INFINITESIMAL ZARISKI CLOSURES OF POSITIVE REPRESENTATIONS

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Abstract. We classify the (semi-simple parts of the) Lie algebra of the Zariski closure of a discrete subgroup of a split simple real-algebraic Lie group, whose limit sets are minimal and such that the limit set in the space of full flags contains a positive triple of flags (as in Lusztig [20]). We then apply our result to obtain a new proof of Guichard's classification [14] of Zariski closures of Hitchin representations into $\mathsf{PSL}_d(\mathbb{R})$.

Contents

1. Introduction

Let us consider the vector space $\mathbb{R}^d$ equipped with its canonical ordered basis $\mathcal{E} = \{e_1, \ldots, e_d\}$ and let $\mathsf{GL}_d(\mathbb{R})$ be the group of invertible matrices. A *minor* of $g \in \mathsf{GL}_d(\mathbb{R})$ is the determinant of a square matrix obtained from g by deleting some lines and columns from it. Minors appear naturally when one considers the exterior powers of $\mathbb{R}^d$. Indeed, theses spaces carry also a natural basis

$$\wedge^k \mathcal{E} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}$$

defined from $\mathcal{E}$, and the coefficients of $\wedge^k g$ in this basis are the $k \times k$ minors of g.

As introduced by Schoenberg [22] and Gantmacher-Krein [9], a matrix is *totally positive* if all its minors are positive[1]. If $g \in \mathsf{GL}_d(\mathbb{R})$ is such a matrix, then, since all its entries are positive, it preserves the sharp convex cone of $\mathbb{R}^d$

$$\mathscr{C}_{\mathcal{E}} = \{(x_1, \ldots, x_d) : x_i \geqslant 0\},$$

consisting on vectors all of whose entries in $\mathcal{E}$ are non-negative. By the preceding paragraph more is true: the same holds for every exterior power of g replacing $\mathcal{E}$ by $\wedge^k \mathcal{E}$.

The author was partially financed by ANR DynGeo ANR-16-CE40-0025.

[1] Let us convene throughout the paper that 0 is not a positive real number.

An application of the classical Perron-Frobenius Theorem implies then that $\wedge^k g$ has a unique attracting fixed line in the interior of this cone,

$$g_{+,k} \in \operatorname{int} \mathscr{C}_{\wedge^k \mathcal{E}}, \tag{1.1}$$

and the collection $(g_{+,k})_1^d$ is an attracting complete flag[2] of g. If we denote by $\vec{\mathcal{E}}$ the complete flag

$$\vec{\mathcal{E}} = \big(\operatorname{span}(e_1 \oplus \cdots \oplus e_k)\big)_1^d$$

then the inclusion (1.1) readily implies that the *lower triangular matrix* $\check{u}_g$ sending $\vec{\mathcal{E}}$ to $(g_{+,k})$ has all minors below the diagonal positive. Such a semi-group will be denoted by $\check{U}_{>0}$. If one is keener on *upper triangular matrices* then one should replace $\vec{\mathcal{E}}$ by $\overleftarrow{\mathcal{E}} = \big(\operatorname{span}(e_d \oplus \cdots \oplus e_{d-k+1})\big)_1^d$ to obtain analogous $U_{>0}$. The subspace of *positive flags* is then defined by

$$\mathcal{F}_{>0} = \check{U}_{>0} \cdot \vec{\mathcal{E}} = U_{>0} \cdot \overleftarrow{\mathcal{E}}.$$

Several implicit choices, other than the pair of flags $(\vec{\mathcal{E}}, \overleftarrow{\mathcal{E}}) \leftrightarrow \mathcal{E}$, have been done to define $\mathcal{F}_{>0}$, but we will not enter this matter at the moment.

The above (very quick) picture has been generalized to the real points of an arbitrary (Zariski-connected) reductive split real-algebraic group G by Lusztig [20]. We refer the reader to §5.1 for the precise definitions and we reuse the notation $\mathcal{F}_{\mathsf{G}} = \mathcal{F}$ as the complete flag space of G and $\mathcal{F}_{>0}$ for the subset of positive flags associated to a pair of fixed opposite Borel subgroups B and $\check{B}$ (and a *pinning*, see §5.1). Let us say that a triple of pairwise transverse flags (x, y, z) is *positive*, if there exists $g \in \mathsf{G}$ such that $g \cdot x = [\check{B}]$, $g \cdot z = [B]$ and $g \cdot y \in \mathcal{F}_{>0}$.

Le us consider more generally a *partial flag* $\mathcal{F}_\theta$ of G, these are indexed by subsets of the set of simple roots Δ, with $\mathcal{F}_\Delta = \mathcal{F}$. An element $g \in \mathsf{G}$ is *proximal on* $\mathcal{F}_\theta$ if it has an attracting fixed point on $\mathcal{F}_\theta$, i.e. there exists $g_{+,\theta} \in \mathcal{F}_\theta$ fixed by g and an open neighborhood V of $g_{+,\theta}$ such that $g\overline{V} \subset \operatorname{int} V$. In this situation one has $\bigcap_{n \in \mathbb{N}} g^n V = \{g_{+,\theta}\}$. Elements that are proximal on $\mathcal{F}$ are often called *purely loxodromic*.

If $\Lambda < \mathsf{G}$ is a discrete subgroup then its *limit set on* $\mathcal{F}_\theta$ is defined as

$$\mathbf{L}_{\Lambda,\theta} = \overline{\{g_{+,\theta} : g \in \Lambda \text{ proximal on } \mathcal{F}_\theta\}} \subset \mathcal{F}_\theta.$$

A result by Benoist[3] [2] asserts that if Λ is Zariski dense, then $\mathbf{L}_{\Lambda,\theta}$ is non-empty and contained in any closed non-empty Λ-invariant set. We will assume a slightly weaker version of this property. Let us say that $\mathbf{L}_{\Lambda,\theta}$ is *minimal* if the only closed Λ-invariant subsets of $\mathbf{L}_{\Lambda,\theta}$ are either the empty set or $\mathbf{L}_{\Lambda,\theta}$ itself.

Definition 1.1. Let $\Lambda < \mathsf{G}$ be a discrete group. We say that

- Λ has *minimal limit sets* if $\mathbf{L}_{\Lambda,\{\sigma\}}$ is minimal for every $\sigma \in \Delta$,
- $\mathbf{L}_{\Lambda,\Delta}$ *contains a positive loxodromic triple* if there exists $g_0 \in \Lambda$ proximal on $\mathcal{F}$ and $x_0 \in \mathbf{L}_{\Lambda,\Delta}$ such that (g_+, x_0, g_-) is a positive triple.

Recall that a reductive Lie algebra $\mathfrak{h}$ splits as the sum $\mathfrak{h} = \mathfrak{h}_{ss} \oplus \mathfrak{Z}(\mathfrak{h})$ where $\mathfrak{Z}(\mathfrak{h})$ is its center and $\mathfrak{h}_{ss} = [\mathfrak{h}, \mathfrak{h}]$ is semi-simple. Recall also that, as $\mathfrak{g}$ is split, it contains a special conjugacy class of sub-algebras isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ called *the principal* $\mathfrak{sl}_2(\mathbb{R})$*'s*, see §2.1.1 for the definition.

[2]Recall that a *complete flag* of $\mathbb{R}^d$ is a sequence of vector subspaces $(V_i)_1^d$ such that $\dim V_i = i$ and $V_i \subset V_{i+1}$.

[3](that holds for G an arbitrary reductive real-algebraic Lie group of non-compact type)

The main purpose of this paper is to prove the following.

Theorem A. *Let* G *be the real points of a Zariski connected, simple split, real-algebraic group and* $\Lambda < \mathsf{G}$ *a subgroup with reductive Zariski closure* H*, minimal limit sets and such that* $\mathbf{L}_{\Lambda,\Delta}$ *contains a positive loxodromic triple. Then* $\mathfrak{h}_{ss}$ *is either* $\mathfrak{g}$*, a principal* $\mathfrak{sl}_2(\mathbb{R})$*, or* $\operatorname{Int}\mathfrak{g}$*-conjugated to one of the possibilities listed in table 1.*

We would like to stress the fact that only one positive (loxodromic) triple in the limit set $\mathbf{L}_{\Lambda,\Delta}$ is required.

$\mathfrak{g}$	$\mathfrak{h}_{ss}$	$\phi : \mathfrak{h}_{ss} \to \mathfrak{g}$
$\mathfrak{sl}_{2n}(\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{R})$	defining representation
$\mathfrak{sl}_{2n+1}(\mathbb{R})$	$\mathfrak{so}(n,n+1)\ \forall n$	defining representation
	$\mathfrak{g}_2$ if $n=3$	fundamental for the short root
$\mathfrak{so}(3,4)$	$\mathfrak{g}_2$	fundamental for the short root
$\mathfrak{so}(n,n)$	$\mathfrak{so}(n-1,n)\ \forall n \geqslant 3$	stabilizer of a non-isotropic line
	$\mathfrak{so}(3,4)$ if $n=4$	fundamental for the short root
	$\mathfrak{g}_2$ if $n=4$	stabilizes a non-isotropic line L and is fundamental for the short root on $L^{\perp}$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\operatorname{Fix}(\operatorname{inv}_0)$ (see example 3.2)

TABLE 1. The statement of Theorem A, if a simple split algebra $\mathfrak{g}$ is not listed in the first column then the only possibilities for $\mathfrak{h}_{ss}$ are $\mathfrak{g}$ or a principal $\mathfrak{sl}_2(\mathbb{R})$. The notations $\mathfrak{e}_6, \mathfrak{f}_4$ and $\mathfrak{g}_2$ refer to the split real forms of the corresponding exceptional complex Lie algebras. Observe that there are two non $\operatorname{Int}\mathfrak{so}(n,n)$-conjugated embeddings $\mathfrak{so}(n,n-1) \to \mathfrak{so}(n,n)$ that stabilize a non-isotropic line.

The use of Lusztig's positivity to study discrete groups seems to have originated in Fock-Goncharov's [7] work, where the notion of *positive representation* of a surface group was introduced. A similar approach simultaneously originated in Labourie [19]. Both works focus on understanding a special connected component of the character variety $\mathfrak{X}(\pi_1 S, \mathsf{G}) = \hom(\pi_1 S, \mathsf{G})/\mathsf{G}$, for a closed connected orientable surface S of genus $\geqslant 2$ and a center-free split simple group G, introduced by Hitchin [15]. These *Hitchin components* are defined as those components that contain a discrete and faithful representation $\pi_1 S \to \mathsf{G}$ whose Zariski closure is a principal $\mathsf{PSL}_2(\mathbb{R})$ in G.

Combining *loc. cit.* together with Guichard [13] one has the following geometric characterization of Hitchin representations. Recall that the Gromov boundary of $\pi_1 S$ is homeomorphic to a circle and carries a $\pi_1 S$-invariant cyclic order.

Theorem 1.2 ([7, 13, 19])**.** *A representation* $\rho : \pi_1 S \to \mathsf{G}$ *lies in a Hitchin component if and only if there exists a continuous equivariant map* $\xi : \partial \pi_1 S \to \mathcal{F}$ *sending cyclically ordered triples to positive triples of flags.*

In this paper we deal with a weaker notion than the one required in the above result. We replace $\pi_1 S$ with any discrete group acting on a Gromov-hyperbolic space and relax the "order preserving" condition.

If X is a proper Gromov-hyperbolic space and $\Gamma < \operatorname{Isom}(X)$ is a discrete subgroup, then we denote by ∂X_{Γ} its limit set on the visual boundary of X. It is a compact Γ-invariant subset and Γ is *non-elementary* if ∂X_Γ contains at least 3 points. If this is the case, then Γ is non-solvable and ∂X_Γ is characterized by being the smallest non-empty Γ-invariant closed subset of ∂X. We refer the reader to Ghys-de la Harpe [10, Chapitre 8] for these (and other) general facts we will require. Unless Γ is convex co-compact, the limit set ∂X_Γ is not an intrinsic object associated to the group structure of Γ.

We will consider the following representations.

Definition 1.3. Let X be a proper Gromov-hyperbolic space and Γ be a non-elementary discrete isometry group. A representation $\rho : \Gamma \to \mathsf{G}$ is *partially positive* if there exists a ρ-equivariant continuous map $\xi : \partial X_\Gamma \to \mathcal{F}$ such that for every pair $x \neq z$ in ∂X_Γ, there exists $y \in \partial X_\Gamma$ such that $\big(\xi(x), \xi(y), \xi(z)\big)$ is a positive triple.

It is implicit in the definition that distinct pairs of ∂X_Γ are mapped to transverse flags.

The second main result of this paper is the following. Recall that a non-solvable Lie algebra $\mathfrak{l}$ is a semi-direct product $\mathfrak{l}_{ss} \oplus_\pi \operatorname{Rad} \mathfrak{l}$, where $\mathfrak{l}_{ss}$ is semi-simple and $\operatorname{Rad} \mathfrak{l}$ is solvable[4], and that the Zariski closure of a non-solvable subgroup $\Lambda < \mathsf{G}$ has non-solvable Lie algebra.

Theorem B. *Let X be a proper Gromov-hyperbolic space, $\Gamma < \operatorname{Isom} X$ a non-elementary discrete subgroup and $\rho : \Gamma \to \mathsf{G}$ a partially positive representation with Zariski closure L. Then the semi-simple part $\mathfrak{l}_{ss}$ is either $\mathfrak{g}$, a principal $\mathfrak{sl}_2(\mathbb{R})$, or* $\operatorname{Int} \mathfrak{g}$*-conjugated to one of the possibilities listed in table 1.*

The challenge here is to show that $\mathbf{L}_{\rho(\Gamma),\Delta} = \xi(\partial X_\Gamma)$ and that for every $\sigma \in \Delta$, it projects surjectively to every $\mathbf{L}_{\rho(\Gamma),\{\sigma\}}$ under the natural projection $\mathcal{F} \to \mathcal{F}_{\{\sigma\}}$.

Let us remark that, in contrast with Theorem A, we do not require the Zariski closure of $\rho(\Gamma)$ to be reductive. We emphasize this by stating the following consequence of Theorem B.

Corollary 1.4. *Assume that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R}), \mathfrak{sp}_{2n}(\mathbb{R}), \mathfrak{so}(n, n+1)$ or $\mathfrak{g}_2$ and let $\rho : \Gamma \to \mathsf{G}$ be partially positive, then its corresponding action on $\mathbb{R}^n, \mathbb{R}^{2n}, \mathbb{R}^{2n+1}$ or $\mathbb{R}^7$ respectively is (strongly) irreducible.*

Theorem B together with Theorem 1.2 give a new proof of the following classification result by Guichard (the argument is postponed to §6). As before, $\mathfrak{g}_2$ is the split real form of the corresponding complex exceptional Lie algebra and $\mathsf{G}_2 = \operatorname{Int} \mathfrak{g}_2$.

Corollary 1.5 (Guichard [14])**.** *Let $\rho : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ be a representation in the Hitchin component, then the Zariski closure of ρ is either $\mathsf{PSL}_d(\mathbb{R})$, a principal $\mathsf{PSL}_2(\mathbb{R})$ or conjugated to one of the following:*

- $\mathsf{PSp}_{2n}(\mathbb{R})$ *if $d = 2n$ for all $n \geqslant 1$,*
- $\mathsf{PSO}(n, n+1)$ *if $d = 2n+1$ for all $n \geqslant 1$,*
- *the fundamental representation for the short root of G_2 if $d = 7$.*

Corollary 1.5 plays a central role in Corollary 11.8 of Bridgeman-Canary-Labourie-S. [4] and in the recent work by Danciger-Zhang [6], allowing the authors to reduce the general problem to the group $\mathsf{PSO}(n, n+1)$.

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[4]See Knapp [17, Chapter B.1].

1.1. **Final remarks.** It is unclear whether all possibilities stated in Theorem B might actually occur. When $\Gamma = \pi_1 S$ (S as above) then Hitchin's Theorem [15] implies this is actually the case. However, a recent result by Alessandrini-Lee-Schaffhauser [1] provides many examples of locally rigid positive representations of groups with torsion.

1.2. **Organization of the paper.** In §2 we recall some facts on representation theory of real reductive Lie algebras of non-compact type. In §3 we introduce the Hasse diagram of a representation of such a Lie algebra, this is nothing but the usual Hasse diagram of a partially order set (here to be the set of restricted weights of the representation). We introduce maps between diagrams and notably study the existence of a surjective map between two Hasse diagrams. There is a case by case proof that is postponed to appendix §A.

In §4 we study Zariski closures of discrete groups verifying a coherence condition with respect to the position of their eigenspaces, and relate these to maps between Hasse diagrams of the Zariski closure and the ambient group. The key point is Proposition 4.8 that, in light of the previous section, classifies Zariski closures of these groups, provided it is reductive.

Section 5 begins by recalling total positivity introduced by Lusztig [20], we prove then that groups whose limit sets contains a positive loxodromic triple verify the coherence condition studied in §4. This proves Theorem A. Theorem B is also proved in this section. In §6 we focus on the $\mathsf{SL}_d(\mathbb{R})$ situation and prove Guichard's classification (Corollary 1.5).

The paper is written rather linearly so one has the following diagram representing dependence between sections:

§A
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§1 — §2 — §3 — §4 — §5 — §6

Acknowledgements. The author would like to thank Olivier Guichard and Maria Beatrice Pozzetti for enlightening discussions and careful reading of this article.

2. Review on Lie Theory

2.1. **Semi-simple Lie algebras.** Let $\mathfrak{g}$ be a semi-simple real Lie algebra of the non-compact type and fix a Cartan involution $o : \mathfrak{g} \to \mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and let $\Phi \subset \mathfrak{a}^*$ be the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. For $\alpha \in \Phi$ let us denote by

$$\mathfrak{g}_\alpha = \{u \in \mathfrak{g} : [a, u] = \alpha(a)u \, \forall a \in \mathfrak{a}\}$$

its associated root space. One has the (restricted) root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$.

Fix a Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ and let Φ^+ and Δ be, respectively, the associated sets of positive roots and of simple roots. One has that $\Phi = \Phi^+ \cup -\Phi^+$ and that if $\alpha \in \Phi^+$ then, upon writing

$$\alpha = \sum_{\sigma \in \Delta} k_\sigma \sigma,$$

every coefficient k_σ is a non-negative integer. The *height* of α is $\mathrm{ht}(\alpha) = \sum_\sigma k_\sigma$.

Let us denote by $(\cdot,\cdot)$ the Killing form of $\mathfrak{g}$, its restriction to $\mathfrak{a}$, and its associated dual form in the dual of $\mathfrak{a}$, $\mathfrak{a}^*$. For $\chi,\psi \in \mathfrak{a}^*$ define

$$\langle \chi,\psi\rangle = 2\frac{(\chi,\psi)}{(\psi,\psi)}.$$

The *Weyl group* of Φ, denoted by W, is the group generated by, for each $\alpha\in\Phi$, the reflection $r_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$ on the hyperplane $\alpha^\perp$,

$$r_\alpha(\chi) = \chi - \langle\chi,\alpha\rangle\alpha.$$

It is a finite group with a unique *longest* element w_0 (w.r.t. the word metric on the generating set $\{r_\alpha : \alpha\in\Delta\}$). This longest element sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$.

Recall that the *Dynkin diagram* of the root system Φ consists on a graph whose vertices are the elements of Δ and such that $\alpha,\beta\in\Delta$ are joined by $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$ edges. If two simple roots are joined by more than one edge then an arrow is added pointing to the shortest (in norm $(\cdot,\cdot)$) root. One speaks indistinctively of the Dynkin diagram of $\mathfrak{g}$, Φ or of Δ.

We will require the following notion:

Definition 2.1. An element of Δ is *extremal* if it is connected to exactly one root in the Dynkin diagram of Φ.

The root systems of type D and E have 3 extremal roots, while the others only have two.

2.1.1. *Some $\mathfrak{sl}_2$'s of $\mathfrak{g}$.* For $\alpha\in\Phi$ let $t_\alpha, h_\alpha\in\mathfrak{a}$ be defined such that, for all $v\in\mathfrak{a}$ and all $\varphi\in\mathfrak{a}^*$, one has

$$\alpha(v) = (v,t_\alpha) \text{ and } \varphi(h_\alpha) = \langle\varphi,\alpha\rangle.$$

These two vectors are related by the simple formula $h_\alpha = 2t_\alpha/(t_\alpha,t_\alpha)$. Recall that for $x\in\mathfrak{g}_\alpha$ and $y\in\mathfrak{g}_{-\alpha}$ one has $[x,y]=(x,y)t_\alpha$. Thus, for each $\alpha\in\Phi^+$ and $\mathsf{x}_\alpha\in\mathfrak{g}_\alpha$ there exists $\mathsf{y}_\alpha\in\mathfrak{g}_{-\alpha}$ such that

$$\begin{aligned} e &= \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \mapsto \mathsf{x}_\alpha\\ f &= \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \mapsto \mathsf{y}_\alpha\\ h &= \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) \mapsto h_\alpha\end{aligned}$$

is a Lie algebra isomorphism between $\mathfrak{sl}_2(\mathbb{R})$ and the span of $\{\mathsf{x}_\alpha,\mathsf{y}_\alpha,h_\alpha\}$. Let us fix such a choice of x_α and y_α from now on.

One says that $\mathfrak{g}$ is *split* if the complexification $\mathfrak{a}\otimes\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}\otimes\mathbb{C}$. Equivalently, $\mathfrak{g}$ is split if the centralizer $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$ vanishes.

Assume that $\mathfrak{g}$ is split. Following Kostant [18, §5], consider the dual basis of $\{t_\sigma : \sigma\in\Delta\}$ relative to $(\cdot,\cdot)$: $(\epsilon_\alpha, t_\beta) = \delta_{\alpha\beta}$, and let $\epsilon_0 = \sum_{\sigma\in\Delta}\epsilon_\sigma\in\mathfrak{a}$. The element ϵ_0 is the semi-simple element of a 3-dimensional simple subalgebra of $\mathfrak{g}$. Such a subalgebra, or any of its $\operatorname{Int}\mathfrak{g}$-conjugates, will be called *a principal* $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{g}$.

Let us denote by $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$.

Theorem 2.2 (Kostant [18, Thm 5.3])**.** *Let $\mathfrak{g}$ be a split Lie algebra and consider an element*

$$e = \sum_{\alpha\in\Phi^+} a_\alpha \mathsf{x}_\alpha \in \mathfrak{n}.$$

Then e lies in a principal $\mathfrak{sl}_2(\mathbb{R})$ if and only if $a_\sigma\neq 0$ for all $\sigma\in\Delta$.

2.2. **Reductive groups.** A Lie algebra $\mathfrak{g}$ is *reductive* if every $\operatorname{ad}\mathfrak{g}$-invariant subspace of $\mathfrak{g}$ has an $\operatorname{ad}\mathfrak{g}$-invariant complement. It is a standard fact (see Knapp [17, Chapter I. §7]) that such an algebra splits as

$$\mathfrak{g} = \mathfrak{Z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss},$$

where $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$ is semi-simple and $\mathfrak{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

A *reductive Lie group*[5] G is a 4-tuple $\big(\mathsf{G}, \mathsf{K}, \sigma, (\cdot,\cdot)\big)$, where K is a compact subgroup of G, σ is a Lie algebra involution of $\mathfrak{g}$ and $(\cdot,\cdot)$ is a σ-invariant, $\operatorname{Ad}\mathsf{G}$-invariant non-degenerate bilinear form on $\mathfrak{g}$ such that:

- $\mathfrak{g}$ is a reductive Lie algebra,
- the Lie algebra $\mathfrak{k}$ of K is the set of fixed points of σ,
- if $\mathfrak{p} = \{x \in \mathfrak{g} : \sigma(x) = -x\}$ then $\mathfrak{k}$ and $\mathfrak{p}$ are $(\cdot,\cdot)$-orthogonal and $(\cdot,\cdot)$ is positive definite on $\mathfrak{p}$,
- the map $\mathsf{K} \times \mathfrak{p} \to \mathsf{G}$, $(k, x) \mapsto k \exp x$, is a surjective diffeomorphism.
- every automorphism of the form $\operatorname{Ad}(h)$, for $h \in \mathsf{G}$, of the complexification $\mathfrak{g} \otimes \mathbb{C}$ is of the form $\operatorname{Ad}(x)$ for some $x \in \operatorname{Int}(\mathfrak{g} \otimes \mathbb{C})$.

Given a reductive group G and a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, one can form, as in the semi-simple case, a restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{g}}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [a, x] = \alpha(a)x \,\forall a \in \mathfrak{a}\}$.

The relation between the restricted roots $\Phi_{\mathfrak{g}}$ and the restricted roots of $\mathfrak{g}_{ss}$ is as follows: the elements of $\Phi_{\mathfrak{g}}$ can be obtained by considering the restricted root space decomposition of $\mathfrak{g}_{ss}$ relative to $\mathfrak{a}_{ss} = \mathfrak{a} \cap \mathfrak{g}_{ss}$ and extending these roots to $\mathfrak{a}$ as being zero on $\mathfrak{a} \cap \mathfrak{Z}(\mathfrak{g})$.

2.3. **Basic facts on representation theory of semi-simple Lie algebras.** Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{R}$ without compact factors. We record here some standard facts about irreducible real representations of $\mathfrak{g}$, see for example Humphreys [16].

The *restricted weight lattice* is defined by

$$\Pi = \{\varphi \in \mathfrak{a}^* : \langle \varphi, \alpha \rangle \in \mathbb{Z} \,\forall \alpha \in \Phi\},$$

it is spanned by the *fundamental weights*: $\{\varpi_\sigma : \sigma \in \Delta\}$ where ϖ_σ is defined by

$$\langle \varpi_\sigma, \beta \rangle = d_\sigma \delta_{\sigma\beta}$$

for every $\sigma, \beta \in \Delta$, where $d_\sigma = 1$ if $2\sigma \notin \Phi^+$ and $d_\sigma = 2$ otherwise. The set Π_+ of *dominant restricted weights* is defined by $\Pi_+ = \Pi \cap (\mathfrak{a}^+)^*$.

Given $\chi, \psi \in \Pi$ one says that $\chi \succ \psi$ if $\chi - \psi$ has non-negative integer coefficients in Δ. A subset $\pi \subset \Pi$ is *saturated* if for every $\chi \in \pi$ and $\alpha \in \Phi$ the *string*

$$\chi - i\alpha \qquad i \text{ between } 0 \text{ and } \langle \chi, \alpha \rangle$$

is entirely contained in π. Such a set is necessarily W-invariant. We say that π has *highest weight* $\mu \in \pi$ if for every $\chi \in \pi$ one has $\chi \prec \mu$. One has the following lemma, see Humphreys [16, §13.4 Lemma B].

Lemma 2.3. *Let π be a saturated set of weights with highest weight μ, then every $\chi \in \Pi_+$ with $\chi \prec \mu$ belongs to π.*

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[5] see for example Knapp [17, Chapter VII. §2.]

Let $\phi : \mathfrak{g} \to \mathfrak{sl}(V)$ be an irreducible representation. The sub-algebra $\phi(\mathfrak{a})$ is self-adjoint for an inner product of V and thus the space V decomposes as a sum $V = \bigoplus_{\chi \in \Pi(\phi)} V^{\chi}$, where

$$V^{\chi} = \{v \in V : \phi(a)v = \chi(a)v \; \forall a \in \mathfrak{a}\}$$

are the common eigen-spaces, called *restricted weight spaces*, and

$$\Pi(\phi) = \big\{\chi \in \mathfrak{a}^{*} : V^{\chi} \neq \{0\}\big\}$$

is called the *set of restricted weights* of ϕ. It is a W-invariant set. The *multiplicity* of $\chi \in \Pi(\phi)$ is denoted by $\mathsf{m}_{\phi}(\chi)$ and defined as the dimension of its restricted weight space, $\mathsf{m}_{\phi}(\chi) = \dim V^{\chi}$. We will often omit the subscript and write $\mathsf{m}(\chi)$ if there no ambiguity in ϕ.

Proposition 2.4. *Let (V, ϕ) be an irreducible representation of $\mathfrak{g}$. Consider $\chi \in \Pi(\phi)$ and $\alpha \in \Phi$, then the elements of $\Pi(\phi)$ of the form $\chi + i\alpha$, $i \in \mathbb{Z}$ form an unbroken string*

$$\chi + i\alpha,\; i \in [\![-r, q]\!]$$

and $r - q = \langle \chi, \alpha \rangle$.

There is a unique maximal element χ_{ϕ} of $\Pi(\phi)$ for $\succ$, called the *the highest restricted weight* of ϕ, and Proposition 2.4 implies that $\Pi(\phi)$ is saturated with highest weight χ_{ϕ}.

By definition, for every $a \in \mathfrak{a}^{+}$ $\chi_{\phi}(a)$ is the spectral radius $\lambda_1\big(\phi(a)\big)$ of $\phi(a)$. The restricted weight space associated to χ_{ϕ} is $V^{+} = V_{\chi_{\phi}} = \big\{v \in V : \phi(\mathfrak{n})v = \{0\}\big\}$. One has the following (to simplify notation, for $\alpha \in \Phi^{+}$ we write $\check{\mathfrak{g}}_{\alpha} = \mathfrak{g}_{-\alpha}$.).

Remark 2.5. The subspaces of the form $\phi(\check{\mathfrak{g}}_{\beta_{\ell}}) \cdots \phi(\check{\mathfrak{g}}_{\beta_0}) V^{+}$ with $\beta_i \in \Delta$ (repetitions allowed) that do not identically vanish are in direct sum. Indeed, such a space is contained the restricted weight space associated to

$$\chi_{\phi} - \sum_{i=0}^{k} \beta_i .$$

Every weight of ϕ is obtained in this fashion, moreover, by construction every weight $\chi \in \Pi(\phi)$ can be written as $\chi = \chi_{\phi} - \beta_0 - \cdots - \beta_{\ell}$, with $\beta_j \in \Delta$, in such a way that all the partial sums

$$\chi = \chi_{\phi} - \beta_0 - \cdots - \beta_j \quad j \in [\![1, \ell]\!]$$

are weights of ϕ.

3. Hasse diagrams for representations

If $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation of a real semi-simple Lie algebra $\mathfrak{g}$ without compact factors, then its set of weights carries the partial order $\succ$ previously defined: $\chi \succ \psi$ if the coefficients of $\chi - \psi$ in Δ are non-negative integers.

One defines then the *Hasse diagram* of the representation ϕ as a graph whose vertices are the elements of $\Pi(\phi)$, and one draws an edge between χ and ψ if and only if $\chi - \psi \in \Delta$. Because of the non-symmetry of $\succ$, the edge should be a directed arrow, however we prefer to forget the arrow and draw ψ *below* χ. It is also convenient to label the edge with the simple root $\chi - \psi$.

These Hasse diagrams carry a natural grading or *levels* defined by the function

$$\operatorname{level}\big(\chi_{\phi} - \sum_{\sigma \in \Delta} k_{\sigma}\sigma\big) = 1 + \sum k_{\sigma} .$$

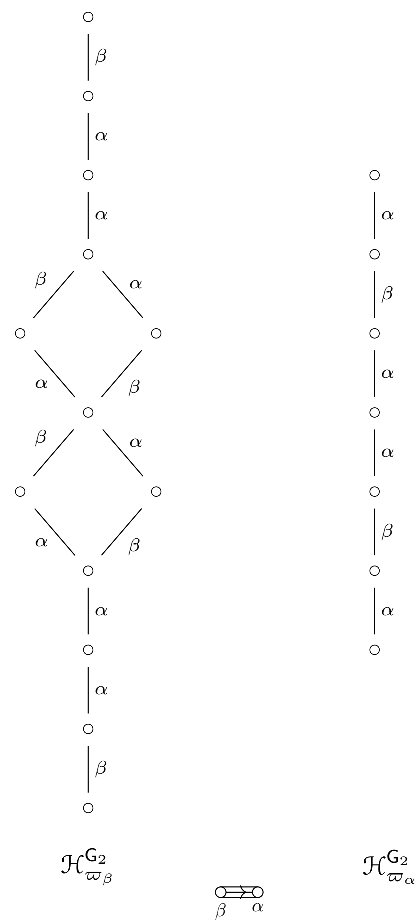


FIGURE 1. Hasse diagrams for fundamental weights of (extremal) roots of G_2.

By means of Remark 2.5 one can draw the Hasse diagram of a given representation *level by level*, starting from it's highest weight and inductively checking, for a given weight $\chi \in \Pi(\phi)$ the set of simple roots $\sigma \in \Delta$ such that $\phi(\check{\mathfrak{g}}_\sigma)V^\chi = \{0\}$. This in turn can be directly computed from the root system Φ using Proposition 2.4: one computes $\langle \chi, \sigma \rangle$ and, since all lower levels of the diagram are assumed to be known, one knows whether $\chi + \sigma$ (down one level) belongs to $\Pi(\phi)$ or not.

It is more convenient then to define the Hasse diagram as depending only on the type of the root system Φ, and of a given dominant weight $\chi \in \Pi_+$ that will play the role of the highest weight of an irreducible representation.

Definition 3.1. The Hasse diagram of a root system of type L and a given dominant weight $\chi \in \Pi_+$ will be denoted by $\mathcal{H}^{\mathsf{L}}_\chi$.

Figure (1) depicts the Hasse diagrams of the exceptional root system G_2 for both its fundamental weights, the Dynkin diagram is added to the picture.

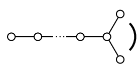
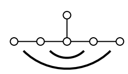
3.1. **Maps between diagrams.** Given two root systems of types J and L, consider a function $f : \Delta_{\mathsf{L}} \to \Delta_{\mathsf{J}}$. We will define a *diagram map with labeling* f, in short a *diagram map*, between two Hasse diagrams as a function $\mathbb{T}^f : \mathcal{H}^{\mathsf{L}}_{\chi} \to \mathcal{H}^{\mathsf{J}}_{\chi'}$ such that if $\psi_0, \psi_1 \in \mathcal{H}^{\mathsf{L}}_{\chi}$ then

$$\psi_0 - \psi_1 \in \Delta_{\mathsf{L}} \text{ implies } \mathbb{T}^f(\psi_0) - \mathbb{T}^f(\psi_1) = f(\psi_0 - \psi_1) \in \Delta_{\mathsf{J}}.$$

Such a map is thus order preserving, level and labeling equivariant. We say that $\mathbb{T}^f$ is *surjective* if it is set-wise surjective. If this is the case, then necessarily f is surjective and both diagrams have the same total number of levels.

Let us emphasize that the function f is merely a set-wise function, no condition on the associated function between the Dynkin diagrams is required.

Example 3.2. Consider the following Dynkin diagrams that carry a non-trivial involution, $\mathrm{inv}_0 : \Delta_{\mathsf{L}} \to \Delta_{\mathsf{L}}$ say,

- the middle point symmetry in A_ℓ: ,
- D_n : ,
- the middle axis symmetry in E_6: .

The quotient by the orbits of inv_0 provides a labeling

- $f : \Delta_{\mathsf{A}_{2n+1}} \to \Delta_{\mathsf{B}_n}$,
- $f : \Delta_{\mathsf{D}_n} \to \Delta_{\mathsf{C}_n}$,
- $f : \Delta_{\mathsf{E}_6} \to \Delta_{\mathsf{F}_4}$,

which induces surjective maps between the Hasse diagrams of the fundamental weight ϖ_σ of a given simple root and the fundamental weight of $f(\sigma)$. Figure (9) in the appendix depicts the E_6 case for one of the extremal roots.

Not every example comes from the fixed point set of an involution, as the fundamental representation $\bar{\phi}_{\varpi_\alpha} : \mathfrak{g}_2 \to \mathfrak{sl}_7(\mathbb{R})$ of the real split Lie algebra $\mathfrak{g}_2$ shows. This is depicted in Figure (2).

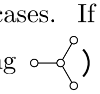
The existence of a surjective map between Hasse diagrams is of course very restrictive as the following lemma shows.

Lemma 3.3. *Consider two irreducible reduced root systems of types* J *and* L. *Assume there exists*

- $f : \Delta_{\mathsf{L}} \to \Delta_{\mathsf{J}}$ *such that* $f(\alpha)$ *is extremal for every extremal* $\alpha \in \Delta_{\mathsf{L}}$,
- *for every extremal* α *a surjective diagram map* $\mathbb{T}^f : \mathcal{H}^{\mathsf{L}}_{\varpi_\alpha} \to \mathcal{H}^{\mathsf{J}}_{\varpi_{f(\alpha)}}$ *with labeling* f.

Then, besides f =*identity, the only possibilities for* J, L, *and* f *are listed in table 2.*

Proof. The proof is a case by case verification. In Appendix A we draw the Hasse diagrams for the fundamental weights of the extremal roots of all irreducible reduced root systems and the non-existence verification is also proven. ☐

To end this section we remark that when $\mathsf{L} = \mathsf{D}_4$, in spite of the apparent symmetry of the B_3's given in table (2), these correspond to different cases. If one considers the complex algebras $\mathfrak{so}(7, \mathbb{C})$ and $\mathfrak{so}(8, \mathbb{C})$, then the labelling 

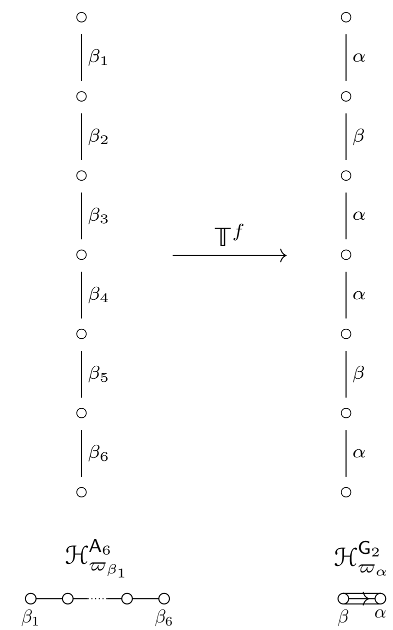


FIGURE 2. The surjective map $\mathcal{H}^{\mathsf{A}_6}_{\varpi_{\beta_1}} \to \mathcal{H}^{\mathsf{G}_2}_{\varpi_{\alpha}}$.

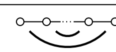
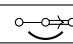
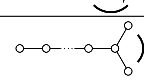
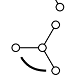
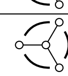
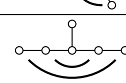
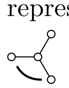
L	J	fibers of f
A_{2n}	B_n $\forall n$	
	G_2 if $n = 3$	Figure (2)
A_{2n-1}	C_{2n}	
B_3	G_2	
D_n	B_{n-1} $\forall n \geqslant 3$	
	B_3 if $n = 4$	
	G_2 if $n = 4$	
E_6	F_4	

TABLE 2

corresponds to the representation $\mathfrak{so}(7,\mathbb{C}) \to \mathfrak{so}(8,\mathbb{C})$ that stabilizes a line in $\mathbb{C}^8$, whilst the labelling  corresponds to the fundamental representation of $\mathfrak{so}(7,\mathbb{C})$ associated to the short root of B_3. This is an irreducible representation with image in $\mathfrak{so}(8,\mathbb{C})$ called *the spin representation*, see Fulton-Harris [8, Lecture 20, Ex. 20.38].

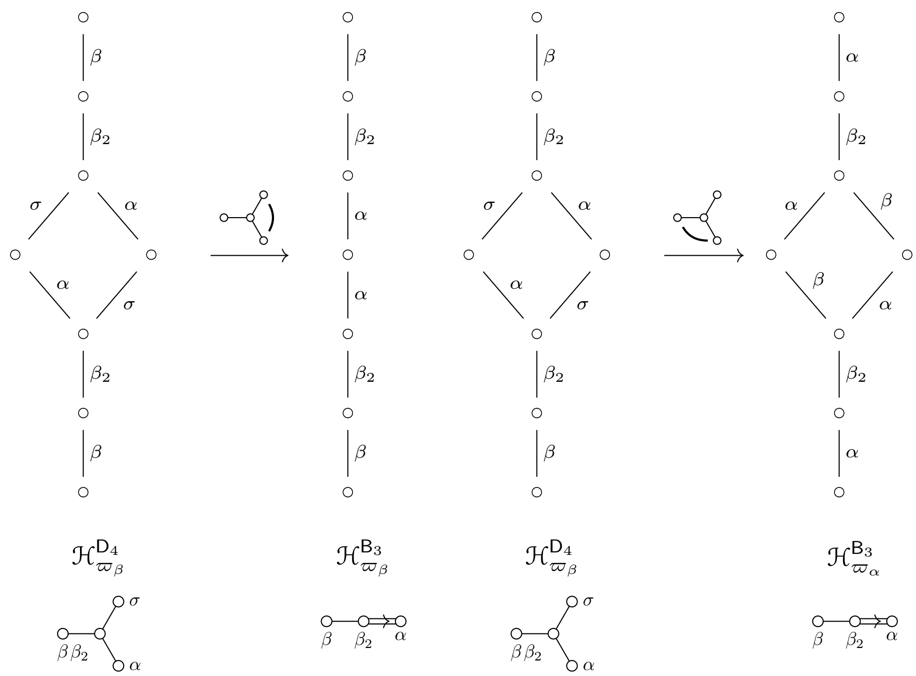


FIGURE 3. The surjective maps $\mathcal{H}^{\mathsf{D}_4}_{\varpi_\beta} \to \mathcal{H}^{\mathsf{B}_3}_{\varpi_\beta}$ and $\mathcal{H}^{\mathsf{D}_4}_{\varpi_\beta} \to \mathcal{H}^{\mathsf{B}_3}_{\varpi_\alpha}$

4. Discrete subgroups satisfying a coherence condition w.r.t. eigenspaces

4.1. **Review on Lie group representations.** Let G be a reductive real algebraic Lie group. If $\bar{\phi} : \mathsf{G} \to \mathsf{GL}(V)$ is a rational representation then we denote by $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ the induced representation on its Lie algebra and we speak indistinctively of highest restricted weight, restricted weight spaces, etc of ϕ and $\bar{\phi}$.

One has the following proposition from Tits [23] that guarantees existence of certain representations of G. We say that ϕ is *proximal* if $\dim V^+ = 1$.

Proposition 4.1 (Tits [23])**.** *For every $\sigma \in \Delta$ there exists an irreducible proximal representation of G whose highest restricted weight is $l\varpi_\sigma$ for some $l \in \mathbb{Z}_{\geqslant 1}$. If $\mathfrak{g}$ is split then one can choose $l = 1$.*

Definition 4.2. We will fix and denote by $\bar{\phi}_\sigma : \mathsf{G} \to \mathsf{GL}(V_\sigma)$ such a set of representations.

Let $\check{\mathfrak{n}} = \bigoplus_{\alpha \in \Phi^+} \check{\mathfrak{g}}_\alpha$ and consider the opposite minimal parabolic algebras $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{n}$ and $\check{\mathfrak{b}} = \mathfrak{g}_0 \oplus \check{\mathfrak{n}}$. The *minimal parabolic subgroups* are denoted by B and $\check{B}$ and defined as the normalizers in G of $\mathfrak{b}$ and $\check{\mathfrak{b}}$ respectively. The groups B and $\check{B}$ are conjugated. The *complete flag space of* G is defined by $\mathcal{F} = \mathsf{G}/B$. The G-orbit of

$$\big([B], [\check{B}]\big) \in \mathcal{F} \times \mathcal{F}$$

is the unique open orbit of G and is denoted by $\mathcal{F}^{(2)}$.

If (ϕ, V) is a proximal irreducible representation, then one has a $\bar{\phi}$-equivariant algebraic map

$$\Phi = \Phi_{\bar{\phi}} : \mathcal{F} \to \mathbb{P}(V)$$

defined by $\Phi_{\bar{\phi}}(g[B]) = \bar{\phi}(g)V^+$. The $\phi(\mathfrak{a})$-invariant complement

$$V^- = \bigoplus_{\chi \in \Pi(\phi) - \{\chi_\phi\}} V^\chi$$

is stabilized by $\check{B}$, giving also $\check{\Phi} = \check{\Phi}_{\bar{\phi}} : \mathcal{F} \to \mathbb{P}(V^*)$ defined by $\check{\Phi}(g \cdot [\check{B}]) = \bar{\phi}(g)V^-$.

4.2. **Jordan-Kostant-Lyapunov's projection and Benoist's limit cone.** Recall that every element $h \in \mathsf{G}$ can be uniquely written as a commuting product $h = h_e h_{ss} h_n$ where h_e is conjugate to an element in K, h_{ss} is conjugate to an element in $\exp(\mathfrak{a}^+)$ and h_n is unipotent. The *Jordan-Kostant-Lyapunov projection* $\lambda = \lambda_{\mathsf{G}} : \mathsf{G} \to \mathfrak{a}^+$ is defined such that h_{ss} is conjugated to $\exp\big(\lambda(h)\big)$.

If $\Lambda \subset \mathsf{G}$ is a discrete subgroup, then its *limit cone* is denoted by $\mathcal{L}_\Lambda$ and is defined as the smallest closed cone that contains $\{\lambda(g) : g \in \Lambda\}$. One has the following fundamental result by Benoist. Recall that $\mathfrak{a}_{ss} = \mathfrak{a} \cap \mathfrak{g}_{ss}$.

Theorem 4.3 (Benoist [2])**.** *Let $\Lambda < \mathsf{G}$ be a Zariski dense subgroup. Then the limit cone $\mathcal{L}_\Lambda$ is convex and the intersection $\mathcal{L}_\Lambda \cap \mathfrak{a}_{ss}$ has non-empty interior in $\mathfrak{a}_{ss}$.*

4.3. **Coherent subgroups.** For $g \in \mathsf{GL}_d(\mathbb{R})$ let us denote by

$$\lambda(g) = \big(\lambda_1(g), \cdots, \lambda_d(g)\big) \in \mathfrak{a}^+$$

its Jordan projection. If $\lambda_1(g) > \lambda_2(g)$ we say that g is *proximal*. Equivalently, the generalized eigenspace associated to the greatest (in modulus) eigenvalue of g is 1-dimensional. We will denote by $g_+ \in \mathbb{P}(\mathbb{R}^d)$ this attracting eigenline and by g_- its g-invariant complement.

A discrete subgroup $\Lambda < \mathsf{PGL}_d(\mathbb{R})$ is *proximal* if it contains a proximal element. One defines then its *limit set* by

$$\mathbf{L}^{\mathbb{P}}_\Lambda = \overline{\{g_+ : g \in \Lambda \text{ proximal}\}}.$$

Recall from the introduction that $\mathbf{L}^{\mathbb{P}}_\Lambda$ is *minimal* if the only closed Λ-invariant subsets are $\{\varnothing, \mathbf{L}^{\mathbb{P}}_\Lambda\}$.

Lemma 4.4. *Let $\Lambda < \mathsf{PGL}_d(\mathbb{R})$ be proximal with minimal $\mathbf{L}^{\mathbb{P}}_\Lambda$. If Λ acts totally reducibly in $\mathbb{R}^d$ then* $\operatorname{span} \mathbf{L}^{\mathbb{P}}_\Lambda$ *is an irreducible factor of Λ.*

Proof. Let $g \in \Lambda$ be proximal and V an irreducible factor. If $v \in V$ does not lie in g_- then $g^n(\mathbb{R} \cdot v) \to g_+$. Consequently, since V is closed and g-invariant, if $g_+ \notin V$ one concludes $V \subset g_-$. Thus, g_+ necessarily belongs to an irreducible factor of Λ, W say. The subset $\mathbf{L}^{\mathbb{P}}_\Lambda \cap \mathbb{P}(W)$ is then non-empty, closed and Λ-invariant. Minimality completes the proof. $\square$

Definition 4.5. A discrete subgroup $\Lambda < \mathsf{PGL}_d(\mathbb{R})$ is *coherent* if

- there exists a proximal $g_0 \in \Lambda$ such that $\wedge^2 g_0$ is proximal and the eigenline associated to $\lambda_2(g_0)$ belongs to $\operatorname{span} \mathbf{L}^{\mathbb{P}}_\Lambda$,
- the limit sets $\mathbf{L}^{\mathbb{P}}_\Lambda$ and $\mathbf{L}^{\mathbb{P}}_{\wedge^2\Lambda}$ are minimal.

Lemma 4.6. *Let $\Lambda < \mathsf{PGL}_d(\mathbb{R})$ be a coherent subgroup with reductive Zariski closure H. Then there exists $\sigma \in \Delta_{\mathfrak{h}}$ with $\dim \mathfrak{h}_\sigma = 1$ such that for every $g \in \Lambda$ one has*

$$\sigma\big(\lambda_{\mathsf{H}}(g)\big) = \lambda_1(g) - \lambda_2(g).$$

Proof. By Lemma 4.4 the representations $\mathsf{H}|\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}$ and $\mathsf{H}|\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\wedge^2\Lambda}$ are irreducible. Denoting by χ_1 and χ_2 their highest restricted weights, $2\chi_1-\chi_2$ verifies that for all $g\in\Lambda$ one has $2\chi_1-\chi_2(\lambda_{\mathsf{H}}(g))=\lambda_1(g)-\lambda_2(g)$.

Denote by $\{W_i\}$ the irreducible factors of Λ. For $g\in\Lambda$ with $\wedge^2 g$ proximal, denote by $V_2(g)$ either the eigenline associated to $\lambda_2(g)$ if g is proximal, either the 2-dimensional Jordan block associated to $\lambda_1(g)$ otherwise. One readily sees that, in both situations, the vector space $V_2(g)$ necessarily intersects one of the W_i's.

We can identify $\mathbf{L}^{\mathbb{P}}_{\wedge^2\Lambda}$ as a subset of $\operatorname{Gr}_2(\mathbb{R}^d)$ and thus consider the subsets

$$\widetilde{W_i}=\big\{P\in\mathbf{L}^{\mathbb{P}}_{\wedge^2\Lambda}:P\cap W_i\neq\{0\}\big\},$$

these are closed and $\wedge^2\Lambda$-invariant and the same holds for any intersection $\widetilde{W_i}\cap\widetilde{W_j}$. By minimality, each intersection is either empty or $\mathbf{L}^{\mathbb{P}}_{\wedge^2\Lambda}$. However, by assumption the space $\widetilde{\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}}$, associated to the irreducible factor $\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}$, is non-empty, and $V_2(g_0)\subset\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}$. One concludes that $\widetilde{\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}}=\mathbf{L}^{\mathbb{P}}_{\wedge^2\Lambda}$ and that all intersections $\widetilde{\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}}\cap\widetilde{W_j}$ are empty. Equivalently, $V_2(g)\subset\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}$ for every $g\in\Lambda$ with proximal $\wedge^2 g$.

Applying §2.3 to $\mathsf{H}|\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\Lambda}$ together with the preceding paragraph, one has that for every $g\in\Lambda$ there exists $\alpha_g\in\Delta_{\mathfrak{h}}$ such that $\alpha_g(\lambda(g))=\lambda_1(g)-\lambda_2(g)$. Since the limit cone $\mathcal{L}_\Lambda$ has non-empty interior on $\mathfrak{a}\cap\mathfrak{h}_{ss}$, (Benoist's Theorem 4.3) and $\Delta_{\mathfrak{h}}$ is a finite set, there exists an open sub-cone $\mathscr{C}\subset\mathcal{L}_\Lambda$ and a root $\sigma\in\Delta_{\mathfrak{h}}$ such that for every $v\in\mathscr{C}$

$$\sigma(v)=\big(2\chi_1-\chi_2\big)(v).$$

Since both functions are linear and coincide on an open set, they must coincide and σ is the required root. $\square$

Definition 4.7. Let G be a reductive group and Λ a discrete subgroup. Then Λ is *totally coherent* if for every $\sigma\in\Delta$ the subgroup $\bar{\phi}_\sigma(\Lambda)$ is coherent.

The following is the main result of this section.

Proposition 4.8. *Let G be a real-algebraic simple group and $\Lambda<\mathsf{G}$ a totally coherent discrete subgroup with reductive Zariski closure H. Then $\mathfrak{h}_{ss}$ is simple split. Moreover, there exists a surjective function $f:\Delta_{\mathfrak{g}}\to\Delta_{\mathfrak{h}}$ and, for every $\alpha\in\Delta_{\mathfrak{g}}$, a surjective map with labeling f between the diagrams*

$$\mathbb{T}^f:\mathcal{H}^{\mathfrak{g}}_{\ell_\alpha\varpi_\alpha}\to\mathcal{H}^{\mathfrak{h}}_{n_\alpha\varpi_{f(\alpha)}},$$

for some $n_\alpha\in\mathbb{Z}_{\geqslant 1}$. If α is extremal then $f(\alpha)$ is extremal, if moreover $\operatorname{rank}\mathfrak{h}_{ss}>1$ *and $\ell_\alpha=1$ then $n_\alpha=1$.*

Proof. Since Λ is totally coherent, applying Lemma 4.6 to each representation $\bar{\phi}_\sigma$ of G provides a function $f:\Delta_{\mathfrak{g}}\to\Delta_{\mathfrak{h}}$ such that for every $g\in\Lambda$ and $\sigma\in\Delta_{\mathfrak{g}}$ one has

$$f(\sigma)\big(\lambda_{\mathsf{H}}(g)\big)=\sigma\big(\lambda_{\mathsf{G}}(g)\big).\tag{4.1}$$

Consider then $\alpha\in\Delta_{\mathfrak{g}}$ and the associated fundamental representation $\bar{\phi}_\alpha:\mathsf{G}\to\mathsf{GL}(V)$. Since H is reductive, Lemma 4.4 implies that $W=\operatorname{span}\mathbf{L}^{\mathbb{P}}_{\bar{\phi}_\alpha\Lambda}$ is an irreducible factor of $\bar{\phi}_\alpha\mathsf{H}$. Let $\phi:\mathfrak{h}\to\mathfrak{gl}(W)$ the representation of $\mathfrak{h}$ defined by $\phi=\phi_\alpha(\mathfrak{h})|W$ and $\chi_\phi\in\Pi_{\mathfrak{h}}$ its highest restricted weight.

As stated in Remark 2.5 every element of $\Pi_{\mathfrak{g}}(\phi_\alpha)$ is of the form

$$\ell_\alpha \varpi_\alpha - \sum_{\sigma \in \Delta_{\mathfrak{g}}} k_\sigma \sigma, \tag{4.2}$$

where for every σ $k_\sigma \in \mathbb{Z}_{\geqslant 0}$. Let us consider then the function $\mathbb{T}^f : \Pi_{\mathfrak{g}}(\phi_\alpha) \to \Pi_{\mathfrak{h}}$ defined by

$$\mathbb{T}^f(\chi) = \chi_\phi - \sum_{\sigma \in \Delta_{\mathfrak{g}}} k_\sigma f(\sigma)$$

if χ is as in equation (4.2). Observe that for every $\chi \in \Pi_{\mathfrak{g}}(\phi_\alpha)$ and $\beta \in \Phi_{\mathfrak{h}}$ one has

$$\langle \mathbb{T}^f(\chi), \beta \rangle = \langle \chi_\phi, \beta \rangle - \sum_{\sigma \in \Delta_{\mathfrak{g}}} k_\sigma \langle f(\sigma), \beta \rangle \in \mathbb{Z},$$

so $\mathbb{T}^f(\chi)$ is indeed a weight of $\mathfrak{h}$.

Observe also that for every $g \in \Lambda$ one has, by equation (4.1), that

$$\mathbb{T}^f(\chi)\big(\lambda_{\mathsf{H}}(g)\big) = \chi\big(\lambda_{\mathsf{G}}(g)\big),$$

so that for every $v \in \mathcal{L}_\Lambda \subset (\mathfrak{a}_{\mathfrak{h}})^+ \subset (\mathfrak{a}_{\mathfrak{g}})^+$ one has $\mathbb{T}^f(\chi)(v) = \chi(v)$. Thus, since $\mathcal{L}_\Lambda \cap \mathfrak{a}^+_{\mathfrak{h},ss}$ has non-empty interior in $\mathfrak{a}_{\mathfrak{h},ss}$ (Benoist's Theorem 4.3), one has

$$\mathbb{T}^f(\chi) = \chi|\mathfrak{a}_{\mathfrak{h},ss}. \tag{4.3}$$

Consequently, for any $v \in \mathfrak{a}_{\mathfrak{h},ss}$ the eigenspace decomposition of $\phi_\alpha(v)$ is

$$V = \bigoplus_{\chi \in \Pi_{\mathfrak{g}}(\phi_\alpha)} (V)_{\mathbb{T}^f(\chi)} \tag{4.4}$$

hence,

$$\Pi_{\mathfrak{h}}(\phi) \subset \mathbb{T}^f\big(\Pi_{\mathfrak{g}}(\phi_\alpha)\big).$$

To show equality, one observes that, by equation (4.4), the highest weight of any other irreducible factor of $\phi_\alpha(\mathfrak{h})$ is a dominant weight $\prec \chi_\phi$ and thus also belongs to $\Pi_{\mathfrak{h}}(\phi)$ (Lemma 2.3). By W-invariance of the set of restricted weights of irreducible representations, one concludes that all the restricted weights of other irreducible factors also belong to $\Pi_{\mathfrak{h}}(\phi)$, consequently

$$\Pi_{\mathfrak{h}}(\phi) = \mathbb{T}^f\big(\Pi_{\mathfrak{g}}(\phi_\alpha)\big).$$

Clearly $\mathbb{T}^f$ is level preserving.

From surjectivity of $\mathbb{T}^f$, and since there is only one weight of ϕ_α of level 2 (i.e. $\ell_\alpha\varpi_\alpha - \alpha$) one has that for every $\beta \in \Delta_{\mathfrak{h}} - \{f(\alpha)\}$ the linear form $\chi_\phi - \beta$ is not a weight, hence $\langle \chi_\phi, \beta \rangle = 0$ and thus $\chi_\phi = n_\alpha \varpi_{f(\alpha)}$ for some $n_\alpha \in \mathbb{Z}_{\geqslant 1}$.

Since G is simple, ϕ_α is injective and thus, since any weight of $\phi_\alpha(\mathfrak{h})$ is contained in $\Pi_{\mathfrak{h}}(\phi)$, $\phi_\alpha(\mathfrak{h}_{ss})$ is simple and thus $\mathfrak{h}_{ss}$ is. Consequently, f is surjective and, since $\dim(\mathfrak{h}_{ss})_{f(\alpha)} = 1$ for every α (Lemma 4.6), $\mathfrak{h}_{ss}$ is split.

Let us assume from now on that α is an extremal root of $\Delta_{\mathfrak{g}}$, so that the only weights of level 3 of ϕ_α are $\ell_\alpha\varpi_\alpha - \alpha - \beta$ for a unique root $\beta \in \Delta_{\mathfrak{g}}$, and $\ell_\alpha\varpi_\alpha - 2\alpha$ (only if $\ell_\alpha \geqslant 2$). This implies that the only weights of level 3 of ϕ are $n_\alpha\varpi_{f(\alpha)} - f(\alpha) - f(\beta)$, and possibly $n_\alpha\varpi_{f(\alpha)} - 2f(\alpha)$.

Hence $\langle n_\alpha\varpi_{f(\alpha)} - f(\alpha), \sigma \rangle = 0$ for every $\sigma \in \Delta_{\mathfrak{h}} - \{f(\alpha), f(\beta)\}$ from which $f(\alpha)$ is an extremal root of $\Delta_{\mathfrak{h}}$. Moreover, either

- $f(\alpha) = f(\beta)$ i.e. for every $\sigma \in \Delta_{\mathfrak{h}} - \{f(\alpha)\}$ one has
$$0 = \langle n_\alpha \varpi_{f(\alpha)} - f(\alpha), \sigma \rangle = -\langle f(\alpha), \sigma \rangle$$
and thus $\mathfrak{h}_{ss}$ has rank 1,
- either $f(\alpha) \neq f(\beta)$.

In the latter case, if one assumes moreover that $\ell_\alpha = 1$, then $n_\alpha \varpi_{f(\alpha)} - 2f(\alpha) \notin \Pi_{\mathfrak{h}}(\phi)$ and hence $n_\alpha = 1$. □

4.4. **Classification of Zariski closures of totally coherent groups.** Throughout this section, $\mathfrak{g}$ is a simple split real Lie algebra, G is a real-algebraic Zariski connected Lie group with Lie algebra $\mathfrak{g}$ and $\Lambda < \mathsf{G}$ is a totally coherent discrete subgroup with reductive Zariski closure H. The purpose is to classify the pairs $(\mathfrak{h}_{ss}, \phi)$ where $\phi : \mathfrak{h}_{ss} \to \mathfrak{g}$ is the representation induced by the inclusion $\mathsf{H} \subset \mathsf{G}$. By Proposition 4.8 $\mathfrak{h}_{ss}$ is simple split.

One begins by the following:

Corollary 4.9. *If $\mathfrak{h}_{ss}$ has rank 1 then it is a principal $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{g}$.*

Proof. From Proposition 4.8 one deduces that if one composes $\mathfrak{h}_{ss}$ with any fundamental representation ϕ_σ of $\mathfrak{g}$, the highest weight space of $\mathfrak{g}$ is also the highest weight space V^χ for some irreducible factor of $\phi_\alpha(\mathfrak{h}_{ss})$. Moreover, any other irreducible factor of $\phi_\alpha(\mathfrak{h}_{ss})$ has highest weight $\prec \chi$. This is to say, if one writes $\mathsf{f} \in \mathfrak{h}_{ss} \cap \check{\mathfrak{n}}$ as $\mathsf{f} = \sum_{\alpha \in \Phi^+} b_\alpha \mathsf{y}_\alpha$, then $b_\sigma \neq 0$ for every $\sigma \in \Delta$. Kostant's Theorem 2.2 asserts then that $\mathfrak{h}_{ss}$ is a principal $\mathfrak{sl}_2(\mathbb{R})$. □

If the rank of $\mathfrak{h}_{ss}$ is $\geqslant 2$ then, since the fundamental representations of $\mathfrak{g}$ verify $\ell_\alpha = 1$ for all $\alpha \in \Delta_{\mathfrak{g}}$, Proposition 4.8 provides a surjective function $f : \Delta_{\mathfrak{g}} \to \Delta_{\mathfrak{h}}$ such that the image of an extremal root is an extremal root, and for every $\alpha \in \Delta_{\mathfrak{g}}$ a surjective map $\mathbb{T}^f : \mathcal{H}^{\mathfrak{g}}_{\varpi_\alpha} \to \mathcal{H}^{\mathfrak{h}}_{\varpi_{f(\alpha)}}$ between the corresponding Hasse diagrams. Applying the table (2) given by Lemma 3.3 one concludes at once the following Corollary.

Corollary 4.10. *If $\operatorname{rank} \mathfrak{h}_{ss} \geqslant 2$ then either $\mathfrak{h}_{ss} = \mathfrak{g}$, either the only possibilities for $\phi : \mathfrak{h}_{ss} \to \mathfrak{g}$ are, up to $\operatorname{Int} \mathfrak{g}$-conjugation, the ones listed in table 3.*

$\mathfrak{g}$	$\mathfrak{h}_{ss}$	$\phi : \mathfrak{h}_{ss} \to \mathfrak{g}$
$\mathfrak{sl}_{2n+1}(\mathbb{R})$	$\mathfrak{so}(n, n+1)$ $\forall n$	defining representation
	$\mathfrak{g}_2$ if $n = 3$	fundamental for the short root
$\mathfrak{sl}_{2n}(\mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{R})$	defining representation
$\mathfrak{so}(3,4)$	$\mathfrak{g}_2$	fundamental for the short root
$\mathfrak{so}(n,n)$	$\mathfrak{so}(n-1, n)$ $\forall n \geqslant 3$	stabilizer of a non-isotropic line
	$\mathfrak{so}(3,4)$ if $n = 4$	fundamental for the short root
	$\mathfrak{g}_2$ if $n = 4$	stabilizes a non-isotropic line L and is fundamental for the short root on $L^\perp$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathrm{Fix}(\mathrm{inv}_0)$ (example 3.2)

TABLE 3. Statement of Corollary 4.10

5. Total positivity

Throughout this section G denotes the real points of a Zariski connected real-algebraic simple split group.

5.1. **Lusztig's total positivity.** Let us fix, for each simple root $\sigma \in \Delta$, algebraic group isomorphisms $x_\sigma : \mathbb{R} \to \exp \mathfrak{g}_\sigma$, $y_\sigma : \mathbb{R} \to \check{\mathfrak{g}}_\sigma$ and $h_\sigma : \mathbb{R} \to \exp(\mathbb{R} \cdot h_\sigma)$ so that

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mapsto x_\sigma(t),\ \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) \mapsto y_\sigma(t),\ \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \mapsto h_\sigma(t),$$

defines a morphism $\mathsf{SL}_2(\mathbb{R}) \to \mathsf{G}$. The collection $\mathrm{O} = \{x_\sigma : \sigma \in \Delta\}$ is called a *pinning* of G and two pinnings are conjugated by G.

Let $U = \exp \mathfrak{n}$ and $\check{U} = \exp \check{\mathfrak{n}}$ be the unipotent radicals of B and $\check{B}$ respectively and let $A = \exp \mathfrak{a}$.

Let $w_0 \in W$ be the longest element and consider a reduced expression $w_0 = r_N \cdots r_1$ as a product of reflections associated to simple roots. Let us denote, for each r_i the associated simple root by $\sigma_i \in \Delta$. The number N equals $|\Phi^+|$, but we will not require this fact.

Consider the maps $\Psi^{\mathrm{O}} : (\mathbb{R}_{>0})^N \to U$ and $\check{\Psi}^{\mathrm{O}} : (\mathbb{R}_{>0})^N \to \check{U}$ defined by

$$\begin{aligned} \Psi^{\mathrm{O}}(a_1, \cdots, a_N) &= x_{\sigma_N}(a_N) \cdots x_{\sigma_1}(a_1), \\ \check{\Psi}^{\mathrm{O}}(a_1, \cdots, a_N) &= y_{\sigma_N}(a_N) \cdots y_{\sigma_1}(a_1). \end{aligned} \tag{5.1}$$

We summarize several results from Lusztig [20, §2] in the following theorem.

Theorem 5.1 (Luzztig [20, §2])**.** *The images $U_{>0} = \Psi^{\mathrm{O}}\big((\mathbb{R}_{>0})^N\big)$ and $\check{U}_{>0} = \check{\Psi}^{\mathrm{O}}\big((\mathbb{R}_{>0})^N\big)$ are semi-groups independent of the chosen reduced expression of w_0. The product*

$$\mathsf{G}_{>0} = \check{U}_{>0} A U_{>0} = U_{>0} A \check{U}_{>0}$$

is also a semi-group and every element $g \in \mathsf{G}_{>0}$ has a unique expression of the form $g = \check{u}tv$ with $\check{u} \in \check{U}_{>0}$, $t \in A$ and $v \in U_{>0}$.

Even though we omit the pinning notation on the semi-groups $U_{>0}$, $\check{U}_{>0}$ and $\mathsf{G}_{>0}$, they do depend on the pinning O.

5.2. **Positivity of flags.** The positive semi-group $\mathsf{G}_{>0}$ determines a special subset $\mathcal{F}_{>0} \subset \mathcal{F}$ defined by

$$\mathcal{F}_{>0} = \mathsf{G}_{>0} \cdot [B] = \check{U}_{>0} \cdot [B] = U_{>0} \cdot [\check{B}].$$

Let us say that an ordered triple $(x_1, x_2, x_3) \in \mathcal{F}^3$ is *generic* if $(x_i, x_j) \in \mathcal{F}^{(2)}$. Then one has the following.

Proposition 5.2 (Lusztig [20, Prop. 8.14])**.** *The subset $\mathcal{F}_{>0}$ is a connected component of*

$$\Big\{x \in \mathcal{F} : \big([B], x, [\check{B}]\big) \text{ is generic}\Big\}.$$

In particular it is an open subset of $\mathcal{F}$.

One then defines positivity on triples flags as being G-equivariant, consequently the notion will not depend on the pinning:

Definition 5.3. A generic triple of flags (x, y, z) is *positive* if there exists $g \in \mathsf{G}$ such that $gx_1 = [B], gx_3 = [\check{B}]$ and $gx_2 \in \mathcal{F}_{>0}$.

5.3. **Simply laced G.** Recall that $\mathfrak{g}$ is *simply laced* if for every pair $\sigma,\alpha\in\Delta$ one has $\langle\sigma,\alpha\rangle=\langle\alpha,\sigma\rangle$. Equivalently, the Dynkin diagram of $\mathfrak{g}$ does not contain a double or triple arrow. Assume moreover that G is *simply connected* in the algebraic sense, i.e. every finite covering from a real algebraic group onto G is trivial, equivalently the group $\mathsf{G}_{\mathbb{C}}$ of $\mathbb{C}$-points of G is simply connected in the topological sense.

Proposition 5.4 (Lusztig [20, §3.1 and Prop. 3.2])**.** *Assume that G is simply laced and simply connected. Let $\bar\phi:\mathsf{G}\to\mathsf{GL}(V)$ be an irreducible real representation, then there exists a basis $\mathbf{B}_{\bar\phi}$ of V such that*

- *each element of $\mathbf{B}_{\bar\phi}$ is contained in a restricted weight space of ϕ,*
- *for every $g\in\mathsf{G}_{>0}$, the map $\bar\phi(g):V\to V$ has >0 coefficients in $\mathbf{B}_{\bar\phi}$.*

5.4. **Theorem A for simply laced G.** We devote this section to the proof of Theorem A when G is moreover simply laced and simply connected (as in §5.3). We prove that a discrete subgroup verifying the hypothesis of Theorem A is totally coherent.

Corollary 5.5. *Let G be simply laced and simply connected and Λ a subgroup with minimal limit sets and with a positive loxodromic triple. Then Λ is totally coherent.*

Proof. Consider $\sigma\in\Delta$ and a fundamental representation $\bar\phi_\sigma:\mathsf{G}\to\mathsf{GL}(V)$. By minimality one has that

$$\Phi\big(\mathbf{L}_{\Lambda,\Delta}\big)=\mathbf{L}^{\mathbb{P}}_{\bar\phi_\sigma(\Lambda)}.$$

Moreover, since the only second level weight of $\bar\phi_\sigma$ is $\varpi_\sigma-\sigma$, the representation $\wedge^2\bar\phi_\sigma$ of G is proximal (though maybe reducible). Denote by $\Omega:\mathsf{G}\to\mathsf{GL}(V')$ the G-irreducible factor containing the highest weight of $\wedge^2\bar\phi_\sigma$, it contains the attracting points of $\wedge^2 g$ for every $g\in\mathsf{G}$ proximal on $\mathcal{F}$. Let $\vartheta\subset\Delta$ be the type of the stabilizer of $V_{\varpi_\sigma}\wedge V_{\varpi_\sigma-\sigma}$ in G. The limit set $\mathbf{L}_{\Lambda,\vartheta}$ is also minimal and one has $\Phi_\Omega\big(\mathbf{L}_{\Lambda,\vartheta}\big)=\mathbf{L}^{\mathbb{P}}_{\wedge^2\bar\phi_\sigma(\Lambda)}$ and the latter is thus minimal.

Finally, consider $g_0\in\Lambda$ proximal on $\mathcal{F}$ and $x_0\in\mathbf{L}_{\Lambda,\Delta}$ so that (g_{0+},x_0,g_{0-}) is a positive triple. We can assume that $g_{0+}=[B]$ and $g_{0-}=[\check B]$ so that $\Phi(g_{0+})=V^+$ and $\check\Phi(g_{0-})=V^-$. We aim to show then that $V_{\varpi_\sigma-\sigma}$ belongs to $\operatorname{span}\Phi(\mathbf{L}_{\Lambda,\Delta})$.

Let $g\in\mathsf{G}_{>0}$ be such that

$$\Phi(x_0)=\Phi\big(g\cdot[B]\big)=\bar\phi_\sigma(g)\cdot\Phi(g_{0+}).$$

Consider then the 2-dimensional subspace $P_{x_0}=\Phi(g_{0+})\oplus\bar\phi_\sigma(g)\Phi(g_{0+})$ and let $\ell_{x_0}\in\mathbb{P}(V)$ be the intersection

$$\ell_{x_0}=P_{x_0}\cap V^-.$$

Since G is simply laced, Lusztig's Proposition 5.4 applies to give that $\bar\phi_\sigma(g)$ has positive coefficients in $\mathbf{B}_{\bar\phi_\sigma}$. In particular, if $v\in V^+-\{0\}$ the vector $\bar\phi_\sigma(g)v$ has positive coefficients in $\mathbf{B}_{\bar\phi_\sigma}$. The line ℓ_{x_0} is thus not contained in any subspace spanned by a partial sum of weights in $\Pi(\phi_\sigma)-\{\varpi_\sigma\}$, i.e. ℓ_{x_0} is not contained in any $\bar\phi_\sigma(g)$-invariant subspace of V^-. Consequently, the sequence $\bar\phi_\sigma({g_0}^n)\cdot\ell_{x_0}$ approaches, as $n\to+\infty$, the $\bar\phi_\sigma(g_0)$-invariant subspace of V^- associated to the top eigenvalue of $\bar\phi(g_0)|V^-$, which is $V_{\varpi_\sigma-\sigma}$. This completes the proof. ☐

Corollary 5.5 gives thus the following.

Corollary 5.6. *Let G be simply laced and simply connected and let $\Lambda<\mathsf{G}$ have reductive Zariski closure H and be as in Corollary 5.5. Then $\mathfrak{h}_{ss}$ is either $\mathfrak{g}$, a principal $\mathfrak{sl}_2(\mathbb{R})$ or $\operatorname{Int}\mathfrak{g}$-conjugated to the possibilities listed in table 4.*

$\mathfrak{g}$	$\mathfrak{h}_{ss}$	$\phi : \mathfrak{h}_{ss} \to \mathfrak{g}$
$\mathfrak{sl}_{2n+1}(\mathbb{R})$	$\mathfrak{so}(n, n+1)$ $\forall n$	defining representation
	$\mathfrak{g}_2$ if $n = 3$	fundamental for the short root
$\mathfrak{sl}_{2n}(\mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{R})$	defining representation
$\mathfrak{so}(n, n)$	$\mathfrak{so}(n-1, n)$ $\forall n \geqslant 3$	stabilizer of a non-isotropic line
	$\mathfrak{so}(3, 4)$ if $n = 4$	fundamental for the short root
	$\mathfrak{g}_2$ if $n = 4$	stabilizes a non-isotropic line L and is fundamental for the short root on $L^\perp$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathrm{Fix}(\mathrm{inv}_0)$ (example 3.2)

TABLE 4. Statement of Corollary 5.6

5.5. **Descent.** The purpose of this section is to briefly explain how to bypass the simply-laced hypothesis in Corollary 5.5. We use a standard technique called *descent*. It consists on observing that every simple split Lie algebra $\mathfrak{g}$ is the fixed point set of an automorphism $\bar{\varkappa} : \dot{\mathfrak{g}} \to \dot{\mathfrak{g}}$ of a simply laced split simple Lie algebra $\dot{\mathfrak{g}}$. One requires also that the action of $\bar{\varkappa}$ on the simple roots of $\dot{\mathfrak{g}}$ is such that if $\alpha, \beta \in \Delta_{\dot{\mathfrak{g}}}$ are in the same $\bar{\varkappa}$-orbit then $\langle \alpha, \beta \rangle = 0$. See table 5.

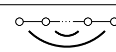
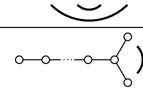
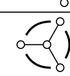
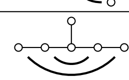
type of $\dot{\mathfrak{g}}$	type of $\mathfrak{g}$	orbits of $\bar{\varkappa}$
A_{2n-1}	C_n	
D_n	B_{n-1}	
D_4	G_2	
E_6	F_4	

TABLE 5

With these considerations, one has the following proposition from Lusztig.

Proposition 5.7 (Lusztig [20, §8.8])**.** *Let* G *be simply connected. Then there exists a simply laced, simply connected, simple split group* $\dot{\mathsf{G}}$ *and a rational representation* $\Omega : \mathsf{G} \to \dot{\mathsf{G}}$ *together with an equivariant map* $\Phi : \mathcal{F}_{\mathsf{G}} \to \mathcal{F}_{\dot{\mathsf{G}}}$ *such that*

$$\Phi\Big(\big(\mathcal{F}_{\mathsf{G}}\big)_{>0}\Big) \subset \Big(\big(\mathcal{F}_{\dot{\mathsf{G}}}\big)_{>0}\Big).$$

We can now conclude the proof of Theorem A.

Corollary 5.8. *Let* G *be the real points of a real-algebraic, Zariski connected, simple split group. Let* $\Lambda < \mathsf{G}$ *be as in Theorem A. Then the semi-simple part* $\mathfrak{h}_{ss}$ *is either* $\mathfrak{g}$, *a principal* $\mathfrak{sl}_2(\mathbb{R})$ *or* $\mathrm{Int}\,\mathfrak{g}$*-conjugated to the possibilities listed in table 3.*

Proof. By passing to a finite cover we can assume that G is simply connected, the pre-image of Λ under this covering has again minimal limit sets and its limit set on $\mathcal{F}$ contains a positive loxodromic triple. From Proposition 5.7 one finds a simply-laced $\dot{\mathsf{G}}$ and a rational representation $\Omega : \mathsf{G} \to \dot{\mathsf{G}}$ such that $\Omega\Lambda$ is partially positive. Applying Corollary 5.6 to $\Omega\Lambda$ gives de required result. $\square$

5.6. **Partially positive representations preserve type.** Recall that G is the real points of a real algebraic, Zariski connected, simple split group.

Let X be a proper Gromov-hyperbolic space and $\Gamma < \mathrm{Isom}(X)$ a non-elementary discrete subgroup, then one has the following facts from Ghys-de la Harpe [10, §8.2]:

i) every $\gamma \in \Gamma$ is either
   - of finite order (called *elliptic*),
   - *proximal*, i.e. has two fixed points $\gamma_-, \gamma_+ \in \partial X_\Gamma$ such that for every $x \in \partial X_\Gamma - \{\gamma_-\}$ one has $\gamma^n x \to \gamma_+$ as $n \to +\infty$,
   - *parabolic*, i.e. has a unique fixed point $x_\gamma \in \partial X_\Gamma$ and every $x \in \partial X_\Gamma$ converges to x_γ under the iterates γ^n as $n \to +\infty$ (some points will drift away from x_γ before coming back though).

ii) The attracting points of proximal elements are dense in ∂X_Γ.

iii) ∂X_Γ is the smallest closed Γ-invariant subset of ∂X, it is thus minimal.

Let us fix throughout this subsection a partially positive representation $\rho : \Gamma \to \mathsf{G}$ with equivariant map $\xi : \partial X_\Gamma \to \mathcal{F}$. We begin by showing that it is type preserving.

Proposition 5.9. *If $\gamma \in \Gamma$ is proximal then $\rho(\gamma)$ is proximal on $\mathcal{F}$ with attracting flag $\xi(\gamma_+)$ and repelling flag $\xi(\gamma_-)$. If $h \in \Gamma$ is parabolic then there exists $k \in \mathbb{N}_{\geqslant 1}$ such that $\rho(h^k)$ belongs to the unipotent radical of $\xi(x_h)$, moreover, there exists an open set $\mathcal{O} \subset \mathcal{F}$ such that $h^n z \to \xi(x_h)$ for every $z \in \mathcal{O}$.*

Proof. We divide the proof into Lemmas 5.10 and 5.11 below. □

Let M be the centralizer in K of $\exp \mathfrak{a}$, as $\mathfrak{g}$ is split this is a finite group. For $\sigma \in \Delta$, let us denote by $\bar{\phi} = \bar{\phi}_\sigma : \mathsf{G} \to \mathsf{SL}(V)$ and by $\Phi : \mathcal{F} \to \mathbb{P}(V)$, $\check{\Phi} : \mathcal{F} \to \mathbb{P}(V^*)$ the corresponding $\bar{\phi}$-equivariant maps.

Lemma 5.10. *For every proximal $\gamma \in \Gamma$, $\bar{\phi}\rho(\gamma)$ is proximal with attracting line $\Phi\xi(\gamma_+)$ and repelling hyperplane $\check{\Phi}\xi(\gamma_-)$.*

Proof. By passing to a finite cover we can assume that G is simply connected. In view of Proposition 5.7 we can also assume that G is simply laced and thus make use of Lusztig's canonical basis $\mathbf{B}_{\bar{\phi}_\sigma}$ (Proposition 5.4).

Without loss of generality we may assume that $\xi(\gamma_+) = [B]$ and that $\xi(\gamma_-) = [\check{B}]$. Since $\rho(\gamma)$ fixes both complete flags $\xi(\gamma_+)$ and $\xi(\gamma_-)$, it can be written as

$$\rho(\gamma) = m_{\rho(\gamma)} \exp(a_\gamma) \tag{5.2}$$

for a unique $a_\gamma \in \mathfrak{a}$ and $m_{\rho(\gamma)} \in \mathsf{M}$.

The composition $\Phi\xi : \partial X_\Gamma \to \mathbb{P}(V)$ is a continuous $\bar{\phi}\rho$-equivariant map. By the assumptions $\xi(\gamma_+) = [B]$ and $\xi(\gamma_-) = [\check{B}]$, one has

$$\Phi\xi(\gamma_+) = V^+ \text{ and } \check{\Phi}\xi(\gamma_-) = V^- = \bigoplus_{\chi \in \Pi(\phi) - \{\varpi_\sigma\}} V^\chi$$

respectively.

By definition there exists $x \in \partial X_\Gamma$ distinct from γ_+ and γ_- and $g \in \mathsf{G}_{>0}$ such that $\xi(x) = g\xi(\gamma_+)$. Lusztig's Proposition 5.4 states, in particular, that if $v \in V^+$ is non-zero then $\bar{\phi}_\sigma(g)v = \sum_{\mathbf{e} \in \mathbf{B}_{\bar{\phi}}} c_{\mathbf{e}} \mathbf{e}$ with $c_{\mathbf{e}} > 0$ for all $\mathbf{e}$.

On the other hand, equation (5.2) implies that $\bar{\phi}\rho(\gamma)$ is the commuting product of a matrix diagonal in $\mathbf{B}_{\bar{\phi}}$ and a finite order element. Let us denote thus by $\Omega_{\mathbf{e}}(\gamma)$ the (possibly complex) eigenvalue of $\bar{\phi}\rho(\gamma)$ of the vector $\mathbf{e} \in \mathbf{B}_{\bar{\phi}}$.

If k is the order of $m_{\rho(\gamma)}$, then $\bar{\phi}\big(\rho(\gamma)^k\big)$ is diagonal in $\mathbf{B}_{\bar{\phi}}$, so that $\Omega_{\mathbf{e}}(\gamma)^k \in \mathbb{R}$ and one has for all $n \in \mathbb{N}$

$$\frac{1}{\lambda_1\big(\bar{\phi}\rho(\gamma)\big)^{nk}}\big(\bar{\phi}\rho(\gamma^{nk})\big)(gv) = \sum_{\mathbf{e}\in\mathbf{B}_{\bar{\phi}_\sigma}}\left(\frac{\Omega_{\mathbf{e}}(\gamma)}{\lambda_1\big(\bar{\phi}\rho(\gamma)\big)}\right)^{nk} c_{\mathbf{e}}\mathbf{e}. \tag{5.3}$$

Since $\gamma^n x \to \gamma_+$, equivariance implies $\bar{\phi}\rho(\gamma^n)(gV^+) \to V^+$. Consequently, given that $c_{\mathbf{e}} > 0$, equation 5.3 implies then

$$|\Omega_{\mathbf{e}}(\gamma)| < |\lambda_1\big(\bar{\phi}\rho(\gamma)\big)|,$$

for every $\mathbf{e}$ except V^+ and thus the spectral radius of $\bar{\phi}\rho(\gamma)$ is attained at (and only at) V^+. Consequently $\bar{\phi}\rho(\gamma)$ is proximal. □

Lemma 5.11. *Let $h \in \Gamma$ be parabolic with fixed point x_h, then there exists $k \in \mathbb{N}_{\geqslant 1}$ such that $\rho(h^k)$ belongs to the unipotent radical of $\xi(x_h)$, moreover, there exists an open set $\mathcal{O} \subset \mathcal{F}$ such that $h^n z \to \xi(x_h)$ for every $z \in \mathcal{O}$.*

Proof. Again we can assume that G is simply laced and simply connected. We assume moreover that $\xi(x_h) = [B]$ and that $[\check{B}] = \xi(z_0)$ for some auxiliary point $z_0 \in \partial X_\Gamma$. one has then $\Phi\xi(x_h) = V^+$. Let us write

$$\rho(h) = m_{\rho(h)}\exp(a_h)u_h \tag{5.4}$$

where $m_h \in \mathsf{M}$ has finite order, commutes with $\exp a_h \in A$ and normalizes $u_h \in U$.

Since every element of $\mathbf{e} \in \mathbf{B}_{\bar{\phi}}$ belongs to a restricted weight space $V_{\chi_{\mathbf{e}}}$ of $\bar{\phi}$, we can order $\mathbf{B}_{\bar{\phi}}$ so that $\mathbf{e} \geqslant \mathbf{f}$ if $\chi_{\mathbf{e}} \succ \chi_{\mathbf{f}}$, (the order between elements lying in the same weight space, or between weight spaces of the same level, is not relevant for the following). The elements of $A \cdot U$ are upper triangular in $\mathbf{B}_{\bar{\phi}}$, so if k is the order of $m_{\rho(h)}$ then the transformation $\bar{\phi}\rho(h^k)$ is upper triangular in $\mathbf{B}_{\bar{\phi}}$.

Let us denote by $\lambda_1 = \exp\lambda_1\big(\bar{\phi}\rho(h^k)\big) \geqslant 1$ the spectral radius of $\bar{\phi}\rho(h^k)$ and by V_{λ_1} the sum of Jordan blocks of $\bar{\phi}\rho(h^k)$ associated to λ_1. By equation (5.4) and the definition of $\mathbf{B}_{\bar{\phi}}$ the intersection $V_{\lambda_1} \cap \mathbf{B}_{\bar{\phi}}$ is a basis of V_{λ_1}. Denote by $\pi : V \to V_{\lambda_1}$ the projection parallel to the vector space spanned by the remaining elements of $\mathbf{B}_{\bar{\phi}}$. If $\ell \in \mathbb{P}(V)$ is not contained in $\ker\pi$ then one has

$$d_{\mathbb{P}}\big(\bar{\phi}\rho(h)^{kn}\cdot\ell, \mathbb{P}(V_{\lambda_1})\big) \to 0 \tag{5.5}$$

as $n \to \infty$.

By definition, there exists $x \in \partial X_\Gamma - \{x_h\}$ and $g \in \check{U}_{>0}$ such that $\xi(x) = g\cdot[B]$. As before, if $v \in V^+$ is non-zero then $\bar{\phi}(g)v$ has positive coefficients in $\mathbf{B}_{\bar{\phi}}$. This implies, in particular, that $\Phi\xi(x) = \bar{\phi}(g)V^+ \not\subset \ker\pi$. Since $h^n x \to x_h$ one has $\bar{\phi}\rho(h)^n\big(\Phi\xi(x)\big) \to \Phi\xi(x_h)$, which combined with equation (5.5) gives $\Phi\xi(x_h) \in V_{\lambda_1}$.

On the other hand, since h is parabolic, $h^{-n}x$ also converges to x_h. So the above argument applied to h^{-1} gives that $\Phi\xi(x_h)$ is also contained in the generalized eigenspace of $\bar{\phi}\rho(h^{-k})$ associated to its spectral radius. Since[6]

$$\|\bar{\phi}\rho(h^{-k})v\|/\|v\| = \lambda_1^{-1},$$

one concludes that the spectral radius of $\bar{\phi}\rho(h^{-k})$ is $\lambda_1^{-1} \leqslant 1$.

Since $\bar{\phi}$ has values in $\mathsf{SL}(V)$ (because G is simple) one concludes that $\lambda_1 = 1$ and that $\bar{\phi}\rho(h)^k$ is upper triangular on $\mathbf{B}_{\bar{\phi}}$ with 1's in the diagonal, i.e. $\rho(h^k) \in U$.

[6] for any auxiliary norm.

Considering $x \in \partial X_\Gamma - \{x_h\}$ and $g \in \check{U}_{>0}$ as before; one has that $\Phi\xi(x) = \bar{\phi}(g)V^+$ does not belong to a $\bar{\phi}\rho(h)$-invariant subspace. Consequently, since $\bar{\phi}\rho(h)^n\Phi\xi(x) \to \Phi\xi(x_h)$, the same holds on a neighborhood of $\Phi\xi(x)$ and the lemma is proved. □

5.7. **Proof of Theorem B.** Proposition 5.9 readily implies that if $\rho : \Gamma \to \mathsf{G}$ is partially positive with limit map ξ_ρ, then it has minimal limit sets and contains a positive loxodromic triple. Indeed, by the descent method (§5.7) we can assume that G is simply laced and apply Proposition 5.9 to obtain that the limit set $\mathbf{L}_{\rho(\Gamma),\Delta} = \xi(\partial X_\Gamma)$ and moreover that $\mathbf{L}_{\rho(\Gamma),\sigma} = p_\sigma\big(\xi(\partial X_\Gamma)\big)$, where $p_\sigma : \mathcal{F} \to \mathcal{F}_{\{\sigma\}}$ is the canonical projection.

Theorem A would then complete the proof provided the Zariski closure of $\rho(\Gamma)$ where reductive. The purpose of this subsection is thus to bypass the 'reductive Zariski closure' assumption. Consequently, Proposition 5.13 below and Theorem A prove Theorem B.

We begin by recalling the following lemma. It is a well known fact that the reader may check in Guéritaud-Guichard-Kassel-Wienhard [12, §2.5.4] or in Benoist's lecture notes [3].

Lemma 5.12. *Let Λ be a group and let $\rho \in \hom(\Lambda, \mathsf{G})$ have non-solvable Zariski closure L. Let $\mathfrak{l} = \mathfrak{h} \oplus_\pi R_u(\mathfrak{h})$ be a Levy decomposition of the Lie algebra of L as a semi-direct product, with $\mathfrak{h}$ reductive and $R_u(\mathfrak{h})$ its unipotent radical. Then there exists $\eta \in \hom(\Lambda, \mathsf{G})$ whose Zariski closure has Lie algebra $\mathfrak{h}$ and a sequence $(g_n) \in \mathsf{G}$ with $g_n\rho g_n^{-1} \to \eta$.*

As in Guéritaud-Guichard-Kassel-Wienhard [12, §2.5.4], we say that η is the *semi-simplification* of ρ (regardless its Zariski closure is reductive and not necessarily semi-simple, and regardless of any uniqueness issues). We then prove the following.

Proposition 5.13. *If $\rho : \Gamma \to \mathsf{G}$ is partially positive then its semi-simplification η has minimal limit sets and contains a positive loxodromic triple.*

Proof. By continuity of the Jordan projection and Proposition 5.9, one has that $\eta(\gamma)$ is purely loxodromic for every proximal $\gamma \in \Gamma$, and that for every parabolic $h \in \Gamma$ there exists $k = k_h$ such that $\eta(h)^k$ is unipotent.

The argument from Guéritaud-Guichard-Kassel-Wienhard [12, Proposition 4.13] works verbatim in this situation to give a η-equivariant continuous map $\xi_\eta : \partial X_\Gamma \to \mathcal{F}$ such that for every proximal $\gamma \in \Gamma$ $\xi_\eta(\gamma_+)$ and $\xi_\eta(\gamma_-)$ are respectively the attracting and repelling flags of $\eta(\gamma)$. The limit set

$$\mathbf{L}_{\eta(\Gamma),\Delta} = \xi_\eta\big(\partial X_\Gamma\big)$$

is thus minimal, and since every element of $\eta(\Gamma)$ is either purely loxodromic, unipotent (up to a finite power) or elliptic, for every $\sigma \in \Delta$ the limit set $\mathbf{L}_{\eta(\Gamma),\sigma}$ is also minimal.

Since the pairs $\{(\gamma_-, \gamma_+) : \gamma \in \Gamma \text{ proximal}\}$ are dense in $\partial X_\Gamma^{(2)} = \partial X_\Gamma \times \partial X_\Gamma -$ diagonal, continuity of ξ_η implies moreover that it is *transverse*, i. e. for every $x \neq y \in \partial X_\Gamma$ the flags $\xi_\eta(x)$ and $\xi_\eta(y)$ are in general position.

In order to find a positive loxodromic triple in $\xi_\eta(\partial X_\Gamma)$, we observe that for every proximal $\gamma \in \Gamma$ $g_n\xi_\rho(\gamma_+) \to \xi_\eta(\gamma_+)$ as $n \to \infty$. Indeed, for every $\sigma \in \Delta$ the line $\Phi_\sigma\big(g_n \cdot \xi_\rho(\gamma)\big)$ is the eigenline of $\bar{\phi}_\sigma(g_n\rho(\gamma)g_n^{-1})$ associated to its spectral radius

$$\lambda_1\big(\bar{\phi}_\sigma(g_n\rho(\gamma)g_n^{-1})\big) = \lambda_1\big(\bar{\phi}_\sigma(\rho(\gamma))\big) = \lambda_1\big(\bar{\phi}_\sigma(\eta(\gamma))\big).$$

Consequently, any accumulation point of $\{\Phi_\sigma(g_n \cdot \xi_\rho(\gamma))\}$ is an eigenline associated to $\lambda_1(\bar\phi_\sigma(\eta(\gamma)))$; since $\bar\phi_\sigma(\eta(\gamma))$ is proximal, this eigenline is $\Phi_\sigma(\xi_\eta(\gamma_+))$.

By assumption, there exists $x \in \partial X_\Gamma$ such that $(\xi_\rho(\gamma_+), \xi_\rho(x), \xi_\rho(\gamma_-))$ is a positive triple of flags. By Proposition 5.2, $\mathcal{F}_{>0}$ is an open subset of $\mathcal{F}$, thus, since attracting points of proximal elements are dense in ∂X_Γ, there exists a proximal $h \in \Gamma$ such that $(\xi_\rho(\gamma_+), \xi_\rho(h_+), \xi_\rho(\gamma_-))$ is also a positive triple.

We claim that $(\xi_\eta(\gamma_+), \xi_\eta(h_+), \xi_\eta(\gamma_-))$ is a positive triple. Indeed, let us asume with out loss of generality that $\xi_\eta(\gamma_+) = [B]$ and that $\xi_\eta(\gamma_-) = [\check B]$. One has the convergence

$$g_n \cdot (\xi_\rho(\gamma_+), \xi_\rho(h_+), \xi_\rho(\gamma_-)) \to ([B], \xi_\eta(h_+), [\check B])$$

and the triple $g_n \cdot (\xi_\rho(\gamma_+), \xi_\rho(h_+), \xi_\rho(\gamma_-))$ is positive by definition. We may then also assume that for every n, $g_n \cdot \xi_\rho(h_+) \in g_n \cdot \mathcal{F}_{>0}$. The limit $\xi_\eta(h_+)$ of the sequence $g_n \cdot \xi_\rho(h_+)$ lies thus in the topological closure $\overline{\mathcal{F}_{>0}}$. Proposition 5.2 states that every element in the topological boundary of $\mathcal{F}_{>0}$ is not transverse to either $[B]$ or $[\check B]$. However, as was observed earlier, $\xi_\eta(h_+)$ is both transverse to $[B]$ and $[\check B]$ and thus necessarily lies in $\mathcal{F}_{>0}$, the topological interior of $\overline{\mathcal{F}_{>0}}$. □

5.8. **Hyperconvexity.** To end this section we record the following remark that will be useful in Bridgeman-Pozzetti-Wienhard-S. [5].

Remark 5.14. Assume that ∂X_Γ is homeomorphic to a circle, and that a partially positive $\rho : \Gamma \to \mathsf{G}$ verifies the extra condition that ξ sends positive ordered triples on ∂X_Γ to positive triples of flags. Then for every $\sigma \in \Delta$ and $x, y, z \in \partial X_\Gamma$ pairwise distinct one has

$$(\Phi\xi(x) \oplus \Phi\xi(y)) \cap \check\Phi_{\wedge^2 \bar\phi_\sigma}\xi(z) = \{0\}.$$

Here we interpret $\check\Phi_{\wedge^2\bar\phi_\sigma}\xi(z)$ as a $\dim V_\sigma - 2$-dimensional subspace of V_σ. In the language of Pozzetti-S.-Wienhard [21], the remark states that the curve $\Phi\xi(\partial X_\Gamma)$ is $(1,1,2)$-hyperconvex.

Proof. We can assume that G is simply laced and simply connected. We may also assume that $\xi(x) = [\check B]$, $\xi(z) = [B]$ and that $\xi(y) = g\xi(x)$ for a $g \in \check U_{>0}$. We mimic now the proof of Corollary 5.5. Since $\bar\phi_\sigma(g)$ has positive coefficients in the basis $\mathbf{B}_{\bar\phi_\sigma}$, the intersection of the plane

$$P_y = \Phi\xi(x) \oplus \Phi\xi(y) = \Phi\xi(x) \oplus \bar\phi_\sigma(g)\Phi\xi(x) = V^+ \oplus \bar\phi_\sigma(g)V^+$$

with V^-, is not contained in any partial sum of restricted weight subspaces, in particular it is not contained in

$$\sum_{\chi \in \Pi(\phi_\sigma) - \{\varpi_\sigma, \varpi_\sigma - \sigma\}} V^\chi = \check\Phi_{\wedge^2\bar\phi_\sigma}\xi(z)$$

as required. □

6. Group level

Let us consider now a non-elementary discrete subgroup $\Gamma < \mathrm{Isom}(X)$ of a proper Gromov-hyperbolic space X, a simple split G and the space

$$\hom_{\gtrsim}(\Gamma, \mathsf{G}) = \{\rho : \Gamma \to \mathsf{G} \text{ partially positive}\}.$$

In view of Proposition 5.9, if $\rho \in \hom_{\gtrsim}(\Gamma, \mathsf{G})$ and $\gamma \in \Gamma$ is non-torsion, then the *elliptic component*

$$\rho(\gamma) \mapsto m_{\rho(\gamma)},$$

as in equations (5.2) and (5.4) according to the type of γ, is a locally constant well defined map. The image of this map is thus an invariant of the connected component of $\hom_{\gtrsim}(\Gamma, \mathsf{G})$ containing ρ.

We will use this map to decide whether the Zariski closure of a given ρ is connected or not. Indeed, let us consider $\rho \in \hom_{\gtrsim}\big(\Gamma, \mathsf{SL}_d(\mathbb{R})\big)$ and denote by H its Zariski closure. By Corollary 1.4 H has finite center. Thus, H_0, the connected component of the identity of H, is (conjugated to) one of the groups in table 6.

- $\mathsf{SL}_d(\mathbb{R})$,
- a principal $\mathsf{SL}_2(\mathbb{R})$,
- $\mathsf{Sp}_{2n}(\mathbb{R})$ if $d = 2n$ for all $n \geqslant 1$,
- $\mathsf{SO}_0(n, n+1)$ if $d = 2n+1$ for all $n \geqslant 1$,
- the fundamental representation for the short root of $\overline{\mathsf{G}_2}$ if $d = 7$.

Table 6. Connected component of the identity of the Zariski closure of an element $\rho \in \hom_{\gtrsim}\big(\Gamma, \mathsf{SL}_d(\mathbb{R})\big)$, $\overline{\mathsf{G}_2}$ denotes the two-fold covering of G_2.

To decide if H is connected, one can observe that for every non-torsion $\gamma \in \Gamma$ the elliptic component $m_{\rho(\gamma)} \in \mathsf{M} \cap \mathsf{H}$. This latter finite group is nothing but the centralizer in K_H of $\exp \mathfrak{a}_\mathsf{H}$, so if $m_{\rho(\gamma)} \in \mathsf{H}_0$ then $\rho(\gamma) \in \mathsf{H}_0$.

Definition 6.1. A discrete and faithful representation $\Gamma \to \mathsf{SL}_d(\mathbb{R})$ is *principal*[7] if its Zariski closure is a principal $\mathsf{SL}_2(\mathbb{R})$. We denote by $\mathfrak{H}\big(\Gamma, \mathsf{SL}_d(\mathbb{R})\big)$ a connected component of $\hom_{\gtrsim}\big(\Gamma, \mathsf{SL}_d(\mathbb{R})\big)$ that contains a principal representation.

Corollary 6.2. *Assume Γ is torsion free, then every element of $\mathfrak{H}\big(\Gamma, \mathsf{SL}_d(\mathbb{R})\big)$ has connected Zariski closure (and is thus an element of table (6)).*

Proof. Let $\tau : \mathsf{SL}_2(\mathbb{R}) \to \mathsf{SL}_d(\mathbb{R})$ be a principal embedding. Observe that the group

$$M := \mathsf{M}_{\tau\left(\mathsf{SL}_2(\mathbb{R})\right)} = \left\{ \tau\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \tau\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$$

is contained in any group in the list 6. If $\rho \in \mathfrak{H}\big(\Gamma, \mathsf{SL}_d(\mathbb{R})\big)$ has Zariski closure H, then for every $\gamma \in \Gamma$ $m_{\rho(\gamma)} \in M \subset \mathsf{H}_0$ and the proof is complete. □

Finally, let S be a closed connected orientable surface of genus $\geqslant 2$ and $\rho : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ in a Hitchin component. Assume first that ρ lifts to a representation $\widetilde{\rho} : \pi_1 S \to \mathsf{SL}_d(\mathbb{R})$. Then Theorem 1.2 assures that $\widetilde{\rho} \in \mathfrak{H}(\pi_1 S, \mathsf{SL}_d(\mathbb{R}))$ and Corollary 6.2 implies that the Zariski closure of ρ is the projectivisation of a group in the table 6.

To prove that ρ lifts, recall that $\pi_1 S$ has the presentation

$$\langle a_1, b_1, \cdots, a_g, b_g : \prod [a_i, b_i] = 1 \rangle.$$

[7]This is usually referred to as *Fuchsian* in the literature.

If $\eta : \pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ is a representation and one considers a lift for each generator $\widetilde{\eta}(a_i), \widetilde{\eta}(b_i) \in \mathsf{SL}_2(\mathbb{R})$, one readily sees that the commutator product

$$\prod \left[\widetilde{\eta(a_i)}, \widetilde{\eta(b_i)}\right] \in \left\{ \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$$

is independent of the chosen lifts.

If η is discrete and faithful then (see for example Goldman [11]) the above product equals id and the representation lifts. This shows that if $\rho_0 : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ is principal, then it lifts to $\mathfrak{H}\big(\pi_1 S, \mathsf{SL}_d(\mathbb{R})\big)$. On the other hand it is clear that the above product of commutators is an invariant of the connected component in $\hom\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ so any Hitchin representation lifts. This completes the proof of Guichard's classification (Corollary 1.5).

Appendix A. The Hasse diagrams for extremal roots

In this appendix we prove Lemma 3.3. To this end we compute the Hasse diagrams for the extremal roots of irreducible reduced root systems and compute, in a case by case manner, the existence/non-existence of surjective maps between them. Let us simplify notation and denote, for a simple root $x \in \Delta_{\mathsf{J}}$ of some root system J, by $\mathcal{H}^{\mathsf{J}}_x$ the Hasse diagram $\mathcal{H}^{\mathsf{J}}_{\varpi_x}$ for the fundamental weight ϖ_x.

Most of the situations are ruled out by the following simple facts. If $f : \Delta_{\mathsf{L}} \to \Delta_{\mathsf{J}}$ is surjective and $\mathbb{T}^f : \mathcal{H}^{\mathsf{L}}_\alpha \to \mathcal{H}^{\mathsf{J}}_{f(\alpha)}$ is a surjective diagram map with labeling f then:

- rank $\mathsf{J} \leqslant$ rank L,
- both $\mathcal{H}^{\mathsf{L}}_\alpha$ and $\mathcal{H}^{\mathsf{J}}_{f(\alpha)}$ have the same total amount of levels,
- if χ is the only vertex at a given level, then the number of arrows pointing downwards in $\mathcal{H}^{\mathsf{L}}_\alpha$ is greater than that of $\mathbb{T}^f(\chi)$ in $\mathcal{H}^{\mathsf{J}}_{f(\alpha)}$,
- to show non-existence of such f, it it sufficient to find one extremal root of L whose Hasse diagram does not surject to any diagram of J (for extremal roots).

We refer the reader to the corresponding figures for the labeling of simple roots for each Dynkin diagram.

Lemma A.1. *Leaving aside the case $f =$identity, one has the following.*

- *Type* A : *The only surjective diagram maps $\mathbb{T}^f : \mathcal{H}^{\mathsf{A}_d}_{\beta_1} \to \mathcal{H}^{\mathsf{J}}_x$ with x extremal are*
  - *$d = 2n$ and $\mathsf{J} = \mathsf{B}_n$ and $x = \beta$ for all n and moreover G_2 and $x = \alpha$ if $d = 6$,*
  - *$d = 2n - 1$, $\mathsf{J} = \mathsf{C}_n$ and $x = \beta$.*
- *Type* B : *The only surjective diagram maps $\mathbb{T}^f : \mathcal{H}^{\mathsf{B}_n}_{\beta} \to \mathcal{H}^{\mathsf{J}}_x$ with x extremal is $n = 3$ and $\mathsf{J} = \mathsf{G}_2$ and $x = \alpha$.*
- *Type* C : *There is no surjective diagram map $\mathbb{T}^f : \mathcal{H}^{\mathsf{C}_n}_{\beta} \to \mathcal{H}^{\mathsf{J}}_x$ with x extremal.*
- *Type* D : *The only surjective diagram maps $\mathbb{T}^f : \mathcal{H}^{\mathsf{D}_n}_{\beta} \to \mathcal{H}^{\mathsf{J}}_x$ with x extremal are*
  - *$\mathsf{J} = \mathsf{B}_{n-1}$ with $x = \beta$ for all n,*
  - *moreover $\mathsf{J} = \mathsf{B}_3$ with $x = \alpha$ and $\mathsf{J} = \mathsf{G}_2$ with $x = \alpha$ if $n = 4$.*

Proof. Observe that all Hasse diagrams $\mathcal{H}^{\mathsf{A}_n}_{\beta_1}$ (Figure (2)), $\mathcal{H}^{\mathsf{B}_n}_{\beta}$ and $\mathcal{H}^{\mathsf{C}_n}_{\beta}$ consist on exactly one arrow exiting each vertex. By restricting the total amount of levels given by the existence of $\mathbb{T}^f$ together with the fact that rank $\mathsf{J} \leqslant n$ (in each case) one completes the proof. A similar argument works for $\mathcal{H}^{\mathsf{D}_n}_{\beta_1}$ (see also Figure 3). □

We now treat the type E family, we will show that there is no surjective diagram map from $\mathcal{H}_\alpha^{\mathsf{E}_k}$ for $k = 6$, 7 or 8 to any other Hasse diagram $\mathcal{H}_x^{\mathsf{J}}$ with extremal x. Except for $\mathcal{H}_\alpha^{\mathsf{E}_6} \to \mathcal{H}_\alpha^{\mathsf{F}_4}$ (as shown in Figure (9)).

Lemma A.2. *There is no surjective map $\mathbb{T}^f$ from $\mathcal{H}_\alpha^{\mathsf{E}_k}$ for $k = 6$, 7 or 8 onto any of $\mathcal{H}_\beta^{\mathsf{A}_n} \approx \mathcal{H}_\alpha^{\mathsf{A}_n}$, $\mathcal{H}_\beta^{\mathsf{B}_n}$, $\mathcal{H}_\beta^{\mathsf{C}_n}$, for $n \leqslant 8$ nor onto $\mathcal{H}_\beta^{\mathsf{G}_2}$ or $\mathcal{H}_\alpha^{\mathsf{G}_2}$.*

Proof. The non-existence of such map comes from the fact that $\mathcal{H}_\alpha^{\mathsf{E}}$ has too many levels (compared to the fact that n must be smaller than k), observe that Figure (6) depicts $\mathcal{H}_\alpha^{\mathsf{E}_k}$ up to levels 9, 10 and 11 respectively for $k = 6$, 7 or 8. The case $\mathcal{H}_\alpha^{\mathsf{G}_2}$ is readily discarded since it has 7 levels.

We now treat $\mathcal{H}_x^{\mathsf{J}}$ for $\mathsf{J} = \mathsf{A}_n$, B_n, C_n and $x = \beta$. Since these diagrams consist on only one arrow pointing downwards at each level, from Figure (6) one sees that if such a $\mathbb{T}^f$ existed then necessarily

$$f(\beta_2) = f(\sigma) = f(\beta) = f(\beta_3) = f(\beta_4).$$

Since f is surjective, the above equalities imply that J has rank $\leqslant k - 4$, that is $n \leqslant k - 4 \leqslant 4$. However $\mathcal{H}_\beta^{\mathsf{A}_4}$ has 5 levels, $\mathcal{H}_\beta^{\mathsf{B}_4}$ has 9 levels and $\mathcal{H}_\beta^{\mathsf{C}_4}$ has 8 levels, but $\mathcal{H}_\alpha^{\mathsf{E}}$ has at least 9 levels (actually at least 17 as seen in Figure (9)).

Finally, from Figure (8) one sees that $\mathcal{H}_\beta^{\mathsf{G}_2}$ has 14 levels but Figure (9) shows that $\mathcal{H}_\alpha^{\mathsf{E}}$ has at least 17 levels. □

Lemma A.3.

- *There is no surjective map $\mathbb{T}^f$ from $\mathcal{H}_\alpha^{\mathsf{E}_k}$ $k = 6$, 7 or 8 onto $\mathcal{H}_\alpha^{\mathsf{B}_n}$, $\mathcal{H}_\alpha^{\mathsf{C}_n}$, $\mathcal{H}_\alpha^{\mathsf{D}_n} \approx \mathcal{H}_\sigma^{\mathsf{D}_n}$, $\mathcal{H}_\sigma^{\mathsf{E}_j}$ ($j = 6$, 7 or 8), $\mathcal{H}_\alpha^{\mathsf{E}_{k-1}}$, (if $k = 7$ or 8) $\mathcal{H}_\alpha^{\mathsf{E}_{k-2}}$ (if $k = 8$).*
- *There is no surjective map $\mathbb{T}^f$ from $\mathcal{H}_\alpha^{\mathsf{E}_7}$ or $\mathcal{H}_\alpha^{\mathsf{E}_8}$ onto $\mathcal{H}_\alpha^{\mathsf{E}_6}$, $\mathcal{H}_\beta^{\mathsf{E}_j}$ ($j = 6$, 7 or 8), $\mathcal{H}_\beta^{\mathsf{F}_4}$ and $\mathcal{H}_\alpha^{\mathsf{F}_4}$.*

Proof. In $\mathcal{H}_\alpha^{\mathsf{E}_k}$ the first level with more than one exiting arrow is at least 4, however the diagrams appearing in the first item have 2 exiting arrows at the third level. Similarly the first level with more than one exiting arrow in $\mathcal{H}_\alpha^{\mathsf{E}_7}$ or $\mathcal{H}_\alpha^{\mathsf{E}_8}$ is at least 5, but the diagrams listed in the second item have earlier multiple exiting arrows. □

The E family is thus achieved with the next Lemma.

Lemma A.4. *There is no surjective map $\mathbb{T}^f$ from $\mathcal{H}_\alpha^{\mathsf{E}_k}$ for $k \in \{6, 7, 8, \}$ onto $\mathcal{H}_\beta^{\mathsf{D}_n}$.*

Proof. Since in $\mathcal{H}_\beta^{\mathsf{D}_n}$ there is only one arrow starting at each node for every level up to $n - 2$, if such a $\mathbb{T}^f$ exists then one must have $n - 2 = k - 3$. However, by looking at levels after the first rombus in Figure (6) one sees that

$$f(\beta) = f(\sigma) = (\beta_3),$$

thus $n \leqslant k - 2$, which is a contradiction with $n = k - 1$. □

The remaining F_4 and G_2 cases are easily discarded since the other reduced root systems with rank $\mathsf{J} \leqslant 4$ and $\leqslant 2$ respectively do not have enough levels.

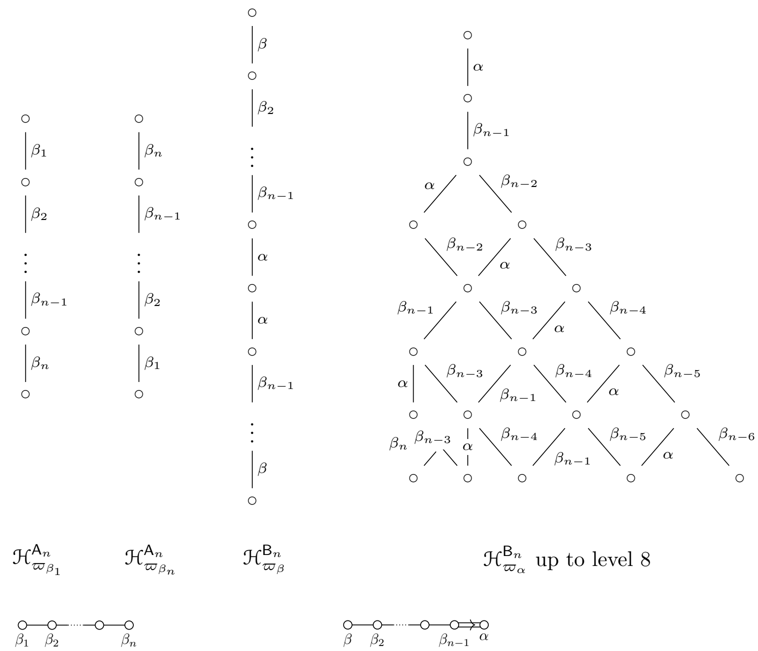


FIGURE 4. Hasse for extremal roots of A_n (left) and B_n (right)

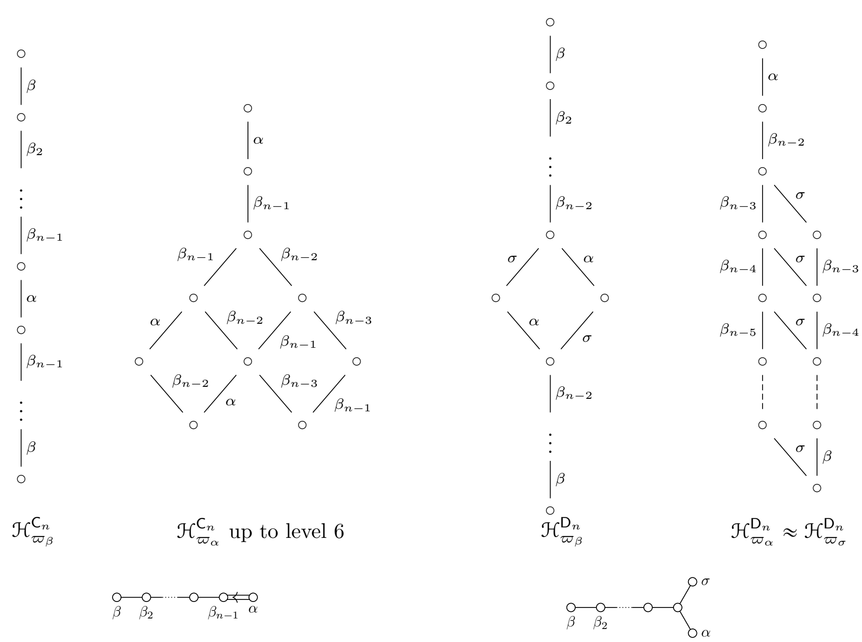


FIGURE 5. Hasse for extremal roots of C_n (left) and D_n (right)

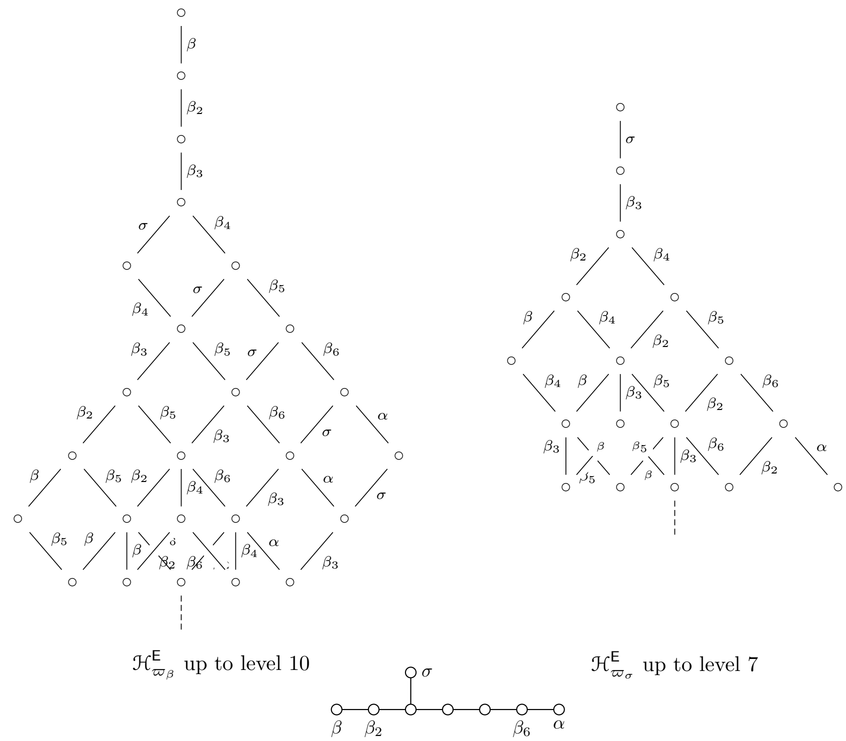


FIGURE 6. Hasse for extremal roots of the E family

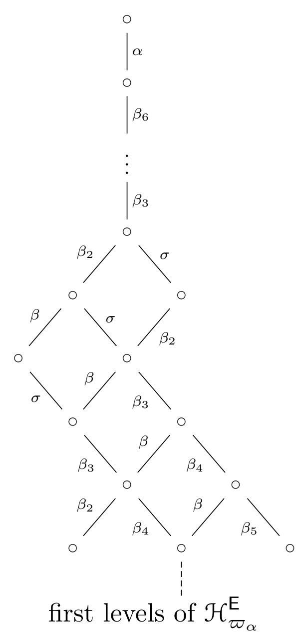
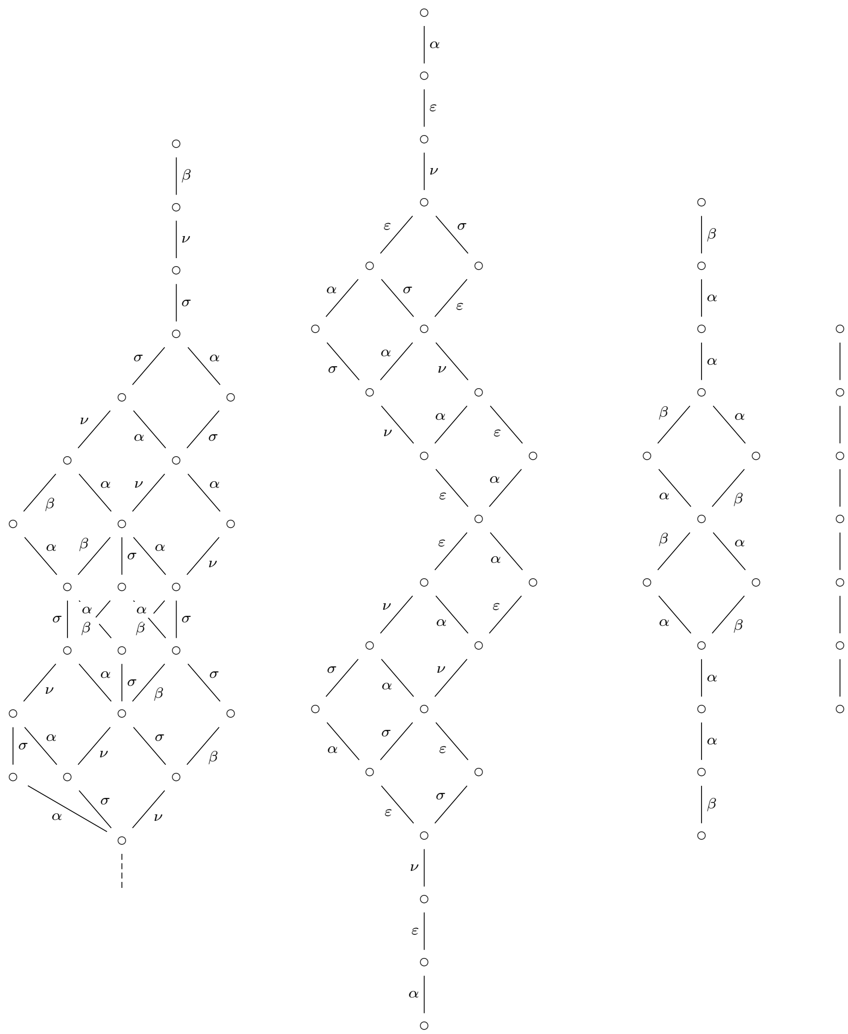


FIGURE 7. Hasse for extremal roots of the E family, continued



$\mathcal{H}^{\mathsf{F}_4}_{\varpi_\beta}$ up to level 12 $\qquad$ $\mathcal{H}^{\mathsf{F}_4}_{\varpi_\alpha}$, it has 17 levels. $\qquad$ $\mathcal{H}^{\mathsf{G}_2}_{\varpi_\beta}$ $\qquad$ $\mathcal{H}^{\mathsf{G}_2}_{\varpi_\alpha}$

FIGURE 8. Hasse for extremal roots of F_4 (left) and G_2 (right)

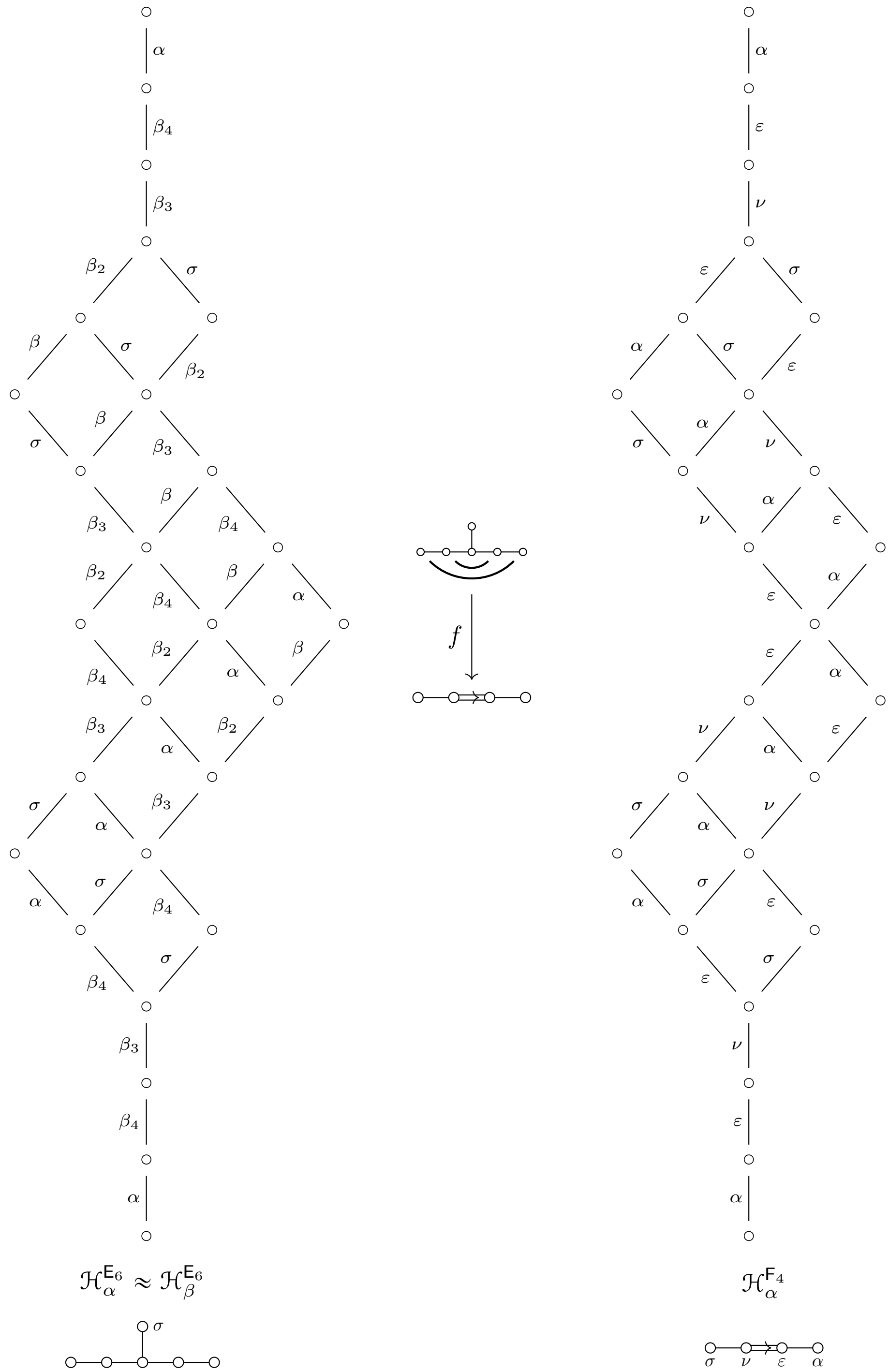


FIGURE 9. A surjective map $\mathbb{T}^f : \mathcal{H}_\alpha^{\mathsf{E}_6} \to \mathcal{H}_\alpha^{\mathsf{F}_4}$ with labeling f.

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