 - \omega_n$ and $\varphi^s = n\omega_n - \omega_1$.

Consider the expansion rate $\lambda^u : \mathsf{T}^1(\rho(\Gamma)\backslash\Omega) \to \mathbb{R}_+$ of the geodesic flow ϕ. A standard computation (for example Benoist [5, Lemma 6.5]) shows that if $\gamma \in \Gamma$ is primitive then

$$\int_{[\gamma]} \lambda^u = \sum_{i=2}^n (\lambda_1 - \lambda_i)(\rho\gamma) = n\omega_1(\lambda(\rho\gamma)) - \omega_n(\lambda(\rho\gamma)) = \varphi^u(\lambda_\theta(\rho\gamma)).$$

Corollary 2.13 and the last computation immediately imply the following.

Corollary 7.1 *Let $\rho : \Gamma \to \mathrm{PGL}(n+1, \mathbb{R})$ be a Benoist representation, then* $h_\rho^{\varphi^u} = h_\rho^{\varphi^s} = 1$.

Let L be the positive cone generated by $\{\varphi^u, \varphi^s\}$. Consider $\varphi \in \operatorname{int} L$ and let $c(\varphi) \in \mathbb{R}_+$ be such that $c(\varphi)\varphi$ is a convex combination of φ^u, φ^s.

Theorem 7.2 *For $\varphi \in \operatorname{int} L$ one has that $h_\rho^\varphi \leq c(\varphi)$ and equality holds if and only if Ω_ρ is an ellipsoid.*

Proof For a given ρ, we know that $\mathcal{D}_\rho^\theta$ is a convex set (Fig. 4) whose boundary contains φ^u and φ^s. This implies the inequality.

If equality holds then Proposition 4.11 implies that ρ is arithmetic in $\mathfrak{a}_\theta$, hence it is not Zariski-dense. Benoist's Theorem [3, Theorem 3.6] implies that Ω is an ellipsoid. □

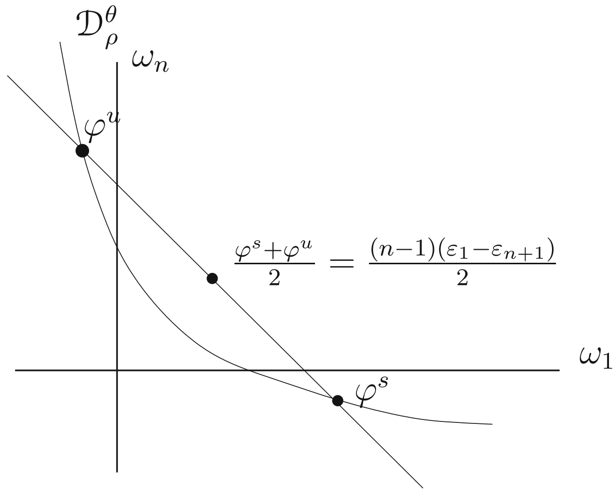

Fig. 4 The set $\mathcal{D}_\rho^\theta$ for a Benoist representation

Notice that $(n-1)\overline{\varphi} = \frac{\varphi^u+\varphi^s}{2}$, hence we obtain:

$$h_\rho^{\overline{\varphi}} \leq n-1.$$

We end this section by observing that for $n+1=3$ one has $\mathfrak{a}_\theta = \mathfrak{a}$ and $\mathcal{D}_\rho^\theta = \mathcal{D}_\rho$. Moreover, since $a_1+a_2+a_3=0$ one has $\varphi^u = \sigma_1$ and $\varphi^s = \sigma_2$. Hence Theorem B is proved for Hitchin$(\Sigma, 3)$.

8 Theorem D: Regularity of the Frenet curve

This section is devoted to the proof of Theorem D which states that if the Frenet equivariant curve ζ_1 of a Hitchin representation ρ is C^∞, then ρ is Fuchsian.

We divide the proof in two steps: Proposition 8.1 states that if ζ_1 is of class C^∞ and ρ belongs to a certain neighborhood of the Fuchsian locus then it is Fuchsian; the proof is completed with Proposition 8.2 which proves that if ζ_1 is of class C^∞ then necessarily ρ belongs to this open set.

In both cases, the proof uses the regularity to show that a certain Anosov flow preserves a volume form via a theorem of Ghys [17]. Hence, Theorem 2.14 applies and one obtains relations between the eigenvalues of a given element. This idea is reminiscent of Benoist [5, Section 6.2].

Recall that $U_i = \{\rho \in \mathrm{Hitchin}(\Sigma, d) : \mathscr{L}_\rho \cap \ker \varepsilon_i = \{0\}\}$ and $U = \bigcap_{i \neq (d+1)/2} U_i$.

Proposition 8.1 *Let ρ be a Hitchin representation in the open set U. Assume moreover that ζ_1 is of class C^∞, then ρ is Fuchsian.*

Proof Since ζ_1 is C^∞, one has that

$$\zeta_i = \zeta_1 \oplus \zeta_1' \oplus \cdots \oplus \zeta_1^{(i-1)} \tag{6}$$

(Labourie [27]). The map ζ_i is thus C^∞ and therefore the manifold L_ρ^i is C^∞.

Moreover, from the formula of the bundles E^u and E^s we deduce that they are smooth bundles too. Applying a result of Ghys [17, Lemme 3.3][13] we deduce that the flow ϕ^i preserves a volume form and hence λ^u and λ^s are Livšic-cohomologous (Theorem 2.14).

One concludes that for all $\gamma \in \pi_1\Sigma$ and $i \in \{2, \ldots, d-1\}$ one has $\sigma_{i-1}(\lambda(\rho\gamma)) = \sigma_i(\lambda(\rho\gamma))$. This implies that $\mathfrak{a}_{G_\rho} = \mathfrak{a}_{\tau_d(\mathrm{PSL}(2,\mathbb{R}))}$, hence ρ is Fuchsian.

Proposition 8.2 *Let* $\rho \in \mathrm{Hitchin}(\Sigma, d)$ *be such that* ζ_1 *is of class* C^∞. *Then for all* $i \in \{1, \ldots, d\}$ *with* $i \neq (d+1)/2$ *one has* $\mathscr{L}_\rho \cap \ker \varepsilon_i = \{0\}$.

Proof Consider $2 \leq i < (d+1)/2$ and consider the projective Anosov representation given by $\Lambda^i\rho : \pi_1\Sigma \to \mathrm{PSL}(\Lambda^i\mathbb{R}^d)$. Its equivariant maps are given by $\xi = \Lambda^i\zeta_i : \partial\pi_1\Sigma \to \mathbb{P}(\Lambda^i\mathbb{R}^d)$ and $\xi^* = \Lambda^{d-i}\zeta_{d-i} : \partial\pi_1\Sigma \to \mathbb{P}((\Lambda^i\mathbb{R}^d)^*)$ (recall $\Lambda^{d-i}\mathbb{R}^d$ is canonically isomorphic to $(\Lambda^i\mathbb{R}^d)^*$).

Equation (6) implies that $\xi(x) = \mathbb{R}(v_1 \wedge \cdots \wedge v_i)$ where $v_j \in \zeta_1^{(j)}(x)$. Since ζ_1 is of class C^∞ we can compute ξ' and one obtains (applying the product rule and observing that all terms but one have repetitions)

$$\xi'(x) = \mathbb{R}(v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1}).$$

Consequently, by Remark 4.2 the tangent space

$$T_{\xi(x)}\xi(\partial\pi_1\Sigma) = \hom(\xi(x), \xi'(x)).$$

The geodesic flow of $\Lambda^i\rho$ (recall Theorem 3.3) is a C^∞ Anosov flow with C^∞ distributions, namely

$$E^u(x, y, (\varphi, v)) = \hom(\xi(y), \xi'(y))$$
$$\text{and } E^s(x, y, (\varphi, v)) = \hom(\xi^*(x), (\xi^*)'(x)).$$

A computation analogous to that of Proposition 6.2 gives

$$\int_{[\gamma]} \lambda^u = \sigma_i(\lambda(\rho\gamma)) \text{ and } \int_{[\gamma]} \lambda^s = -\sigma_{d-i}(\lambda(\rho\gamma)) = \sigma_i \circ \mathrm{i}(\lambda(\rho\gamma)).$$

[13] The result of Ghys only requires C^2-regularity of the bundles (see [17, Section 6]) to provide a volume (contact) form invariant by the flow. This allows to reduce the required regularity for the rigidity. Nevertheless, we do not know if this reduction is optimal.

Since the distributions are smooth, Ghys's result [17, Lemme 3.3] implies that the geodesic flow preserves a volume form and hence λ^u and λ^s are Livšic-cohomologous, this implies that for all $\gamma \in \pi_1\Sigma$ and $i \neq (d+1)/2$ one has

$$\sigma_i(\lambda(\rho\gamma)) = \sigma_i \circ \mathrm{i}(\lambda(\rho\gamma)),$$

hence for all $j \in \{1, \ldots, d\}$ one has $\varepsilon_j(\lambda(\rho\gamma)) = -\varepsilon_{d-j}(\lambda(\rho\gamma))$.

Since $\mathscr{L}_\rho \subset \operatorname{int} \mathfrak{a}^+$ (Proposition 4.6) one deduces that $\mathscr{L}_\rho \cap \ker \varepsilon_i = \{0\}$ for all $i \neq (d+1)/2$. $\square$

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