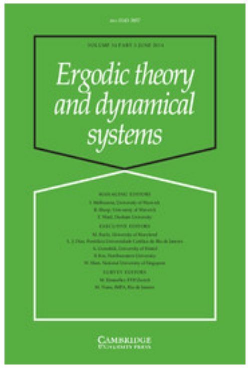

Hyperconvex representations and exponential growth

A. SAMBARINO

Ergodic Theory and Dynamical Systems / Volume 34 / Issue 03 / June 2014, pp 986 - 1010
DOI: 10.1017/etds.2012.170, Published online: 25 January 2013

Ergod. Th. & Dynam. Sys. (2014), **34**, 986–1010
doi:10.1017/etds.2012.170

Hyperconvex representations and exponential growth

A. SAMBARINO

Département de Mathématiques, Université Paris Sud, F-91405 Orsay, France
(e-mail: andres.sambarino@gmail.com)

(*Received* 1 *March* 2012 *and accepted in revised form* 30 *October* 2012)

Abstract. Let G be a real algebraic semi-simple Lie group and Γ be the fundamental group of a closed negatively curved manifold. In this article we study the limit cone, introduced by Benoist [Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.* **7**(1) (1997), 1–47], and the growth indicator function, introduced by Quint [Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.* **77** (2002), 503–608], for a class of representations $\rho : \Gamma \to G$ admitting an equivariant map from $\partial\Gamma$ to the Furstenberg boundary of the symmetric space of G, together with a transversality condition. We then study how these objects vary with the representation.

Contents

1. *Introduction*

Consider a discrete subgroup of isometries Γ of a negatively curved space X. The exponential growth rate

$$\limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : d(o, \gamma o) \leq s\}}{s}$$

plays a crucial role in understanding the asymptotic properties of the group Γ. In nice situations this exponential growth rate coincides with the topological entropy of the geodesic flow on $\Gamma\backslash X$ on its non-wandering set and with the Hausdorff dimension of the limit set of Γ on the visual boundary of X. Recall the work of Margulis [**15**], Patterson [**16**] and Sullivan [**25**], to name just a few.

An important difference appears when one considers higher rank geometry. Let us briefly explain the work of Benoist [**4**] and Quint [**18**].

Consider G to be a connected real semi-simple algebraic group and consider a discrete subgroup Δ of G. Let K be a maximal compact subgroup of G and τ the Cartan involution on $\mathfrak{g}$ whose fixed point set is the Lie algebra of K. Consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and $\mathfrak{a}$, a maximal abelian subspace contained in $\mathfrak{p}$.

Let Σ be the set of roots of $\mathfrak{a}$ on $\mathfrak{g}$ and consider $\mathfrak{a}^+$ to be the closed Weyl chamber. Denote Σ^+ as the set of positive roots associated to $\mathfrak{a}^+$ and Π as the set of simple roots determined by Σ^+. Denote by $a : G \to \mathfrak{a}^+$ the Cartan projection and fix some norm $\| \ \|$ on $\mathfrak{a}$, invariant under the Weyl group.

If $\| \ \|$ is induced by a G-invariant Riemannian distance on X (the symmetric space of G) then $\| \ \|$ is Euclidean and $\|a(g)\|$ is the distance between o and $g \cdot o$ for some $o \in X$. If $\| \ \|$ is not Euclidean then $\|a(g)\|$ can still be interpreted as $d(o, g \cdot o)$ for some G-invariant Finsler metric on X. Hence, one is interested in the exponential growth rate

$$h_\Delta^{\| \ \|} := \limsup_{s \to \infty} \frac{\log \#\{g \in \Delta : \|a(g)\| \leq s\}}{s}.$$

Nevertheless, the space $\mathfrak{a}$ being higher dimensional, one can consider the directions in $\mathfrak{a}^+$ where the points $\{a(g) : g \in \Delta\}$ are. Benoist [**4**] has shown that the asymptotic cone of $\{a(g) : g \in \Delta\}$, i.e., the limit points of sequences $t_n a(g_n)$ where $t_n \in \mathbb{R}$ goes to zero and g_n belongs to Δ, coincides with the closed cone generated by the spectrum $\{\lambda(g) : g \in \Delta\}$, where $\lambda : G \to \mathfrak{a}^+$ is the Jordan projection. One inclusion is trivial since

$$\frac{a(g^n)}{n} \to \lambda(g)$$

when $n \to \infty$ (cf. Benoist [**4**]).

THEOREM 1.1. (Benoist [**4**]) *Assume Δ is Zariski dense in G; then the asymptotic cone generated by $\{a(g) : g \in \Delta\}$ coincides with the closed cone generated by $\{\lambda(g) : g \in \Delta\}$. This cone is convex and has non-empty interior.*

This cone is called the *limit cone* of Δ and denoted $\mathscr{L}_\Delta$. Quint [**18**] is then interested in *how many* elements of $\{a(g) : g \in \Delta\}$ are in each direction of $\mathscr{L}_\Delta$. Given an open cone $\mathscr{C} \subset \mathfrak{a}^+$ consider the exponential growth rate

$$h_{\mathscr{C}}^{\| \ \|} := \limsup_{s \to \infty} \frac{\log \#\{g \in \Delta : a(g) \in \mathscr{C} \text{ with } \|a(g)\| \leq s\}}{s};$$

the *growth indicator of* Δ is the function $\psi_\Delta : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ defined as

$$\psi_\Delta(v) := \|v\| \inf\{h_{\mathscr{C}}^{\| \ \|} : \mathscr{C} \text{ open cone with } v \in \mathscr{C}\}.$$

Note that ψ_Δ is homogeneous and independent of the norm $\| \ \|$ chosen.

THEOREM 1.2. (Quint [**18**]) *Let Δ be a Zariski dense discrete subgroup of G. Then ψ_Δ is concave and upper semi-continuous, and the set*

$$\{v \in \mathfrak{a} : \psi_\Delta(v) > -\infty\}$$

is the limit cone $\mathscr{L}_\Delta$ of Δ. The function ψ_Δ is non-negative on $\mathscr{L}_\Delta$ and positive on its interior.

Quint [18] shows that the exponential growth rate for a given norm $\| \ \|$ is then retrieved as

$$\sup_{v\in\mathfrak{a}-\{0\}} \frac{\psi_\Delta(v)}{\|v\|} = h_\Delta^{\| \ \|}.$$

Assume for a moment that the norm $\| \ \|$ is Euclidean. Concavity of ψ_Δ and strict convexity of balls for $\| \ \|$ imply that there is a unique direction $\tau_\Delta^{\| \ \|} \in \mathscr{L}_\Delta$ such that the supremum of $\psi_\rho/\| \ \|$ is realized; this direction is called the *growth direction* of Δ for the norm $\| \ \|$. By definition the set of points in $\{a(g) : g \in \Delta\}$ outside a given open cone containing τ_Δ has exponential growth rate strictly smaller than $h_\Delta^{\| \ \|}$.

Deciding whether the growth direction is contained in the interior of the Weyl chamber $\mathfrak{a}^+$ plays a role in understanding the orbital counting problem for Δ, see Thirion [26]. Nevertheless, there are no known examples of Zariski dense discrete subgroups Δ such that τ_Δ belongs to the boundary of $\mathfrak{a}^+$.

This work consists of a deeper study of these objects for hyperconvex representations. This notion has its origin in the work of Labourie [12].

Consider Γ to be a discrete co-compact torsion free isometry group of a negatively curved Hadamard manifold $\widetilde{M}$ and denote $\mathscr{F} = G/P$, where P is a minimal parabolic subgroup of G. The space $\mathscr{F} \times \mathscr{F}$ has a unique open G-orbit denoted by $\mathscr{F}^{(2)}$.

Definition 1.3. A representation $\rho : \Gamma \to G$ is *hyperconvex*† if it admits a Hölder continuous ρ-equivariant map $\zeta : \partial\Gamma \to \mathscr{F}$ such that whenever $x, y \in \partial\Gamma$ are distinct the pair $(\zeta(x), \zeta(y))$ belongs to $\mathscr{F}^{(2)}$.

The main example of a hyperconvex representation is the following. Consider Σ to be a closed orientable surface of genus $g \geq 2$ and say that a representation $\pi_1(\Sigma) \to \mathrm{PSL}(d, \mathbb{R})$ is *Fuchsian* if it factors as

$$\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R}),$$

where $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R})$ is the irreducible linear action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{R}^d$ (unique modulo conjugation by $\mathrm{PSL}(d, \mathbb{R})$) and $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ is discrete and faithful. A *Hitchin component* of $\mathrm{PSL}(d, \mathbb{R})$ is a connected component of

$$\hom(\pi_1(\Sigma), \mathrm{PSL}(d, \mathbb{R})) = \{\text{morphisms } \rho : \pi_1(\Sigma) \to \mathrm{PSL}(d, \mathbb{R})\}$$

containing a Fuchsian representation.

THEOREM. (Labourie [12]) *A representation in a Hitchin component of* $\mathrm{PSL}(d, \mathbb{R})$ *is hyperconvex.*

In this work we begin by showing the following property of the limit cone of a hyperconvex representation.

PROPOSITION. (See Corollary 3.13 for the proof) *Consider* $\rho : \Gamma \to G$ *to be a Zariski dense hyperconvex representation. Then the limit cone* $\mathscr{L}_{\rho(\Gamma)}$ *is contained in the interior of the Weyl chamber* $\mathfrak{a}^+$.

† Note that this is slightly different from Labourie's [12] terminology.

Recall that $\mathscr{L}_\Delta$ is by definition closed, so the statement of the last proposition is stronger than '$\lambda(\rho\gamma)$ belongs to the interior of the Weyl chamber for every $\gamma \in \Gamma$'.

The last proposition, together with [**23**, Theorem C], imply directly the following precise counting result. For g in $\mathrm{PGL}(d, \mathbb{R})$ denote by $\lambda_1(g) \geq \lambda_2(g) \cdots \geq \lambda_d(g)$ the logarithm of the modulus of the eigenvalues (counted with multiplicity) of a lift $\widetilde{g} \in \mathrm{GL}(d, \mathbb{R})$ of g, with determinant in $\{-1, 1\}$.

COROLLARY. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a Zariski dense hyperconvex representation, and fix some $i \in \{1, \ldots, d-1\}$. Then there exists some positive $h = h_i$ such that*

$$hte^{-ht}\#\{[\gamma] \in [\Gamma] : \lambda_i(\rho\gamma) - \lambda_{i+1}(\rho\gamma) \leq t\} \to 1$$

when $t \to \infty$, where $[\gamma]$ is the conjugacy class of γ.

Concerning the growth indicator function, we show the following theorem inspired on the work of Quint [**20**] for Schottky groups of G. We will say that $\psi_{\rho(\Gamma)}$ has *vertical tangent* at some point x if, for every $\varphi \in \mathfrak{a}^*$ such that $\varphi \geq \psi_{\rho(\Gamma)}$, one has $\varphi(x) > \psi_{\rho(\Gamma)}(x)$.

THEOREM A. (See Corollary 4.9 for the proof) *Let $\rho : \Gamma \to G$ be a Zariski dense hyperconvex representation. Then the growth indicator $\psi_{\rho(\Gamma)} : \mathfrak{a} \to \mathbb{R}$ is strictly concave, analytic on the interior of $\mathscr{L}_{\rho(\Gamma)}$ and has vertical tangent on its boundary.*

Fix some hyperconvex representation $\rho : \Gamma \to G$ and denote ψ_ρ as its growth indicator. Since ψ_ρ is strictly concave, the uniqueness of a growth direction for $\rho(\Gamma)$ (for a given norm $\| \ \|$) is guaranteed, even if $\| \ \|$ is not Euclidean. Moreover, this direction belongs to the interior of the limit cone of $\rho(\Gamma)$.

In order to prove Theorem A we use dual objects associated to $\mathscr{L}_\rho$ and ψ_ρ. If a linear functional $\varphi \in \mathfrak{a}^*$ verifies $\varphi \geq \psi_\rho$ then

$$\|\varphi\| = \sup_{v:\|v\|=1} \varphi(v) \geq \sup_{\|v\|=1} \psi_\rho(v) = h_\rho^{\| \ \|}.$$

One is then led to consider the set

$$D_\rho = \{\varphi \in \mathfrak{a}^* : \varphi \geq \psi_\rho\}.$$

This set is a subset of the dual cone $\mathscr{L}_\rho^* = \{\varphi \in \mathfrak{a}^* : \varphi|\mathscr{L}_\rho \geq 0\}$ since $\psi_\rho|\mathscr{L}_\rho \geq 0$.

We then relate the set $D_{\rho(\Gamma)}$ with the thermodynamic formalism of the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$. This idea is already present in the work of Quint [**20**]; nevertheless, the way to find this relation is different and this method has the advantage of extending to (for example) Hitchin representations of surface groups.

We now briefly explain this relation.

Recall that periodic orbits of the geodesic flow $\phi_t : \Gamma\backslash T^1\widetilde{M} \to \Gamma\backslash T^1\widetilde{M}$ are in correspondence with conjugacy classes $[\gamma] \in [\Gamma]$, and recall that the *pressure* of some potential $f : \Gamma\backslash T^1\widetilde{M} \to \mathbb{R}$ is defined as

$$P(f) = \sup\left\{h(\phi_t, m) + \int f \, dm : m \ \phi_t\text{-invariant probability}\right\},$$

where $h(\phi_t, m)$ is the metric entropy of ϕ_t with respect to the measure m. A probability maximizing $P(f)$ is called an *equilibrium state* of f. The equilibrium state of f is unique provided that f is Hölder continuous. See Bowen and Ruelle [**10**] or Proposition 2.5 below.

Following Ledrappier [13], Quint [19] and [23] one finds a Γ-invariant Hölder continuous function $F_\rho : T^1\widetilde{M} \to \mathfrak{a}$, such that

$$\int_{[\gamma]} F_\rho = \lambda(\rho\gamma).$$

We then show the following.

PROPOSITION. (See Proposition 4.7 for the proof) *Let $\rho : \Gamma \to G$ be a Zariski dense hyperconvex representation. Then the set $D_{\rho(\Gamma)}$ is the set of functionals $\varphi \in \mathfrak{a}^*$ which are non-negative on the limit cone and such that $P(-\varphi(F_\rho)) \leq 0$.*

We then find the following nice dynamical interpretation of the growth indicator. For $\varphi \in \mathfrak{a}^*$ denote by m_φ the equilibrium state of $-\varphi(F_\rho) : \Gamma\backslash T^1\widetilde{M} \to \mathbb{R}$. A linear functional $\varphi \in \mathfrak{a}^*$ is *tangent* to ψ_ρ at x if $\varphi(x) = \psi_{\rho(\Gamma)}(x)$ and $\varphi \in D_{\rho(\Gamma)}$.

COROLLARY. (See Corollary 4.9 for the proof) *Consider $\varphi_0 \in \mathfrak{a}^*$ tangent to ψ_ρ. Then the direction where φ_0 and ψ_ρ are tangent is given by the vector $\int F_\rho \, dm_{\varphi_0}$ and the value of ψ_ρ in this vector is the metric entropy of the geodesic flow for the equilibrium state m_{φ_0}*

$$\psi_\rho\left(\int F_\rho \, dm_{\varphi_0}\right) = h(\phi_t, m_{\varphi_0}).$$

In the last section of this work we study continuity properties of these objects when the representation ρ varies.

Say that $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *strictly convex* if it is irreducible and admits two Hölder continuous ρ-equivariant maps $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\eta : \partial\Gamma \to \mathbb{P}(\mathbb{R}^{d*})$, such that

$$\xi(x) \oplus \ker \eta(y) = \mathbb{R}^d$$

whenever x and y are distinct.

PROPOSITION. (Proposition 3.8) *The functions*

$$\rho \mapsto \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\}}{s}$$

and

$$\rho \mapsto \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) \leq s\}}{s}$$

are continuous among strictly convex representations.

Consider a closed hyperbolic oriented surface Σ and denote by $\mathrm{Hitchin}(\Sigma, d)$ the Hitchin components of the space

$$\hom(\pi_1(\Sigma), \mathrm{PGL}(d, \mathbb{R}))/\,\mathrm{PGL}(d, \mathbb{R}).$$

Since Hitchin representations are hyperconvex and irreducible (Labourie [12]) they are, in particular, strictly convex.

COROLLARY. *The function* $\mathrm{Hitchin}(\Sigma, d) \to \mathbb{R}$*,*

$$\rho \mapsto \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : (\lambda_1 - \lambda_d)(\rho\gamma) \leq s\}}{s},$$

is continuous.

This particular function is shown to be analytic by Pollicott and Sharp [**17**]. In fact, what stops us from obtaining more regularity is that the equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ varies (only?) continuously with the representation. This is a consequence of the Anosov property, which has been shown to hold by Guichard and Wienhard [**11**].

We return now to Zariski dense hyperconvex representations. We note that the work of Guichard and Wienhard [**11**] implies that Zariski dense hyperconvex representations form an open set of the space of all representations $\Gamma \to G$.

We show in Corollary 3.21 that the limit cone varies continuously with the representation in the Hausdorff topology on compact subsets of $\mathbb{P}(\mathfrak{a})$. Hence, if one fixes a Zariski dense hyperconvex representation ρ and an open cone $\mathscr{C}$ contained in the interior of $\mathscr{L}_\rho$ the cone $\mathscr{C}$ will remain in the interior of the limit cone of all representations nearby. One can thus study the continuity of the growth indicator.

Consider the uniform topology on the space $C(X, \mathbb{R})$ of continuous real-valued functions on some space X.

THEOREM B. (See Theorem 5.1 for the proof) *Let $\rho_0 : \Gamma \to G$ be a Zariski dense hyperconvex representation and fix some closed cone $\mathscr{C}$ in the interior of the limit cone $\mathscr{L}_{\rho_0}$ of ρ_0. Consider some neighborhood U of ρ_0 such that $\mathscr{C}$ is contained in* $\mathrm{int}(\mathscr{L}_\rho)$ *for every $\rho \in U$. Then the function shows that $U \to C(\mathscr{C}, \mathbb{R})$ defined by*

$$\rho \mapsto \psi_\rho|\mathscr{C}$$

is continuous.

We then find the following corollary.

COROLLARY. (See Corollary 5.3 for the proof) *The function that associates to a Zariski dense hyperconvex representation ρ the exponential growth rate $h^{\|\ \|}_{\rho(\Gamma)}$ is continuous.*

2. *Anosov flows and Hölder cocycles*

Reparametrizations. Let X be a compact metric space, $\phi_t : X \to X$ a continuous flow on X without fixed points and $f : X \to \mathbb{R}$ a positive continuous function. Set $\kappa : X \times \mathbb{R} \to \mathbb{R}$ as

$$\kappa(x, t) = \int_0^t f(\phi_s x)\, ds \tag{1}$$

if t is positive, and $\kappa(x, t) := -\kappa(\phi_t x, -t)$ if t is negative. Thus, κ has the cocycle property $\kappa(x, t + s) = \kappa(\phi_t x, s) + \kappa(x, t)$ for every $t, s \in \mathbb{R}$ and $x \in X$.

Since $f > 0$ and X is compact, f has a positive minimum and $\kappa(x, \cdot)$ is an increasing homeomorphism of $\mathbb{R}$. We then have an inverse $\alpha : X \times \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(x, \kappa(x, t)) = \kappa(x, \alpha(x, t)) = t \tag{2}$$

for every $(x, t) \in X \times \mathbb{R}$.

Definition 2.1. The *reparametrization* of ϕ_t by f is the flow $\psi_t : X \to X$ defined as $\psi_t(x) := \phi_{\alpha(x,t)}(x)$. If f is Hölder continuous we shall say that ψ_t is a Hölder reparametrization of ϕ_t.

We say that a function $U : X \to \mathbb{R}$ is C^1 *in the direction of the flow* if, for every $p \in X$, the function $t \mapsto U(\phi_t(p))$ is of class C^1 and the function

$$p \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t(p))$$

is continuous. Two Hölder potentials $f, g : X \to \mathbb{R}$ are then said to be *Livšic cohomologous* if there exists a continuous $U : X \to \mathbb{R}$, C^1 in the flow's direction, such that for all $p \in X$ one has

$$f(p) - g(p) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t(p)).$$

Remark 2.2. When two Hölder potentials $f, g : X \to \mathbb{R}_+^*$ are Livšic cohomologous the reparametrization of ϕ_t by f is conjugated to the reparametrization by g, i.e., there exists a homeomorphism $h : X \to X$ such that, for all $p \in X$ and $t \in \mathbb{R}$,

$$h(\psi_t^f p) = \psi_t^g(hp).$$

Proof. This is standard. It suffices to show the remark for f Livšic cohomologous to 1, i.e. when

$$\int_0^t f(\phi_s x)\, ds - t = U(\phi_t x) - U(x).$$

A direct computation shows $h : X \to X$ defined as $h(x) = \phi_{U(x)}(x)$ is the desired conjugating map. □

If m is a ϕ_t-invariant probability measure on X and ψ_t is the reparametrization of ϕ_t by f, then the probability measure m' defined by $dm'/dm(\cdot) = f(\cdot)/m(f)$ is ψ_t-invariant. In particular, if τ is a periodic orbit of ϕ_t, then it is also periodic for ψ_t and the new period is

$$\int_\tau f = \int_0^{p(\tau)} f(\phi_s x)\, ds,$$

where $p(\tau)$ is the period of τ for ϕ_t and $x \in \tau$. This relation between invariant probabilities induces a bijection and Abramov [**1**] relates the corresponding metric entropies:

$$h(\psi_t, m') = h(\phi_t, m) \Big/ \int f\, dm. \tag{3}$$

Denote by $\mathcal{M}^{\phi_t}$ the set of ϕ_t-invariant probability measures. The *pressure* of a continuous function $f : X \to \mathbb{R}$ is defined as

$$P(\phi_t, f) = \sup_{m \in \mathcal{M}^{\phi_t}} h(\phi_t, m) + \int_X f\, dm.$$

A probability measure m such that the supremum is attained is called an *equilibrium state* of f. An equilibrium state for the potential $f \equiv 0$ is called a probability measure with maximal entropy and its entropy is called the *topological entropy* of ϕ_t, denoted by $h_{\mathrm{top}}(\phi_t)$.

LEMMA 2.3. [**23**, §2] *Consider $\psi_t : X \to X$ to be the reparametrization of $\phi_t : X \to X$ by $f : X \to \mathbb{R}_+^*$, and assume that $h_{\mathrm{top}}(\psi_t)$ is finite. Then $m \mapsto m'$ induces a bijection between equilibrium states of $-h_{\mathrm{top}}(\psi_t) f$ and probability measures of maximal entropy of ψ_t.*

Anosov flows. Assume from now on that X is a compact manifold and that the flow $\phi_t : X \to X$ is C^1. We say that ϕ_t is *Anosov* if the tangent bundle of X splits as a sum of three $d\phi_t$-invariant bundles,

$$TX = E^s \oplus E^0 \oplus E^u,$$

and there exist positive constants C and c such that E^0 is the direction of the flow and for every $t \geq 0$ one has, for every $v \in E^s$,

$$\|d\phi_t v\| \leq Ce^{-ct}\|v\|,$$

and, for every $v \in E^u$, $\|d\phi_{-t} v\| \leq Ce^{-ct}\|v\|$.

One can compute the topological entropy of a reparametrization of an Anosov flow as the exponential growth rate of its periodic orbits.

PROPOSITION 2.4. (Bowen [**8**]) *Let $\psi_t : X \to X$ be a reparametrization of an Anosov flow. Then the topological entropy of ψ_t is*

$$h_{\text{top}}(\psi_t) = \limsup_{s\to\infty} \frac{\log \#\{\tau \text{ periodic} : p(\tau) \leq s\}}{s},$$

where $p(\tau)$ is the period of τ for ψ_t.

As shown by Bowen [**9**], transitive Anosov flows admit Markov partitions and thus the ergodic theory of suspension of subshifts of finite type extends to these flows.

PROPOSITION 2.5. (Bowen and Ruelle [**10**]) *Let $\phi_t : X \to X$ be a transitive Anosov flow. Then, given a Hölder potential $f : X \to \mathbb{R}$ there exists a unique equilibrium state for f.*

PROPOSITION 2.6. (Cf. Ruelle [**22**] and Ratner [**21**]) *Let $\phi_t : X \to X$ be a transitive Anosov flow and let $f, g : X \to \mathbb{R}$ be Hölder continuous. Then the function $t \mapsto P(f - tg)$ is analytic and*

$$\left.\frac{\partial P(f-tg)}{\partial t}\right|_{t=0} = -\int g\, dm_f,$$

where m_f is the equilibrium state of f. If $\int g\, dm_f = 0$ and

$$\left.\frac{\partial^2 P(f-tg)}{\partial t^2}\right|_{t=0} = 0,$$

then g is cohomologous to zero. Thus, if g is not cohomologically trivial and $\int g\, dm_f = 0$, then $t \mapsto P(f - tg)$ is strictly convex.

We will need the following lemma of Ledrappier [**13**].

LEMMA 2.7. (Ledrappier [**13**, p. 106]) *Consider some potential $f : X \to \mathbb{R}$ such that $\int_\tau f \geq 0$ for every periodic orbit τ. If the number*

$$h := \limsup_{s\to\infty} \frac{\log \#\{\tau \text{ periodic} : \int_\tau f \leq s\}}{s}$$

belongs to $(0, \infty)$ then $P(-hf) = 0$. Conversely, if $P(-s_0 f) = 0$ for some $s_0 \in (0, \infty)$ then

$$s_0 = \limsup_{s\to\infty} \frac{\log \#\{\tau \text{ periodic} : \int_\tau f \leq s\}}{s} = h.$$

If this is the case,

$$0 < \inf_{\tau \text{ periodic}} \frac{1}{p(\tau)} \int_\tau f \leq \sup_{\tau \text{ periodic}} \frac{1}{p(\tau)} \int_\tau f < \infty.$$

Hölder cocycles on $\partial\Gamma$. Let Γ be a discrete co-compact torsion free isometry group of a negatively curved complete simply connected manifold $\widetilde{M}$. The group Γ is thus a hyperbolic group and its boundary at infinity $\partial\Gamma$ is naturally identified with the visual boundary of $\widetilde{M}$.

We will now focus on Hölder cocycles on $\partial\Gamma$.

Definition 2.8. A *Hölder cocycle* is a function $c : \Gamma \times \partial\Gamma \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $x \in \partial\Gamma$, and where $c(\gamma, \cdot)$ is a Hölder map for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Given a Hölder cocycle c we define the *periods* of c as the numbers

$$\ell_c(\gamma) := c(\gamma, \gamma_+),$$

where γ_+ is the attractive fixed point of γ in $\Gamma - \{e\}$. The cocycle property implies that the period of an element γ only depends on its conjugacy class $[\gamma] \in [\Gamma]$.

The main result we shall use on Hölder cocycles is the following theorem of Ledrappier [**13**] which relates them to Hölder potentials on $T^1\widetilde{M}$.

THEOREM 2.9. (Ledrappier [**13**, p. 105]) *For each Hölder cocycle c there exists a Hölder continuous* Γ*-invariant function* $F_c : T^1\widetilde{M} \to \mathbb{R}$ *such that for every* $\gamma \in \Gamma$ *one has*

$$\ell_c(\gamma) = \int_{[\gamma]} F_c,$$

where $[\gamma]$ *denotes the periodic orbit associated to* γ *of the geodesic flow.*

Recall that $\int_{[\gamma]} F_c$ denotes the integral of F_c along the periodic orbit associated to γ.

One can find an explicit formula for such F_c as follows (Ledrappier [**13**, p. 105]). Fix some point $o \in \widetilde{M}$ and consider a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with compact support, such that $f(0) = 1$, $f'(0) = f''(0) = 0$ and $f(t) > 1/2$ if $|t| \leq 2 \sup\{d(p, \Gamma \cdot o) : p \in \widetilde{M}\}$.

We can assume that $t \mapsto f(d(\phi_t(p, x)_b, q))$ is differentiable in t for every $p, q \in \widetilde{M}$, where $\phi_t(p, x)_b \in \widetilde{M}$ is the base point of $\phi_t(p, x) \in T^1\widetilde{M}$.

Recall that $T^1\widetilde{M}$ can be identified with $\widetilde{M} \times \partial\widetilde{M}$ by considering $(p, v) \mapsto (p, v_\infty)$, where $v_\infty \in \partial\widetilde{M}$ is the endpoint of the geodesic through p with speed v. Set $A : \widetilde{M} \times \partial\widetilde{M} \to \mathbb{R}$ to be

$$A(p, x) = \sum_{\gamma \in \Gamma} f(d(p, \gamma o)) e^{-c(\gamma^{-1}, x)}, \tag{4}$$

then the function $F_c : \widetilde{M} \times \partial\widetilde{M} \to \mathbb{R}$,

$$F_c(p, x) = -\frac{d}{dt}\bigg|_{t=0} \log A(\phi_t(p, x)_b, x), \tag{5}$$

is Γ-invariant and satisfies $\int_{[\gamma]} F_c = c(\gamma, \gamma_+)$.

From the explicit formula for F_c one can deduce some regularity properties. Denote $\text{Holder}^\alpha(X)$ to be the set of Hölder continuous real-valued functions $f : X \to \mathbb{R}$ with exponent α, where X is some compact metric space. For $f \in \text{Holder}^\alpha(X)$ denote $\|f\|_\infty := \max |f|$ and

$$K_f = \sup \frac{|f(p) - f(q)|}{d(p, q)^\alpha};$$

one then defines the norm $\|f\|_\alpha$ as $\|f\|_\alpha := \|f\|_\infty + K_f$. The vector space $(\text{Holder}^\alpha(X), \| \, \|_\alpha)$ is a Banach space.

If c is a Hölder cocycle with exponent α and $\gamma \in \Gamma$, then define $\|c(\gamma, \cdot)\|_\alpha$ as its Hölder norm on $\text{Holder}^\alpha(\partial\Gamma)$. Fix a finite generating set $\mathcal{A}$ of Γ. Note that a Hölder cocycle c is uniquely determined by

$$\{c(\gamma, \cdot) : \gamma \in \mathcal{A}\};$$

hence, we can define the norm of a Hölder cocycle c with exponent α as

$$\|c\| = \sup\{\|c(\gamma, \cdot)\|_\alpha : \gamma \in \mathcal{A}\}.$$

The vector space $\mathcal{C}^\alpha$ of Hölder cocycles with exponent α is then a Banach space with this norm. It is clear that the topology of $\mathcal{C}^\alpha$ does not depend on the (finite) generating set of Γ.

Let I be an open interval in the real line and X a Banach space. We say that $f : I \to X$ is *analytic* if for every linear continuous functional $\varphi \in X^*$ the function $\varphi \circ f$ is analytic. A function between Banach spaces $T : X \to Y$ is *analytic* if $\varphi \circ T \circ f$ is analytic for every continuous linear functional $\varphi \in Y^*$ and every analytic $f : I \to X$. See Bhatia and Parthasarathy [**7**].

COROLLARY 2.10. *The function* $\mathcal{C}^\alpha \to \text{Holder}^\alpha(\Gamma\backslash T^1\widetilde{M})$,

$$c \mapsto F_c,$$

given by formula (5) is analytic.

Proof. Consider a compact fundamental domain of Γ acting on $\widetilde{M}$ and let W be a small neighborhood of this compact set.

It is then clear that the function given by formula (4), $c \mapsto \log A(\cdot, \cdot)|W \times \partial\widetilde{M}$, is analytic since only finitely many elements of Γ appear in the sum (note also that this set is a generating set of Γ). An explicit formula for the derivative

$$t \mapsto -\frac{d}{dt}\bigg|_{t=0} \log A(\phi_t(p, x)_b, x)$$

shows analyticity of $c \mapsto F_c|W \times \partial\widetilde{M}$. Since F_c is Γ-invariant and W contains a fundamental domain we obtain that $c \mapsto F_c$ is analytic. □

Livšic's theorem [**14**] implies that the set of Γ-invariant Hölder functions $F : T^1\widetilde{M} \to \mathbb{R}$ cohomologous to zero is a closed subspace of $\text{Holder}^\alpha(\Gamma\backslash T^1\widetilde{M})$, and hence one obtains the following.

COROLLARY 2.11. *The function* $\mathcal{C}^\alpha \to \text{Holder}^\alpha(\Gamma\backslash T^1\widetilde{M})/\{$*Livšic cohomology*$\}$,

$$c \mapsto \textit{the cohomology class of } F_c,$$

is analytic.

We will assume from now on that the periods of a Hölder cocycle c are positive, i.e. $\ell_c(\gamma) > 0$ for every γ. For such a cocycle one defines the exponential growth rate as

$$h_c := \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \ell_c(\gamma) \leq s\}}{s} \in (0, \infty]$$

(it is a consequence of Ledrappier's theorem that h_c is always positive).

LEMMA 2.12. [**23**, §2] *Let $c : \Gamma \times \partial\Gamma \to \mathbb{R}$ be a Hölder cocycle with finite exponential growth rate. Then the function F_c is cohomologous to a positive function.*

Denote $\mathrm{Holder}^{\alpha}_{+}(\Gamma\backslash T^1\widetilde{M})$ to be the subset of $\mathrm{Holder}^{\alpha}(\Gamma\backslash T^1\widetilde{M})$ of functions cohomologous to a positive function.

LEMMA 2.13. *The function $h : \mathrm{Holder}^{\alpha}_{+}(\Gamma\backslash T^1\widetilde{M}) \to \mathbb{R}$ given as a solution to the equation*

$$P(-h(F)F) = 0$$

is analytic. Moreover, the function $F \mapsto$ equilibrium state of $-h(F)F$ is also analytic.

Proof. This is a direct consequence of the implicit function theorem for Banach spaces (see, for example, Akerkar [**2**]), and of the formula

$$\left.\frac{\partial P(f - tg)}{\partial t}\right|_{t=0} = \int g \, dm_f,$$

where m_f is the equilibrium state of f. □

Denote $\mathcal{C}^{\alpha}_{+}$ to be the open cone of $\mathcal{C}^{\alpha}$ of Hölder cocycles with positive periods and such that $h_c \in (0, \infty)$. We obtain the following proposition.

PROPOSITION 2.14. *The exponential growth rate function $h : \mathcal{C}^{\alpha}_{+} \to \mathbb{R}$,*

$$c \mapsto h_c,$$

is analytic.

Proof. Consider some $c \in \mathcal{C}^{\alpha}_{+}$. Since h_c is finite and positive, Lemma 2.12 implies that the function F_c belongs to $\mathrm{Holder}^{\alpha}_{+}(\Gamma\backslash T^1\widetilde{M})$. One then applies Corollary 2.10 together with Lemma 2.13. □

3. *Convex representations*

This section is devoted to the study of the limit cone of convex representations. We first work on strictly convex representations, i.e. irreducible morphisms $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ admitting equivariant mappings to $\mathbb{P}(\mathbb{R}^d)$ and $\mathbb{P}(\mathbb{R}^{d^*})$ with a transversality condition. We then use these representations to study Zariski dense hyperconvex representations.

Strictly convex representations. Recall that Γ is the fundamental group of a compact negatively curved manifold. Fix some finite-dimensional real vector space V.

Definition 3.1. We shall say that an irreducible representation $\rho : \Gamma \to \mathrm{PGL}(V)$ is *strictly convex* if there exist two Hölder ρ-equivariant mappings $\xi : \partial\Gamma \to \mathbb{P}(V)$ and $\eta : \partial\Gamma \to \mathbb{P}(V^*)$ such that, for every pair of distinct points $x \neq y$ on $\partial\Gamma$, the line $\xi(x)$ does not belong to the kernel of $\eta(y)$.

We say that $g \in \mathrm{PGL}(V)$ is *proximal* if any lift of g to $\mathrm{GL}(V)$ has a unique complex eigenvalue of maximal modulus, and its generalized eigenspace is one-dimensional. This eigenvalue is necessarily real and its modulus is equal to $\exp \lambda_1(g)$. We will denote g_+ as the g-fixed line of V consisting of eigenvectors of this eigenvalue and denote g_- as the g-invariant complement of g_+ (this is $V = g_+ \oplus g_-$). The line g_+ is an attractor on $\mathbb{P}(V)$ for the action of g and g_- is the repelling hyperplane.

LEMMA 3.2. [**23**, §5] *Let $\rho : \Gamma \to \mathrm{PGL}(V)$ be a strictly convex representation with ρ-equivariant maps ξ and η. Then, for every $\gamma \in \Gamma$, the matrix $\rho\gamma$ is proximal with $\xi(\gamma_+)$ as its attracting fixed line and* $\ker \eta(\gamma_-)$ *as the repelling hyperplane.*

From this lemma, one deduces that for every $x \in \partial\Gamma$ one has $\xi(x) \subset \eta(x)$.

Fix some strictly convex representation $\rho : \Gamma \to \mathrm{PGL}(V)$. The choice of a norm $\| \ \|$ on V induces a Hölder cocycle $\beta_1 : \Gamma \times \partial\Gamma \to \mathbb{R}$ defined as

$$\beta_1(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|},$$

where v belongs to the line $\xi(x)$. We note that Lemma 3.2 implies that the period $\beta_1(\gamma, \gamma_+)$ is exactly $\lambda_1(\rho\gamma)$, the logarithm of the spectral radius of $\rho\gamma$.

The following proposition is key in this work; it states that the cocycle β_1 has finite exponential growth rate.

PROPOSITION 3.3. [**23**, §5] *Let $\rho : \Gamma \to \mathrm{PGL}(V)$ be strictly convex. Then the cocycle β_1 has finite exponential growth rate, i.e.*

$$\limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\}}{s} < \infty.$$

Let $\mathfrak{v}$ be the Cartan algebra

$$\mathfrak{v} = \{(w_1, \ldots, w_d) \in \mathbb{R}^d : w_1 + \cdots + w_d = 0\}$$

of $\mathrm{PGL}(d, \mathbb{R})$ and consider the Weyl chamber

$$\mathfrak{v}^+ = \{(w_1, \ldots, w_d) \in \mathfrak{v} : w_1 \geq w_2 \geq \cdots \geq w_d\}.$$

We will show that the limit cone of a strictly convex representation does not intersect the walls $\{w \in \mathfrak{v}^+ : w_1 = w_2\}$ and $\{w \in \mathfrak{v}^+ : w_{d-1} = w_d\}$. The following lemma is from Benoist [**6**].

LEMMA 3.4. (Benoist [**6**]) *Let $g \in \mathrm{PGL}(V)$ be proximal and let $V_{\lambda_2(g)}$ be the sum of the characteristic spaces of g whose eigenvalue is of modulus* $\exp \lambda_2(g)$. *Then, for every $v \notin \mathbb{P}(g_-)$ with non-zero component in $V_{\lambda_2(g)}$ with respect to the g-invariant decomposition*

$$V = g_+ \oplus V_{\lambda_2(g)} \oplus W,$$

one has

$$\lim_{n\to\infty} \frac{\log d_{\mathbb{P}}(g^n(v), g_+)}{n} = \lambda_2(g) - \lambda_1(g).$$

Proof. Consider $u \in g_+$ and a the eigenvalue of u. By definition one has $\lambda_1(g) = \log |a|$. We then consider $T : g_- \to \mathbb{P}(\mathbb{R}^d)$ as $Tw = \mathbb{R}(w + u)$; T identifies the hyperplane g_- to the complement of $\mathbb{P}(g_-)$ in $\mathbb{P}(V)$. The action of g on $\mathbb{P}(V)$ is read, via this identification, as $\hat{g} : g_- \to g_-$,

$$\hat{g}(w) = \frac{1}{a} g w.$$

One then finds, with a linear algebra argument, that

$$\frac{1}{n} \log \frac{\|g^n w\|}{|a|^n} \to \lambda_2(g) - \lambda_1(g)$$

for every $w \in g_-$ that is not contained in the characteristic spaces of eigenvalue with modulus $< \exp \lambda_2(g)$. □

LEMMA 3.5. (Cf. Yue [**28**]) *There exist two positive constants a and b such that, for every $\gamma \in \Gamma$ and any point $x \in \partial\Gamma - \{\gamma_-\}$, one has*

$$-a|\gamma| \leq \lim_{n\to\infty} \frac{\log d_o(\gamma^n x, \gamma_+)}{n} \leq -b|\gamma|,$$

where d_o is a visual metric on $\partial\Gamma$.

One obtains the following corollary.

COROLLARY 3.6. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be strictly convex. Then there exists $k > 0$ such that, for any $\gamma \in \Gamma$, one has*

$$\frac{\lambda_1\rho(\gamma) - \lambda_2\rho(\gamma)}{\lambda_1(\rho\gamma)} > k.$$

Consequently, the limit cone of $\rho(\Gamma)$ does not intersect the walls $\{w \in \mathfrak{v}^+ : w_1 = w_2\}$ and $\{w \in \mathfrak{v}^+ : w_{d-1} = w_d\}$.

Proof. Since $\rho(\gamma)$ is proximal and its attractive line is $\xi(\gamma_+)$ one finds, by Lemma 3.4 and the fact that ρ is irreducible, that

$$\lim_{n\to\infty} \frac{\log d_{\mathbb{P}}(\rho(\gamma)^n \xi(x), \xi(\gamma_+))}{n} = \lambda_2\rho(\gamma) - \lambda_1\rho(\gamma)$$

for a point $x \in \partial\Gamma - \{\gamma_-\}$. The fact that ξ is Hölder then implies that

$$\begin{aligned}\lambda_2\rho(\gamma) - \lambda_1\rho(\gamma) &\leq \lim_{n\to\infty} \frac{1}{n} \log C d_o(\gamma^n x, \gamma_+)^\kappa \\ &= \lim_{n\to\infty} \kappa \frac{1}{n} \log d_o(\gamma^n x, \gamma_+).\end{aligned}$$

Lemma 3.5 implies that this quantity is smaller than $-\kappa b|\gamma|$. In order to finish the proof we need to compare $|\gamma|$ with $\lambda_1(\rho\gamma)$. To do so we apply Proposition 3.3 together with Ledrappier's Lemma 2.7 to the cocycle β_1:

$$0 < \frac{1}{m} < \inf_{[\gamma]} \frac{\lambda_1(\rho(\gamma))}{|\gamma|} \leq \sup_{[\gamma]} \frac{\lambda_1(\rho(\gamma))}{|\gamma|} < m$$

for some constant $m > 1$. □

PROPOSITION 3.7. (Guichard and Wienhard [**11**, Proposition 4.10], Labourie [**12**, Proposition 2.1]) *The function that associates to a strictly convex representation of its equivariant maps is continuous and the Hölder exponent of the equivariant maps can be chosen locally constant.*

Let us briefly explain why the Hölder exponent can be chosen locally constant. The equivariant map can be thought of as the stable foliation of a locally maximal hyperbolic set. The Hölder exponent of such a foliation is regulated by the contracting and expanding constants for the flow (see Barreira and Pesin [**3**, Appendix A]). Hence one can choose a Hölder exponent that works on a sufficiently small open set.

We are now able to prove the following proposition. Recall that $\lambda_d(g)$ is the logarithm of the modulus of the smallest eigenvalue of g.

PROPOSITION 3.8. *The functions*

$$\rho \mapsto h_1(\rho) := \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\}}{s}$$

and

$$\rho \mapsto h_{1d}(\rho) := \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) \leq s\}}{s}$$

are continuous among strictly convex representations.

Proof. Since the cocycle β_1 has finite exponential growth rate, Proposition 3.7 together with Proposition 2.14 imply directly the continuity of $h_1(\rho)$.

We focus then on $h_{1d}(\rho)$. The dual representation $\rho^* : \Gamma \to \mathrm{PGL}(\mathbb{R}^{d^*})$ given by $\rho^*(\gamma)\varphi = \varphi \circ \rho(\gamma^{-1})$ is also strictly convex. The cocycle associated to ρ^*,

$$\beta_d(\gamma, x) = \log \frac{\|\rho^*(\gamma)\varphi\|}{\|\varphi\|},$$

where $\varphi \in \eta(x)$, has periods

$$\beta_d(\gamma, \gamma_+) = \lambda_1(\rho\gamma^{-1}) = -\lambda_d(\rho\gamma).$$

Now consider the Hölder cocycle $\beta_{1d} : \Gamma \times \partial\Gamma \to \mathbb{R}$ defined as

$$\beta_{1d}(\gamma, x) = \beta_1(\gamma, x) + \beta_d(\gamma, x).$$

The periods of β_{1d} are

$$\beta_{1d}(\gamma, \gamma_+) = \lambda_1(\rho\gamma) + \lambda_1(\rho\gamma^{-1}) = \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) > 0$$

for every $\gamma \in \Gamma$. Again by Propositions 3.7 and 2.14 it is sufficient to prove that the cocycle β_{1d} has finite exponential growth rate, but this is clear from the inequality

$$\lambda_1(g) - \lambda_d(g) \geq \lambda_1(g)$$

for every $g \in \mathrm{PGL}(d, \mathbb{R})$ together with the fact that $h_1(\rho)$ is finite. This finishes the proof. □

Convex representations on some flag space. Strictly convex representations are then used to study Zariski dense representations $\Gamma \to G$ which have equivariant maps to G/P, where P is some parabolic subgroup of G.

Consider G a connected real semi-simple algebraic group and consider a discrete subgroup Δ of G. Let K be a maximal compact subgroup of G and τ the Cartan involution on $\mathfrak{g}$ whose fixed point set is the Lie algebra of K. Consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and $\mathfrak{a}$ a maximal abelian subspace contained in $\mathfrak{p}$.

Let Σ be the set of roots of $\mathfrak{a}$ on $\mathfrak{g}$. Consider $\mathfrak{a}^+$ a closed Weyl chamber, Σ^+ the set of positive roots associated to $\mathfrak{a}^+$ and Π the set of simple roots determined by Σ^+. To each subset θ of Π one associates a parabolic subgroup P_θ of G whose Lie algebra is, by definition,

$$\mathfrak{p}_\theta = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \langle \Pi - \theta \rangle} \mathfrak{g}_{-\alpha},$$

where $\langle \theta \rangle$ is the set of positive roots generated by θ and

$$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : [v, w] = \alpha(v) w \ \forall v \in \mathfrak{a}\}.$$

Every parabolic subgroup of G is conjugated to a unique P_θ.

Set W to be the Weyl group of Σ and denote by $u_0 : \mathfrak{a} \to \mathfrak{a}$ the longest element in W, where u_0 is the unique element in W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined as $\mathrm{i} := -u_0$.

From now on fix $\theta \subset \Pi$ to be a subset of simple roots of G and write $\mathscr{F}_\theta$ for G/P_θ. We also consider $P_{\mathrm{i}(\theta)}$, the parabolic group associated to

$$\mathrm{i}(\theta) := \{\alpha \circ \mathrm{i} : \alpha \in \theta\}.$$

The set $\mathscr{F}_{\mathrm{i}(\theta)} \times \mathscr{F}_\theta$ possesses a unique open G-orbit, which we will denote by $\mathscr{F}_\theta^{(2)}$.

Definition 3.9. We shall say that a representation $\rho : \Gamma \to G$ is *θ-convex* if there exist two ρ-equivariant Hölder maps $\xi : \partial\Gamma \to \mathscr{F}_\theta$ and $\eta : \partial\Gamma \to \mathscr{F}_{\mathrm{i}(\theta)}$ such that if $x \neq y$ are distinct points in $\partial\Gamma$, then the pair $(\eta(y), \xi(x))$ belongs to $\mathscr{F}_\theta^{(2)}$.

A Π-convex representation (i.e., when the set θ is the full set of simple roots and thus the parabolic group P_Π is minimal) is said to be *hyperconvex*. The set $\mathscr{F}_\Pi$ is the Furstenberg boundary $\mathscr{F}$ of the symmetric space of G.

Consider $\{\omega_\alpha\}_{\alpha \in \Pi}$ to be the set of fundamental weights of Π.

PROPOSITION 3.10. (Tits [**27**]) *For each $\alpha \in \Pi$ there exists a finite-dimensional proximal irreducible representation $\Lambda_\alpha : G \to \mathrm{PGL}(V_\alpha)$ such that the highest weight χ_α of Λ_α is an integer multiple of the fundamental weight ω_α.*

Fix some θ and consider some $\alpha \in \theta$. Consider also $\Lambda_\alpha : G \to \mathrm{PGL}(V_\alpha)$, a representation given by Tits's proposition. Since Λ_α is proximal and $\alpha \in \theta$, one obtains an equivariant mapping $\xi_\alpha : \mathscr{F}_\theta \to \mathbb{P}(V_\alpha)$.

The highest weight of the dual representation $\Lambda_\alpha^* : G \to \mathbb{P}(V_\alpha^*)$ is $\chi_\alpha \circ \mathrm{i}$; one thus obtains an equivariant mapping $\eta_\alpha : \mathscr{F}_{\mathrm{i}\,\theta} \to \mathbb{P}(V_\alpha^*)$. Moreover, for a pair $(x, y) \in \mathscr{F}_\theta^{(2)}$ one has

$$\eta_\alpha(x) | \xi_\alpha(y) \neq 0.$$

One deduces the following remark.

Remark 3.11. If $\rho : \Gamma \to G$ is Zariski dense and θ-convex then the composition $\Lambda_\alpha \circ \rho : \Gamma \to \mathrm{PGL}(V_\alpha)$ (where Λ_α is Tits's representation for $\alpha \in \theta$) is strictly convex.

An element $g \in G$ is said to be *proximal on* $\mathscr{F}_\theta$ if it has an attracting fixed point on $\mathscr{F}_\theta$. This point is denoted as g_+^θ. One finds that g also has a fixed point g_-^θ on $\mathscr{F}_{\mathrm{i}\,\theta}$ and for every point $x \in \mathscr{F}_\theta$, such that the pair (g_-^θ, x) belongs to $\mathscr{F}_\theta^{(2)}$, one has $g^n x \to g_+^\theta$. The point g_-^θ is called the repelling point of g on $\mathscr{F}_{\mathrm{i}\,\theta}$.

Tits's Proposition 3.10 together with Lemma 3.2 (and the remark preceding it) imply directly the following corollary.

COROLLARY 3.12. *Let $\rho : \Gamma \to G$ be a Zariski dense θ-convex representation. Then, for every $\gamma \in \Gamma$, $\rho(\gamma)$ is proximal on $\mathscr{F}_\theta$, $\xi(\gamma_+)$ is its attracting fixed point and $\eta(\gamma_-)$ is the repelling point.*

Remark 3.11 together with Corollary 3.6 imply the following corollary.

COROLLARY 3.13. *Let $\rho : \Gamma \to G$ be a Zariski dense θ-convex representation. Then the limit cone $\mathscr{L}_{\rho(\Gamma)}$ of $\rho(\Gamma)$ does not intersect the walls $\{v \in \mathfrak{a} : \alpha(v) = 0\}$ for every $\alpha \in \theta \cup \mathrm{i}(\theta)$.*

In particular, the limit cone of a Zariski dense hyperconvex representation is contained in the interior of the Weyl chamber $\mathfrak{a}^+$.

Proof. As observed before, if $\alpha \in \theta$ the composition $\Lambda_\alpha \rho : \Gamma \to \mathrm{PGL}(V_\alpha)$ is strictly convex. Applying Corollary 3.6 for the representation $\Lambda_\alpha \rho$ implies the existence of some $\kappa_\alpha > 0$ such that

$$\frac{\alpha(\lambda(\rho\gamma))}{\chi_\alpha(\lambda(\rho\gamma))} = \frac{\lambda_1(\Lambda_\alpha \rho\gamma) - \lambda_2(\Lambda_\alpha \rho\gamma)}{\lambda_1(\Lambda_\alpha \rho\gamma)} > \kappa_\alpha. \qquad \square$$

Busemann cocycle. To a θ-convex representation $\rho : \Gamma \to G$ one naturally associates a (vector-valued) Hölder cocycle on the boundary of Γ. In order to do so we need *Busemann's cocycle* on G introduced by Quint [**19**].

The set $\mathscr{F}$ is K-homogeneous with stabilizer M. One then defines $\sigma_\Pi : G \times \mathscr{F} \to \mathfrak{a}$ to satisfy the equation

$$gk = l \exp(\sigma_\Pi(g, kM))n$$

following Iwasawa's decomposition of $G = K \exp(\mathfrak{a})N$, where N is the unipotent radical of P.

In order to obtain a cocycle only depending on the set $\mathscr{F}_\theta$ (and G) one considers

$$\mathfrak{a}_\theta := \bigcap_{\alpha \in \Pi - \theta} \ker \alpha,$$

the Lie algebra of the center of the reductive group $P_\theta \cap \overline{P_\theta}$, where $\overline{P_\theta}$ is the opposite parabolic group of P_θ. Consider also $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$, the only projection invariant under the group $W_\theta = \{w \in W : w(\mathfrak{a}_\theta) = \mathfrak{a}_\theta\}$.

Remark 3.14. One easily verifies the following relation: $p_{\mathrm{i}(\theta)} = \mathrm{i} \circ p_\theta \circ \mathrm{i}$.

Quint [**19**] shows the following lemma.

LEMMA 3.15. (Quint [**19**, Lemma 6.1]) *The function $p_\theta \circ \sigma_\Pi$ factors through a function $\sigma_\theta : G \times \mathscr{F}_\theta \to \mathfrak{a}_\theta$. The function σ_θ has the cocycle relation: for every $g, h \in G$ and $x \in \mathscr{F}_\theta$, one has*

$$\sigma_\theta(gh, x) = \sigma_\theta(g, hx) + \sigma_\theta(h, x).$$

The cocycle one naturally associates to a θ-convex representation is then $\beta_\theta^\rho = \beta_\theta : \Gamma \times \partial\Gamma \to \mathfrak{a}_\theta$, which is defined as

$$\beta_\theta(\gamma, x) = \sigma_\theta(\rho(\gamma), \xi(x)).$$

LEMMA 3.16. *Let $\rho : \Gamma \to G$ be Zariski dense and θ-convex. Then the period of β_θ for $\gamma \in \Gamma$ is*

$$\beta_\theta(\gamma, \gamma_+) = p_\theta(\lambda(\rho\gamma)) := \lambda_\theta(\rho\gamma).$$

Proof. The proof follows exactly as [**23**, Lemma 7.5]. □

Consider the cone $\mathscr{L}_\rho^\theta$, which is the closed cone generated by

$$\{\lambda_\theta(\rho\gamma) : \gamma \in \Gamma\},$$

and consider its dual cone

$$\mathscr{L}_\rho^{\theta *} := \{\varphi \in \mathfrak{a}_\theta^* : \varphi | \mathscr{L}_\rho^\theta \geq 0\}.$$

Since $\mathscr{L}_\rho^\theta$ is contained in $p_\theta(\mathfrak{a}^+)$ it does not contain any line, and the dual cone is thus $\mathscr{L}_\rho^{\theta *}$ with non-empty interior.

LEMMA 3.17. *Let $\rho : \Gamma \to G$ be a Zariski dense θ-convex representation and consider some φ in the interior of the dual cone $\mathscr{L}_\rho^{\theta *}$. Then the cocycle $\varphi \circ \beta_\theta : \Gamma \times \partial\Gamma \to \mathbb{R}$ has finite exponential growth rate.*

Proof. The proof follows exactly as [**23**, Lemma 7.7]. □

For a θ-convex representation $\rho : \Gamma \to G$ consider $F_\rho^\theta : T^1\widetilde{M} \to \mathfrak{a}$ given by Ledrappier's Theorem 2.9 for the cocycle $\beta^\theta : \Gamma \times \partial\Gamma \to \mathfrak{a}_\theta$. The following corollary is a direct consequence of the previous lemma and Lemma 2.12.

COROLLARY 3.18. *Let $\rho : \Gamma \to G$ be a Zariski dense θ-convex representation and fix some φ in the interior of the dual cone $\mathscr{L}_\rho^{\theta *}$. Then the function $\varphi(F_\rho^\theta) : T^1\widetilde{M} \to \mathbb{R}$ is cohomologous to a positive function.*

Continuity of $\mathscr{L}_\rho^\theta$. Recall that $\phi_t : \Gamma\backslash T^1\widetilde{M} \to \Gamma\backslash T^1\widetilde{M}$ is the geodesic flow and denote $\mathcal{M}^{\phi_t}$ as the set of ϕ_t-invariant probability measures (for all t).

LEMMA 3.19. *The set*

$$\left\{\int F_\rho^\theta \, dm : m \in \mathcal{M}^{\phi_t}\right\}$$

is compact, does not contain $\{0\}$ *and generates the cone* $\mathscr{L}_\rho^\theta$.

Proof. Compactness is immediate since $\mathcal{M}^{\phi_t}$ is compact. Considering some φ in the interior of the dual cone $\mathscr{L}_\rho^{\theta *}$ and applying Lemma 3.17 together with Ledrappier's Lemma 2.7 one finds that zero does not belong to $\{\int F_\rho^\theta \, dm : m \in \mathcal{M}^{\phi_t}\}$.

Lemma 3.16 implies that, for every $[\gamma] \in [\Gamma]$, one has $\lambda_\theta(\rho\gamma) = \int_{[\gamma]} F_\rho^\theta$, and hence $\mathscr{L}_\rho^\theta$ is the closed cone generated by

$$\left\{ \int_{[\gamma]} F_\rho^\theta : [\gamma] \in [\Gamma] \right\}.$$

Since convex combinations of periodic orbits are dense in $\mathcal{M}^{\phi_t}$ (Anosov's closing lemma, cf. Shub [**24**]) the last statement follows. □

Denote $\hom_\theta^Z(\Gamma, G)$ as the set of Zariski dense θ-convex representations endowed with the topology as a subset of $G^{\mathcal{A}}$, where $\mathcal{A}$ is a finite generating set of Γ.

Proposition 3.7 together with Corollary 2.10 give the following proposition.

PROPOSITION 3.20. *The function that associates to every $\rho \in \hom_\Pi^Z(\Gamma, G)$ the cohomology class of $F_\rho^\theta : T^1\widetilde{M} \to \mathfrak{a}_\theta$ is continuous.*

One directly obtains the continuity of the cone $\mathscr{L}_\rho^\theta$.

COROLLARY 3.21. *The function $\hom_\theta^Z(\Gamma, G) \to \{$compact subsets of $\mathbb{P}(\mathfrak{a}_\theta)\}$ given by $\rho \mapsto \mathbb{P}(\mathscr{L}_\rho^\theta)$ is continuous.*

Proof. The statement is obvious from Lemma 3.19 and Proposition 3.20. □

COROLLARY 3.22. *Fix some Zariski dense θ-convex representation ρ_0 and consider some φ in the interior of the dual cone $\mathscr{L}_{\rho_0}^{\theta *}$. Then the function*

$$\rho \mapsto h_\varphi(\rho) := \lim_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda_\theta(\rho\gamma)) \leq s\}}{s}$$

is continuous in a neighborhood U of ρ_0 such that $\varphi|\mathscr{L}_\rho^\theta - \{0\} > 0$ for every $\rho \in U$.

Proof. Since the equivariant maps vary continuously with the representation, the function $\rho \mapsto \varphi \circ \beta^\rho : \Gamma \times \partial\Gamma \to \mathbb{R}$ is continuous. The exponential growth rate of $\varphi \circ \beta_\theta^\rho$ is $h_\varphi(\rho)$ and thus the corollary is a consequence of Proposition 2.14. □

4. *The growth indicator is strictly concave*

We shall now consider the *growth indicator function* introduced by Quint [**18**]. Recall that $a : G \to \mathfrak{a}^+$ is the Cartan projection and fix some norm $\| \ \|$ on $\mathfrak{a}$ invariant under the Weyl group.

Consider Δ to be a discrete Zariski dense subgroup of G. For an open cone $\mathscr{C}$ on $\mathfrak{a}^+$ consider the exponential growth rate

$$h_{\mathscr{C}} := \limsup_{s\to\infty} \frac{\log \#\{g \in \Delta : a(g) \in \mathscr{C} \text{ and } \|a(g)\| \leq s\}}{s}.$$

One then sets $\psi_\Delta : \mathfrak{a}^+ \to \mathbb{R}$ as

$$\psi_\Delta(v) := \|v\| \inf_{\mathscr{C} \text{ open cone}: v \in \mathscr{C}} h_{\mathscr{C}}.$$

Note that ψ_Δ is homogeneous and does not depend on the norm chosen.

THEOREM 4.1. (Quint [**18**]) *The function* ψ_Δ *is concave and upper semi-continuous, positive on* $\mathscr{L}_\Delta$ *and strictly positive on its relative interior. The set* $\{v \in \mathfrak{a} : \psi_\Delta(v) > -\infty\}$ *coincides with the limit cone* $\mathscr{L}_\Delta$.

We need the following lemma of Quint [**18**].

LEMMA 4.2. (Quint [**18**, Lemma 3.1.3]) *Let* Δ *be a Zariski dense subgroup of* G *and consider* $\varphi \in \mathfrak{a}^*$. *If* $\varphi(v) > \psi_\Delta(v)$ *for every* $v \in \mathfrak{a} - \{0\}$ *then the Poincaré series*

$$\sum_{g\in\Delta} e^{-\varphi(a(g))}$$

is convergent; if there exists v *such that* $\varphi(v) < \psi_\Delta(v)$ *then it is divergent.*

Fix a Zariski dense hyperconvex representation $\rho : \Gamma \to G$ and denote ψ_ρ as its growth indicator function. If $\varphi \in \mathfrak{a}^*$ verifies $\varphi \geq \psi_\rho$ then

$$\|\varphi\| \geq \sup \frac{\psi_\rho(v)}{\|v\|} = h_{\rho(\Gamma)}.$$

One is then interested in the set

$$D_\rho := \{\varphi \in \mathfrak{a}^* : \varphi \geq \psi_\rho\}.$$

Since ψ_ρ is non-negative on the limit cone $\mathscr{L}_\rho$, the set D_ρ is contained in the dual cone ${\mathscr{L}_\rho}^*$. For $\varphi \in {\mathscr{L}_\rho}^*$ define

$$h_\varphi = \lim_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq s\}}{s}.$$

Note that h_φ is the critical exponent of the Poincaré series

$$\sum_{\gamma\in\Gamma} e^{-s\varphi(a(\rho\gamma))}.$$

Quint's Lemma 4.2 implies the following characterization of the set D_ρ.

LEMMA 4.3. *The interior of the set* D_ρ *is the set of* $\varphi \in {\mathscr{L}_\rho}^*$ *such that* $h_\varphi < 1$, *and its boundary coincides with the set of linear functionals such that* $h_\varphi = 1$.

We are now interested in showing that the growth indicator function ψ_ρ of $\rho(\Gamma)$ is strictly concave with vertical tangent on the boundary of the limit cone $\mathscr{L}_\rho$.

One (trivial) consequence of [**23**, Theorem C] is the following corollary.

COROLLARY 4.4. *If* $\varphi \in {\mathscr{L}_\rho}^*$ *then the exponential growth rate of the Hölder cocycle* $\varphi \circ \beta$ *coincides with the exponential growth rate of* $\{a(\rho\gamma) : \gamma \in \Gamma\}$, *i.e.,*

$$\limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s} = \limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq s\}}{s} = h_\varphi.$$

This corollary allows us to link the growth indicator function with the thermodynamic formalism on the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$. We will give a description of the set D_ρ by considering the pressure function of potentials on $\Gamma\backslash T^1\widetilde{M}$. Fix from now on a Γ-invariant function $F_\rho : T^1\widetilde{M} \to \mathfrak{a}$ given by Ledrappier's Theorem 2.9 for the Busemann cocycle $\beta = \beta_\Pi : \Gamma \times \partial\Gamma \to \mathfrak{a}$.

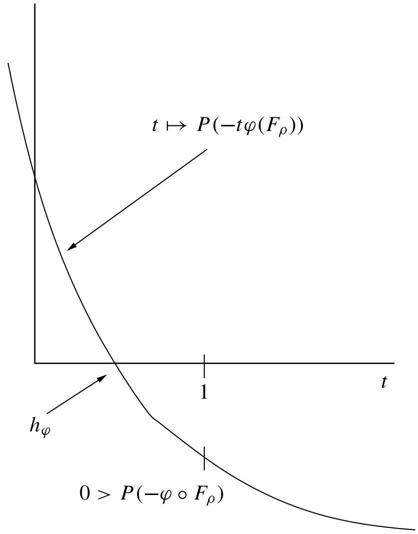

FIGURE 1. The function $t \mapsto P(-t\varphi(F_\rho))$ when $\varphi \circ F_\rho$ is cohomologous to a positive function.

PROPOSITION 4.5. *Let $\rho : \Gamma \to G$ be a Zariski dense hyperconvex representation. Then $D_\rho = \{\varphi \in \mathfrak{a}^* : P(-\varphi \circ F_\rho) \leq 0\}$. The interior of D_ρ is then the set*

$$\{\varphi \in \mathfrak{a}^* : P(-\varphi \circ F_\rho) < 0\}.$$

Proof. Corollary 4.4 states that for every $\varphi \in \mathscr{L}_\rho{}^*$ one has

$$\begin{aligned} h_\varphi &= \limsup_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq s\}}{s} \\ &= \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}. \end{aligned}$$

Recall that Lemma 4.3 states that the interior of D_ρ is the set of linear functionals $\varphi \in \operatorname{int} \mathscr{L}_\rho{}^*$ with $0 < h_\varphi < 1$.

Consider then a linear functional φ in the interior of D_ρ, that is, $\varphi(v) > \psi_\rho(v)$ for all $v \in \mathfrak{a} - \{0\}$. We want to show that $P(-\varphi \circ F_\rho) < 0$.

Quint's Theorem 4.1 states that ψ_ρ is positive on the interior of the limit cone and thus φ belongs to the interior of the dual cone $\mathscr{L}_\rho{}^*$, that is, $\varphi|\mathscr{L}_\rho - \{0\} > 0$. Moreover, one has $h_\varphi < 1$.

Corollary 3.18 states that $\varphi \circ F_\rho$ is cohomologous to a strictly positive function; Proposition 2.6 then implies that $t \mapsto P(-t\varphi(F_\rho))$ has strictly negative derivative. Thus

$$t \mapsto P(-t\varphi(F_\rho))$$

is strictly decreasing. Ledrappier's Lemma 2.7 implies that $P(-h_\varphi \varphi(F_\rho)) = 0$. One then finds that $P(-\varphi(F_\rho)) < 0$ (see Figure 1).

Conversely, fix a linear functional $\varphi \in \mathfrak{a}^*$ such that $P(-\varphi(F_\rho)) < 0$ and consider again the function $t \mapsto P(-t\varphi(F_\rho))$. Since $P(0) = h_{\mathrm{top}}(\phi_t) > 0$ and $P(-\varphi(F_\rho)) < 0$, there exists some $0 < h < 1$ with $P(-h\varphi(F_\rho)) = 0$.

By the definition of pressure one has

$$P(-\varphi(F_\rho)) = \sup_{m\in\mathcal{M}^{\phi_t}} h(m,\phi_t) - \int \varphi(F_\rho)\,dm < 0,$$

which implies that, for every $\gamma \in \Gamma$,

$$\int_{[\gamma]} \varphi(F_\rho) > 0,$$

that is, $\varphi \circ \beta$ has positive periods. We can thus apply Ledrappier's Lemma 2.7 and conclude that such h is necessarily h_φ and thus φ belongs to the interior of D_ρ. □

We will now deduce properties for ψ_ρ from properties of the pressure function. We need Benoist's theorem below.

THEOREM 4.6. (Benoist [**5**]) *Consider Δ a Zariski dense subgroup of G. Then the group generated by $\{\lambda(g) : g \in \Delta\}$ is dense in $\mathfrak{a}$.*

PROPOSITION 4.7. *Let $\rho : \Gamma \to G$ be a Zariski dense hyperconvex representation. Then the set D_ρ is strictly convex and its boundary is an analytic submanifold of $\mathfrak{a}^*$.*

Proof. Fix $F_\rho : T^1\widetilde{M} \to \mathfrak{a}$ given by Ledrappier's Theorem 2.9 and consider the function $\overline{P} : \mathfrak{a}^* \to \mathbb{R}$ given by $\overline{P}(\varphi) = P(-\varphi(F_\rho))$. Proposition 2.6 implies that this function is analytic and its derivative $d\overline{P} : \mathfrak{a}^* \to \mathfrak{a}$ is given by the formula

$$d\overline{P}(\varphi) = -\int F_\rho\, dm_\varphi,$$

where m_φ is the equilibrium state of $-\varphi(F_\rho)$.

If $\varphi \in \mathfrak{a}^*$ is such that $\overline{P}(\varphi) = 0$, then Proposition 4.5 implies that φ belongs to the boundary of the set D_ρ; in particular, $\varphi \in \mathscr{L}_\rho^*$ and $h_\varphi = 1$. One deduces that $\varphi(F_\rho)$ is cohomologous to a positive function (Corollary 3.18) and thus

$$\int \varphi(F_\rho)\, dm_\varphi \neq 0.$$

Hence the vector

$$d\overline{P}(\varphi) = \int F_\rho\, dm_\varphi \neq 0.$$

We conclude that 0 is a regular value of $\overline{P}$ and thus $\partial D_\rho = \overline{P}^{-1}\{0\}$ is an analytic submanifold of $\mathfrak{a}^*$.

The tangent space to ∂D_ρ at $\varphi_0 \in \partial D_\rho$ is

$$T_{\varphi_0}\partial D_\rho = \left\{\varphi \in \mathfrak{a}^* : \int \varphi(F_\rho)\, dm_{\varphi_0} = 0\right\}.$$

Then consider $\varphi \in T_{\varphi_0}\partial D_\rho$. Since the periods of F_ρ generate a dense subgroup of $\mathfrak{a}$ (Benoist's Theorem 4.6) the function $\varphi(F_\rho)$ is not cohomologous to zero. Proposition 2.6 then implies that the function

$$t \mapsto \overline{P}(\varphi_0 - t\varphi)$$

is strictly convex with one critical point at 0. Thus $\varphi_0 + T_{\varphi_0}\partial D_\rho$ does not intersect ∂D_ρ (except at φ_0) and thus D_ρ is strictly convex. □

The following lemma is a consequence of Hahn–Banach's theorem; one can find a proof in Quint [**20**, §4.1]. Let V be a finite-dimensional vector space and $\Psi : V \to \mathbb{R} \cup \{-\infty\}$ a concave homogeneous upper semi-continuous function. Set

$$D_\Psi = \{\Phi \in V^* : \Phi \geq \Psi\} \quad \text{and} \quad L_\Psi = \{x \in V : \Psi(x) > -\infty\}.$$

A linear functional $\Phi \in V$ is *tangent* to Ψ at x if $\Phi \in D_\Psi$ and $\Phi(x) = \Psi(x)$. The function Ψ has *vertical tangent* at some point x if, for every $\Phi \in D_\Psi$, one has $\Phi(x) > \Psi(x)$.

LEMMA 4.8. (Duality lemma) *Let V be a finite-dimensional vector space and $\Psi : V \to \mathbb{R} \cup \{-\infty\}$ a concave homogeneous upper semi-continuous function. Suppose that D_Ψ and L_Ψ have non-empty interior. Then*

- *for every $x \in L_\Psi$ one has*

$$\Psi(x) = \inf_{\Phi \in D_\Psi} \Phi(x);$$

- *the set D_Ψ is strictly convex if and only if Ψ is differentiable on the interior of L_Ψ and with vertical tangent on the boundary;*
- *the boundary ∂D_Ψ is differentiable if and only if the function Ψ is strictly concave.*

When these conditions are satisfied, the derivative induces a bijection between the set of directions in the interior of L_Ψ and ∂D_Ψ.

Let us briefly explain the bijection between the set of directions in the interior of L_Ψ and ∂D_Ψ. If $\Phi \in \partial D_\Psi$ then Φ is tangent to Ψ at some point x in the interior of L_Ψ. The function Ψ is then differentiable at x and its derivative at x is Φ. Since Ψ is strictly concave and homogeneous, the functions Φ and Ψ coincide (only) in the direction determined by x. This is the bijection.

We find the following corollary.

COROLLARY 4.9. *The growth indicator ψ_ρ of a Zariski dense hyperconvex representation ρ is strictly concave, analytic on the interior of the limit cone and has vertical tangent on its boundary. If $P(-\varphi_0(F_\rho)) = 0$ then φ_0 is tangent to ψ_ρ in the direction given by the vector*

$$\int F_\rho \, dm_{\varphi_0}$$

and the value

$$\psi_\rho\left(\int F_\rho \, dm_{\varphi_0}\right) = h(\phi_t, m_{\varphi_0})$$

is the metric entropy of the geodesic flow for the equilibrium state m_{φ_0}.

Proof. Proposition 4.7 together with the duality Lemma 4.8 imply that:

(i) since D_ρ is strictly convex, ψ_ρ is of class C^1 on the interior of the cone of $\mathscr{L}_\rho$ but has vertical tangent on its boundary;

(ii) since ∂D_ρ is of class C^1, ψ_ρ is strictly concave.

The formula

$$d\overline{P}(\varphi) = -\int F_\rho \, dm_\varphi$$

together with the first point of the duality Lemma 4.8 imply that $\varphi_0 \in \partial D_\rho$ is tangent to ψ_ρ in the direction given by the vector $\int F_\rho \, dm_{\varphi_0}$. Moreover, one has

$$\varphi_0\bigg(\int F_\rho \, dm_{\varphi_0}\bigg) = \psi_\rho\bigg(\int F_\rho \, dm_{\varphi_0}\bigg).$$

The bijection between ∂D_ρ and interior directions of $\mathscr{L}_\rho$ is $\varphi_0 \mapsto \mathbb{R}_+ \cdot d\overline{P}(\varphi_0)$; an analogous reasoning to Proposition 4.7 implies that its derivative is invertible. Thus, since ∂D_ρ is analytic, the derivative of ψ_ρ is analytic on the interior of $\mathscr{L}_\rho$ and thus ψ_ρ is analytic.

Since $\varphi_0(F_\rho)$ is cohomologous to a positive function (Corollary 3.18), we can consider $\sigma_t : \Gamma\backslash T^1\widetilde{M} \to \Gamma\backslash T^1\widetilde{M}$ to be the reparametrization of the geodesic flow by $\varphi_0(F_\rho)$. Applying Lemma 2.3 and Abramov's formula (3) we have that the topological entropy of σ_t satisfies

$$h_{\text{top}}(\sigma_t) = h(\phi_t, m_{\varphi_0}) \bigg/ \int \varphi_0(F_\rho) \, dm_{\varphi_0}.$$

Proposition 2.4 implies that the topological entropy of σ_t coincides with the exponential growth rate of its periodic orbits, i.e., $h_{\text{top}}(\sigma_t) = h_{\varphi_0}$. This last quantity is equal to 1 since $\varphi_0 \in \partial D_\rho$ and thus

$$\psi_\rho\bigg(\int F_\rho dm_{\varphi_0}\bigg) = \int \varphi_0(F_\rho) \, dm_{\varphi_0} = h(\phi_t, m_{\varphi_0}).$$

This finishes the proof. □

5. *Continuity properties*

In this section we are interested in showing that certain objects one associates to a Zariski dense hyperconvex representation vary continuously with the representation. We are concerned with the growth indicator ψ_ρ of ρ and the exponential growth rate $h_\rho^{\| \, \|}$.

For a Zariski dense hyperconvex representation $\rho : \Gamma \to G$ denote by $F_\rho : T^1\widetilde{M} \to \mathfrak{a}$ the function given by Ledrappier's Theorem 2.9 for the cocycle β^ρ (this choice is only valid modulo Livšic cohomology).

Consider the uniform topology on the space $C(X, \mathbb{R})$ of continuous real-valued functions on some space X. We can now show Theorem B.

THEOREM 5.1. *Let $\rho_0 : \Gamma \to G$ be a Zariski dense hyperconvex representation and fix some open cone $\mathscr{C}$ such that its closure is contained in the interior of the limit cone $\mathscr{L}_{\rho_0}$ of ρ_0. Consider a neighborhood U of ρ_0 such that $\mathscr{C}$ is contained in* $\operatorname{int}(\mathscr{L}_\rho)$ *for every $\rho \in U$. Then the function $U \to C(\mathscr{C}, \mathbb{R})$ given by*

$$\rho \mapsto \psi_\rho | \mathscr{C}$$

is continuous.

Proof. Recall that, from the duality Lemma 4.8, the growth indicator ψ_ρ is uniquely determined by the boundary ∂D_ρ. Hence, it suffices to prove that for some fixed $\varphi \in \partial D_{\rho_0}$ and a given neighborhood W of φ there exists a neighborhood U of ρ_0 such that, for every $\rho \in U$, one has that ∂D_ρ intersects W.

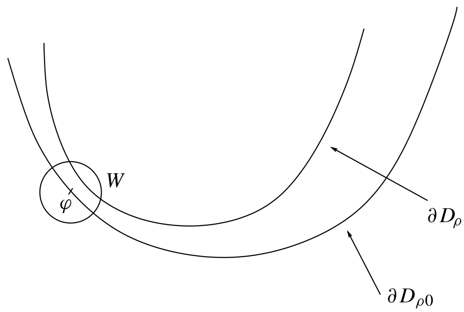

FIGURE 2. The set D_ρ after perturbation.

Proposition 4.5 states that ∂D_ρ is the set of linear functionals in $\mathscr{L}_\rho{}^*$ such that $P(-\varphi(F_\rho)) = 0$; Lemma 2.13 together with Proposition 3.20 thus imply the result. □

Strict concavity of ψ_ρ together with the last theorem imply the following corollaries. Fix some norm $\| \ \|$ on $\mathfrak{a}$ invariant under the Weyl group. The *growth form* $\Theta_\rho^{\| \ \|} \in \mathfrak{a}^*$ of ρ for the norm $\| \ \|$ is the unique linear form tangent to ψ_ρ in the direction where $\psi_\rho/\| \ \|$ attains its maximum. This direction is called the *growth direction* of ρ for $\| \ \|$.

COROLLARY 5.2. *The function* $\hom_\Pi^Z(\Gamma, G) \to \mathfrak{a}^*$ *that associates to* ρ *the growth form* $\Theta_\rho^{\| \ \|}$ *is continuous.*

Proof. Since ψ_{ρ_0} is strictly concave, the growth direction belongs to the interior of $\mathscr{L}_{\rho_0}$. Fix a small open cone $\mathscr{C}$ containing this growth direction. Theorem 5.1 implies that in a neighborhood of ρ_0 the function

$$\rho \mapsto \max_{\mathscr{C}} \frac{\psi_\rho}{\| \ \|}$$

is continuous. Again, since ψ_ρ is strictly concave, this is the global maximum of $\psi_\rho/\| \ \|$. This finishes the proof. □

Recall that we have denoted $h_\rho^{\| \ \|}$ as the exponential growth rate of $\rho(\Gamma)$ on the symmetric space of G,

$$h_\rho^{\| \ \|} = \lim_{s\to\infty} \frac{\log \#\{\gamma \in \Gamma : \|a(\rho\gamma)\| \leq s\}}{s}.$$

COROLLARY 5.3. *The function* $\hom_\Pi^Z(\Gamma, G) \to \mathbb{R}$,

$$\rho \mapsto h_{\rho(\Gamma)}^{\| \ \|},$$

is continuous.

Proof. Since $h_\rho^{\| \ \|}$ coincides with the norm of the growth form (this is a general fact, see Quint [**18**]),

$$\|\Theta_\rho^{\| \ \|}\| = h_{\rho(\Gamma)}^{\| \ \|}.$$

The result then follows from the previous corollary. □

Acknowledgements. The author is extremely grateful to Jean-François Quint for useful discussions and guidance, and to Rafael Potrie for useful discussions, in particular for providing reference [**3**].

REFERENCES

[1] L. M. Abramov. On the entropy of a flow. *Dokl. Akad. Nauk SSSR* **128** (1959), 873–875.

[2] R. Akerkar. *Nonlinear Functional Analysis*. Narosa Publishing House, 1999.

[3] L. Barreira and Ya. Pesin. Lectures on Lyapunov exponents and smooth ergodic theory. *Smooth Ergodic Theory and its Applications (Seattle, 1999) (Proc. Symp. in Pure Mathematics)*. American Mathematical Society, Providence, RI, 2000.

[4] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.* **7**(1) (1997), 1–47.

[5] Y. Benoist. Propriétés asymptotiques des groupes linéaires II. *Adv. Stud. Pure Math.* **26** (2000), 33–44.

[6] Y. Benoist. Convexes divisibles I. *Algebraic Groups and Arithmetic*. Tata Institute of Fundamental Research, 2004, pp. 339–374.

[7] R. Bhatia and K. R. Parthasarathy. *Lecture on Functional Analysis*. MacMillan, Delhi, 1977.

[8] R. Bowen. Periodic orbits of hyperbolic flows. *Amer. J. Math.* **94** (1972), 1–30.

[9] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973), 429–460.

[10] R. Bowen and D. Ruelle. The ergodic theory of axiom A flows. *Invent. Math.* **29** (1975), 181–202.

[11] O. Guichard and A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.* **190** (2012), 357–438.

[12] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.* **165** (2006), 51–114.

[13] F. Ledrappier. Structure au bord des variétés à courbure négative. *Sémin. Théor. Spectr. Géom.* **71** (1994−1995), 97–122.

[14] A. N. Livšic. Cohomology of dynamical systems. *Math. USSR Izv.* **6** (1972), 1278–1301.

[15] G. Margulis. Applications of ergodic theory to the investigation of manifolds with negative curvature. *Funct. Anal. Appl.* **3** (1969), 335–336.

[16] S.-J. Patterson. The limit set of a Fuchsian group. *Acta Math.* **136** (1976), 241–273.

[17] M. Pollicott and R. Sharp. Length asymptotics in higher Teichmüller theory. *Preprint*, 2012.

[18] J.-F. Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.* **77** (2002), 503–608.

[19] J.-F. Quint. Mesures de Patterson–Sullivan en rang supérieur. *Geom. Funct. Anal.* **12** (2002), 776–809.

[20] J.-F. Quint. L'indicateur de croissance des groupes de Schottky. *Ergod. Th. & Dynam. Sys.* **23** (2003), 249–272.

[21] M. Ratner. The central limit theorem for geodesic flows on n-manifolds of negative curvature. *Israel J. Math.* **16** (1973), 181–197.

[22] D. Ruelle. *Thermodynamic Formalism*. Addison-Wesley, London, 1978.

[23] A. Sambarino. Quantitative properties of convex representations. *Comm. Math. Helv.*, to appear, arXiv:1104.4705v1.

[24] M. Shub. *Global Stability of Dynamical Systems*. Springer, New York, 1987.

[25] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 171–202.

[26] X. Thirion. *Propriétés de mélange du flot des chambres de Weyl des groupes de Ping-Pong (Bulletin de la SMF, 137)*. Société mathématique de France, 2009, pp. 387–421.

[27] J. Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconqe. *J. Reine Angew. Math.* **247** (1971), 196–220.

[28] C. Yue. The ergodic theory of discrete isometry groups on manifolds of variable negative curvature. *Trans. Amer. Math. Soc.* **348**(12) (1996), 4965–5005.