COHOMOLOGICAL INJECTIVITY QUESTIONS ON BRUHAT-TITS SUBGROUPS

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Abstract. The aim of this article is to refine certain aspects of the cohomological study of Bruhat-Tits subgroups. It is intended to be complementary to the work done in the 1987 article. As an application, we obtain a result "à la Grothendieck-Serre" for the Bruhat-Tits subgroups of a simply connected semisimple group and the exact computation of the obstruction in the case of quasi-split adjoint groups.

Keywords: Group schemes, Algebraic groups, Reductive groups, Bruhat-Tits theory, Galois cohomology, Discrete valuation rings.

MSC: 20G10, 20G15, 14L10, 14L15.

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Introduction

This article aims to develop the cohomological aspects of Bruhat-Tits theory. It can be seen as a complement to [BT87].

Recall that the cohomological study of [BT87] focuses on anisotropic cocycles (cf. [BT87, 3.6.]) and the resulting decomposition results (cf. [BT87, 3.12. Theorem.]). This article addresses a completely different problem, which we explain below.

Consider a henselian discrete valuation ring R, with field of fractions K. Denote by R^{unr} its strict henselization and by K^{unr} the field of fractions of R^{unr}. This is the maximal unramified extension of K. It is Galois, and we denote its Galois group by Γ^{unr}. The residue field of R is denoted by κ, and is not assumed to be perfect. Denote also by I the inertia subgroup of K, i.e. the absolute Galois group of K^{unr}.

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Given a reductive group G over K, we ask whether it is possible to understand the following kernel:

$$\operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$$

where $\widetilde{\mathcal{F}}$ is a Γ^{unr}-invariant facet of the building $\mathscr{B}(G_{K^{\mathrm{unr}}})$ and $G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}$ is its stabilizer under the action of $G(K^{\mathrm{unr}})$.

More generally, we introduce the notion of a *global subgroup*. A subgroup of $G(K)$ is called global if it is open for the adic topology, and if it contains $G(K)^+$, i.e., the subgroup generated by the K-points of the root subgroups of G. The study of the action of these subgroups on the building is of great interest. Indeed, they act transitively on pairs $(\mathcal{A}, \mathcal{C})$ such that $\mathcal{A}$ is an apartment of $\mathscr{B}(G)$ and $\mathcal{C}$ is a chamber of $\mathcal{A}$ (cf. Lemma 2.5).

One can also consider a global subgroup $\widetilde{H}$ of $G(K^{\mathrm{unr}})$ invariant under the action of Γ^{unr}. These objects are then a generalization of the subgroups considered by Tits in [BT87, 3.5.] (cf. Remark (2) of 2.6).

In this case, as in [BT87], one can be interested in more general questions involving $\widetilde{H}$. In summary, one can study the kernel:

$$\operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, \widetilde{H}_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, \widetilde{H})\right)$$

where $\widetilde{H}$ is this time a Γ^{unr}-invariant global subgroup of $G(K^{\mathrm{unr}})$ and $\widetilde{H}_{\widetilde{\mathcal{F}}}$ is the stabilizer of $\widetilde{\mathcal{F}}$ under the action of $\widetilde{H}$.

Some techniques from classical group cohomology allow us to show the bijection (cf. point (1) (a) of Theorem 4.3):

$$(\operatorname{Orb}(\widetilde{\mathcal{F}})_{\widetilde{H}})^{\Gamma^{\mathrm{unr}}}/H \xrightarrow{\sim} \operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, \widetilde{H}_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, \widetilde{H})\right). \tag{$*$}$$

where we have set $H := \widetilde{H}^{\Gamma^{\mathrm{unr}}}$. In other words, the kernel corresponds to the Γ^{unr}-invariant elements of the orbit $\widetilde{\mathcal{F}}$ under $\widetilde{H}$, modulo the action of H.

Therefore, we note that Bruhat and Tits had already implicitly addressed the question in [BT84a]. For example, the result [BT84a, 5.2.10.(ii) Proposition.] means, among other things, that $(\operatorname{Orb}(\widetilde{\mathcal{F}})_{G(K^{\mathrm{unr}})})^{\Gamma^{\mathrm{unr}}}/G(K)$ is trivial when G is semisimple simply connected, quasi-split over K^{unr}. Consequently, $\operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$ is trivial in this case (cf. Remark 4.6).

Also, in [BT84a, 5.2.13.], Bruhat and Tits give a case where $(\operatorname{Orb}(\widetilde{\mathcal{F}})_{G(K^{\mathrm{unr}})})^{\Gamma^{\mathrm{unr}}}/G(K)$ is non-trivial. In this example, G is quasi-split and adjoint.

By observing this example carefully, we see that Bruhat and Tits essentially reason at the level of types, and thus at the level of affine Dynkin diagrams. It turns out that this phenomenon is quite general.

Indeed, we prove in this article that the bijection $(*)$ always holds if we reduce it to the level of types. We then obtain (cf. point (2) (a) of Theorem 4.3):

$$\left(\{\omega \cdot \widetilde{\mathcal{T}} \prec \widetilde{\mathcal{T}}_{\max} \mid \omega \in \widetilde{H}\}^{\Gamma^{\mathrm{unr}}}\right)/H \xrightarrow{\sim} \operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, \widetilde{H}_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, \widetilde{H})\right). \tag{$*'$}$$

Let us explain the objects involved. Recall that the affine Tits index of G is the data of its affine Dynkin diagram over K^{unr}, an action of Γ^{unr}, and a Γ^{unr}-stable set of vertices, which we denote by $\widetilde{\mathcal{T}}_{\max}$. This is also the type of the largest Γ^{unr}-invariant facet in $\mathscr{B}(G_{K^{\mathrm{unr}}})$ (also called the Γ^{unr}-chamber). The type $\widetilde{\mathcal{T}}$ is defined as the type of $\widetilde{\mathcal{F}}$.

Consequently, the bijection $(*')$ gives an explicit and combinatorial way to compute the kernel, depending only on the affine Tits index of G equipped with the natural action of $\widetilde{H}$ and H. This is the main theoretical result of this article.

From this theorem, we immediately deduce that $\operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, \widetilde{H}_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, \widetilde{H})\right)$ is trivial when $\widetilde{H}$ acts trivially on the affine Tits index. This is notably the case when G is semisimple simply connected and quasi-split over K^{unr} according to [BT84a, 5.2.10.(i) Proposition.]. We thus recover the result of Bruhat and Tits mentioned above.

Furthermore, thanks to the bijection $(*')$, the case where G is quasi-split and adjoint can be fully understood, thus generalizing the example [BT84a, 5.2.13.] of Bruhat and Tits. We thus prove in this article:

Theorem (cf. Theorem 6.15)**.** *Let G be a semisimple adjoint group quasi-split over K. Also let $\widetilde{\mathcal{F}}$ be a Γ^{unr}-invariant facet of the building $\mathscr{B}(G_{K^{\mathrm{unr}}})$. Then the kernel:*

$$\operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$$

has cardinality 2^k where k is an integer bounded above by the number of factors that are a Weil restriction of an absolutely almost simple group of type ${}^2D_n$ (for $n \geq 4$) or ${}^2A_{4n+3}$ (for $n \geq 0$) split by an unramified extension.

That being said, one can also be interested in the following kernel:

$$\operatorname{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^0_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$$

where $G(K^{\mathrm{unr}})^0$ is the subgroup generated by the parahoric subgroups over K^{unr}, also called the residually neutral component of $G_{K^{\mathrm{unr}}}$.

This question is significantly more delicate. Despite our efforts and our exploration of the literature, we do not know if there are situations where it is non-trivial.

The case where $\widetilde{\mathcal{F}}$ is a hyperspecial point (cf. Definition 5.1) is in fact trivial according to the Grothendieck-Serre conjecture in the case of a henselian discrete valuation ring. Again, the result has in fact already been proved by Bruhat and Tits when G is semisimple in [BT84a, 5.2.14. Proposition.] using the bijection $(*)$. We also show in this article that the proof can be adjusted to directly prove the reductive case. This is the subject of Proposition 5.5.

Another case where we can prove triviality is, once again, the quasi-split adjoint case. This is the subject of Theorem 6.8.

To conclude, let us make an observation on the hypotheses of the article. The residue field κ of R is not assumed to be perfect, contrary to the article [BT87] by Bruhat and Tits. The group G is also not assumed to be quasi-split over K^{unr}, although this is the setting of the theorems of [BT84a] (recall moreover that if κ is perfect, then G is quasi-split over K^{unr} as mentioned in [BT84a, 5.1.1.]). It is therefore necessary to recall some aspects of Bruhat-Tits theory in this generality, in particular to explain why the building of G exists. This is the purpose of Section 1.

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Notations and conventions

For any field k, the notation k^s denotes a separable closure of k.

We use the definition of reductive group by Chevalley and Borel (cf. [Bor91]). In particular, they are affine, smooth, and connected.

Note that the maximal unramified extension of a complete field is not always complete. For example, the maximal unramified extension of $\kappa((t))$ is not $\kappa^s((t))$ if κ^s/κ is infinite.

1. Some reminders on the Bruhat-Tits building

Let us recall some points about the building and its existence. To be as general as possible, we assume in this section that K is not necessarily henselian. The results of this section will in fact be available in the forthcoming book [Rou]. For convenience, we propose to present this section independently of this reference.

In [Rou77, Définitions 2.1.12], Rousseau proposes a definition of a building associated to a reductive group G over K. As proven in [Rou77, Théorème 2.1.14], this building exists if and only if the building in the sense of [BT72], i.e., constructed from a valued root datum, exists; and it is unique up to isomorphism.

Furthermore, as indicated in [Rou77, Théorème 2.1.14.2)c)] and [Rou77, Théorème 2.1.15.c)], it is possible to canonize this building by constructing it as the product of the building of the derived group $D(G)$ with a vector part given by the radical $R(G)$. A building in this form is called a *centered building*. A centered building is unique up to unique isomorphism (of centered buildings).

Note that the building constructed from a valued root datum associated to G is identified with the building of $D(G)$.

The building proposed by Rousseau for G is exactly the extended building under modern terminology. We denote it by $\mathscr{B}^e(G)$. As said before, it decomposes into a product $\mathscr{B}(G) \times V_G$ where V_G is the vector part of the building and where $\mathscr{B}(G)$ is the building of $D(G)$ (or the building in the sense of a valued root datum of G as in [BT72]). The part $\mathscr{B}(G)$ is thus the (reduced) building of G under modern terminology.

In particular, when G is semisimple, $\mathscr{B}^e(G) = \mathscr{B}(G)$ is unique up to unique isomorphism and corresponds to the building in the sense of a valued root datum of G.

Another important point to consider is whether the building $\mathscr{B}^e(G)$ is bornological, i.e., whether the stabilizers of bounded subsets are bounded, or more precisely whether it satisfies the equivalent conditions given in [Rou77, Théorème 2.2.11]. It turns out that according to [Rou77, Corollaire 5.2.4.], an (extended) building is always bornological when K is henselian.

As indicated in [Rou77, Exemples 2.2.14.f)], the question of existence reduces to the almost-simple case and the case of tori. Now, the case of tori, when K is henselian, is treated in [Rou77, Proposition 2.4.8.2)]. Furthermore, [Rou77, Proposition 2.3.9.] tells us that the question reduces to the case where K is complete.

Note that the proof of [Rou77, Proposition 5.1.5.] shows exactly that the building of a separable Weil restriction of a group is naturally a building of that latter group. In particular, the almost-simple case reduces to the absolutely almost-simple case.

Finally, recall from Struyve ([MSV14] and [Str14, Main Corollary.]) that the conjecture [Tit86, 13. Conjecture] is verified. This therefore means that every absolutely almost simple algebraic group over an arbitrary discretely valued complete field admits a valued root datum compatible with the field's valuation. We thus deduce a building associated to this type of group according to [BT72], and hence a building in the sense of Rousseau by the previous discussion.

In conclusion, we have:

Proposition 1.1. *Let G be a reductive group over a henselian discretely valued field K. Then G admits an extended building, unique up to unique isomorphism. It is moreover bornological.*

We can in fact improve this result thanks to [Rou77, Proposition 2.3.5]:

Theorem 1.2. *Let G be a reductive group over a discretely valued field K. Suppose that G has the same relative rank over K and over its henselization (or alternatively its completion $\widehat{K}$). Then G admits an extended building, unique up to unique isomorphism. It is furthermore bornological.*

This building is canonically identified with that of $G_{\widehat{K}}$ and its apartments are the $\widehat{K}$-apartments corresponding to the $\widehat{K}$-maximal split tori defined and split over K.

Remark 1.3. However, it is still not known whether every reductive group over an arbitrary valued field admits a building, not necessarily bornological.

Let us now give some information about the vector part V_G and the apartments.

Note $D := G/D(G)$, the coradical torus of G. The vector part V_G is given by $X_*(R(G)) \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(D) \otimes_{\mathbb{Z}} \mathbb{R}$ and $G(K)$ acts by translation via $g \mapsto (\chi \mapsto -v(\chi(g)))$ from $G(K)$ to $\mathrm{Hom}(X^*(G), \mathbb{R}) = \mathrm{Hom}(X^*(D), \mathbb{R}) = X_*(D) \otimes_{\mathbb{Z}} \mathbb{R}$. It is viewed both as an affine space over itself and as a vector space. We denote by $G(K)^1$ the pointwise stabilizer in $G(K)$ of V_G (or equivalently, the stabilizer of a point of V_G). In other words, it is the kernel of the morphism $g \mapsto (\chi \mapsto -v(\chi(g)))$.

We immediately deduce that $G(K)^1$ is normal in $G(K)$ and that the quotient is isomorphic to $\mathbb{Z}^r$ where r is the rank of the group of K-characters of G (hence of D or of $R(G)$).

The definition of $G(K)^1$ is functorial in G and its construction is compatible with Galois extensions: for any Galois extension of valued fields L/K with Galois group Γ (the valuation on L being assumed Γ-invariant), the group $G(L)^1$ is Γ-invariant and $(G(L)^1)^\Gamma = G(L)^1 \cap G(K) = G(K)^1$. Furthermore, $G(K)^1$ can also be defined as the inverse image of $D(K)^1$ under $G(K) \mapsto D(K)$ (cf. [KP23, Lemma 2.6.16]). Consequently, since D is isogenous to $R(G)$, we have $G(K)^1 = G(K)$ if and only if $R(G)$ (or D) contains no K-split subtorus (this is in particular the case for semisimple groups).

More generally, as explained in [Rou77, 2.1.7-2.1.11], given a maximal split torus S of G, an apartment $\mathcal{A}(S)$ of $\mathscr{B}^e(G)$ associated to S is an affine space under $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with an action $\nu : N_G(S)(K) \to \mathrm{Aut}_{\mathrm{aff}}(\mathcal{A}(S))$ satisfying the conditions of definition [Rou77, 2.1.8.a)]. It is unique up to isomorphism. Moreover, every apartment of $\mathscr{B}^e(G)$ is of this form.

In particular, the restriction to $Z(K)$ (where $Z := Z_G(S)$) is an action by translation defined by $z \mapsto (\chi \mapsto -v(\chi(z)))$ mapping to:

$$\mathrm{Hom}(X^*(Z), \mathbb{R}) = \mathrm{Hom}(X^*(Z/D(Z)), \mathbb{R}) = X_*(Z/D(Z)) \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Furthermore, the kernel of ν is equal to $Z(K)^1$.

It turns out that an apartment $\mathcal{A}(S)$ can also be given by $\mathcal{A}(S') \times V_G$, where $\mathcal{A}(S')$ is an apartment of $\mathscr{B}(G')$ associated to the maximal split torus $S' := S \cap D(G)$. In this form, $\mathcal{A}(S)$ is called the *centered apartment* of G associated to S. Such an apartment has an affine structure, but also a vectorial structure given by V_G. It is unique up to unique isomorphism of centered apartments. In the following, the notation $\mathcal{A}(S)$ denotes the centered apartment of G associated to S. Such an apartment of $\mathscr{B}^e(G)$ of course respects the decomposition $\mathscr{B}^e(G) = \mathscr{B}(G) \times V_G$.

Let us finally look at facets and types. We now suppose that K is henselian.

Recall that a facet $\mathcal{F}$ of $\mathscr{B}(G)$ denotes the open geometric realization of the polysimplex it represents. Its topological closure $\overline{\mathcal{F}}$ in $\mathscr{B}(G)$ is exactly the (disjoint) union of its open sub-polysimplices (hence its sub-facets) according to [BT72, (2.5.10.)]. A facet $\mathcal{F}$ is said to be incident to a facet $\mathcal{F}'$ if we have the inclusion $\overline{\mathcal{F}} \subseteq \overline{\mathcal{F}'}$. We then write $\mathcal{F} \prec \mathcal{F}'$.

Furthermore, every facet is contained in the closure of a chamber (which by definition is a maximal facet for incidence, or even of maximal dimension).

The closure of a chamber is in natural correspondence with the Dynkin diagram of the affine root system (or equivalently of the *échelonnage*, cf. [BT72, I.4.]) of G. This graph is called in [KP23] the *relative affine Dynkin diagram*, and in [Tit79] the *relative local Dynkin diagram* or even the *K-residual graph* in [BT87].

The type of a facet is then defined as being its image under this correspondence. This image does not depend on the choice of the closure of a chamber in which the facet is included. Consequently, two facets of the same type in the same closure of a chamber are equal.

A type $\mathcal{T}$ is said to be incident to a type $\mathcal{T}'$ if we have the inclusion $\mathcal{T} \subset \mathcal{T}'$ (seen as sets of points of the Dynkin diagram). We then write $\mathcal{T} \prec \mathcal{T}'$. Take two facets $\mathcal{F}$ and $\mathcal{F}'$ respectively of type $\mathcal{T}$ and $\mathcal{T}'$. If $\mathcal{F} \prec \mathcal{F}'$, then we have $\mathcal{T} \prec \mathcal{T}'$.

As indicated previously, the existence of a building $\mathscr{B}(G)$ for G is equivalent to the existence of a valued root datum for G. The latter allows one to deduce a double Tits system equipped with an adapted morphism whose associated building is exactly $\mathscr{B}(G)$ according to [BT72, 6.5. Théorème.].

We thus deduce a type morphism, denoted ξ, from $G(K)$ to the group of automorphisms of the relative affine Dynkin diagram according to [BT72, 1.2.16]. Its image is denoted Ξ and its kernel is denoted $G(K)^c$. There is thus an isomorphism $G(K)/G(K)^c \cong \Xi$. We can also restrict this morphism to $G(K)^1$. Its image is denoted Ξ^1 and its kernel is denoted $G(K)^b := G(K)^c \cap G(K)^1$. We deduce an isomorphism $G(K)^1/G(K)^b \cong \Xi^1$.

The interested reader can analyze the case of GL_n for examples. Indeed, there is a way to interpret the building and the types in this case through lattice chains (cf. [BT84b]).

The type morphism measures how the type of a facet changes under the action of an element of $G(K)$. In other words, a facet $\mathcal{F}$ of type $\mathcal{T}$ is such that $g \cdot \mathcal{F}$ is of type $\xi(g) \cdot \mathcal{T}$ for all $g \in G(K)$. Denote by $\Xi_{\mathcal{T}}$ the subgroup of $w \in \Xi$ such that $w \cdot \mathcal{T} = \mathcal{T}$. One can also verify that $G(K)_{\mathcal{F}}$ surjects onto $\Xi_{\mathcal{T}}$ and that its kernel is $G(K)^c_{\mathcal{F}}$, whence an isomorphism $G(K)_{\mathcal{F}}/G(K)^c_{\mathcal{F}} \cong \Xi_{\mathcal{T}}$. We also define all this analogously for $G(K)^1$ and Ξ^1. Cf. [BT72, 1.2.13 - 1.2.20] and [BT72, 2.7.].

The type of a facet can also be seen as the orbit of this facet under a group acting transitively on the chambers while preserving the types. This is notably the case for $G(K)^c$, $G(K)^b$ and even $G(K)^+$ (cf. Lemma 2.4).

Remark 1.4. We do not necessarily have equality between Ξ^1 and Ξ. To simplify, consider the case where κ is perfect. In order to realize the counterexample efficiently, we use the Kottwitz morphism (cf. [KP23, 11.5]). It is a morphism $G(K) \to \pi_1(G)_I$ (functorial in G) whose kernel is $G(K)^0$, the subgroup generated by the parahoric subgroups, also called the residually neutral component of $G(K)$, (cf. [KP23, Proposition 11.5.4]), and whose inverse image of the torsion elements is $G(K)^1$ (cf. [KP23, Lemma 11.5.2]). The $\mathrm{Gal}(K^s/K)$-module $\pi_1(G)$ is the algebraic fundamental group, defined in [KP23, 11.3]. Consequently, $\pi_1(G)_I$ denotes the Γ^{unr}-module obtained by taking coinvariants.

Consider the case where $G = \mathrm{GL}_2$. An immediate calculation shows that $\pi_1(G)_I = \mathbb{Z}$ and is thus torsion-free. We deduce that $G(K)^0 = G(K)^1$. Since $G(K)^0$ acts trivially on the types (cf. [BT84a, 5.2.12.(i) Proposition.]), the same holds for $G(K)^1$. Hence $\Xi^1 = 0$.

On the other hand, $G(K)$ does not act trivially on the types. Indeed, the following two parahoric subgroups are associated to points of different types (cf. [KP23, Chapter 3.1]):

$$\begin{pmatrix} R & R \\ R & R \end{pmatrix} \text{ and } \begin{pmatrix} R & tR \\ t^{-1}R & R \end{pmatrix}$$

whereas they are conjugate by the matrix $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in G(K)$.

To finish, let us prove that Ξ is finite abelian. For this, we need a few results:

Lemma 1.5. *Let Z be a Levi subgroup of G. We have $D(G(K)) = G(K)^+ D(Z(K))$.*

Proof. Since $G(K)^+$ is perfect ([BT73, 6.4. Corollaire.]), we have $G(K)^+ \subset D(G(K))$. We deduce the inclusion $G(K)^+ D(Z(K)) \subset D(G(K))$. Conversely, since $G(K) = G(K)^+ Z(K)$ ([BT73, 6.11.(i) Proposition.]), we can take an element $d \in D(G(K))$ of the form $g_1 z_1 g_2 z_2 (g_1 z_1)^{-1} (g_2 z_2)^{-1}$, with g_1, g_2 in $G(K)^+$ (resp. z_1, z_2 in $Z(K)$). Since $G(K)^+$ is normal in $G(K)$, we can consider the quotient $G(K)/G(K)^+$ and see that the image of d in $G(K)/G(K)^+$ is equal to that of $z_1 z_2 z_1^{-1} z_2^{-1}$, hence $d \in G(K)^+ D(Z(K))$. Since $D(G(K))$ is generated by this type of elements, we deduce the inclusion $D(G(K)) \subset G(K)^+ D(Z(K))$ as desired. □

Proposition 1.6. *$G(K)^b$ is a normal subgroup of $G(K)$ whose quotient is abelian of finite type and whose number of generators is bounded by the relative rank of G.*

Proof. Take Z a minimal Levi subgroup of G. By Lemma 1.5, we have $D(G(K)) = G(K)^+ D(Z(K))$. Now, on one hand $G(K)^b$ contains $G(K)^+$ thanks to Lemma 2.4, and on the other hand $D(Z(K)) \subset D(Z)(K) \subset Z(K)^1 \subset G(K)^b$. The subgroup $G(K)^b$ therefore contains $D(G(K))$ and is thus normal with abelian quotient.

Furthermore, there is a surjection $Z(K)/Z(K)^1 \to G(K)/G(K)^+ Z(K)^1$ such that $G(K)/G(K)^+ Z(K)^1$ is abelian of finite type since $Z(K)/Z(K)^1$ is. Its number of generators is thus bounded by that of $Z(K)/Z(K)^1$, which is simply the relative rank of G. We then use Lemma 2.8 which tells us that $G(K)^b = G(K)^+ Z(K)^1$ to conclude. □

We then deduce what we wanted:

Proposition 1.7. *The group* $G(K)/G(K)^c \cong \Xi$ *(and hence* $G(K)^1/G(K)^b \cong \Xi^1$*) is finite abelian.*

Proof. Note that we have the isomorphism $G(K)/G(K)^c \cong \Xi$. Furthermore, since $G(K)^b \subset G(K)^c$, Corollary 1.6 gives that Ξ is abelian. On the other hand, the fact that there are finitely many ways to permute a finite number of vertices implies that Ξ is finite. Consequently, the same holds for $G(K)^1/G(K)^b \cong \Xi^1 \subset \Xi$. □

2. Global subgroups and new notions

We consider a possibly infinite unramified Galois extension L/K, with Galois group Γ. We can thus assume the inclusion $L \subset K^{\mathrm{unr}}$.

In what follows, we consider the following subgroups of $G(K)$:

Definition 2.1. *Let* H *be an open subgroup of* $G(K)$*.*

- *We say that* H *is a* ***global subgroup*** *of* $G(K)$ *if* $G(K)^+ \subset H$*.*
- *Furthermore, we say that* H*, assumed to be global, is* ***L-conformal*** *(or just* ***conformal*** *if* $L = K$*) if* H *preserves the* L*-types, or equivalently if* $H \subset G(L)^c$*. Also,* H *is said to be* ***very conformal*** *if* H *is* K^{unr}*-conformal. We also say that* H *is* ***uniform*** *if* H *fixes a point of* V_G*, or equivalently, if* $H \subset G(K)^1$*.*
- *We say that* H *is* ***L-good*** *(or just* ***good*** *if* $L = K$*) if* H *is uniform and* L*-conformal, or equivalently, if* $H \subset G(L)^b$*. Also,* H *is said to be* ***very good*** *if* H *is* K^{unr}*-good.*
- *We also define* H^1*,* H^b*,* H^{tb}*,* H^c*, and* H^{tc} *as the subgroups obtained by taking the intersection of* H *with* $G(K)^1$*,* $G(K)^b$*,* $G(K^{\mathrm{unr}})^b$*,* $G(K)^c$*, and* $G(K^{\mathrm{unr}})^c$ *respectively.*
- *For any subset* Ω *of* $\mathscr{B}(G)$*, we denote by* H_Ω *(resp.* H^{f}_Ω*) the* ***stabilizer*** *(resp.* ***pointwise stabilizer****) of* Ω *under* H*. If we take several subsets* $(\Omega_i)_{i\in I}$*, we denote* $H_{(\Omega_i)_{i\in I}} := \bigcap_{i\in I} H_{\Omega_i}$*. This latter subgroup is called the* ***multistabilizer*** *of* $(\Omega_i)_{i\in I}$ *under* H*.*

Observe that

$$H^b = (H^1)^c = (H^c)^1 = H^1 \cap H^c \text{ and } H^{tb} = (H^1)^{tc} = (H^{tc})^1 = H^1 \cap H^{tc}.$$

Indeed, it suffices to verify the result when $H = G(K)$. In this case, it follows from the definitions.

As we will see later in Corollary 3.6, a global subgroup that is L-conformal (resp. L-good) is conformal (resp. good), but the converse is false.

Remarks 2.2.

(1) We do not take the same convention as Bruhat and Tits in [BT84a], and as Prasad in [KP23]. For Prasad, $G(K)^1_\Omega$ denotes the pointwise stabilizer of Ω under the action of $G(K)^1$, while $G(K)^\dagger_\Omega$ denotes the stabilizer of Ω under the action of $G(K)^1$. Bruhat and Tits take an analogous convention.

(2) The convoluted notations "1" and "b" were already present in the literature. We have therefore chosen to give them names so that they are easier to remember ("1" is associated with the "uniform" property and "b" is associated with the "good" property, which is translated "bon" in French). We have also added the notion of "conformity" (associated with "c"), which, although convenient, was not present in the literature. Note also that "t" in "tb" and "tc" stands for "très" in French, which means "very".

(3) One could have defined a notion of "L-uniform" (or "very uniform") property. However, $(G(K^{\mathrm{unr}})^1)^{\Gamma^{\mathrm{unr}}} = G(K)^1$ according to the end of Section 1. This is therefore equivalent to the notion of "uniform" property.

Remark 2.3. The pointwise stabilizer of any subset Ω of $\mathscr{B}(G)$ under the action of a global subgroup H is the multistabilizer of $(x)_{x\in\Omega}$ under H. If moreover Ω is a finite union of facets, it is also the multistabilizer under H of the family given by the vertices incident to Ω (in finite number).

The main result concerning global subgroups is the following:

Lemma 2.4. *A global subgroup acts transitively on the pairs $(\mathcal{A},\mathcal{C})$ of apartments and chambers contained in that apartment. Moreover, $G(K)^+$ is a good global subgroup (i.e. $G(K)^+ \subset G(K)^b$).*

Proof. It suffices to prove the result for $G(K)^+$. One can also reduce to the semisimple adjoint case. Indeed, the action on the building factors through $Z(G)(K)$ and [BT73, Corollary 6.3.] implies that $G(K)^+ \to G^{\mathrm{ad}}(K)^+$ is surjective.

We know that there exists a valued root datum associated to $G(K)$. The parahoric subgroups (in the sense of [BT72], i.e., the stabilizers of facets under the action of $G(K)^c$) are described in [BT72, (7.1.1.)] and thus generate $G(K)^+Z(K)^1$ ("H" equals $Z(K)^1$, with Z a minimal Levi subgroup, since G is semisimple. Also, the affine root groups generate $G(K)^+$). But by definition, this group also equals $G(K)^c$ (and even $G(K)^b$ since G is assumed semisimple). It therefore acts transitively on the apartment-chamber pairs that interest us (cf. [BT72, (2.2.6)]). Since $G(K)^+ \subset G(K)^+Z(K)^1 = G(K)^b$, we deduce in particular that $G(K)^+$ is good.

Let $\mathcal{A}$ and $\mathcal{A}'$ be apartments and $\mathcal{C} \subset \mathcal{A}$ and $\mathcal{C}' \subset \mathcal{A}'$ be chambers of these apartments. Let $g \in G(K)^b$ such that $g \cdot (\mathcal{A},\mathcal{C}) = (\mathcal{A}',\mathcal{C}')$. Let Z relative to $\mathcal{A}$ and write the decomposition $g = g^+z$ given by $G(K)^b = G(K)^+Z(K)^1$. Since $Z(K)^1$ fixes $\mathcal{A}$ (and hence $\mathcal{C}$), we have:

$$g^+ \cdot (\mathcal{A},\mathcal{C}) = g^+ \cdot (z \cdot (\mathcal{A},\mathcal{C})) = g \cdot (\mathcal{A},\mathcal{C}) = (\mathcal{A}',\mathcal{C}').$$

Hence the result. □

We then deduce:

Proposition 2.5. *Choose an apartment $\mathcal{A}$ and a chamber $\mathcal{C} \subset \mathcal{A}$. Any conformal global subgroup K of $G(K)$ defines a saturated BN-pair by setting $B = K_{\mathcal{C}}$ and $N = K_{\mathcal{A}}$, the stabilizers of $\mathcal{C}$ and $\mathcal{A}$ under the action of K. The associated building is exactly $\mathscr{B}(G)$ and its Weyl group is the affine Weyl group of the building.*

Up to conjugation by K, this BN-pair does not depend on the choice of the pair $(\mathcal{A},\mathcal{C})$.

Proof. According to the previous lemma, we fall into the framework of application of [Tit74, 3.11. Proposition], which gives us the result. □

Remarks 2.6.

(1) The terminology *global subgroup* is actually inspired by Proposition 2.5: a global subgroup is large enough to determine a sufficiently rich set of "local" subgroups given by the stabilizers of bounded subsets of the building $\mathscr{B}(G)$.

(2) The notion of a global subgroup of $G(K^{\mathrm{unr}})$ invariant under Γ^{unr} encompasses that of the subgroups considered in [BT87, 3.5.] in the case where the residue field κ is perfect. Indeed, Tits rather imposes to contain $G(K^{\mathrm{unr}})^0$, the subgroup generated by the parahoric subgroups over K^{unr}, also called the residually neutral component of $G_{K^{\mathrm{unr}}}$, instead of $G(K^{\mathrm{unr}})^+$, and $G(K^{\mathrm{unr}})^+ \subset G(K^{\mathrm{unr}})^0$ (cf. 3rd paragraph of [BT84a, 5.2.11.]).

From this, we deduce some elementary results concerning facets and global subgroups:

Proposition 2.7. *Let H be a global subgroup of $G(K)$ and two facets $\mathcal{F}$ and $\mathcal{F}'$ in $\mathscr{B}(G)$. We have:*

(1) The subgroup $H_{\mathcal{F}}$ acts transitively on the apartments containing $\mathcal{F}$.
(2) If $H_{\mathcal{F}'} \subset H_{\mathcal{F}}$, then $\overline{\mathcal{F}} \subset \overline{\mathcal{F}'}$. The converse is true if H is moreover conformal.
(3) We have $H_{\mathcal{F}} = H_{\mathcal{F}}$ if and only if $\mathcal{F} = \mathcal{F}'$.

Proof.

(1) Since H is global, it suffices to show the result for $H = G(K)^+$. This is [KP23, Proposition 1.5.13.(1)] applied to the Tits system of $G(K)^+$ (cf. Proposition 2.5).

(2) and (3) Observe that $H_{\mathcal{F}'} \subset H_{\mathcal{F}}$ implies $G(K)^+_{\mathcal{F}'} \subset G(K)^+_{\mathcal{F}}$. Since $G(K)^+$ induces a Tits system whose building is exactly $\mathscr{B}(G)$ (cf. Proposition 2.5), there is a correspondence between the parabolic subgroups of the Tits system for inclusion (which are the stabilizers of facets) and the facets of the building for incidence. Hence $\overline{\mathcal{F}} \subset \overline{\mathcal{F}'}$ if and only if $G(K)^+_{\mathcal{F}'} \subset G(K)^+_{\mathcal{F}}$. The same reasoning also applies to an arbitrary conformal subgroup.

□

We also have decomposition results:

Lemma 2.8. *Let H be a global subgroup of $G(K)$. We have $H = G(K)^+ H_{(\mathcal{A},\mathcal{C})}$, where $\mathcal{A}$ is an apartment of $\mathscr{B}(G)$ and $\mathcal{C}$ is a chamber in $\mathcal{A}$. Furthermore, $H^c_{(\mathcal{A},\mathcal{C})} = H^{\mathrm{f}}_{\mathcal{A}}$ and $H^b_{(\mathcal{A},\mathcal{C})} = H^{1,\mathrm{f}}_{\mathcal{A}}$. In particular, we have $G(K)^b = G(K)^+ Z(K)^1$ for Z a minimal Levi subgroup of G.*

Proof. The reverse inclusion is obvious. Let us study the direct inclusion.

Let $h \in H$. By transitivity of $G(K)^+$ on apartment-chamber pairs, there exists $g \in G(K)^+$ such that $g \cdot \mathcal{C} = h \cdot \mathcal{C}$ and $g \cdot \mathcal{A} = h \cdot \mathcal{A}$. So $h' := g^{-1}h \in H_{(\mathcal{A},\mathcal{C})}$. So $h = gh'$.

Observe that $H^b_{(\mathcal{A},\mathcal{C})} = H^{b,\mathrm{f}}_{\mathcal{A}}$ since H^b fixes the types, hence $\mathcal{C}$, and hence all of $\mathcal{A}$ since the vertices of $\mathcal{C}$ determine an affine basis of $\mathcal{A}$. Furthermore, $H^{b,\mathrm{f}}_{\mathcal{A}} = H^{1,\mathrm{f}}_{\mathcal{A}}$ because $H^{1,\mathrm{f}}_{\mathcal{A}}$ fixes $\mathcal{C}$ and thus acts trivially on the types. The same reasoning proves that $H^c_{(\mathcal{A},\mathcal{C})} = H^{\mathrm{f}}_{\mathcal{A}}$.

We deduce therefore from Section 1 that $G(K)^b_{(\mathcal{A},\mathcal{C})}$ is exactly the pointwise stabilizer of the extended apartment $\mathcal{A} \times V_G \subset \mathscr{B}^e(G)$, i.e., $Z(K)^1$, where Z is the Levi subgroup associated to $\mathcal{A}$. Hence the last decomposition. □

This last result allows us to deduce a more precise Bruhat decomposition:

Proposition 2.9 (Bruhat decomposition)**.** *Take $\mathcal{A}$ an apartment of $\mathscr{B}(G)$ and $\mathcal{C}$ a chamber of $\mathcal{A}$. Let H be a global subgroup of $G(K)$. We have: $H = G(K)^+_{\mathcal{C}}\, H_{\mathcal{A}}\, G(K)^+_{\mathcal{C}}$.*
In particular, $G(K) = G(K)^+_{\mathcal{C}}\, N(K)\, G(K)^+_{\mathcal{C}}$, where N is the normalizer of the maximal split torus of G associated to $\mathcal{A}$.

Proof. According to [KP23, Proposition 1.4.5.(1)], there exists a Bruhat decomposition for $G(K)^+$ (since the latter determines a Tits system, cf. Proposition 2.5). Consequently, $G(K)^+ = G(K)^+_{\mathcal{C}}\, G(K)^+_{\mathcal{A}}\, G(K)^+_{\mathcal{C}}$.

Observe that $H_{\mathcal{C}} = H_{(\mathcal{A},\mathcal{C})}\, G(K)^+_{\mathcal{C}} = G(K)^+_{\mathcal{C}}\, H_{(\mathcal{A},\mathcal{C})}$ since $G(K)^+_{\mathcal{C}}$ acts transitively on the apartments containing $\mathcal{C}$ according to Lemma 2.4. Then, note that $H_{\mathcal{A}} = H_{(\mathcal{A},\mathcal{C})}\, G(K)^+_{\mathcal{A}}$ since $G(K)^+_{\mathcal{A}}$ acts transitively on the chambers of $\mathcal{A}$ according to also Lemma 2.4. Since by Lemma 2.8, $H = H_{(\mathcal{A},\mathcal{C})}\, G(K)^+$, we therefore finally have:

$$H = H_{(\mathcal{A},\mathcal{C})}\,(G(K)^+_{\mathcal{C}}\, G(K)^+_{\mathcal{A}}\, G(K)^+_{\mathcal{C}}) = G(K)^+_{\mathcal{C}}\left(H_{(\mathcal{A},\mathcal{C})}\, G(K)^+_{\mathcal{A}}\right) G(K)^+_{\mathcal{C}} = G(K)^+_{\mathcal{C}}\, H_{\mathcal{A}}\, G(K)^+_{\mathcal{C}}.$$

□

Let us now prove some compatibility results of global subgroups with unramified extensions. Before that, we need to show the following elementary lemma:

Lemma 2.10. *Let G' be a reductive group over a field K' and L'/K', a Galois extension with Galois group Γ'. The subgroup $G'(L')^+$ is Γ'-invariant and we have:*

$$G'(K')^+ \subset (G'(L')^+)^{\Gamma'} = G'(L')^+ \cap G'(K').$$

Proof. The first assertion comes from [BT73, 6.1.]. Indeed, $\sigma \in \Gamma'$ defines an isomorphism $\sigma : G'_{L'} \to G'_{L'}$, and thus sends $G'(L')^+$ to $G(L')^+$. Hence the Γ'-invariance.

On the other hand, [BT73, 6.1.] also gives $G'(K')^+ \subset G'(L')^+$. We therefore have the result using the Γ'-invariance. □

This allows us to obtain:

Proposition 2.11. *We have:*

1. *(1) Every global subgroup H admits a largest global subgroup that is respectively uniform, good, L-good, conformal, L-conformal given respectively by H^1, H^b, $H \cap G(L)^b\, H^c$, $H \cap G(L)^c$ (and thus in particular a largest global subgroup that is respectively very good and very conformal given by H^{tb} and H^{tc}).*
2. *(2) If $\widetilde{H}$ is a global subgroup respectively uniform, good, conformal, Γ-invariant of $G(L)$, then $\widetilde{H}^\Gamma$ is a global subgroup respectively uniform, L-good, L-conformal of $G(K)$.*
3. *(3) If $\widetilde{H}$ is a global Γ-invariant subgroup of $G(L)$, then $\widetilde{H}^1$, $\widetilde{H}^b$ and $\widetilde{H}^c$ are also Γ-invariant.*

Proof.

(1) For the first point, it suffices to show that the subgroups in question are global. We have according to Lemmas 2.4, 2.10 and the first point of Corollary 3.6:

$$\begin{array}{ccccccccc}
 & & & & G(L)^c \cap H & \xhookrightarrow[3.6.(1)]{} & H^c & & \\
 & & & & \uparrow & & \uparrow & & \\
G(K)^+ & \xhookrightarrow[2.10]{} & G(L)^+ \cap H & \xhookrightarrow[2.4]{} & G(L)^b \cap H & \xhookrightarrow[3.6.(1)]{} & H^b & \longrightarrow & H^1.
\end{array}$$

(2) Since $\widetilde{H}^\Gamma \subset \widetilde{H}$, it suffices only to show that $G(K)^+ \subset \widetilde{H}^\Gamma$. But we have $G(K)^+ \underset{2.10}{\subset} (G(L)^+)^\Gamma \subset \widetilde{H}^\Gamma$. Hence the result.

(3) The third point reduces to the case where $\widetilde{H} = G(L)$. For $G(L)^1$, this has already been done at the end of Section 1. For the rest, let us use Lemma 2.8. Given an apartment $\mathcal{A} \subset \mathscr{B}(G_L)$ and an L-chamber $\mathcal{C} \subset \mathcal{A}$, we have: $G(L)^* = G(L)^+ G(L)^*_{(\mathcal{A},\mathcal{C})}$ for $* \in \{b, c\}$. Now, $G(L)^+$ is Γ-invariant according to Lemma 2.10. It therefore suffices to show that the orbit under Galois of $G(L)^*_{(\mathcal{A},\mathcal{C})}$ is in $G(L)^*$ for all $* \in \{b, c\}$.

Now, Lemma 2.8 also shows that $G(L)^b_{(\mathcal{A},\mathcal{C})} = G(L)^{1,\mathrm{f}}_{\mathcal{A}}$ and that $G(L)^c_{(\mathcal{A},\mathcal{C})} = G(L)^{\mathrm{f}}_{\mathcal{A}}$. But for $\sigma \in \Gamma$, $\sigma(G(L)^{1,\mathrm{f}}_{\mathcal{A}}) = G(L)^{1,\mathrm{f}}_{\sigma(\mathcal{A})} \subset G(L)^b$, and similarly $\sigma(G(L)^{\mathrm{f}}_{\mathcal{A}}) = G(L)^{\mathrm{f}}_{\sigma(\mathcal{A})} \subset G(L)^c$. This therefore gives the result as desired.

□

Let us introduce some additional notations that will be useful later:

Definition 2.12. *Let H be a global subgroup of $G(K)$ and $\mathcal{F}$ a facet of type $\mathcal{T}$. Denote:*

- *Ξ_H, the image of H by ξ (which thus induces $H/H^c \cong \Xi_H$).*
- *$\Xi_{H,\mathcal{T}}$, the image of $H_{\mathcal{F}}$ by ξ (which thus induces $H_{\mathcal{F}}/H^c_{\mathcal{F}} \cong \Xi_{H,\mathcal{T}}$). It is also the set $\{w \in \Xi_H \mid w \cdot \mathcal{T} = \mathcal{T}\}$ since $H^c \subset H$ acts transitively and conformally on the chambers (cf. definition below).*
- *$\mathrm{Orb}(\mathcal{F})_H$, the orbit of $\mathcal{F}$ by H.*
- *$\mathrm{Orb}(\mathcal{T})_{\Xi_H}$ (or even $\mathrm{Orb}(\mathcal{T})_H$), the orbit of $\mathcal{T}$ by Ξ_H.*

Let us now generalize the notion of facet and the associated objects. This generalization is inexpensive for what follows and adds additional richness to our general problem.

Definition 2.13. *Let us call a* ***multifacet*** *any union of facets included in the same closure of a chamber. For such an object, we can define the* ***type*** *(or* ***multitype****, to emphasize that this is relative to a multifacet) as the set of the types of the different facets composing it. A facet is then in particular a multifacet, and his type can naturally be identified as his multitype.*

We say that a multifacet is ***strongly invariant*** *under the action of a group if each of the facets composing it is invariant (it is therefore not sufficient that the multifacet is invariant as a geometric object). We define the same notion for the multitypes.*

We also say that a group acting on $\mathscr{B}(G)$ by polysimplicial automorphisms acts ***conformally*** *on a multifacet $\mathcal{F}$ if it sends it to multifacets of the same type.*

If $\mathcal{F}$ is a multifacet with decomposition into facets $\bigsqcup_{i\in I} \mathcal{F}_i$, then for any global subgroup H of $G(K)$, we denote $H_{(\mathcal{F})} := H_{(\mathcal{F}_i)_{i\in I}} := \bigcap_{i\in I} H_{\mathcal{F}_i}$. This group is called the ***multistabilizer subgroup of the multifacet $\mathcal{F}$ relative to H****.*

More generally, we use the notation $(\mathcal{F})$ to specify that we are considering $\mathcal{F}$ as a multifacet and not as a subset of the building (we do the same for the multitypes).

Remark 2.14. We see therefore that the use of multifacets gives rise to a larger family of subgroups than just the stabilizers of facets. In particular, this gives access to the pointwise stabilizers of facets, by taking for example the multifacet associated to the vertices incident to a facet.

Note however that, in the conformal case, stabilizing a facet and preserving its type implies in fact pointwise stabilizing it. In this case, the multistabilizer of a multifacet whose type is preserved is none other than the pointwise stabilizer of the union of the facets composing it: the notion of multistabilizer is only interesting if one considers non-conformal global subgroups.

Remark 2.15. As for facets, the topological closure of a multifacet $\mathcal{F} := \bigsqcup_i \mathcal{F}_i$ is exactly the union of the subfacets of the $\mathcal{F}_i$. Indeed, this is a consequence of the fact that $\overline{\mathcal{F}} = \overline{\bigsqcup_i \mathcal{F}_i} = \bigcup_i \overline{\mathcal{F}}_i$. We also define the incidence relation as an inclusion at the level of closures. We also say that a multitype $\mathcal{T} = \{\mathcal{T}_1, ..., \mathcal{T}_n\}$ is incident to a multitype $\mathcal{T}' = \{\mathcal{T}'_1, ..., \mathcal{T}'_m\}$, which we denote by $\mathcal{T} \prec \mathcal{T}'$, if for every $\mathcal{T}_i$, there exists $\mathcal{T}'_j$ such that $\mathcal{T}_i \prec \mathcal{T}'_j$ (in the sense of the usual types). One can check that, as for the case of facets, taking the type of a multifacet is compatible with the incidence relation.

3. Some complements on unramified descent

Recall that, according to Rousseau in [Rou77, Proposition 2.4.6], the Galois group Γ acts by polysimplicial automorphisms on $\mathscr{B}(G_L)$ in a manner compatible with the action of $G(L)$ (i.e., $\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$ for any $\sigma \in \Gamma, g \in G(L), x \in \mathscr{B}(G_L)$). According to the tamely ramified descent theorem ([Rou77, Proposition 5.1.1.]), the set of fixed points is uniquely identified with $\mathscr{B}(G)$. One can also choose a Γ-invariant metric on $\mathscr{B}(G_L)$ (cf. [Rou77, §2.2]) such that Γ acts by isometries ([Rou77, Remark 2.4.7.(f)]), and hence such that $\mathscr{B}(G) \subset \mathscr{B}(G_L)$ is an isometric embedding. Under this choice, $\mathscr{B}(G)$ is also a closed convex subset of $\mathscr{B}(G_L)$. Indeed, Γ acts continuously on $\mathscr{B}(G_L)$ (since it acts by isometries), hence the closedness. The convexity then follows from the uniqueness of the geodesic connecting two points (since Γ acts by isometries).

In particular, since Γ acts by polysimplicial automorphisms, it sends facets to facets. Furthermore, this action on facets factors through an action on the types (and even on the relative affine Dynkin diagram). It suffices to see that, given an L-facet $\mathcal{F}$ and $g \in G(L)^c$, the facets $\sigma(\mathcal{F})$ and $\sigma(g \cdot \mathcal{F})$ have the same type. Since $\sigma(g \cdot \mathcal{F}) = \sigma(g) \cdot \sigma(\mathcal{F})$ and $G(L)^c$ is Γ-invariant (cf. point (3) of Proposition 2.11), we get the result. This observation extends of course to multifacets.

Let us then introduce the following definition (already present in [KP23, 9.2.4]):

Definition 3.1. *A* $\mathbf{\Gamma}$***-multifacet*** *is an L-multifacet that is strongly Γ-invariant. In particular, a* $\mathbf{\Gamma}$***-facet*** *is an L-facet that is Γ-invariant.*

One also defines a $\mathbf{\Gamma}$***-vertex*** *(resp. a* $\mathbf{\Gamma}$***-chamber****) as a Γ-facet that is minimal (resp. maximal) among Γ-facets.*

Recall also that the unramified descent theorem was originally proved by Bruhat and Tits (in [BT84a, 5.]) and generalized by Prasad (in [Pra20, Theorem 3.8.]). This theorem provides a more precise dictionary than the tamely ramified descent theorem (notably a strong compatibility at the level of facets and parahoric subgroups. For example, parahoric group schemes commutes with unramified base change, but not with tame ones).

We propose to develop some complements to this theorem. Before that, we need to prove the following lemma:

Lemma 3.2. *Let $\mathscr{F}$ be a Γ-multifacet of $\mathscr{B}(G_L)$.*
We have the equality: $\overline{\mathscr{F} \cap \mathscr{B}(G)} = \overline{\mathscr{F}} \cap \mathscr{B}(G)$.

Proof. Let us first prove the case where $\mathscr{F}$ is a facet.
Observe first that $\mathscr{F} \cap \mathscr{B}(G) \subset \overline{\mathscr{F}} \cap \mathscr{B}(G)$. Consequently, $\overline{\mathscr{F} \cap \mathscr{B}(G)} \subset \overline{\mathscr{F}} \cap \mathscr{B}(G)$.
Let us show the reverse inclusion. Take $x \in \mathscr{F} \cap \mathscr{B}(G)$ and $y \in \overline{\mathscr{F}} \cap \mathscr{B}(G)$. Since $\mathscr{F}$ is convex, the geodesic $[x, y] \subset \overline{\mathscr{F}}$ is such that the half-open geodesic $[x, y[$ is included in $\mathscr{F}$ (cf. [Bou81, II.§2.6. Proposition 16.]).

Furthermore, since $\mathscr{B}(G)$ is convex and x and y are in $\mathscr{B}(G)$, the geodesic $[x, y]$ is in fact included in $\mathscr{B}(G)$. Consequently, $[x, y[$ is included in $\mathscr{F} \cap \mathscr{B}(G)$. This implies that y is in $\overline{\mathscr{F} \cap \mathscr{B}(G)}$. Hence the reverse inclusion.

Let us now prove the general case. Write $\mathscr{F} = \bigsqcup_i \mathscr{F}_i$ as the decomposition of $\mathscr{F}$ into facets. We have:

$$\begin{aligned}\overline{\mathscr{F} \cap \mathscr{B}(G)} &= \overline{(\bigcup_i \mathscr{F}_i) \cap \mathscr{B}(G)} \\ &= \overline{\bigcup_i (\mathscr{F}_i \cap \mathscr{B}(G))} = \bigcup_i \overline{\mathscr{F}_i \cap \mathscr{B}(G)} \underset{\substack{\text{case of}\\ \text{facets}}}{=} \bigcup_i (\overline{\mathscr{F}_i} \cap \mathscr{B}(G)) = (\bigcup_i \overline{\mathscr{F}_i}) \cap \mathscr{B}(G) \\ &= \overline{\mathscr{F}} \cap \mathscr{B}(G).\end{aligned}$$

Hence the result. □

We thus have:

Proposition 3.3. *We have the following $G(K)$-equivariant correspondence, increasing for the inclusion and for the incidence:*

$$\left\{\begin{matrix}\Gamma\text{-}\mathit{multifacets} \\ \mathit{of}\ \mathscr{B}(G_L)\end{matrix}\right\} \cong \left\{\begin{matrix}K\text{-}\mathit{multifacets} \\ \mathit{of}\ \mathscr{B}(G)\end{matrix}\right\}$$

$$\bigsqcup_i \mathscr{F}_i \overset{\alpha}{\mapsto} \bigsqcup_i \mathscr{F}_i^{\Gamma} = \bigsqcup_i \mathscr{F}_i \cap \mathscr{B}(G)$$

$$\bigsqcup_i \widetilde{\mathcal{F}}_i \overset{\beta}{\leftarrow\!\shortmid} \bigsqcup_i \mathcal{F}_i$$

where $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ associates to a K-facet the unique L-facet containing its barycenter.

In particular, under this correspondence, a Γ-vertex corresponds to a K-vertex and a Γ-chamber corresponds to a K-chamber.

Proof. To lighten the proof, we only write the case of facets. It suffices to reason facet by facet to get the case of multifacets.

Remark [Rou, 5.1.5.1 Remark (c)] explicitly states the well-definedness and even the surjectivity of the direct arrow at the level of facets. Conversely, for a K-facet $\mathcal{F}$, the L-facet $\widetilde{\mathcal{F}}$ is Γ-invariant because it is the unique L-facet containing the barycenter of $\mathcal{F}$, which is itself fixed by Γ. Hence the well-definedness of the inverse arrow.

Take a Γ-facet $\mathscr{F}$. Observe then that $\widetilde{\mathscr{F}^{\Gamma}} = \mathscr{F}$ because both K-facets contain the barycenter of $\mathscr{F}^{\Gamma}$. Conversely, $(\widetilde{\mathscr{F}})^{\Gamma} = \mathscr{F}$ because both L-facets contain the barycenter of $\mathscr{F}$.

Note that both sets are $G(K)$-stable. The direct arrow is obviously $G(K)$-equivariant since every element of $G(K)$ is fixed by Γ and since the action of Γ on $\mathscr{B}(G_L)$ is compatible with the action of $G(L)$. The inverse arrow is therefore also equivariant.

The increasing property for the inclusion is of course obvious in both directions.

Let's look at the incidence for the direct arrow. According to Lemma 3.2, we have $\overline{\mathscr{F}} \cap \mathscr{B}(G) = \overline{\mathscr{F} \cap \mathscr{B}(G)}$. Consequently, if a Γ-invariant facet $\mathscr{F}'$ is in $\overline{\mathscr{F}}$, then $\mathscr{F}' \cap \mathscr{B}(G) \subset \overline{\mathscr{F}} \cap \mathscr{B}(G) = \overline{\mathscr{F} \cap \mathscr{B}(G)}$. In other words, ${\mathscr{F}'}^{\Gamma}$ is incident to $\mathscr{F}^{\Gamma}$. This is what we wanted.

For the inverse arrow, if $\overline{\mathcal{F}} \subset \overline{\mathcal{F}'}$, then the barycenter of $\mathcal{F}$ is contained in $\overline{\mathcal{F}'} \subset \overline{\widetilde{\mathcal{F}'}}$. Therefore $\widetilde{\mathcal{F}} \subset \overline{\widetilde{\mathcal{F}'}}$ since $\widetilde{\mathcal{F}}$ is the unique L-facet containing the barycenter. □

Recall the following definitions:

Definition 3.4.

1. *We say that G is* ***residually split*** *if G and $G_{K^{\mathrm{unr}}}$ have the same relative semisimple rank.*
2. *We say that G is* ***residually quasi-split*** *if there exists a Γ^{unr}-invariant K^{unr}-chamber in $\mathscr{B}(G_{K^{\mathrm{unr}}})$ (or equivalently if there exists a Γ^{unr}-chamber which is a K^{unr}-chamber).*

We also have a correspondence at the level of the types:

Proposition 3.5.

1. *The correspondence from Proposition 3.3 preserves the types.*
   In particular, the orbit under the action of a conformal global subgroup of $G(K)$ of a Γ-multifacet (resp. a K-multifacet) describes exactly the Γ-multifacets of the same L-type (resp. the K-multifacets of the same K-type).
2. *Let $\widetilde{\mathcal{T}}_{\max}$ denote the type of a Γ-chamber (which is independent of the choice of the Γ-chamber). We thus have the following natural Ξ-equivariant bijections, increasing for the inclusion and the incidence (in an obvious sense):*

$$\left\{\begin{array}{c}\textit{L-multitypes strongly}\\ \Gamma\textit{-inv. of } \mathscr{B}(G_L) \textit{ in } \widetilde{\mathcal{T}}_{\max}\end{array}\right\} \overset{\sim}{\leftarrow} \left\{\begin{array}{c}\textit{Sets of } \Gamma\textit{-multifacets of}\\ \mathscr{B}(G_L) \textit{ of same L-multitype}\end{array}\right\} \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} \left\{\begin{array}{c}\textit{K-multitypes}\\ \textit{of } \mathscr{B}(G)\end{array}\right\}$$

In particular, if G is residually quasi-split, the set on the left is exactly that of strongly Γ-invariant L-multitypes of $\mathscr{B}(G_L)$.

Proof.

1. Consider two Γ-multifacets $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ and take H an L-conformal subgroup of $G(K)$ (for example $G(K)^+$). Also denote $\mathcal{F} := \widetilde{\mathcal{F}}^\Gamma$ and $\mathcal{F}' := (\widetilde{\mathcal{F}}')^\Gamma$.

   Suppose that $\mathcal{F}$ and $\mathcal{F}'$ have the same K-type. Then, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ have the same L-type. Indeed, there exists $g \in H$ such that $g \cdot \mathcal{F} = \mathcal{F}'$. By bijectivity and $G(K)$-equivariance of the correspondence in Proposition 3.3, we have $g \cdot \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}'$. Hence the result since H does not change the L-types.

   Suppose now that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ have the same L-type. We know there exists $g \in H$ such that $g \cdot \mathcal{F}$ and $\mathcal{F}'$ lie in the closure of the same K-chamber. Let $\widetilde{\mathcal{C}}$ be the corresponding Γ-chamber. Now, $(g \cdot \widetilde{\mathcal{F}})^\Gamma = g \cdot \mathcal{F}$. This means that $g \cdot \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are in the closure of $\widetilde{\mathcal{C}}$ by the increasing property. In particular, they lie in the closure of the same L-chamber. Since g does not change the L-types, this means that $g \cdot \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}'$. In particular, $g \cdot \mathcal{F}$ and $\mathcal{F}'$ are equal. Since g also does not change the K-types, we deduce that $\mathcal{F}$ and $\mathcal{F}'$ have the same K-type. Hence the result.
2. The correspondence in point (2) given by $\overline{\alpha}$ and $\overline{\beta}$ is then obtained by factoring the maps α and β from Proposition 3.3 at the level of orbits under H. Indeed, on one hand, the orbit of a K-multifacet under H corresponds to the K-multitypes. On the other hand, the orbit of a Γ-multifacet under H describes Γ-multifacets which are also of the same L-type by the L-conformal property. Conversely, the Γ-multifacets of the same L-type are all described according to point (1).

Let us now prove the first bijection of point (2). Take a Γ-chamber $\widetilde{\mathcal{C}}$. Since $\widetilde{\mathcal{C}}$ corresponds to the set of L-multitypes in $\widetilde{\mathcal{T}}_{\max}$, we can lift a strongly Γ-invariant L-multitype $\widetilde{\mathcal{T}}$ living in $\widetilde{\mathcal{T}}_{\max}$ to an L-multifacet $\widetilde{\mathcal{F}}$ in the closure of $\widetilde{\mathcal{C}}$. But since $\widetilde{\mathcal{C}}$ is Γ-invariant, the orbit of $\widetilde{\mathcal{F}}$ under Γ remains in $\widetilde{\mathcal{C}}$. Since $\widetilde{\mathcal{T}}$ is strongly Γ-invariant, Γ acts conformally on $\widetilde{\mathcal{F}}$, and thus $\widetilde{\mathcal{F}}$ is strongly Γ-invariant. This shows the surjectivity, injectivity being of course obvious.

If G is residually quasi-split, then $\widetilde{\mathcal{T}}_{\max}$ is the type of an L-chamber. This gives the result.

Finally, note that these correspondances are of course $G(K)$-equivariant. Since $G(K)^c$ acts trivially on the K-multitypes, it also acts trivially on the other sets and the action therefore factors everywhere through $G(K)/G(K)^c \cong \Xi$. This shows in particular that the orbit under the action of a conformal global subgroup of $G(K)$ of a Γ-multifacet describes exactly the Γ-multifacets of the same L-type, hence the second remark of the proposition.

□

Let us now establish some corollaries to Proposition 3.5:

Proposition 3.6.

(1) Every global subgroup of $G(K)$ that is L-conformal is conformal. In particular, given H, a global subgroup of $G(K)$, we have $H \cap G(L)^c \subset H^c$ and $H \cap G(L)^b \subset H^b$.

(2) A global subgroup of $G(K)$ is conformal if and only if it acts conformally on the Γ-multifacets of $\mathscr{B}(G_L)$.

Remark 3.7. Careful! It is possible that the inclusion $G(K)^{tc} \subset G(K)^c$ is strict, and hence that a conformal global subgroup is not very conformal. A counterexample where G is the unique inner form of PGL_2 over $\mathbb{Q}_p$, with p prime (since $H^2(\mathbb{Q}_p, \mu_2) = \mathbb{Z}/2\mathbb{Z}$), is given in [KP23, Example 2.6.31]. It is thus adjoint, anisotropic, and residually quasi-split. In other words, $G(K)$ permutes the vertices of the K^{unr}-grading, i.e., A_1 (•—•).

Note also the following result:

Proposition 3.8. *Let $\widetilde{H}$ be a Γ-invariant global subgroup of $G(L)$. Take $\widetilde{\mathcal{F}}$ a Γ-multifacet of $\mathscr{B}(G_L)$. Set $H := \widetilde{H}^\Gamma$ and $\mathcal{F} := \widetilde{\mathcal{F}}^\Gamma$. Then*

$$\widetilde{H}_{(\widetilde{\mathcal{F}})} \cap H = H_{(\widetilde{\mathcal{F}})} = H_{(\mathcal{F})}.$$

Proof. The result obviously reduces to the case of facets. Let $h \in H_{\mathcal{F}}$. Since $\emptyset \neq \mathcal{F} \subset h \cdot \widetilde{\mathcal{F}} \cap \widetilde{\mathcal{F}}$, we have $h \cdot \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}$. So $h \in H_{\widetilde{\mathcal{F}}}$. Conversely, if $h \in H_{\widetilde{\mathcal{F}}}$, take $x \in \mathcal{F}$. Then $h \cdot x \in \widetilde{\mathcal{F}}$. But $\sigma(h \cdot x) = \sigma(h) \cdot \sigma(x) = h \cdot x$ for all $\sigma \in \Gamma$. So $h \cdot x \in \mathcal{F}$ and $h \in H_{\mathcal{F}}$. □

From this, we can introduce the following definition:

Definition 3.9. *Take $\widetilde{H}$ a Γ^{unr}-invariant global subgroup of $G(K^{\mathrm{unr}})$. Set $H := \widetilde{H}^{\Gamma^{\mathrm{unr}}}$.*

(1) Consider $\mathcal{F}$, a facet of $\mathscr{B}(G)$ and $\widetilde{\mathcal{F}}$ its associated Γ^{unr}-facet via the correspondence in Proposition 3.3.

(a) We say that a smooth separated R-model of G having as R^{unr}-points the group $\widetilde{H}_{\widetilde{\mathcal{F}}}$ (resp. $\widetilde{H}^{\mathrm{f}}_{\widetilde{\mathcal{F}}}$) is a ***stabilizer (resp. pointwise stabilizer) group scheme of $\mathcal{F}$ relative to H****. It is also called a* ***Bruhat-Tits model of $H_{\mathcal{F}}$ (resp. $H^{\mathrm{f}}_{\mathcal{F}}$)****.*

   (b) Its group of R-points is given by $(\widetilde{H}_{\widetilde{\mathcal{F}}})^{\Gamma^{\mathrm{unr}}} = H_{\widetilde{\mathcal{F}}} = H_{\mathcal{F}}$ *(resp.* $(\widetilde{H}^{\mathrm{f}}_{\widetilde{\mathcal{F}}})^{\Gamma^{\mathrm{unr}}} = H^{\mathrm{f}}_{\widetilde{\mathcal{F}}} = H^{\mathrm{f}}_{\mathcal{F}}$*) according to Proposition 3.8.*

(2) Suppose this time that $\mathcal{F}$ is a multifacet.
   *(a) We say that a smooth separated R-model of G having as R^{unr}-points the group $\widetilde{H}_{(\widetilde{\mathcal{F}})}$ is a **multistabilizer group scheme of $\mathcal{F}$ relative to H**. It is also called a **Bruhat-Tits model of $H_{(\mathcal{F})}$**.*
   (b) Its group of R-points is given by $(\widetilde{H}_{(\widetilde{\mathcal{F}})})^{\Gamma^{\mathrm{unr}}} = H_{(\widetilde{\mathcal{F}})} = H_{(\mathcal{F})}$ *according to Proposition 3.8.*

If $\widetilde{H} = G(K^{\mathrm{unr}})$, relative to H *may be omitted in the preceding definitions.*

Remark 3.10. The preceding definitions are of course compatible with unramified Galois algebraic extensions $K^{\mathrm{unr}}/L/K$ in an obvious sense.

Remark 3.11. A smooth affine R-scheme is unique up to isomorphism if its R^{unr}-points are fixed (cf. [KP23, Corollary 2.10.11]). Consequently, using the notations of the definition, there is at most one affine Bruhat-Tits model given the choice of $\widetilde{H}$ and $\mathcal{F}$.

The question of uniqueness in the case where the model is not affine will be discussed in a future article.

Remark 3.12. This definition includes in particular the group schemes defined by Bruhat and Tits in [BT84a] and also the group schemes defined in [KP23]. It also includes the Néron models of tori (which thus gives an example of a situation where the model is not necessarily affine). The question of existence, under certain hypotheses, of Bruhat-Tits models (especially when they are not affine) will be addressed in a future article.

4. Theoretical cohomological results

In all that follows, we denote by $\widetilde{\xi}$ the type morphism associated to G_L. We also denote by $\mathscr{D}$ (resp. $\widetilde{\mathscr{D}}$) the relative affine Dynkin diagram of G (resp. G_L).

Consider also Ξ^{ext} (resp. Ξ^{ext}_L), the subgroup of Dynkin automorphisms of $\mathscr{D}$ (resp. $\widetilde{\mathscr{D}}$) induced by the polysimplicial automorphisms of an apartment of $\mathscr{B}(G)$ (resp. $\mathscr{B}(G_L)$) that vectorially induce an element of the vectorial Weyl group (cf. [KP23, Definition 1.3.71]). More precisely, this construction is indicated in [KP23, Remark 1.3.76].

Note that the action of $G(L)$ on $\mathscr{B}(G_L)$ is compatible with the action of $G(L)$ on the vectorial building of G over L, in the sense that the latter gives the underlying vectorial action. Moreover, the vectorial action is, on each apartment, induced by elements of the vectorial Weyl group, and consequently preserves the vectorial types (cf. [Rou, 2.2.16.(c) Theorem.]). This implies that the image of the type morphism over K (resp. L) is included in Ξ^{ext} (resp. Ξ^{ext}_L).

Furthermore, the Galois action on $\mathscr{B}(G_L)$ is compatible with the Galois action on the vectorial building over L, and as in the affine case, the vectorial building over K embeds into the vectorial building over L (cf. [Rou, 2.3.1.(2) Theorem.]).

We also define the *extended type*: to a pair $(\mathcal{F}, *)$ composed of a K-multifacet and a point of V_G, we associate $(\mathcal{T}, *)$, the pair formed by the type of $\mathcal{F}$ and $*$ (thus seen in $\widetilde{\mathscr{D}} \times V_G$). The action of $G(K)$ on $(\mathcal{F}, *)$ induces an action on $(\mathcal{T}, *)$ given by $g \cdot (\mathcal{T}, *) = (\xi(g) \cdot \mathcal{T}, g \cdot *)$ and thus an associated morphism ξ^e, whose kernel is by definition $G(K)^b := G(K)^c \cap G(K)^1$. Given a global subgroup H of $G(K)$, we denote $\Xi^e_H := \xi^e(H) \cong H/H^b$.

Of course, we generalize all this over L, and the action of Γ on $\mathscr{B}^e(G_L)$ factors through $\widetilde{\mathscr{D}} \times V_{G_L}$. We denote by $\widetilde{\xi}^e$ the associated morphism over L.

Let us begin with the following theorem:

Theorem 4.1. *Let $\widetilde{H}$ be a Γ-invariant global subgroup of $G(L)$. Denote $H := \widetilde{H}^\Gamma$. Denote also by $\widetilde{\mathcal{T}}_{\max}$ the type of a Γ-chamber.*

(1) (a) The group $\Xi_{\widetilde{H}}$ is endowed with the action of Γ by conjugation (given by $\sigma \mapsto (\omega \mapsto \sigma \circ \omega \circ \sigma^{-1})$), such that we have the exact sequence of Γ-groups:

$$1 \longrightarrow \widetilde{H}^c \longrightarrow \widetilde{H} \xrightarrow{\widetilde{\xi}} \Xi_{\widetilde{H}} \longrightarrow 1.$$

(b) Similarly, $\Xi^e_{\widetilde{H}}$ is endowed with the action of Γ by conjugation, such that we have the exact sequence of Γ-groups:

$$1 \longrightarrow \widetilde{H}^b \longrightarrow \widetilde{H} \xrightarrow{\widetilde{\xi}^e} \Xi^e_{\widetilde{H}} \longrightarrow 1.$$

(2) The previous exact sequences give rise to the following exact sequences of pointed sets:

(a) $1 \longrightarrow (\Xi_{\widetilde{H}})^\Gamma/\widetilde{\xi}(H) \longrightarrow H^1(\Gamma, \widetilde{H}^c) \longrightarrow H^1(\Gamma, \widetilde{H}) \longrightarrow H^1(\Gamma, \Xi_{\widetilde{H}}).$

(b) $1 \longrightarrow (\Xi^e_{\widetilde{H}})^\Gamma/\widetilde{\xi}^e(H) \longrightarrow H^1(\Gamma, \widetilde{H}^b) \longrightarrow H^1(\Gamma, \widetilde{H}) \longrightarrow H^1(\Gamma, \Xi^e_{\widetilde{H}}).$

(3) We have the following inclusions: $\widetilde{\xi}(H) \subset (\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^\Gamma \subset (\Xi_{\widetilde{H}})^\Gamma$.

(4) The group $(\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^\Gamma$ acts naturally on $\mathscr{D}$ and induces a map $(\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^\Gamma \to \Xi^{\text{ext}}$.

(5) The kernel $\operatorname{Ker}\left((\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^\Gamma \to \Xi^{\text{ext}}\right)$ is given by the elements of $(\Xi_{\widetilde{H}})^\Gamma$ stabilizing each Γ-orbits of $\widetilde{\mathscr{D}}$ in $\widetilde{\mathcal{T}}_{\max}$, or equivalently, stabilizing $\widetilde{\mathcal{T}}_{\max}$ and stabilizing a Γ-orbit descending to a K-special vertex.

(6) If $\widetilde{\mathscr{D}}$ admits a special vertex x in $\widetilde{\mathcal{T}}_{\max}$ (for example if G is residually quasi-split), then an element ω of the kernel writes $\sigma \circ \phi = \phi \circ \sigma$ with $\sigma \in \Gamma$ and $\phi \in \operatorname{Aut}(\widetilde{\mathscr{D}})$, the latter fixing x, the Γ-orbits in $\widetilde{\mathcal{T}}_{\max}$ and sending an arbitrary Γ-orbit to another. Moreover, ω is the unique element of the kernel having a decomposition with σ.

(7) The cardinality of the kernel is bounded by the size of the Γ-orbit of x. In particular, if x is fixed by Γ (for example if it is hyperspecial, cf. Definition 5.1), then the kernel is trivial.

(8) The restriction of the map $(\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^\Gamma \to \Xi^{\text{ext}}$ to $\widetilde{\xi}(H)$ has image Ξ and kernel $\widetilde{\xi}(H^c) \cong H^c/(H \cap \widetilde{H}^c)$. In particular, $H^c = H \cap \widetilde{H}^c$ when G admits a L-special vertex in $\widetilde{\mathcal{T}}_{\max}$ fixed by Γ.

Proof.

(1) (a) By point (3) of Proposition 2.11, $\widetilde{H}^c$ is a Γ-invariant subgroup of $\widetilde{H}$. Consequently, the map $h \mapsto \sigma(h) \mapsto \widetilde{\xi}(\sigma(h))$ from $\widetilde{H}$ to $\Xi_{\widetilde{H}}$ factors through $\Xi_{\widetilde{H}}$. We deduce then that the action of Γ on $\widetilde{H}$ factors into an action of Γ on $\Xi_{\widetilde{H}}$ such that the exact sequence of the statement is realized. The relation $\sigma(h) \cdot \mathcal{F} = \sigma(h \cdot \sigma^{-1}(\mathcal{F}))$ for every facet $\mathcal{F}$, every $\sigma \in \Gamma$ and $h \in \widetilde{H}$, and the fact that every element of Γ induces a Dynkin automorphism on the types, implies the relation $\widetilde{\xi}(\sigma(h)) = \sigma \circ \widetilde{\xi}(h) \circ \sigma^{-1}$.

(b) This point is done similarly to the previous point.

(2) (a) The exact sequence in cohomology then gives:

$$1 \longrightarrow (\widetilde{H}^c)^\Gamma \longrightarrow H \longrightarrow (\Xi_{\widetilde{H}})^\Gamma \longrightarrow H^1(\Gamma, \widetilde{H}^c) \longrightarrow H^1(\Gamma, \widetilde{H}) \longrightarrow H^1(\Gamma, \Xi_{\widetilde{H}}).$$

It then implies the exact sequence:

$$1 \longrightarrow \widetilde{\xi}(H) \longrightarrow (\Xi_{\widetilde{H}})^\Gamma \longrightarrow H^1(\Gamma, \widetilde{H}^c) \longrightarrow H^1(\Gamma, \widetilde{H}) \longrightarrow H^1(\Gamma, \Xi_{\widetilde{H}}).$$

And similarly, the latter implies the exact sequence of the statement.

(b) This point is done in the same manner.

(3) The inclusion $\widetilde{\xi}(H) \subset (\Xi_{\widetilde{H}, \widetilde{\mathcal{T}}_{\max}})^\Gamma$ comes from the fact that H sends a Γ-chamber to a Γ-chamber.

(4) Take a K-apartment $\mathcal{A}$ and a K-chamber $\mathcal{C}$ in $\mathcal{A}$. There exists a K^{unr}-apartment $\widetilde{\mathcal{A}}$ that contains $\mathcal{A}$. Consider the unique Γ-chamber $\widetilde{\mathcal{C}}$ such that $\mathcal{C} = (\widetilde{\mathcal{C}})^\Gamma$. Since $\mathcal{C} \subset \mathcal{A} \subset \widetilde{\mathcal{A}}$, we have that $\widetilde{\mathcal{C}}$ is in $\widetilde{\mathcal{A}}$. Denote by $\mathscr{C}$ a K^{unr}-chamber of $\widetilde{\mathcal{A}}$ such that $\widetilde{\mathcal{C}}$ is incident to it.

Take then $h \in \widetilde{H}$ such that $\widetilde{\xi}(h) \in (\Xi_{\widetilde{H}, \widetilde{\mathcal{T}}_{\max}})^\Gamma$. This means that $\widetilde{\xi}(h) = \widetilde{\xi}(\sigma(h))$ and that $h \cdot \widetilde{\mathcal{C}}$ is of the same type as $\widetilde{\mathcal{C}}$. Up to moving h by an element of $\widetilde{H}^b$, we can assume that $h \cdot (\mathscr{C}, \widetilde{\mathcal{A}}) = (\mathscr{C}, \widetilde{\mathcal{A}})$. This implies that $h \cdot \widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}$.

Let us now study $h^{-1}\sigma(h)$. Observe that:

$$\widetilde{\mathcal{C}} = \sigma(\widetilde{\mathcal{C}}) = \sigma(h \cdot \widetilde{\mathcal{C}}) = \sigma(h) \cdot \sigma(\widetilde{\mathcal{C}}) = \sigma(h) \cdot \widetilde{\mathcal{C}}$$

This implies that $h^{-1}\sigma(h) \cdot \widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}$. But, by hypothesis, $h^{-1}\sigma(h) \in \widetilde{H}^c$. This implies that $\widetilde{\mathcal{C}}$ is fixed by $h^{-1}\sigma(h)$. Consequently:

$$\begin{aligned} h \cdot \mathcal{C} &= \{h \cdot x \in h \cdot \widetilde{\mathcal{C}} \mid \forall \sigma \in \Gamma, \sigma(x) = x\} \\ &= \{h \cdot x \in h \cdot \widetilde{\mathcal{C}} \mid \forall \sigma \in \Gamma, h^{-1}\sigma(h) \cdot \sigma(x) = x\} \\ &= \{h \cdot x \in h \cdot \widetilde{\mathcal{C}} \mid \forall \sigma \in \Gamma, \sigma(h \cdot x) = h \cdot x\} \\ &= (h \cdot \widetilde{\mathcal{C}})^\Gamma = \mathcal{C}. \end{aligned}$$

Therefore, h stabilizes the K-chamber $\mathcal{C}$. This implies in particular that h stabilizes the affine subspace of $\widetilde{\mathcal{A}}$ generated by $\mathcal{C}$, that is the K-apartment $\mathcal{A}$. The element h obviously acts on $\mathcal{A}$ by affine automorphisms.

Now, Let us prove that the action is vectorially by Weyl automorphisms. This will gives us the result. Indeed, recall that $\mathcal{C}$ is in correspondence with $\mathscr{D}$, so that an action on $\mathcal{C}$ coming from an action on $\mathcal{A}$ by affine automorphism that are vectorially by Weyl automorphisms exactly determine an action on $\mathscr{D}$ by automorphisms from Ξ^{ext}.

Consider $\widetilde{\mathcal{V}}$ and $\mathcal{V}$ the vector spaces associated respectively to $\widetilde{\mathcal{A}}$ and $\mathcal{A}$. According to [Rou, 2.3.1.(2) Theorem.], the map between the vectorial buildings over K and over K^{unr} is Weyl compatible. In particular, in our case, that means that the natural embedding $\mathcal{V} \subset \widetilde{\mathcal{V}}$ is such that any element of the K^{unr}-Weyl group restricted to $\mathcal{V}$ is exactly an element of the K-Weyl group (cf. [Rou, 2.4.3.1. Definitions]). Therefore, since h acts on $\widetilde{\mathcal{V}}$ by Weyl automorphisms according to [Rou, 2.1.7.(b) Theorem.], his induced action on $\mathcal{V}$ is still by Weyl automorphisms.

(5) The description of $\operatorname{Ker}\left(\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^{\Gamma} \to \Xi^{\text{ext}}\right)$ comes from the fact that the Γ-orbits of $\widetilde{\mathscr{D}}$ in $\widetilde{\mathcal{T}}_{\max}$ are in correspondence with the vertices of $\mathscr{D}$. The equivalent condition comes from the fact that the only element of Ξ^{ext} fixing a special point is the identity (cf. [KP23, Remark 1.3.76]).

(6) Since ω is in the kernel, it sends x to an element of its Γ-orbit. In other words, there exists $\sigma \in \Gamma$ such that $\omega \cdot x = \sigma \cdot x$. So $\phi := \sigma^{-1} \circ \omega$ fixes x. Since σ and ω fix the Γ-orbits in $\widetilde{\mathcal{T}}_{\max}$, the same holds for ϕ. Moreover, since $\sigma^{-1}\,\omega\,\sigma = \omega$, we also have $\sigma \circ \phi = \phi \circ \sigma$. Finally, since σ and ω send a Γ-orbit to another, the same holds for ϕ.

According to [KP23, Remark 1.3.76], the morphism $\omega' \mapsto \omega' \cdot x$ from $\Xi_{\widetilde{H}}$ to the special points of $\widetilde{\mathscr{D}}$ is injective. We deduce also that ω is the unique element of the kernel having σ in its decomposition since $\omega \cdot x = \sigma \cdot x$.

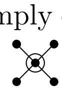
(7) We reuse [KP23, Remark 1.3.76]. Since an element of the kernel sends x to an element of its Γ-orbit, we deduce the bound. If G admits a hyperspecial point, it is Γ-invariant and its orbit is reduced to itself. Hence the result.

(8) Observe that an element $\widetilde{\xi}(h) \in (\Xi_{\widetilde{H}})^{\Gamma}$ (for $h \in H$) is in the kernel $\operatorname{Ker}\left(\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^{\Gamma} \to \Xi^{\text{ext}}\right)$ if and only if it stabilizes each Γ-orbits of $\widetilde{\mathscr{D}}$ in $\widetilde{\mathcal{T}}_{\max}$ according to point (5). This is equivalent to requiring that $\widetilde{\xi}(h)$ stabilizes the K-types according to Proposition 3.5, or even that $h \in H^c$. The kernel is thus given by $\widetilde{\xi}(H^c)$. Note that $\operatorname{Ker}(\widetilde{\xi}) = \widetilde{H}^c$. Moreover, point (1) of Corollary 3.6 gives that $H \cap \widetilde{H}^c \subset H^c$. This implies that $H^c/(H \cap \widetilde{H}^c) \cong \widetilde{\xi}(H^c)$.

According to Proposition 3.5, the group $H \cap \widetilde{H}^c$ acts transitively on the Γ-chambers, since it acts transitively on the K-chambers. Consequently, every element of $\widetilde{\xi}(H)$ comes from an $h \in H$ that stabilizes a certain Γ-chamber $\mathcal{C}$. As seen in point (4), the action of h on $\mathcal{C}^{\Gamma}$ determines the image of $\widetilde{\xi}(h)$ in Ξ^{ext}. But this action is none other than the natural action of h on the K-chamber $\mathcal{C}^{\Gamma}$ in the building $\mathscr{B}(G) \cong \mathscr{B}(G_L)^{\Gamma}$. By definition, this action defines an element of Ξ_H, and of course, every element of Ξ_H is obtained this way.

According to point (7), if G admits a L-special vertex in $\widetilde{\mathcal{T}}_{\max}$ fixed by Γ, then the kernel $\operatorname{Ker}\left(\Xi_{\widetilde{H},\widetilde{\mathcal{T}}_{\max}})^{\Gamma} \to \Xi^{\text{ext}}\right)$ is trivial. Consequently, $\widetilde{\xi}(H^c)$, and thus $H^c/(H \cap \widetilde{H}^c)$ is trivial. This implies the equality $H^c = H \cap \widetilde{H}^c$.

□

Remark 4.2. There exist situations where $\widetilde{\mathcal{T}}_{\max}$ contains no special vertex of $\widetilde{\mathscr{D}}$. Indeed, set $q_0 = X_1^2 + X_2^2 + X_3^2 + X_4^2$ and consider the quadratic form $q = q_0(X_1, X_2, X_3, X_4) + t\, q_0(X_1', X_2', X_3', X_4')$ in $\mathbb{R}((t))$. It is anisotropic of discriminant 1, so $\operatorname{Spin}(q)$ is an anisotropic simply connected absolutely almost simple group of type ${}^1D_4$. Its affine Tits index is given by  , hence the desired counterexample since the central point is not special in D_4 according to [KP23, Table 1.3.5].

Indeed, observe first that there is a natural inclusion of $\operatorname{SO}(q_0) \times \operatorname{SO}(q_0)$ into $\operatorname{SO}(q)$ over $\mathbb{R}((t))$. This inclusion lifts to a morphism at the level of simply connected coverings $\operatorname{Spin}(q_0) \times \operatorname{Spin}(q_0) \to \operatorname{Spin}(q)$ according to [Con14, Exercise 6.5.2.(iii)]. We then observe that the kernel μ of the lift is included in $\operatorname{Ker}(\operatorname{Spin}(q_0) \times \operatorname{Spin}(q_0) \to \operatorname{SO}(q_0) \times \operatorname{SO}(q_0))$: consequently μ is a finite multiplicative central split subgroup.

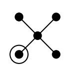
Denote $\mathcal{P}$ the unique parahoric group scheme of $\mathrm{Spin}(q)$ over $\mathbb{R}[[t]]$. Since $\mathrm{Spin}(q)$ is anisotropic over $\mathbb{R}((t))$ and simply connected, we have $\mathcal{P}(\mathbb{R}[[t]]) = \mathrm{Spin}(q)(\mathbb{R}((t)))$ according to [BT84a, 5.2.10.(i) Proposition.]. Moreover, since q_0 is defined and regular over $\mathbb{R}[[t]]$, the group $(\mathrm{Spin}(q_0) \times \mathrm{Spin}(q_0))/\mu$ is defined and reductive over $\mathbb{R}[[t]]$. We denote it then $\mathcal{Q}$.

Let us then show that there exists a morphism $\mathcal{Q} \to \mathcal{P}$ of $\mathbb{R}[[t]]$-group schemes extending the inclusions $(\mathrm{Spin}(q_0) \times \mathrm{Spin}(q_0))/\mu \to \mathrm{Spin}(q)$ and $\mathcal{Q}(\mathbb{R}[[t]]) \subset \mathcal{P}(\mathbb{R}[[t]]) = \mathrm{Spin}(q)(\mathbb{R}((t)))$. It suffices for this to show that $\mathcal{Q}$ is étoffé (cf. [BT84a, 1.7.1. Définition.]).

According to [BT84a, 1.7.2.], it suffices to show that $\mathcal{Q}$ satisfies (ET 1) and (ET 2). [BT84a, 1.7.3.] already gives us that (ET 1) is satisfied. (ET 2) means that the image of $\mathcal{Q}(R)$ to $\mathcal{Q}(\kappa)$ is schematically dense in $\mathcal{Q}_\kappa$. Since $\mathcal{Q}$ is smooth and R is henselian, this reduces to showing that $\mathcal{Q}(\kappa)$ is schematically dense in $\mathcal{Q}_\kappa$ according to Hensel's lemma ([KP23, Lemma 8.1.3]). This is true according to [Mil17, Theorem 17.93].

Since $\mathcal{Q}$ is reductive over $\mathbb{R}[[t]]$ and of rank 4, it admits an $R[[t]]$-maximal torus T of rank 4. The induced map $T \to \mathcal{P}$ has a kernel of multiplicative type according to [SGA3, Exp. IX, Théorème 6.8.]. But the latter is trivial on the generic fiber: it is therefore trivial according to [SGA3, Exp. IX, Remarque 1.4.1.b)]. Consequently, $T \to \mathcal{P}$ and in particular the rank of $\mathcal{P}_\mathbb{C}$ is at least 4.

Also, since the group is anisotropic, its affine index contains only one distinguished orbit. Finally, according to [Tit79, 3.5.2.], the Tits index of the reductive quotient of $\mathcal{P}_\mathbb{R}$ is obtained by removing all the vertices associated to the facet of $\mathcal{P}$ from the affine Tits index of $\mathrm{Spin}(q)$ (moreover, the unique distinguished orbit). Note also that the affine Tits index of $\mathrm{Spin}(q)$ has 5 vertices. The previous observation on the rank shows that at most one vertex is removed, and thus that the distinguished orbit is reduced to a point.

It remains to eliminate the following case:  (modulo rotation). Suppose, by contradiction, that its index is of this form. This means that $\mathrm{Spin}(q)$ admits a hyperspecial point (cf. Definition 5.1) and thus a reductive model G over $\mathbb{R}[[t]]$ (cf. Lemma 5.2). Consider then a regular quadratic form q' over $\mathbb{R}$ such that $G_\mathbb{R} = \mathrm{Spin}(q')$ (G is simply connected). Thanks to the inclusion $\mathbb{R} \to \mathbb{R}[[t]]$, this defines a reductive group $\mathrm{Spin}(q')$ over $\mathbb{R}[[t]]$.

The two groups G and $\mathrm{Spin}(q')$ are then forms of Spin_8 that coincide over $\mathbb{R}$. Now, according to [SGA3, Exp. XXIV, Proposition 8.1.(ii)], since $\mathrm{Aut}(\mathrm{Spin}_8)$ is smooth, we have $H^1(\mathbb{R}[[t]], \mathrm{Aut}(\mathrm{Spin}_8)) \overset{\sim}{\to} H^1(\mathbb{R}, \mathrm{Aut}(\mathrm{Spin}_8))$. We deduce then that G and $\mathrm{Spin}(q')$ are isomorphic.

In particular, we have an isomorphism between $\mathrm{Spin}(q')$ and $\mathrm{Spin}(q)$ over $\mathbb{R}((t))$. Let us then show that this implies that q' and q are equivalent up to homothety by a scalar of $\mathbb{R}((t))$. From this, we then deduce an absurdity because q is not regular when reduced modulo t, and thus not regular over $\mathbb{R}[[t]]$, contrary to q'.

According to [KMRT98, (44.8) Theorem.], there is an equivalence of categories between triality algebras and simply connected groups of type D_4 via $T \mapsto \mathrm{Spin}(T)$. In particular, in the context of groups of type ${}^1D_4$, things simplify greatly. We consider the algebras with involution $(\mathrm{M}_8(\mathbb{R}((t))), *)$ and $(\mathrm{M}_8(\mathbb{R}((t))), *')$ with $* := X \mapsto M_q^{-1}\, {}^tX\, M_q$ and $*' := X \mapsto M_{q'}^{-1}\, {}^tX\, M_{q'}$, where M_q and $M_{q'}$ are respectively the matrices of the quadratic forms q and q'. The associated "Spin" groups are simply $\mathrm{Spin}(q)$ and $\mathrm{Spin}(q')$. The algebras are thus isomorphic. To conclude, we then use [KMRT98, (12.34) Proposition.] which gives that $(\mathrm{M}_8(\mathbb{R}((t))), *)$ and $(\mathrm{M}_8(\mathbb{R}((t))), *')$ are isomorphic algebras with involution if and only if q and q' are equivalent up to homothety.

Let us now show the following theorem, which is at the core of this part:

Theorem 4.3. *Let $\widetilde{\Omega}$ be a disjoint union of parts $\bigsqcup_{i\in I}\widetilde{\Omega_i}$ of $\mathscr{B}(G_L)$ where each $\widetilde{\Omega_i}$ is Γ-invariant. Take $\widetilde{H}\subset G(L)$, a Γ-invariant global subgroup and set $H := \widetilde{H}^\Gamma$. Choose $*$, an arbitrary point of the vector part V_G of $\mathscr{B}^e(G)$.*

(1) We have the natural isomorphisms (where the considered quotients are quotients of actions):
  (a) $(\mathrm{Orb}((\widetilde{\Omega}_i)_{i\in I})_{\widetilde{H}})^\Gamma/H \xrightarrow{\sim} \mathrm{Ker}\left(H^1(\Gamma,\widetilde{H}_{(\widetilde{\Omega}_i)_{i\in I}})\to H^1(\Gamma,\widetilde{H})\right)$.
  (b) $(\mathrm{Orb}((\widetilde{\Omega}_i)_{i\in I},*))_{\widetilde{H}})^\Gamma/H \xrightarrow{\sim} \mathrm{Ker}\left(H^1(\Gamma,\widetilde{H}^1_{(\widetilde{\Omega}_i)_{i\in I}})\to H^1(\Gamma,\widetilde{H})\right)$.
  The action on the families is term-by-term.

(2) Let us now consider a Γ-multifacet $\widetilde{\mathcal{F}}$ of type $\widetilde{\mathcal{T}}$. We have the following isomorphisms induced by passing to the types:
  (a) $(\mathrm{Orb}((\widetilde{\mathcal{F}}))_{\widetilde{H}})^\Gamma/H \xrightarrow{\sim} \left(\{\omega\cdot\widetilde{\mathcal{T}}\prec\widetilde{\mathcal{T}}_{\max}\mid \omega\in\Xi_{\widetilde{H}}\}^\Gamma\right)/\,\Xi_H$.
  (b) $(\mathrm{Orb}((\widetilde{\mathcal{F}}),*)_{\widetilde{H}})^\Gamma/H \xrightarrow{\sim} \left(\{(\omega\cdot\widetilde{\mathcal{T}},\omega\cdot *)\mid \omega\cdot\widetilde{\mathcal{T}}\prec\widetilde{\mathcal{T}}_{\max}\ ,\ \omega\in\Xi^e_{\widetilde{H}}\}^\Gamma\right)/\,\Xi^e_H$.

Proof.

(1) (a) The result comes from [Ser94, I.§5.4., Corollaire 1.]. Indeed, it suffices to apply it to the Γ-morphism $\widetilde{H}_{(\widetilde{\Omega}_i)_{i\in I}}\to\widetilde{H}$, and observe that $\widetilde{H}/\widetilde{H}_{(\widetilde{\Omega}_i)_{i\in I}}\cong\mathrm{Orb}((\widetilde{\Omega}_i)_{i\in I})_{\widetilde{H}}$ as a Γ-set with an action of $\widetilde{H}$ (term-by-term).
  (b) The second point is proved in the same manner. Indeed, it suffices to observe that $\widetilde{H}^1_{(\widetilde{\Omega}_i)_{i\in I}}=\widetilde{H}_{((\widetilde{\Omega}_i)_{i\in I},*)}$.

(2) (a) Note first that the map $(\mathrm{Orb}((\widetilde{\mathcal{F}}))_{\widetilde{H}})^\Gamma\to\{\omega\cdot\widetilde{\mathcal{T}}\prec\widetilde{\mathcal{T}}_{\max}\mid\omega\in\Xi^{\widetilde{H}}\}^\Gamma$ is well-defined since every Γ-multifacet is incident to a Γ-chamber, and this incidence passes to the types. Moreover, by Proposition 3.5, the action of H on strongly Γ-invariant types factors through Ξ^H. This proves the good definition of the map in the statement.
  Let us show injectivity. Take $g,g'\in\widetilde{H}$ such that $g\cdot\widetilde{\mathcal{F}}$ and $g'\cdot\widetilde{\mathcal{F}}$ are strongly Γ-invariant. Suppose also that there exists $h\in H$ such that $h\cdot(g\cdot\widetilde{\mathcal{F}})$ and $g'\cdot\widetilde{\mathcal{F}}$ have the same L-type. By Proposition 3.5, this means that there exists $h_b\in H^b$ such that $(h_bhg)\cdot\widetilde{\mathcal{F}}$ and $g'\cdot\widetilde{\mathcal{F}}$ are equal. Since $h_b h\in H$, this means therefore that $g\cdot\widetilde{\mathcal{F}}$ and $g'\cdot\widetilde{\mathcal{F}}$ are in the same orbit under H. Hence injectivity.
  Let us now show surjectivity. Take $\widetilde{\mathcal{C}}$, a Γ-chamber such that $\widetilde{\mathcal{F}}$ is incident to it. Let $h\in\widetilde{H}$ such that $\xi(h)\cdot\widetilde{\mathcal{T}}$ is a strongly Γ-invariant type incident to $\widetilde{\mathcal{T}}_{\max}$. It lifts to the L-multifacet $h\cdot\widetilde{\mathcal{F}}$, which we can assume incident to $\widetilde{\mathcal{C}}$, up to moving h by an element of $\widetilde{H}^b$.
  Let $\sigma\in\Gamma$. We then have $\sigma(h\cdot\widetilde{\mathcal{F}})$ incident to $\sigma(\widetilde{\mathcal{C}})=\widetilde{\mathcal{C}}$. But, since $\xi(h)\cdot\widetilde{\mathcal{T}}$ is Γ-invariant, $\sigma(h\cdot\widetilde{\mathcal{F}})$ is also of this type. We therefore have $\sigma(h\cdot\widetilde{\mathcal{F}})$ and $h\cdot\widetilde{\mathcal{F}}$ of the same type in $\widetilde{\mathcal{C}}$, so they are equal as multifacets. Consequently, $h\cdot\widetilde{\mathcal{F}}$ is strongly Γ-invariant and surjectivity is proved as desired.
  (b) As before, the map (defined by taking the type on the first factor) is well-defined for the same reasons. Let us also add that the action of H on the set on the left factors through Ξ^e_H because $H^b\subset H^c$ acts trivially on strongly Γ-invariant types (as shown previously), and $H^b\subset H^1$ acts trivially on $*\in V_G$. The proof of injectivity and surjectivity is done, *mutatis mutandis*, as in the previous case.

□

Remark 4.4. By combining point (1) and point (2) in the context of a multifacet, we deduce that the kernels of point (1) depend only on the types, more precisely on the relative affine Dynkin diagram over L equipped with its Galois action and the set of vertices $\widetilde{\mathcal{T}}_{\max}$ (what one could call a L/K-affine Tits index) and the action of $\widetilde{H}$ on it.

To perform calculations in the case where $L = K^{\mathrm{unr}}$, one can notably use the classification of affine Tits indices done in [Tit79, 4. Classification.] for local fields, or even the recent classification of residually quasi-split groups when κ is perfect in [Rou, 6.5.13].

One can also determine the list of affine Tits indices in the hyperspecial case (cf. Definition 5.1), and thus perform a calculation, using the list of (classical) Tits indices in [Tit66] on which one adjoins a hyperspecial point using the list of affine Dynkin diagrams (cf. [KP23, Table 1.3.4 / 1.3.5]).

Remark 4.5. The two sets of point (2) (a) of Theorem 4.3 are therefore finite since there are finitely many distinct types.

Remark 4.6. If G is semisimple simply connected, quasi-split over K^{unr}, it has already been proved in [BT84a, 5.2.10.(ii)] that $(\mathrm{Orb}(\widetilde{\mathcal{F}})_{G(K^{\mathrm{unr}})})^{\Gamma^{\mathrm{unr}}}/G(K) = 1$ for any Γ^{unr}-facet $\mathcal{F}$. One can also deduce it from [BT84a, 5.2.10.(i)] (which says that $G(K^{\mathrm{unr}})$ is conformal) and point (2) (a) of Theorem 4.3.

We deduce from Theorem 4.3 the following particular cases:

Corollary 4.7. *Let $\widetilde{\mathcal{F}}$ be a Γ-multifacet of $\mathscr{B}(G_L)$. Take $\widetilde{H} \subset G(L)$ global, Γ-invariant, and acting conformally on $\widetilde{\mathcal{F}}$. We have:*

$$\mathrm{Ker}\left(H^1(\Gamma, \widetilde{H}_{(\widetilde{\mathcal{F}})}) \to H^1(\Gamma, \widetilde{H})\right) = 1.$$

Proof. Note that $\widetilde{\mathcal{T}}$ is the type of $\widetilde{\mathcal{F}}$. The hypothesis means that $\widetilde{H}$ acts trivially on $\widetilde{\mathcal{T}}$. Consequently, $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\widetilde{H}})^{\Gamma}$ is trivial. We then conclude thanks to Theorem 4.3. □

Corollary 4.8. *Take $\widetilde{H} \subset G(L)$, a Γ-invariant global subgroup. Let $\widetilde{\mathcal{C}}$ be a Γ-chamber of $\mathscr{B}(G_L)$. We have:*

$$\mathrm{Ker}\left(H^1(\Gamma, \widetilde{H}_{\widetilde{\mathcal{C}}}) \to H^1(\Gamma, \widetilde{H})\right) = 1.$$

Proof. Note that $\widetilde{\mathcal{T}}_{\max}$ is the type of $\widetilde{\mathcal{C}}$. By Theorem 4.3, the kernel in the statement is equal to the subset of $(\mathrm{Orb}(\widetilde{\mathcal{T}}_{\max})_{\widetilde{H}})^{\Gamma}$ consisting of types incident to $\widetilde{\mathcal{T}}_{\max}$. This set is therefore obviously reduced to $\{\widetilde{\mathcal{T}}_{\max}\}$. □

Let us now dwell on some results expressing to what extent the extension L/K can be changed for cohomology calculations.

Lemma 4.9. *Assume that G has the same relative rank over L as over K. Take S a K-split maximal torus. Note $Z := Z_G(S)$ and $N := N_G(S)$. Also take a Γ-invariant global subgroup $\widetilde{H}$ of $G(L)$ containing $Z(L)^1$ (or equivalently, $G(L)^b$) and note $H := \widetilde{H}^{\Gamma}$. We have the following assertions:*

1. *The natural map $(H \cap N(K))/Z(K)^1 \to (\widetilde{H} \cap N(L))/Z(L)^1$ is an isomorphism.*
2. *The building $\mathscr{B}^e(G)$ is canonically identified with a subset of $\mathscr{B}^e(G_L)$ which sends a K-apartment to a Γ-invariant apartment over L.*
3. *Consider $\mathcal{A}$ the extended apartment associated to S in $\mathscr{B}^e(G) \subset \mathscr{B}^e(G_L)$. Point (1) also means that $H_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ have the same image in $\mathrm{Aut}_{\mathrm{aff}}(\mathcal{A})$.*
4. *We have the natural isomorphisms $\Xi_H \cong \Xi_{\widetilde{H}}$ and $\Xi^e_H \cong \Xi^e_{\widetilde{H}}$.*

Proof. Note that $\widetilde{H}$ contains $Z(L)^1$ if and only if it contains $G(L)^b$ since the latter equals $G(L)^+Z(L)^1$ by Lemma 2.8.

(1) We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (H\cap N(K))/(H\cap Z(K)) & \longrightarrow & (H\cap N(K))/Z(K)^1 & \longrightarrow & (H\cap Z(K))/Z(K)^1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (\widetilde{H}\cap N(L))/(H\cap Z(K)) & \longrightarrow & (\widetilde{H}\cap N(L))/Z(L)^1 & \longrightarrow & (\widetilde{H}\cap Z(L))/Z(L)^1 & \longrightarrow & 1. \end{array}$$

Note first that $N(K)/Z(K) \to N(L)/Z(L)$ is an isomorphism since N/Z is a finite constant group (cf. [Mil17, 25.16]). Since $HZ(K) = G(K)$ and $\widetilde{H}Z(L) = G(K)$ (since $G(K)^+Z(K) = G(K)$ and $G(L)^+Z(L) = G(L)$ by [BT73, 6.11.(i) Proposition.]), $H \cap N(K)$ (resp. $\widetilde{H} \cap N(L)$) surjects onto $N(K)/Z(K)$ (resp. $N(L)/Z(L)$). This gives that the left vertical arrow is an isomorphism.

Take a basis $\mathcal{B}$ of characters of Z. It is therefore of cardinality r, where r is the relative rank of G over K. Since it has the same relative rank over L, it is also a basis of characters of Z_L. The map $z \mapsto (v(\chi(z)))_{\chi\in\mathcal{B}}$ defines an isomorphism from $Z(K)/Z(K)^1$ to $\mathbb{Z}^r$. Since L/K is unramified, it extends to an isomorphism $Z(L)/Z(L)^1 \cong \mathbb{Z}^r$. Hence a natural isomorphism $Z(K)/Z(K)^1 \cong Z(L)/Z(L)^1$. Given $h \in \widetilde{H} \cap Z(L)$, there exists therefore $z \in Z(K)$ and $z^1 \in Z(L)^1$ such that $h = zz^1$. Since $Z(L)^1 \subset \widetilde{H}$, it turns out that $z \in Z(K) \cap \widetilde{H} = Z(K) \cap H$. This proves therefore that the right vertical arrow is an isomorphism.

In conclusion, the central vertical arrow is an isomorphism since it is the case for the left and right arrows. Hence the desired isomorphism.

(2) This is an immediate consequence of [Rou77, Proposition 2.3.1.] and the unramified descent theorem.

(3) Recall that we have an action map $N(L) \to \mathrm{Aut}_{\mathrm{aff}}(\mathcal{A})$ whose kernel is $Z(L)^1$ (cf. end of section 1). By the previous point, this map is compatible with the one associated to $N(K)$. It then suffices to observe that $H_{\mathcal{A}} = H \cap N(K)$ and $\widetilde{H}_{\mathcal{A}} = \widetilde{H} \cap N(L)$ to conclude thanks to point (1).

(4) Observe first that $H^b = \widetilde{H}^b \cap H$ and that $H^c = \widetilde{H}^c \cap H$. Indeed, this is a consequence of Proposition 3.5 since Γ acts trivially on $\mathscr{B}^e(G_L)$ and thus on types, and the equality of relative ranks means that every L-chamber is a K-chamber. In other words, being conformal over L is equivalent to being conformal over K.

Let us then show that $\widetilde{H} = H\,\widetilde{H}^b$ to conclude. This will of course imply $\widetilde{H} = H\,\widetilde{H}^c$. Since $Z(L)^1 \subset \widetilde{H}$, we in fact have $\widetilde{H}^b = G(L)^b$. Point (1) says that $(H \cap N(K))\,Z(L)^1 = (\widetilde{H} \cap N(L)) = \widetilde{H}_{\mathcal{A}}$. Multiplying by $G(L)^+$ and using Lemma 2.8, we obtain $(H \cap N(K))\,G(L)^b = \widetilde{H}_{\mathcal{A}}\,G(L)^+ = \widetilde{H}$. Hence the result.

□

Proposition 4.10. *Consider L'/K, an unramified Galois extension containing L with Galois group Γ'. Assume that G has the same relative rank over L' as over L. Let $\widetilde{\mathcal{F}}'$, a Γ'-multifacet. It induces a Γ-multifacet which we denote $\widetilde{\mathcal{F}}$.*

Also take a Γ-invariant global subgroup $\widetilde{H}'$ of $G(L')$ containing $G(L')^b$, and note $\widetilde{H} := (\widetilde{H}')^{\mathrm{Gal}(L'/L)}$ and $H := (\widetilde{H}')^{\Gamma'}$. We have the equalities:

(1) *(a)* $\mathrm{Ker}\left(H^1(\Gamma, \widetilde{H}^c) \to H^1(\Gamma, \widetilde{H})\right) = \mathrm{Ker}\left(H^1(\Gamma', (\widetilde{H}')^c) \to H^1(\Gamma', \widetilde{H}')\right)$.

(b) $\mathrm{Ker}\left(H^1(\Gamma, \widetilde{H}^b) \to H^1(\Gamma, \widetilde{H})\right) = \mathrm{Ker}\left(H^1(\Gamma', (\widetilde{H}')^b) \to H^1(\Gamma', \widetilde{H}')\right)$.

(2) *(a)* $\operatorname{Ker}\left(H^1(\Gamma, \widetilde{H}_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma, \widetilde{H})\right) = \operatorname{Ker}\left(H^1(\Gamma', (\widetilde{H}')_{\widetilde{\mathcal{F}}'}) \to H^1(\Gamma', \widetilde{H}')\right)$.
(b) $\operatorname{Ker}\left(H^1(\Gamma, \widetilde{H}^1_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma, \widetilde{H})\right) = \operatorname{Ker}\left(H^1(\Gamma', (\widetilde{H}')^1_{\widetilde{\mathcal{F}}'}) \to H^1(\Gamma', \widetilde{H}')\right)$.

Proof. Observe that L'/L is Galois. By Lemma 4.9, since G has the same relative rank over L' as over L, the inclusion $\mathscr{B}^e(G_L) \subset \mathscr{B}^e(G_{L'})$ sends an apartment to an apartment invariant under $\operatorname{Gal}(L'/L)$. This inclusion is therefore such that Γ and Γ' act in the same way on the L-types (which are thus in correspondence with the L'-types). Moreover, note that a Γ'-chamber is naturally a Γ-chamber, so that the type $\widetilde{\mathcal{T}}_{\max}$ is the same over L and over L'. Note then that the facet $\widetilde{\mathcal{F}}$ is none other than the image of $\widetilde{\mathcal{F}}'$ under this inclusion.

(1) (a) By point (2) of Theorem 4.1 (and its functoriality), it suffices to show that the map $(\Xi_{\widetilde{H}})^{\Gamma}/\widetilde{\xi}(H) \to (\Xi_{\widetilde{H}'})^{\Gamma'}/\widetilde{\xi}'(H)$ is an isomorphism. This is an immediate consequence of the isomorphism $\Xi_{\widetilde{H}} \xrightarrow{\sim} \Xi_{\widetilde{H}'}$ by point (4) of Lemma 4.9, this latter isomorphism also identifying $\widetilde{\xi}(H)$ with $\widetilde{\xi}'(H)$.
(b) This case is treated in the same manner.

(2) (a) We have the following commutative diagram according to point (2) (a) of Theorem 4.3:

$$\begin{array}{ccc}
\operatorname{Ker}\left(H^1(\Gamma, \widetilde{H}_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma, \widetilde{H})\right) & \longrightarrow & \operatorname{Ker}\left(H^1(\Gamma', \widetilde{H}'_{\widetilde{\mathcal{F}}'}) \to H^1(\Gamma', \widetilde{H}')\right) \\
\cong\uparrow & & \uparrow\cong \\
(\operatorname{Orb}(\widetilde{\mathcal{F}})_{\widetilde{H}})^{\Gamma}/H & \longrightarrow & (\operatorname{Orb}(\widetilde{\mathcal{F}'})_{\widetilde{H}'})^{\Gamma'}/H \\
\cong\downarrow & & \downarrow\cong \\
\left(\{\omega \cdot \widetilde{\mathcal{T}} \prec \widetilde{\mathcal{T}}_{\max} \mid \omega \in \Xi_{\widetilde{H}}\}^{\Gamma}\right)/\Xi_H & \longrightarrow & \left(\{\omega \cdot \widetilde{\mathcal{T}} \prec \widetilde{\mathcal{T}}_{\max} \mid \omega \in \Xi_{\widetilde{H}'}\}^{\Gamma'}\right)/\Xi_H
\end{array}$$

and the last horizontal arrow is an isomorphism by point (4) of Lemma 4.9 since the latter gives $\Xi_{\widetilde{H}} \xrightarrow{\sim} \Xi_{\widetilde{H}'}$.
(b) This case is treated analogously.

□

Remark 4.11. One can in particular apply this proposition when $L' = K^{\mathrm{unr}}$ and with a finite Galois extension L/K such that G has the same rank over L as over K^{unr}. Consequently, the kernels are trivial if G is split over K, or if G is semisimple and residually split.

The kernels are also trivial in the case where G is an absolutely almost simple quasi-split group over K and split by a totally ramified extension, because in this case, G is residually split. Indeed, a quasi-split group admits a minimal Galois splitting extension K' (which is none other than the Galois extension with Galois group the kernel of the $*$-action). This extension is totally ramified by hypothesis. Consequently, G is residually split because, otherwise, there would exist an unramified extension between K and K'.

Note that, according to [Gil15, 2.9. Calculs galoisiens.], given an R-group scheme $\mathcal{G}$, any Γ-cocycle in $Z^1(\Gamma, \mathcal{G}(R_L))$ (where R_L is the ring of integers of L) defines a $\mathcal{G}$-torsor over R, and thus an element of $H^1(R, \mathcal{G})$ (in fact, any $\mathcal{G}$-torsor over R trivialized over R_L comes from a unique such cocycle according to [Gil15, Lemme 2.2.1.]). Similarly, a cocycle in $Z^1(\Gamma, G(L))$ defines an element of $H^1(K, G)$. Moreover, two torsors are isomorphic if and only if the associated cocycles are cohomologous.

We therefore define the twist of $\mathcal{G}$ by a cocycle $z \in Z^1(\Gamma, \mathcal{G}(R_L))$, denoted ${}^z\mathcal{G}$, as the twist through the torsor it induces (cf. [Gil15, 2.1.]). We similarly define the twist of G by a cocycle of $Z^1(\Gamma, G(L))$. Two cohomologous cocycles of course induce isomorphic twists.

Note finally that twisting $\mathcal{G}$ by a cocycle $z \in Z^1(\Gamma, \mathcal{G}(R_L))$ is compatible with twisting $\mathcal{G}(R_L)$ by the same cocycle in the following sense: $({}^z\mathcal{G})(R_L)$ is equal to ${}^z(\mathcal{G}(R_L))$ as a Γ-module (it is of course the same for G).

Let us finish this section with some properties about the behavior under twisting by a cocycle of the building, of the facets, and of the stabilizers:

Proposition 4.12. *Take a cocycle $z \in Z^1(\Gamma, G(L))$.*

(1) The twists ${}^zG(L)$ and ${}^z\mathscr{B}(G_L)$ are such that on one hand, ${}^zG(L)$ is $G(L)$ equipped with the action given by $\sigma \star g := z(\sigma)\sigma(g)z(\sigma)^{-1}$, and on the other hand, such that ${}^z\mathscr{B}(G_L)$ is $\mathscr{B}(G_L)$ equipped with the action $\sigma \star x = z(\sigma)\sigma(x)$. Moreover, these two actions are compatible, in other words: $\sigma \star (g \cdot x) = (\sigma \star g) \cdot (\sigma \star x)$. Furthermore, $({}^z\mathscr{B}(G_L))^\Gamma$ is identified with $\mathscr{B}({}^zG)$.

(2) Let z' be a cocycle cohomologous to z via an element $g_0 \in G(L)$ (such that $z' = \sigma \mapsto g_0^{-1}z(\sigma)\sigma(g_0)$). Then we have the following isomorphisms:

$$\begin{array}{c} {}^{z'}G(L) \xrightarrow{\sim} {}^zG(L) \\ g \mapsto g_0 g g_0^{-1} \end{array} \quad \text{and} \quad \begin{array}{c} {}^{z'}\mathscr{B}(G_L) \xrightarrow{\sim} {}^z\mathscr{B}(G_L) \\ x \mapsto g_0 \cdot x \end{array}$$

where the first isomorphism is an isomorphism of Γ-groups, and where the second isomorphism is an isomorphism of $\Gamma \ltimes G(L)$-sets. Moreover, ${}^zG$ and ${}^{z'}G$ are isomorphic.

Proof.

(1) This is an immediate consequence of [Ser94, 5.3. Torsion] and notably of [Ser94, Proposition 34.] for the compatibility. Indeed, $\mathscr{B}(G_L)$ can be seen as a Γ-set with a compatible action of the Γ-group $G(K^{\mathrm{unr}})$.

As for the fixed points, this comes from the tamely ramified descent theorem ([Rou77, Proposition 5.1.1.]), noting that ${}^zG$ is reductive over K since it is over L, and from the fact that $({}^zG)(L) = {}^z(G(L))$ as Γ-groups.

(2) The verification is immediate. As for the isomorphism between ${}^zG$ and ${}^{z'}G$, this is a consequence of the fact that z and z' come from isomorphic torsors since they are cohomologous.

□

Proposition 4.13. *Take $\widetilde{\mathcal{F}}$ a Γ-multifacet of $\mathscr{B}(G_L)$ and $z \in Z^1(\Gamma, G(L)_{(\widetilde{\mathcal{F}})})$. Write $\widetilde{\mathcal{F}} = \bigsqcup_{i \in I} \widetilde{\mathcal{F}_i}$, its decomposition into Γ-facets and note $\mathcal{F} = \bigsqcup_{i\in I} \mathcal{F}_i$ the associated K-multifacet. The cocycle z defines for every $i \in I$ a class in $Z^1(\Gamma, G(L)_{\widetilde{\mathcal{F}_i}})$ and in $Z^1(\Gamma, G(L))$ which we also denote by z. Then:*

(1) The twisted multifacet ${}^z\widetilde{\mathcal{F}}$ is also compatible with ${}^z\mathscr{B}(G_L)$, in the following sense: $\widetilde{\mathcal{F}}$ is strongly Γ-invariant for the action $\star$ introduced in point (1) of Proposition 4.12 and the induced $\Gamma \ltimes G(L)_{(\widetilde{F})}$-set is exactly ${}^z\widetilde{\mathcal{F}}$. The set of fixed points is denoted ${}^z\mathcal{F}$. The latter is a multifacet of $\mathscr{B}({}^zG)$ and admits the decomposition into facets ${}^z\mathcal{F} = \bigsqcup_{i\in I} {}^z\mathcal{F}_i$.

(2) *Let $z' \in Z^1(\Gamma, G(L)_{(\widetilde{\mathcal{F}})})$ be a cocycle cohomologous to z in $Z^1(\Gamma, G(L))$, thus via an element $g_0 \in G(L)$ (such that $z' = \sigma \mapsto g_0^{-1} z(\sigma)\sigma(g_0)$). Then the isomorphism from point (2) of Proposition 4.12 sends ${}^{z'}\widetilde{\mathcal{F}}$ to the facet $g_0\widetilde{\mathcal{F}}$ which is Γ-invariant in ${}^{z}\mathscr{B}(G_L)$. Consequently, ${}^{z}\mathcal{F}$ is sent to $(g_0\widetilde{\mathcal{F}})^{\Gamma}$.*
In particular, if $g_0 \in G(L)_{(\widetilde{F})}$ (and thus z and z' are cohomologous in $Z^1(\Gamma, G(L)_{(\widetilde{\mathcal{F}})})$), then ${}^{z'}\widetilde{\mathcal{F}}$ is sent to ${}^{z}\widetilde{\mathcal{F}}$ (and ${}^{z}\mathcal{F}$ to ${}^{z'}\mathcal{F}$).

Proof.

(1) As before, we use [Ser94, 5.3. Torsion]. Since $\widetilde{\mathcal{F}}$ is strongly Γ-invariant and $G(L)_{(\widetilde{\mathcal{F}})}$ operates on $\widetilde{\mathcal{F}}$ in a manner compatible with Γ, we can consider the twist ${}^{z}\widetilde{\mathcal{F}}$. We similarly obtain the ${}^{z}\widetilde{\mathcal{F}}_i$ for all $i \in I$. Since, for every $i \in I$, we have $\widetilde{\mathcal{F}}_i \subset \widetilde{\mathcal{F}} \subset \mathscr{B}(G_L)$ as $\Gamma \ltimes G(L)_{(\widetilde{F})}$-sets, then similarly, ${}^{z}\widetilde{\mathcal{F}}_i \subset {}^{z}\widetilde{\mathcal{F}} \subset {}^{z}\mathscr{B}(G_L)$ as $\Gamma \ltimes G(L)_{(\widetilde{F})}$-sets. Hence the compatibility and the strong Γ-invariance of ${}^{z}\widetilde{\mathcal{F}}$. Proposition 3.3 then tells us that ${}^{z}\mathcal{F}$ is a multifacet of $\mathscr{B}({}^{z}G)$ with decomposition $\bigsqcup_{i\in I} {}^{z}\mathcal{F}_i$.
(2) The verification is immediate.

□

Proposition 4.14. *Take $\widetilde{H}$, a Γ-invariant global subgroup of $G(L)$ and set $H := \widetilde{H}^{\Gamma}$. Also take $z \in Z^1(\Gamma, \widetilde{H})$. We have:*

(1) *The subgroup ${}^{z}H := ({}^{z}\widetilde{H})^{\Gamma}$ is global in $({}^{z}G)(K)$.*

Suppose furthermore that $z \in Z^1(\Gamma, \widetilde{H}_{(\widetilde{\mathcal{F}})})$ (which of course defines a cocycle in $Z^1(\Gamma, G(L)_{(\widetilde{\mathcal{F}})})$ and in $Z^1(\Gamma, \widetilde{H})$ that we also denote by z). We have:

(2) *The subgroup ${}^{z}H_{(\mathcal{F})} := ({}^{z}\widetilde{H}_{(\widetilde{\mathcal{F}})})^{\Gamma}$ is the multistabilizer of ${}^{z}\mathcal{F}$ in $\mathscr{B}({}^{z}G)$ relative to ${}^{z}H$. In other words, ${}^{z}H_{(\mathcal{F})} = ({}^{z}H)_{({}^{z}\mathcal{F})}$.*
(3) *Assume that $L = K^{\mathrm{unr}}$. If furthermore $H_{(\mathcal{F})}$ admits a Bruhat-Tits model $\mathcal{H}_{(\mathcal{F})}$, then $({}^{z}H)_{({}^{z}\mathcal{F})}$ also does and one is given by ${}^{z}\mathcal{H}_{(\mathcal{F})}$.*

Proof. Observe that the twisted groups ${}^{z}\widetilde{H}_{(\widetilde{\mathcal{F}})}$, ${}^{z}\widetilde{H}_{\mathcal{F}_i}$ for all $i \in I$, ${}^{z}\widetilde{H}$ and ${}^{z}G(L)$ are equipped with compatible Γ-actions. Note also that ${}^{z}\widetilde{H}$ is global since its underlying group is $\widetilde{H}$.

(1) Since ${}^{z}\widetilde{H}$ is global, ${}^{z}H$ is also global by point (2) of Proposition 2.11.
(2) Regarding the multistabilizer, we observe by Proposition 3.8 and the compatibility of the Γ-groups:
$$({}^{z}H)_{({}^{z}\mathcal{F})} = {}^{z}H \cap {}^{z}\widetilde{H}_{(\widetilde{\mathcal{F}})} = ({}^{z}\widetilde{H}_{(\widetilde{\mathcal{F}})})^{\Gamma} = {}^{z}H_{(\mathcal{F})}.$$
(3) Suppose now that $H_{(\mathcal{F})}$ admits a Bruhat-Tits model $\mathcal{H}_{(\mathcal{F})}$. Observe then that the group ${}^{z}\mathcal{H}_{(\mathcal{F})}$ has as R^{unr}-points (with its Γ-action) ${}^{z}\widetilde{H}_{(\widetilde{\mathcal{F}})}$ and thus as R-points ${}^{z}H_{(\mathcal{F})} = ({}^{z}H)_{({}^{z}\mathcal{F})}$. It is therefore indeed a Bruhat-Tits model of $({}^{z}H)_{({}^{z}\mathcal{F})}$.

□

5. Case of hyperspecial points

Let us now focus on the case of hyperspecial points. Recall the definition:

Definition 5.1. *A point $x \in \mathscr{B}(G)$ is called a* ***hyperspecial point*** *of G if G is split over K^{unr} and if x is a special vertex of $\mathscr{B}(G_{K^{\mathrm{unr}}})$ (via the identification $\mathscr{B}(G) \cong \mathscr{B}(G_{K^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$).*

The notion of a hyperspecial point thus depends on the Γ^{unr}-set $\mathscr{B}(G_{K^{\mathrm{unr}}})$, but also on G. Note also that a hyperspecial point is a vertex of $\mathscr{B}(G)$ by Proposition 3.3. Let us now recall some results linking hyperspecial points and reductive models:

Lemma 5.2. *Let G be a reductive group over K and x a hyperspecial point of G (so G is split over K^{unr}). The following statements hold:*

1. *The affine Bruhat-Tits model of $G(K)^1_x$ is reductive (with connected fibers) over R.*
2. *Conversely, every reductive model of G is obtained in this way.*
3. *In particular, if G is an R-reductive group, then $D(G)$ (resp. G) is K-anisotropic if and only if $G(R) = G(K)^1$ (resp. $G(R) = G(K)$).*

Proof.

1. By definition, the point x is special in $\mathscr{B}(G_{K^{\mathrm{unr}}})$. According to [BT84a, 4.6.22.] and [BT84a, 4.6.28.(ii)], the affine model associated to $G(K^{\mathrm{unr}})^1_x$ is reductive (with connected fibers). Since the affine model of $G(K)^1_x$ is simply the descent to R (which exists using the process of [BT84a, 5.1.30.] because x is Γ^{unr}-invariant), we obtain the result.
2. Conversely, [BT84a, 4.6.31.] tells us that a reductive model $\mathcal{G}$ of G is isomorphic over R^{unr} to the scheme associated to $G(K^{\mathrm{unr}})^1_x$ for some special point $x \in \mathscr{B}(G_{K^{\mathrm{unr}}})$. Since $\mathcal{G}$ is defined over R, $G(K^{\mathrm{unr}})^1_x$ is Γ^{unr}-invariant and thus $G(K^{\mathrm{unr}})^1_x = G(K^{\mathrm{unr}})^1_{\sigma(x)}$ for all $\sigma \in \Gamma^{\mathrm{unr}}$. By point (3) of Proposition 2.7, x is therefore also Γ^{unr}-invariant. It thus comes from a point on $\mathscr{B}(G)$ which is therefore hyperspecial and then $\mathcal{G}(R) = G(K)^1_x$.
3. Finally, if $D(G)$ is anisotropic, $G(R)$ is the stabilizer under the action of $G(K)^1$ of the unique (hyperspecial) point of $\mathscr{B}(G)$. It is therefore exactly $G(K)^1$. If moreover G is anisotropic, $G(K)^1 = G(K)$ and we get the result.

   Conversely, if $G(R) = G(K)$, then $G(K)$ is bounded, and thus cannot contain the image of a K-cocharacter (which is unbounded). So G is K-anisotropic. If this time $G(R) = G(K)^1$, then $D(G)(K) \subset G(K)^1$ is bounded and we reason as before.

□

We thus deduce:

Proposition 5.3. *Let G be a reductive group over R. The following properties are equivalent:*

1. *$D(G)$ (resp. G) is anisotropic over κ.*
2. *$D(G)$ (resp. G) is anisotropic over K.*
3. *$G(R) = G(K)^1$ (resp. $G(R) = G(K)$).*

Proof. The equivalence between (2) and (3) is a consequence of Lemma 5.2. Let us now show that (1) and (2) are equivalent.

Recall that G is isogenous to $D(G) \times R(G)$, so that there is a correspondence between non-central cocharacters of G and cocharacters of $D(G)$, and between central cocharacters and cocharacters of $R(G)$.

Recall from the decomposition of [SGA3, Exp. XXVI, Corollaire 3.5.] that the R-scheme of proper parabolic subgroups of G, denoted $\underline{\mathrm{Par}}(G)^+$, is smooth and projective. By the valuative criterion of properness, we have $\underline{\mathrm{Par}}(G)^+(R) = \underline{\mathrm{Par}}(G)^+(K)$. Now, there is a natural map $\underline{\mathrm{Par}}(G)^+(R) \to \underline{\mathrm{Par}}(G)^+(\kappa)$. Consequently, if $\underline{\mathrm{Par}}(G)^+(K)$ is non-empty, then $\underline{\mathrm{Par}}(G)^+(\kappa)$ is also non-empty. So if G has a non-central cocharacter over K, then it has one over κ.

Moreover, smoothness also allows the use of Hensel's lemma, so that the map $\underline{\mathrm{Par}}(G)^+(R) \to \underline{\mathrm{Par}}(G)^+(\kappa)$ is surjective. We therefore finally have:

$$\underline{\mathrm{Par}}(G)^+(K) = \underline{\mathrm{Par}}(G)^+(R) \twoheadrightarrow \underline{\mathrm{Par}}(G)^+(\kappa)$$

Consequently, if G admits no proper parabolic subgroup over K, then it admits none over κ either. In other words, if G has no non-central cocharacter over K, then it admits none over κ either.

Now, let us deal with central characters. We can reduce to the case of a torus T. This is an immediate consequence of the fact that:

$$\mathrm{Hom}_{K^{\mathrm{unr}}}(\mathbb{G}_{m,K^{\mathrm{unr}}}, T_{K^{\mathrm{unr}}}) \xleftarrow{\sim} \mathrm{Hom}_{R^{\mathrm{unr}}}(\mathbb{G}_{m,R^{\mathrm{unr}}}, T_{R^{\mathrm{unr}}}) \xrightarrow{\sim} \mathrm{Hom}_{\kappa^s}(\mathbb{G}_{m,\kappa^s}, T_{\kappa^s})$$

as Γ^{unr}-groups, since T is split over R^{unr}. □

That being said, the case of tori can be understood quite easily thanks to the following lemma. This lemma seems to be known to some specialists, but we haven't found any reference in the literature about it.

Lemma 5.4. *Let T be a K-torus split over K^{unr}. It thus admits a toric model over R which we denote by T. Let us consider the Γ^{unr}-group $\widehat{T}^\circ := \mathrm{Hom}_{R^{\mathrm{unr}}}(\mathbb{G}_{m,R^{\mathrm{unr}}}, T_{R^{\mathrm{unr}}})$.*

(1) There is a canonical isomorphism of Γ^{unr}-modules:

$$T(K^{\mathrm{unr}})^1 \times \widehat{T}^\circ = T(R^{\mathrm{unr}}) \times \widehat{T}^\circ \cong T(K^{\mathrm{unr}}).$$

(2) For all $i \geq 1$, we have:

$$\mathrm{Ker}\left(H^i(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}})^1) \to H^i(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}}))\right) = 0.$$

Proof.

(1) We have the natural exact sequence of Γ^{unr}-groups:

$$0 \longrightarrow (R^{\mathrm{unr}})^\times \longrightarrow (K^{\mathrm{unr}})^\times \longrightarrow \mathbb{Z} \longrightarrow 0.$$

It is split by $1 \mapsto \pi$ (where π is a uniformizer of K, and thus also of K^{unr}). This section is Γ^{unr}-invariant. Tensoring the previous exact sequence by $\widehat{T}^\circ$, we obtain the exact sequence of Γ^{unr}-groups:

$$0 \longrightarrow T(R^{\mathrm{unr}}) \longrightarrow T(K^{\mathrm{unr}}) \longrightarrow \widehat{T}^\circ \longrightarrow 0.$$

because we have canonical isomorphisms $\widehat{T}^\circ \otimes_{\mathbb{Z}} (R^{\mathrm{unr}})^\times \cong T(R^{\mathrm{unr}})$ and $\widehat{T}^\circ \otimes_{\mathbb{Z}} (K^{\mathrm{unr}})^\times \cong T(K^{\mathrm{unr}})$ given by $\theta \otimes x \mapsto \theta(x)$. It is also split by $\theta \mapsto \theta \otimes \pi \cong \theta \mapsto \theta(\pi)$, a section which is also Γ^{unr}-invariant. Hence the isomorphism of Γ^{unr}-modules.

Note also that, by definition of the exact sequence, we have $T(R^{\mathrm{unr}}) = T(K^{\mathrm{unr}})^1$.

(2) This point then follows from the fact that this isomorphism induces the canonical isomorphism:

$$H^i(\Gamma^{\mathrm{unr}}, T(R^{\mathrm{unr}})) \times H^i(\Gamma^{\mathrm{unr}}, \widehat{T}^\circ) \cong H^i(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}})).$$

and hence the desired injectivity.

□

Note that Bruhat and Tits have already treated the case of hyperspecial points when G is semisimple in [BT84a, 5.2.14. Proposition.]. We can in fact adjust their proof to include the reductive case:

Proposition 5.5. *Let x be a hyperspecial point of G. We have:*

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1_x) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right) = 1.$$

Proof. Recall that, since x is hyperspecial, it is a vertex both in $\mathscr{B}(G)$ and in $\mathscr{B}(G_{K^{\mathrm{unr}}})$. Take $\overline{x} = (x, \lambda)$, a point in $\mathscr{B}^e(G)$. Thanks to Theorem 4.3, the question reduces to showing that $(\mathrm{Orb}(\overline{x})_{G(K^{\mathrm{unr}})})^{\Gamma^{\mathrm{unr}}}/G(K) = 1$.

Take then a point $\overline{y} = (y, \mu)$ in $\mathscr{B}^e(G)$ such that there exists $g \in G(K^{\mathrm{unr}})$ with $g \cdot \overline{x} = \overline{y}$. We then observe that y is an hyperspecial vertex. Since $G(K)^+$ acts transitively on the K-chambers and is very conformal, there exists $g' \in G(K)^+$ such that $x' := g' \cdot x$ is still hyperspecial and such that x' and y are in the same closure of a K-chamber (and thus the same closure of a K^{unr}-chamber, which we denote $\mathcal{C}$). Moreover, $g' \cdot \lambda = \lambda$ since $G(K)^+ \subset G(K)^1$. By replacing $\overline{x} := (x, \lambda)$ with (x', λ), we can assume this. We can in fact assume that x, y and even $\mathcal{C}$ lie in a K^{unr}-special apartment (so that the associated K^{unr}-split maximal torus T is defined over K and contains a K-split maximal torus). Since G is split over K^{unr}, the torus T is a K^{unr}-maximal torus, and is thus its own centralizer.

Let $\mathcal{I} := G(K^{\mathrm{unr}})^+_{\mathcal{C}}$. Let $N(K^{\mathrm{unr}})$ be the normalizer associated to the special apartment. The Bruhat decomposition (Proposition 2.9) then gives $G(K^{\mathrm{unr}}) = \mathcal{I}\, N(K^{\mathrm{unr}})\, \mathcal{I}$. We can thus write $g = i\, n\, i'$ with obvious notations. Consequently, $i\, n\, i' \cdot \overline{x} = \overline{y}$. So $n \cdot \overline{x} = \overline{y}$ since $\mathcal{I}$ fixes $\overline{x}$ and $\overline{y}$ (because it pointwise stabilizes the chamber where they lie and $\mathcal{I} \subset G(K^{\mathrm{unr}})^1$).

Moreover, since x is special over K^{unr}, $G(K^{\mathrm{unr}})^b_x \cap N(K^{\mathrm{unr}})$ surjects onto the (vectorial) Weyl group of $G_{K^{\mathrm{unr}}}$, i.e., $N(K^{\mathrm{unr}})/T(K^{\mathrm{unr}})$ (cf. [BT84a, 4.6.22.]). There thus exists $n' \in G(K^{\mathrm{unr}})^b_x \cap N(K^{\mathrm{unr}})$ such that n' and n have the same image in the Weyl group. In other words, $t := n\, n'^{-1} \in T(K^{\mathrm{unr}})$. But $n'^{-1} \cdot \overline{x} = \overline{x}$. So $t \cdot \overline{x} = n \cdot \overline{x}$.

Consider $\sigma \mapsto t^{-1}\, \sigma(t)$. This is a coboundary in $B^1(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}}))$ and also a cocycle in $Z^1(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}})^1)$. Indeed:

$$t \cdot \overline{x} = \overline{y} = \sigma(\overline{y}) = \sigma(t \cdot \overline{x}) = \sigma(t) \cdot \overline{x}$$

since $\overline{x}$ and $\overline{y}$ are Γ^{unr}-invariant. Consequently, $t^{-1}\, \sigma(t)$ fixes $\overline{x}$. But $T(K^{\mathrm{unr}})$ acts by translation on the (extended) apartment. So if it fixes $\overline{x}$, it fixes the (extended) apartment. This means that we actually have $t^{-1}\, \sigma(t) \in T(K^{\mathrm{unr}})^1$. The associated cohomology class thus lives in

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}})^1) \to H^1(\Gamma^{\mathrm{unr}}, T(K^{\mathrm{unr}})\right).$$

This kernel is in fact trivial by Lemma 5.4. Consequently, there exists $t' \in T(K^{\mathrm{unr}})^1$ such that $\sigma \mapsto t^{-1}\, \sigma(t) = \sigma \mapsto t'^{-1}\, \sigma(t')$, or equivalently such that $t\, t'^{-1}$ is Γ^{unr}-invariant, and thus lives in $G(K)$. Therefore, $t\, t'^{-1} \cdot \overline{x} = t \cdot \overline{x} = \overline{y}$. So $\overline{x}$ and $\overline{y}$ are in the same orbit under $G(K)$. □

Remark 5.6. We have the factorization $H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1_x) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))$. The previous theorem thus implies:

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1_x) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1)\right) = 1.$$

An alternative proof of this result can also be obtained by reworking the previous proof with the reduced building.

Remark 5.7. However, it is not always true that $\mathrm{Ker}(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})))$ is trivial, even if G admits a hyperspecial point. Consider the following example:

Let D be a division algebra of degree d over a field k. Consider here $K = k((t))$. Thanks to Proposition 5.3 applied to $\mathrm{GL}_1(D \otimes_k k[[t]])$, the algebra D defines a division algebra $D \otimes_k k((t))$ over $k((t))$. It is moreover split over K^{unr}. Take here $G = \mathrm{GL}_1(D)_K$ (which admits, by the way, the reductive model $G = \mathrm{GL}_1(D)_{k[[t]]}$). The group $\mathrm{GL}_1(D)_{K^{\mathrm{unr}}}$ admits a unique character given by the reduced norm. Consequently, $\mathrm{GL}_1(D)(K^{\mathrm{unr}})^1$ is given by the kernel of the valuation of the reduced norm on K^{unr} (which is surjective since D is split over K^{unr}).

Observe also that, since D is of finite dimension over k, we have the canonical isomorphisms $D \otimes_k k((t)) \cong D((t))$ and $D \otimes_k k[[t]] \cong D[[t]]$.

Now, we have the decomposition:

$$(D \otimes_k k((t)))^\times = k((t))^\times (D \otimes_k k[[t]])^\times.$$

Indeed, an element of $D((t))^\times$ can be written as $t^i x$ with x having non-zero reduction modulo t. We denote by $x_0 \in D^\times$ this reduction. This gives the desired decomposition. Indeed, $t^i \in k((t))^\times$ and x is of the form $x_0(1 - ty)$, with $y \in D[[t]]$, whose inverse is $(\sum_{k=0}^{+\infty} (ty)^k) x_0^{-1} \in D[[t]]$.

Consequently, the image of $(D \otimes_k k((t)))^\times$ under the valuation of the reduced norm is given by $k((t))^\times$ since $(D \otimes_k k[[t]])^\times$ is bounded. Since the norm on $k((t))$ is compatible with the reduced norm of $D_{k((t))}$, the image is therefore $d\mathbb{Z}$ (cf. [TW15, Theorem 1.4.]). The exact sequence in cohomology then implies:

$$\mathrm{Ker}(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^1) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))) = \mathbb{Z}/d\mathbb{Z} \neq 1.$$

Remark 5.8. It turns out that $\mathrm{Ker}(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_x) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})))$ is not always trivial. A counterexample is given in [BT84a, 5.2.15. Remarque.]. Let us precise that.

Take the extension $L/K = \mathbb{C}((t))/\mathbb{R}((t))$, and h the Hermitian form given by $z_1\overline{z_1} - z_2\overline{z_2}$. Take $G = \mathrm{U}(h)$ (also denoted $\mathrm{U}(1,1)$). This is a quasi-split form of GL_2 which satisfies on one hand $D(G) = \mathrm{SU}(h) \cong \mathrm{SL}_2$, and on the other hand $Z(G) \cong \mathrm{R}^1_{L/K}(\mathbb{G}_m)$, which is not split. In fact, G is residually split, so that $\mathscr{B}(G) = \mathscr{B}(G_L) = \mathscr{B}(\mathrm{SL}_2)$.

By Theorem 4.3, it suffices to find two hyperspecial points in the same orbit under $G(L)$, and whose types are not conjugate by $G(K)$ for the kernel to be non-trivial.

As said above, the building of G is exactly that of SL_2. Its relative affine Dynkin diagram is given by $\bullet\!\!-\!\!\bullet$ whose two vertices are special (cf. [BT84a, 4.2.23.] and [BT72, (1.4.6)]).

These two points are however not conjugate in $G(K)$. In fact, the latter acts by preserving the types. Indeed, on one hand we have $G(K) = G(K)^1$ since the radical of G is anisotropic. On the other hand, since $G_L = \mathrm{GL}_2$, the Kottwitz morphism of G (cf. [KP23, Chapter 11]) is obtained by restricting that of GL_2. For the same reasons as in Remark 1.4, we conclude that $G(K)^0 = G(K)^1$ and thus that $G(K)$ acts trivially on the types.

For GL_2, we have the same Dynkin diagram. The two vertices of the diagram are fixed by Galois, because otherwise, unramified descent would tell us that the diagram of G would consist of a single point. The two vertices of the diagram of G are therefore hyperspecial.

It now suffices to find two vertices of different types conjugated by $\mathrm{GL}_2(L)$. This has already been done in Remark 1.4. This concludes.

6. Application: the case of quasi-split adjoint groups

Let us conclude this article by using everything we have shown in the previous parts to compute exactly the kernels

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^0_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$$

and

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$$

for K-groups G that are semisimple, adjoint, and quasi-split over K, and where $\widetilde{\mathcal{F}}$ is a Γ^{unr}-facet of the building $\mathscr{B}(G_{K^{\mathrm{unr}}})$.

Let us first address the case of parahoric subgroups. According to [BT84a, 5.2.12. Proposition.], the parahoric subgroups over K are given by the stabilizers of facets under the action of the residually neutral component $G(K)^0$. Now, since G is quasi-split and adjoint, we have the following lemma:

Lemma 6.1. *We have the equalities:* $G(K)^0 = G(K)^b = G(K)^c$.

Proof. Note that since G is semisimple, the extended building equals the reduced building and thus $G(K)^b = G(K)^c$.

Let T be a K-maximal torus containing a maximal split torus. According to [BT84a, 4.4.16. Proposition.], it is an induced torus. Its canonical scheme (i.e., its finite type Néron model) is therefore smooth and connected. Its R-points are given by $T(K)^1$.

Furthermore, according to [BT84a, 5.2.11.], $G(K)^0$ is generated by $G(K)^+$ and the R-points of the identity component of the canonical scheme of T, which here is $T(K)^1$ by the previous discussion.

However, according to Lemma 2.8, $G(K)^b = G(K)^+\, T(K)^1$. Hence $G(K)^0 = G(K)^b$. □

The question then reduces to asking whether the composite map

$$H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^c_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^c) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))$$

has a trivial kernel for $\widetilde{\mathcal{F}}$, a Γ^{unr}-invariant facet of $\mathscr{B}(G_{K^{\mathrm{unr}}})$.

The first map has trivial kernel according to Corollary 4.7. Let us now focus on the second map.

For this, we need to prove that every quasi-split reductive group is residually quasi-split. This is already known when the residue field κ is perfect (cf. [KP23, Proposition 9.10.5]). It turns out that the result holds in general, but our proof requires the use of the theory of pseudo-reductive groups (i.e. groups with trivial unipotent radical over the base field).

Recall that a pseudo-parabolic subgroup of a pseudo-reductive group is called a pseudo Borel subgroup if it is a minimal pseudo-parabolic over the separable closure. This is actually equivalent to requiring it to be a solvable pseudo-parabolic. A pseudo-reductive group possessing a pseudo Borel subgroup is said to be quasi-split. In this case, all its minimal pseudo-parabolic subgroups are pseudo Borel subgroups since they are conjugate. All this is explained at the beginning of [CP16, Section C.2]).

These definitions extend naturally to the case of smooth connected affine groups since there is a correspondence between their pseudo-parabolic subgroups and those of its maximal pseudo-reductive quotient (cf. [CGP15, Proposition 2.2.10]).

Furthermore, let us introduce (in full generality) the following definition:

Definition 6.2. *Assume that the affine Bruhat-Tits model of $G(K)^1_{\mathcal{F}}$ exists. Denote it by $\mathcal{G}^1_{\mathcal{F}}$. We then define the* ***parahoric group scheme*** *(resp. the* ***parahoric subgroup****) associated to a facet $\mathcal{F}$ of $\mathscr{B}(G)$ as being $\mathcal{G}^0_{\mathcal{F}} := (\mathcal{G}^1_{\mathcal{F}})^\circ$ (resp. $G(K)^0_{\mathcal{F}} := (\mathcal{G}^1_{\mathcal{F}})^\circ(R)$).*

Remark 6.3. Note that the affine group schemes $\mathcal{G}^1_{\mathcal{F}}$ are constructed by Bruhat and Tits in the case where G is quasi-split over K^{unr} in [BT84a, 5.1.30.].

That being said, this definition coincides with the definition of [BT84a] in the quasi-split over K^{unr} case. Indeed, on one hand, in the quasi-split case, [BT84a, 4.6.21. Proposition. (ii)] combined with [BT84a, 4.6.26.] and [BT84a, 4.6.28. Proposition.] implies that $(\mathcal{G}^1_{\mathcal{F}})^\circ$ is indeed the parahoric group scheme associated to the parahoric subgroup of [BT84a, 5.2.6. Définition.]. In general, as indicated in [BT84a, 5.1.30.], the schemes descend over R and its R-points are the parahoric subgroups according to the last paragraph of [BT84a, 5.2.8. Proposition.].

We can thus propose a generalization of [KP23, Proposition 9.10.1], which gives conditions equivalent to being residually quasi-split:

Proposition 6.4. *Let G be a reductive group over K, quasi-split over K^{unr}. The following statements are equivalent:*

(1) There exists a K-chamber $\mathcal{C}$ such that the κ-group $(\mathcal{G}^0_{\mathcal{C}})_\kappa$ is solvable.
(2) There exists a K^{unr}-chamber that is Γ^{unr}-invariant in $\mathscr{B}(G_{K^{\mathrm{unr}}})$.
(3) Every Γ^{unr}-chamber in $\mathscr{B}(G_{K^{\mathrm{unr}}})$ is a Γ^{unr}-invariant K^{unr}-chamber.
(4) For every K-chamber $\mathcal{C}$, the κ-group $(\mathcal{G}^0_{\mathcal{C}})_\kappa$ is solvable.
(5) For every K-facet, the κ-group $(\mathcal{G}^0_{\mathcal{F}})_\kappa$ is quasi-split.

Proof.

(1) $\Longrightarrow$ (2) The K-chamber $\mathcal{C}$ comes from a Γ^{unr}-chamber $\widetilde{\mathcal{C}}$. It comes from a Γ^{unr}-chamber $\widetilde{\mathcal{C}}$ which is therefore a K^{unr}-chamber. This induces a compatibility $(\mathcal{G}^0_{\mathcal{C}})_{R^{\mathrm{unr}}} = \mathcal{G}^0_{\widetilde{\mathcal{C}}}$, hence $(\mathcal{G}^0_{\mathcal{C}})_{\kappa^s} = (\mathcal{G}^0_{\widetilde{\mathcal{C}}})_{\kappa^s}$ and the latter is therefore solvable. It thus possesses no non-trivial parabolic subgroup. According to the parabolic-parahoric correspondence over K^{unr} ([BT84a, 5.1.32.(i) Proposition.]), we deduce that $\widetilde{\mathcal{C}}$ is a K^{unr}-chamber.

(2) $\Longrightarrow$ (3) According to Proposition 3.3, the Γ^{unr}-chambers are $G(K)$-conjugate since the K-chambers are. Consequently, if one Γ^{unr}-chamber is a K^{unr}-chamber, then all Γ^{unr}-chambers are by conjugation.

(3) $\Longrightarrow$ (4) Take a K-chamber $\mathcal{C}$. It comes from a Γ^{unr}-chamber $\widetilde{\mathcal{C}}$ which is therefore a K^{unr}-chamber. As before, we have $(\mathcal{G}^0_{\mathcal{C}})_{\kappa^s} = (\mathcal{G}^0_{\widetilde{\mathcal{C}}})_{\kappa^s}$. This latter group has no non-trivial parabolic subgroup and is therefore solvable according to [CGP15, Proposition 3.5.1.(4)], since its pseudo-reductive quotient is pseudo-split.

(4) $\Longrightarrow$ (1) This is trivial.

(4) $\Longrightarrow$ (5) Let $\mathcal{F}$ be a K-facet and $\mathcal{C}$ a K-chamber containing $\mathcal{F}$. According to the parabolic-parahoric correspondence, the image of $(\mathcal{G}^0_{\mathcal{C}})_\kappa \to (\mathcal{G}^0_{\mathcal{F}})_\kappa$ is a pseudo-parabolic subgroup of $(\mathcal{G}^0_{\mathcal{F}})_\kappa$. By hypothesis, it is solvable. Therefore $(\mathcal{G}^0_{\mathcal{F}})_\kappa$ is quasi-split.

(5) $\Longrightarrow$ (4) Conversely, take a K-chamber $\mathcal{C}$. By hypothesis, $(\mathcal{G}^0_{\mathcal{C}})_\kappa$ is quasi-split. It is therefore solvable because it admits no non-trivial pseudo-parabolic subgroup.

□

Remark 6.5. The hypothesis of quasi-splitness over K^{unr} intervenes notably in the proof to use the parabolic-parahoric correspondence. We do not know if the correspondence remains valid in general, and therefore a fortiori if the hypothesis of quasi-splitness over K^{unr} can be removed.

We then deduce the desired result:

Proposition 6.6. *Every quasi-split K-reductive group is residually quasi-split.*

Proof. Take a reductive group G that is quasi-split. According to Proposition 6.4, it suffices to show that for every K-facet $\mathcal{F}$, the κ-group $(\mathcal{G}_{\mathcal{F}}^0)_\kappa$ is quasi-split.

Take S, a maximal split torus of G. Its centralizer in G is a torus T. According to [BT84a, 4.6.4.(ii) Proposition.] and [BT84a, 4.6.26.], $\mathcal{G}_{\mathcal{F}}^0$ admits a unique closed split subtorus $\mathcal{S}$ with generic fiber S and its centralizer $\mathcal{T}$ in $\mathcal{G}_{\mathcal{F}}^0$ is the identity component of the Néron model of T. In particular, the centralizer of $\mathcal{S}_\kappa$ is $\mathcal{T}_\kappa$, which is commutative. Then take a cocharacter λ such that the centralizer of its image in G is that of $\mathcal{S}_\kappa$, i.e., $\mathcal{T}_\kappa$. The pseudo-parabolic subgroup associated to λ is therefore solvable. This proves that $(\mathcal{G}_{\mathcal{F}}^0)_\kappa$ is quasi-split. □

Let us return to our problem. Note for the sequel ξ^{unr} the type morphism on K^{unr}, and Ξ^{unr} the image of $G(K^{\mathrm{unr}})$ by this morphism. We can then prove the triviality of the kernel of the second map:

Proposition 6.7. *We have the equality $\xi^{\mathrm{unr}}(G(K)) = (\Xi^{\mathrm{unr}})^{\Gamma^{\mathrm{unr}}}$ and the fact that these two groups are canonically isomorphic to Ξ^{ext}. Consequently:*

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^c) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right) = 1.$$

Proof. Note that G is residually quasi-split according to Proposition 6.6. Consequently, by point (4) of Theorem 4.1, there is a canonical morphism $(\Xi^{\mathrm{unr}})^{\Gamma^{\mathrm{unr}}} \to \Xi^{\mathrm{ext}}$. Its restriction to $\xi^{\mathrm{unr}}(G(K))$ has image Ξ, which in our case equals Ξ^{ext} according to [KP23, Proposition 6.6.2]. The previous canonical morphism is therefore surjective.

Furthermore, since G is quasi-split, it admits a special vertex which remains so after passing to any separable extension. Indeed, the split case is obvious. We obtain the general case by quasi-split descent (cf. [BT84a, 4.2.3.-4.2.4.], a Chevalley valuation on a split group represents a special point, and this valuation, hence this point, descends).

Point (7) of Theorem 4.1 then says that $\mathrm{Ker}\left((\Xi^{\mathrm{unr}})^{\Gamma^{\mathrm{unr}}} \to \Xi^{\mathrm{ext}}\right)$ is trivial.

We therefore finally deduce that $\xi^{\mathrm{unr}}(G(K))$ and $(\Xi^{\mathrm{unr}})^{\Gamma^{\mathrm{unr}}}$ are isomorphic to Ξ^{ext} via the same morphism, such that $\xi^{\mathrm{unr}}(G(K)) = (\Xi^{\mathrm{unr}})^{\Gamma^{\mathrm{unr}}}$ as desired. The triviality of the kernel then follows from point (2) of Theorem 4.1. □

From all this, we finally deduce the theorem:

Theorem 6.8. *Let G be a semisimple adjoint group, quasi-split over K. We have:*

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})^0_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right) = 1$$

where $\widetilde{\mathcal{F}}$ is a Γ^{unr}-invariant facet of the building $\mathscr{B}(G_{K^{\mathrm{unr}}})$.

Now let us focus on the case of facet stabilizers. We wish to determine the kernel of the map:

$$H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))$$

for $\widetilde{\mathcal{F}}$, a Γ^{unr}-invariant facet of $\mathscr{B}(G_{K^{\mathrm{unr}}})$.

The strategy is to reduce to the absolutely almost simple case and perform explicit calculations.

Denote $G_K := \prod_{i\in I} G_i$, the decomposition of G_K into a product of K-almost simple groups. We then have the Galois-compatible equivariant bijection: $\mathscr{B}(G_{K^{\mathrm{unr}}}) \cong \prod_{i\in I}\mathscr{B}(G_{i,K^{\mathrm{unr}}})$, and thus a decomposition $\widetilde{\mathcal{F}} = \prod_{i\in I}\widetilde{\mathcal{F}_i}$ into Γ^{unr}-invariant facets. This then gives:

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right) = \prod_{i\in I}\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G_i(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}_i}) \to H^1(\Gamma^{\mathrm{unr}}, G_i(K^{\mathrm{unr}}))\right).$$

The problem thus reduces to the case where G is a K-almost simple K-group. It is written as $G := \mathrm{R}_{L/K}(G')$ where G' is an adjoint absolutely almost simple L-group and L/K is a finite separable extension. The calculation of the kernel is then further reduced to the absolutely almost simple case thanks to the following lemma:

Lemma 6.9. *Let L/K be a finite separable extension and H' a reductive group over L. Denote $\Gamma_L^{\mathrm{unr}} := \mathrm{Gal}(L^{\mathrm{unr}}/L)$ and $H := \mathrm{R}_{L/K}(H')$. Take a Γ^{unr}-invariant facet $\widetilde{\mathcal{F}}$ in $\mathscr{B}(H_{K^{\mathrm{unr}}})$. It induces a K-facet in $\mathscr{B}(H) \cong \mathscr{B}(H')$, and thus corresponds to a Γ_L^{unr}-invariant facet $\widetilde{\mathcal{F}}'$ in $\mathscr{B}(H'_{L^{\mathrm{unr}}})$. We then have the identifications:*

$$H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}})) = H^1(\Gamma_L^{\mathrm{unr}}, H'(L^{\mathrm{unr}}))$$

$$H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) = H^1(\Gamma_L^{\mathrm{unr}}, H'(L^{\mathrm{unr}})_{\widetilde{\mathcal{F}}'})$$

and this, in a functorial manner, such that:

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}}))\right) = \mathrm{Ker}\left(H^1(\Gamma_L^{\mathrm{unr}}, H'(L^{\mathrm{unr}})_{\widetilde{\mathcal{F}}'}) \to H^1(\Gamma_L^{\mathrm{unr}}, H'(L^{\mathrm{unr}}))\right).$$

Proof. Note that $H_{K^{\mathrm{unr}}}$ is given by $\mathrm{R}_{L\otimes_K K^{\mathrm{unr}}/K^{\mathrm{unr}}}(H'_{L\otimes_K K^{\mathrm{unr}}})$.

Set $L_{\mathrm{unr}} := K^{\mathrm{unr}} \cap L$, the maximal unramified extension of K in L. We then have: $L\otimes_{L_{\mathrm{unr}}} K^{\mathrm{unr}} \cong LK^{\mathrm{unr}} = L^{\mathrm{unr}}$. Consider the identification $\Gamma_L^{\mathrm{unr}} := \mathrm{Gal}(L^{\mathrm{unr}}/L) \cong \mathrm{Gal}(K^{\mathrm{unr}}/L_{\mathrm{unr}})$. This is an open subgroup of Γ^{unr}. Set $\Sigma := \mathrm{Hom}_K(L_{\mathrm{unr}}, K^{\mathrm{unr}})$ and observe the following isomorphisms of Γ^{unr}-modules:

$$\begin{aligned} K^{\mathrm{unr}} \otimes_K L \cong (K^{\mathrm{unr}} \otimes_K L_{\mathrm{unr}}) \otimes_{L_{\mathrm{unr}}} L &\cong (\prod_{\sigma\in\Sigma} {}^{\sigma}K^{\mathrm{unr}}) \otimes_{L_{\mathrm{unr}}} L \\ &\cong \prod_{\sigma\in\Sigma} ({}^{\sigma}K^{\mathrm{unr}} \otimes_{L_{\mathrm{unr}}} L) \cong \prod_{\sigma\in\Sigma} {}^{\sigma}L^{\mathrm{unr}} \end{aligned}$$

Now, since $K^{\mathrm{unr}}/L_{\mathrm{unr}}$ is separable, Σ lifts into Γ^{unr}. Still denote Σ one of its lifts. This is then a set of representatives in Γ^{unr} for $\Gamma^{\mathrm{unr}}/\Gamma_L^{\mathrm{unr}}$. We can therefore use Shapiro's lemma (in group cohomology):

$$H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}})) = H^1(\Gamma^{\mathrm{unr}}, H'(K^{\mathrm{unr}}\otimes_K L)) = H^1(\Gamma^{\mathrm{unr}}, \prod_{\sigma\in\Sigma} {}^{\sigma}H'(L^{\mathrm{unr}})) = H^1(\Gamma_L^{\mathrm{unr}}, H'(L^{\mathrm{unr}})).$$

Now let us address $H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}})$. Note that:

$$\mathrm{R}_{L\otimes_K K^{\mathrm{unr}}/K^{\mathrm{unr}}}(H'_{L\otimes_K K^{\mathrm{unr}}}) = \prod_{\sigma\in\Sigma} {}^{\sigma}\mathrm{R}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(H'_{L^{\mathrm{unr}}}).$$

The compatibility of buildings with separable Weil restrictions (cf. proof of [Rou77, Proposition 5.1.5.]) and with products gives the Galois-compatible and equivariant bijections:

$$\mathscr{B}(H_{K^{\mathrm{unr}}}) \cong \prod_{\sigma\in\Sigma}\mathscr{B}({}^{\sigma}\mathrm{R}_{L^{\mathrm{unr}}/K^{\mathrm{unr}}}(H'_{L^{\mathrm{unr}}})) \cong \prod_{\sigma\in\Sigma} {}^{\sigma}\mathscr{B}(H'_{L^{\mathrm{unr}}}).$$

Note that $\widetilde{\mathcal{F}}'$ is the image of $\widetilde{\mathcal{F}}$ in $\mathscr{B}(H'_{L^{\mathrm{unr}}})$ (i.e., by looking at the factor where $\sigma = \mathrm{id}$). In this case, the image of $\widetilde{\mathcal{F}}$ under the above correspondence is $({}^{\sigma}\widetilde{\mathcal{F}}')_{\sigma\in\Sigma}$. This identification then induces the identification: $H(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}} \cong \prod_{\sigma\in\Sigma} {}^{\sigma}H'(L^{\mathrm{unr}})_{{}^{\sigma}\widetilde{\mathcal{F}}'} = \prod_{\sigma\in\Sigma} {}^{\sigma}(H'(L^{\mathrm{unr}})_{\widetilde{\mathcal{F}}'})$. We can once again apply Shapiro's lemma:

$$H^1(\Gamma^{\mathrm{unr}}, H(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) = H^1(\Gamma^{\mathrm{unr}}, \prod_{\sigma\in\Sigma} {}^{\sigma}(H'(L^{\mathrm{unr}})_{\widetilde{\mathcal{F}}'})) = H^1(\Gamma_L^{\mathrm{unr}}, H'(L^{\mathrm{unr}})_{\widetilde{\mathcal{F}}'}).$$

The functoriality of the Shapiro isomorphism allows us to deduce the desired equality of kernels. □

Let us continue our investigation. According to Remark 4.11, semisimple groups that are residually split are such that the kernel

$$H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))$$

is trivial for every Γ^{unr}-facet $\widetilde{\mathcal{F}}$. Since we have reduced to the absolutely almost simple case, and G is quasi-split, this therefore eliminates split groups, and groups of the form ${}^2X_y$ split by a (quadratic) ramified extension (as explained in Remark 4.11). Furthermore, note that groups of type ${}^6D_4$ and ${}^3D_4$ have the same rank, so we can eliminate the situation of a group of type ${}^6D_4$ becoming of type ${}^3D_4$ over K^{unr}. Finally, since a group of type ${}^6D_4$ is split by an extension with Galois group S_3, there is no Galois extension such that it becomes of type ${}^2D_4$.

Thus, only groups of type ${}^2E_6$, ${}^2A_n$ (for $n \geq 1$), ${}^2D_n$ (for $n \geq 4$), ${}^3D_4$ and ${}^6D_4$ that are split over K^{unr} remain to be treated.

Next, since G is quasi-split, it is residually quasi-split according to Proposition 6.6, and therefore $\widetilde{\mathcal{T}}_{\max}$, the type of a Γ^{unr}-chamber, is exactly the type of a K^{unr}-chamber. Point (2) (a) of Theorem 4.3 then simplifies to:

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right) \cong (\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$$

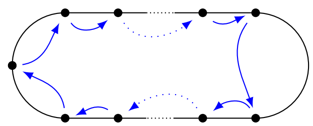
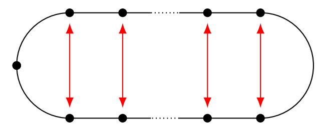
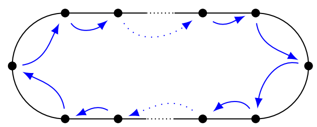
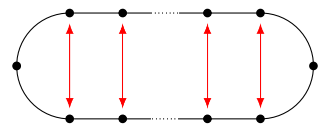
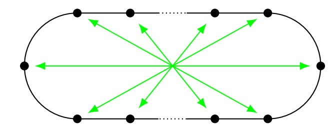
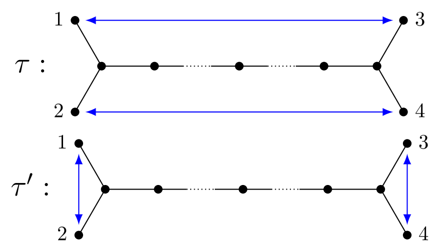
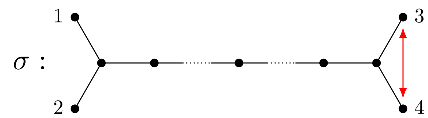
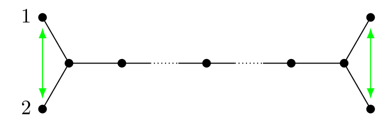
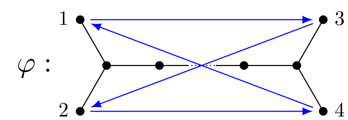
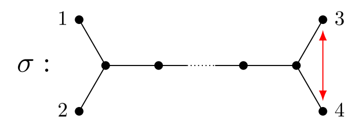
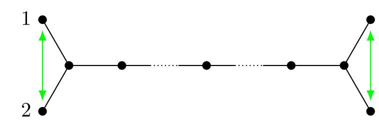
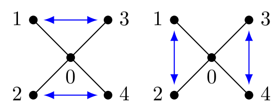
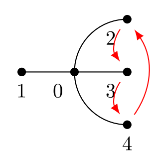
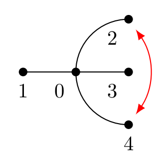
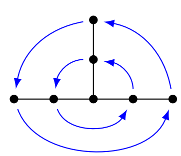
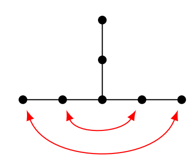
where $\widetilde{\mathcal{T}}$ is the K^{unr}-type of $\widetilde{\mathcal{F}}$.

Furthermore, since G is quasi-split and adjoint, according to Proposition 6.7, we have $\Xi^{\mathrm{unr}} = \Xi^{\mathrm{ext}}_{K^{\mathrm{unr}}}$ and $\Xi = (\Xi^{\mathrm{unr}})^{\Gamma^{\mathrm{unr}}} = \Xi^{\mathrm{ext}}$. We will therefore use the notations Ξ and Ξ^{ext} (resp. Ξ^{unr} and $\Xi^{\mathrm{ext}}_{K^{\mathrm{unr}}}$) interchangeably in the sequel.

Let us gather some data concerning the remaining cases. The list in [BT84a, 4.2.23.] then allows us to determine the affine root system over K, and more precisely the Galois action on the affine root system over K^{unr}, and the list [KP23, Remark 1.3.76] gives the associated affine Dynkin diagrams and the groups Ξ and Ξ^{unr}. We then deduce Table 1.

Note that we have added numbering on some diagrams to facilitate reasoning in the sequel.

TABLE 1. Affine Dynkin diagrams, actions of Ξ^{unr}, Galois actions, and actions of Ξ for groups split over K^{unr}.

Type	Ξ^{unr}	Generators of Ξ^{unr}	Generator of Γ^{unr}	Ξ	Generator of Ξ
$^2A_{2n}$ $(n \geq 1)$	$\mathbb{Z}/2n\mathbb{Z}$			0	$\emptyset$
$^2A_{2n-1}$ $(n \geq 1)$	$\mathbb{Z}/(2n+1)\mathbb{Z}$			$\mathbb{Z}/2\mathbb{Z}$	
$^2D_{2n}$ $(n \geq 2)$	$(\mathbb{Z}/2\mathbb{Z})^2$			$\mathbb{Z}/2\mathbb{Z}$	
$^2D_{2n-1}$ $(n \geq 3)$	$\mathbb{Z}/4\mathbb{Z}$			$\mathbb{Z}/2\mathbb{Z}$	
$^3D_4$ and $^6D_4$	$(\mathbb{Z}/2\mathbb{Z})^2$		 (and  if $^6D_4$)	0	$\emptyset$
$^2E_6$	$\mathbb{Z}/3\mathbb{Z}$			0	$\emptyset$

Let's start with type ${}^2A_n$ (for $n \geq 1$):

Proposition 6.10. *Consider the affine Dynkin diagram of type A_n (for $n \geq 1$) equipped with the Galois action Γ^{unr} given by the axial symmetry in Table 1. Its Ξ^{unr} is then $\mathbb{Z}/(n+1)\mathbb{Z}$ and is given by the rotation described in Table 1. Consider a type $\widetilde{\mathcal{T}}$ of this diagram and let m be the cardinality of its orbit under Ξ^{unr} (so $m \mid n+1$). We have:*

- *If m is odd, then $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is trivial.*
- *If m is even, then $n+1$ is also even, the set $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ contains 2 elements, and we have the following cases:*
  - *If $\frac{n+1}{m}$ is odd, then $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ is trivial.*
  - *Otherwise, if $\frac{n+1}{m}$ is even, then $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ contains 2 elements.*

In particular, $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ is trivial if $n \not\equiv 3 \pmod 4$.

Proof. For type A_n, the automorphism group is given by the dihedral group. It is thus given by the presentation $\langle r, \sigma \mid r^{n+1} = 1, \sigma^2 = 1, \sigma r \sigma = r^{-1}\rangle$. The associated group Ξ^{unr} is then the subgroup generated by r (which is thus $\mathbb{Z}/(n+1)\mathbb{Z}$), and Γ^{unr} acts through the subgroup generated by σ.

Consider a Γ^{unr}-invariant type $\widetilde{\mathcal{T}}$, which is thus given by a subset of vertices. Let's try to see if the orbit of $\widetilde{\mathcal{T}}$ under Ξ^{unr} admits another Γ^{unr}-invariant type. Let $\widetilde{\mathcal{T}}'$ be a potential such type. Let $m \in \mathbb{N}^*$ be the smallest strictly positive integer such that $r^m \cdot \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}$ (this is also the cardinality of the orbit under Ξ^{unr}).

Consider then $k \in \{0, ..., m-1\}$ such that $r^k \cdot \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}'$. Since $\widetilde{\mathcal{T}}'$ is Γ^{unr}-invariant, we have $\sigma \cdot \widetilde{\mathcal{T}}' = \widetilde{\mathcal{T}}'$. We thus have:

$$r^k \cdot \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}' = \sigma \cdot \widetilde{\mathcal{T}}' = \sigma \cdot (r^k \cdot \widetilde{\mathcal{T}}) = \sigma r^k \cdot (\sigma \cdot \widetilde{\mathcal{T}}) = \sigma r^k \sigma \cdot \widetilde{\mathcal{T}} = r^{-k} \cdot \widetilde{\mathcal{T}}$$

We conclude that $r^{2k} \cdot \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}$. Since $2k \in \{0, ..., 2m-2\}$, either $2k = 0$, or $2k = m$ by minimality of m: that is, $k = 0$ or $k = \frac{m}{2}$. If m is odd, the second possibility is ruled out and only $k = 0$ is valid. Otherwise, both possibilities are valid. There is thus 1 element that is Γ^{unr}-invariant in the orbit of $\widetilde{\mathcal{T}}$ under Ξ^{unr} if m is odd, and 2 otherwise.

Since $m \mid n+1$, if n is even, then m is always odd. Consequently, there is thus 1 element that is Γ^{unr}-invariant in the orbit of $\widetilde{\mathcal{T}}$ under Ξ^{unr}. Let us now study the case where $n = 2n' + 1$ is odd and where $m = 2m'$ is even. We thus have $m' \mid n' + 1$.

Consider now Ξ, which, according to Table 1, is none other than the group $\langle r^{n'+1}\rangle$. When are $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}' = r^{m'} \cdot \widetilde{\mathcal{T}}$ conjugate by this rotation? We have:

$$r^{n'+1} \cdot \widetilde{\mathcal{T}} = r^{m'} \cdot \widetilde{\mathcal{T}} \iff r^{m' \frac{n'+1}{m'}} \cdot \widetilde{\mathcal{T}} = r^{m'} \cdot \widetilde{\mathcal{T}}$$

If $\frac{n'+1}{m'} = \frac{n+1}{m}$ is odd, then $r^{m' \frac{n'+1}{m'}} = r^{m'}$ since $r^{2m'} \cdot \widetilde{\mathcal{T}} = r^m \cdot \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}$, and in this case the equality is satisfied. We deduce then that $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}'$ are conjugate by this rotation.

Conversely, if $\frac{n+1}{m}$ is even, we have $r^{m' \frac{n'+1}{m'}} \cdot \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}$. We must then satisfy $\widetilde{\mathcal{T}} = r^{m'} \cdot \widetilde{\mathcal{T}}$. This is impossible by minimality of m. The two types $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}'$ are then not conjugate by this rotation. We have thus considered all cases. ☐

Remark 6.11. [BT84a, 5.2.13] gives an example of a quasi-split adjoint group G of type ${}^2A_3$ split by an unramified extension with a Γ-facet $\widetilde{\mathcal{F}}$ such that $(\mathrm{Orb}(\widetilde{\mathcal{F}})_{G(L)})^{\Gamma}/G(K)$ is non-trivial, and thus, according to point (1) (a) of Theorem 4.3, such that $\mathrm{Ker}\left(H^1(\Gamma, G(L)_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma, G(L)\right)$ is non-trivial.

The previous calculation actually generalizes this result. In the example [BT84a, 5.2.13], one chooses in fact a Γ^{unr}-edge such that its orbit under Ξ^{unr} has cardinality $m = 2$. Since $\frac{n+1}{m} = \frac{4}{2} = 2$, the previous proposition allows us to conclude that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ has two elements.

Let us now address the case of type ${}^2D_n$ (for $n \geq 4$).

For this, we need the morphisms introduced in Table 1, that is, τ, τ', σ and φ. They are defined regardless of whether n is even or odd. More precisely, τ is the symmetry with respect to the central vertical axis, τ' is the symmetry with respect to the horizontal axis (or the rotation of the two extreme branches), σ is the rotation of the branch $3-4$, and φ is in fact $\tau \circ \sigma$. Note also that $\varphi^2 = \tau'$.

Let us also introduce the symbol $\oplus$ to denote "type concatenation". In other words, to two types, it associates the type given by the union of the vertices composing each type.

We can then state the result:

Proposition 6.12. *Consider the affine Dynkin diagram of type D_n (for $n \geq 4$) equipped with the actions indicated in Table 1 (for D_4, we consider the non-trialitarian case). Take $\widetilde{\mathcal{T}}$, a Γ^{unr}-invariant type of this diagram. It can then be written as $\widetilde{\mathcal{S}} \oplus \widetilde{\mathcal{R}}$ where $\widetilde{\mathcal{S}}$ is a type without the four numbered vertices from Table 1 (hence Γ^{unr}-invariant) and $\widetilde{\mathcal{R}}$, a Γ^{unr}-invariant type whose vertices are among the four numbered vertices. Then we have:*

(1) If $\widetilde{\mathcal{R}}$ has zero or four vertices, then:
   - *(a) If $\tau(\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}$, then the sets $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ and $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ are both trivial.*
   - *(b) Otherwise, they are both of cardinality 2.*

(2) If $\widetilde{\mathcal{R}}$ has an odd number of vertices, then the set $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is of cardinality 2 and $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ is trivial.

(3) If $\widetilde{\mathcal{R}}$ has two vertices, then the sets $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ and $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ are both of cardinality 2.

Proof. Of course, since $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{S}}$ are Γ^{unr}-invariant, so is $\widetilde{\mathcal{R}}$. We then observe that the only possibilities for $\widetilde{\mathcal{R}}$ are:

$$\emptyset, (1), (2), (1,2), (3,4), (1,3,4), (2,3,4), (1,2,3,4)$$

We must therefore treat each of these cases.

Observe moreover that the orbit of $\widetilde{\mathcal{T}}$ under $\langle\varphi\rangle$ is the same as under $\langle\tau, \tau'\rangle$. Indeed, this is a consequence of the fact that $\varphi = \tau \circ \sigma$ and $\varphi^2 = \tau'$. Consequently, the calculations are the same regardless of whether n is even or odd.

Quick calculations using Table 1 then yield:

(1) $\widetilde{\mathcal{R}} = \emptyset$. We find that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}} = \{\widetilde{\mathcal{S}}, \tau(\widetilde{\mathcal{S}})\}$. It is thus trivial if and only if $\widetilde{\mathcal{S}} = \tau(\widetilde{\mathcal{S}})$. We then observe that the same holds for $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$.

(2) $\widetilde{\mathcal{R}} \in \{(1), (2)\}$. We find that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}} = \{\widetilde{\mathcal{S}} \oplus (1), \widetilde{\mathcal{S}} \oplus (2)\}$. We then observe that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ is trivial.

(3) $\widetilde{\mathcal{R}} = (1,2)$. We find $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}} = \{\widetilde{\mathcal{S}} \oplus (1,2), \tau(\widetilde{\mathcal{S}}) \oplus (3,4)\}$. We then observe that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ has two elements.

(4) $\widetilde{\mathcal{R}} = (3,4)$. We find $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}} = \{\widetilde{\mathcal{S}} \oplus (3,4), \tau(\widetilde{\mathcal{S}}) \oplus (1,2)\}$. We then observe that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ has two elements.

(5) $\widetilde{\mathcal{R}} \in \{(1,3,4), (2,3,4)\}$. We find $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}} = \{\widetilde{\mathcal{S}} \oplus (1,3,4), \widetilde{\mathcal{S}} \oplus (2,3,4)\}$. We then observe that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$ is trivial.

(6) $\widetilde{\mathcal{R}} = (1,2,3,4)$. We find $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}} = \{\widetilde{\mathcal{S}} \oplus (1,2,3,4), \tau(\widetilde{\mathcal{S}}) \oplus (1,2,3,4)\}$. It is thus trivial if and only if $\widetilde{\mathcal{S}} = \tau(\widetilde{\mathcal{S}})$. We then observe that the same holds for $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$.

This concludes the proof. □

Proposition 6.13. *Consider the affine Dynkin diagram of type D_4 equipped with the Galois action Γ^{unr} given either by the rotation of 3 points, or by all possible permutations of these 3 points (in other words, the trialitarian case of Table 1). Its group Ξ^{unr} is then $(\mathbb{Z}/2\mathbb{Z})^2$. We then have that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is trivial for every type of this diagram.*

Proof. Let's reuse the numbering from Table 1. We identify a type with the n-tuple of its points. Observe then that the only Γ^{unr}-invariant types are (0), (1), $(0,1)$, $(2,3,4)$, $(0,2,3,4)$, $(1,2,3,4)$ and $(0,1,2,3,4)$. Since the action by Ξ^{unr} preserves the size of the types, we can already say that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is trivial for $\widetilde{\mathcal{T}}$ in $\{(0,1),(2,3,4),(0,1,2,3,4)\}$. Since (0) is fixed by Ξ^{unr}, we can also eliminate (0) and (1). Similarly, any type in the orbit of $(0,2,3,4)$ under Ξ^{unr} must contain 0, so $(1,2,3,4)$ cannot be in the orbit. We have thus treated all cases and $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is trivial for every Γ^{unr}-invariant type $\widetilde{\mathcal{T}}$. □

Proposition 6.14. *Consider the affine Dynkin diagram of type E_6 equipped with the Galois action Γ^{unr} given by the axial symmetry in Table 1. Its Ξ^{unr} is $\mathbb{Z}/3\mathbb{Z}$ and is given by the rotation described in Table 1. We then have that $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is trivial for every type of this diagram.*

Proof. The proof is essentially the same as for the A_n case. Let $\widetilde{\mathcal{T}}$ be a type of this diagram. Take $r \in \Xi^{\mathrm{unr}}$ and $\sigma \in \Gamma^{\mathrm{unr}}$. Once again, we have $\sigma \circ r = r^2 \circ \sigma$. If $r \cdot \widetilde{\mathcal{T}}$ is Γ^{unr}-invariant, it is such that:

$$r \cdot \widetilde{\mathcal{T}} = (\sigma \circ r) \cdot \widetilde{\mathcal{T}} = (r^2 \circ \sigma) \cdot \widetilde{\mathcal{T}} = r^2 \cdot (\sigma \cdot \widetilde{\mathcal{T}}) = r^2 \cdot \widetilde{\mathcal{T}}$$

Consequently, $\widetilde{\mathcal{T}} = r \cdot \widetilde{\mathcal{T}} = r^2 \cdot \widetilde{\mathcal{T}}$ and so $(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$ is trivial. □

Let us summarize all this using Table 2 (using the notations of the previous propositions):
We observe in particular that only the values 1 or 2 are present, and that 2 appears only when G is of type ${}^2A_{4n+3}$ (for $n \geq 0$) or ${}^2D_n$ (for $n \geq 4$), split over K^{unr}.

In conclusion, we obtain the following theorem:

Theorem 6.15. *Let G be a semisimple adjoint group quasi-split over K. Let also $\widetilde{\mathcal{F}}$ be a Γ^{unr}-invariant facet of the building $\mathscr{B}(G_{K^{\mathrm{unr}}})$. Then the kernel:*

$$\mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}})_{\widetilde{\mathcal{F}}}) \to H^1(\Gamma^{\mathrm{unr}}, G(K^{\mathrm{unr}}))\right)$$

has cardinality 2^k where k is an integer bounded above by the number of factors that are a Weil restriction of an absolutely almost simple group of type ${}^2D_n$ (for $n \geq 4$) or ${}^2A_{4n+3}$ (for $n \geq 0$) split by an unramified extension.

Remark 6.16. Of course, it is possible to compute this kernel explicitly by reducing to the absolutely almost simple case thanks to the compatibility of the kernel with the product and the Weil restriction (cf. Lemma 6.9) and by using Table 2.

Type of G	$\#(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}$	$\#(\mathrm{Orb}(\widetilde{\mathcal{T}})_{\Xi^{\mathrm{unr}}})^{\Gamma^{\mathrm{unr}}}/\Xi$
${}^2A_n$ split over K^{unr} (for $n \geq 1$)	m odd: 1 m even: 2	m odd: 1 m even and $\frac{n+1}{m}$ odd: 1 m even and $\frac{n+1}{m}$ even: 2
${}^2D_n$ split over K^{unr} (for $n \geq 4$)	$\tau(\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}$ and $\#\widetilde{\mathcal{R}} \in \{0,4\}$: 1 $\tau(\widetilde{\mathcal{S}}) \neq \widetilde{\mathcal{S}}$ and $\#\widetilde{\mathcal{R}} \in \{0,4\}$: 2 $\#\widetilde{\mathcal{R}} \in \{1,2,3\}$: 2	$\tau(\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}$ and $\#\widetilde{\mathcal{R}} \in \{0,4\}$: 1 $\tau(\widetilde{\mathcal{S}}) \neq \widetilde{\mathcal{S}}$ and $\#\widetilde{\mathcal{R}} \in \{0,4\}$: 2 $\#\widetilde{\mathcal{R}} \in \{1,3\}$: 1 $\#\widetilde{\mathcal{R}} = 2$: 2
Other types	1	1

TABLE 2. Summary of the previous calculations.

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