ARITHMETIC OF BRUHAT-TITS GROUP SCHEMES OVER A SEMI-LOCAL DEDEKIND RING

ANIS ZIDANI

Abstract. The aim of this paper is to lay the foundations for the cohomological study of Bruhat-Tits group schemes over a semi-local Dedekind ring. In particular, we obtain a simplified proof of the Grothendieck-Serre conjecture in this case and also an analogous result for Bruhat-Tits group schemes of a semisimple simply connected group.

Keywords: Algebraic groups, Reductive groups, Bruhat-Tits theory, Grothendieck-Serre conjecture, Torsors, Integral models, Semi-local Dedekind rings, Weak approximation.

MSC: 20G10, 20G15, 14L10, 14L15.

Contents

Introduction

The starting point of this article comes from the question posed by Eva Bayer-Fluckiger and Uriya A. First in [BFF17] concerning objects that generalize Bruhat-Tits group schemes over semi-local Dedekind rings.

Let us therefore consider R, a connected semi-local Dedekind ring of dimension 1, and K its field of fractions. By definition, a maximal ideal $\mathfrak{m}$ of R defines, by localization, a discrete valuation ring $R_{\mathfrak{m}}$, whose completion is denoted $\widehat{R}_{\mathfrak{m}}$. Also denote by $\widehat{K}_{\mathfrak{m}}$ the field of fractions of $\widehat{R}_{\mathfrak{m}}$.

Let us introduce the following definition:

Definition 0.1. *Let G be a reductive algebraic group over K and $\mathcal{G}$ a smooth group scheme over R such that $G := \mathcal{G}_K$. We say that $\mathcal{G}$ is a* ***facet stabilizer group scheme*** *(resp.* ***parahoric group scheme****) of G if for every maximal ideal $\mathfrak{m}$ of R, the group scheme $\mathcal{G}_{\widehat{R}_{\mathfrak{m}}}$ is the stabilizer of a facet in the Bruhat-Tits building $\mathscr{B}(G_{\widehat{K}_{\mathfrak{m}}})$, cf. [Zid, Définition 3.9.] (resp. is parahoric, cf. [Zid, Définition 6.2.]).*

Date: February 10, 2026.

Thanks to [Zid26, Proposition A.15.] and [Zid26, Proposition A.16.], this definition is compatible with our definition in the henselian case (i.e. compatible with [Zid, Définition 3.9.] and [Zid, Définition 6.2.]).

Note also that this definition coincides with the one taken by Heinloth in [Hei10], in the semisimple case, and in the case where the base is a smooth projective curve over a field.

Moreover, in the case of tori, a facet stabilizer group scheme can correspond to the Néron model of the torus (knowing that the building of a torus is reduced to a single vertex). Note also that this is an example where the considered model is not necessarily affine.

The question of Bayer-Fluckiger and First on rationally trivial torsors is thus stated as follows:

Question 0.2 ([BFF17, Question 6.4])**.** *Let $\mathcal{G}$ be a group scheme over R such that $G := \mathcal{G}_K$ is reductive. Is the base change morphism:*

$$H^1_{\text{ét}}(R, \mathcal{G}) \to H^1_{\text{ét}}(K, G)$$

injective when $\mathcal{G}$ is:

(1) a facet stabilizer group scheme of G?
(2) a parahoric group scheme of G?

In the article, the authors also assume that the residue fields of R are perfect, but specify that this is simply a simplifying hypothesis.

In [BFFH19], the same authors found a counterexample in the case where the group G is non-connected and has adjoint neutral component. This counterexample is more precisely constructed in [BFFH19, §4.]. This led them to formulate a weaker conjecture in the last paragraph of [BFFH19, §5.]: is the case (1) of Question 0.2 satisfied when the considered facet is a chamber and G is residually quasi-split over each $\widehat{K}_{\mathfrak{m}}$? (cf. [Zid, Définition 3.4.]). This was already known to Bruhat and Tits in the complete case with perfect residue field (cf. [BT87, 3.9. Lemme]). We answer this conjecture positively in this article. This is the subject of Theorem 3.9.

As mentioned in [Zid], it turns out that a counterexample where G is connected had already been found for case (1) of Question 0.2 in the case of a complete discrete valuation ring and a quasi-split adjoint group of type ${}^2A_3$ split by an unramified extension by Bruhat and Tits in [BT84, 5.2.13.].

In the article [Zid], we then generalized this counterexample and computed all possible kernels in the quasi-split and adjoint case over a henselian valued field (cf. [Zid, Théorème 6.15.]). We also showed that the kernel of the morphism in Question 0.2 in case (2) is trivial in this setting. We propose in the present article to generalize these results for any adjoint group G over K and quasi-split over each $\widehat{K}_{\mathfrak{m}}$ (cf. Theorems 3.13 and 3.12).

Despite our efforts, case (2) of Question 0.2 is still an open question when R is a henselian discrete valuation ring. When it is not necessarily henselian, we construct counterexamples in [Zid26, Chapter 3].

Note also that Question 0.2 is a generalization of the Grothendieck-Serre conjecture in the case of a discrete valuation ring. Indeed, it is the case where the parahoric group scheme is associated to a hyperspecial vertex (in this case, the group scheme is reductive, cf. [Zid, Lemme 5.2.]).

The first attempt to prove this case is due to Nisnevich in his thesis [Nis82]. The idea is to use patching techniques to show that the problem reduces to the complete case and to a decomposition problem.

In our setting, using the notations of Question 0.2, a decomposition problem would amount to asking whether the following equality is satisfied:

$$\prod_{\mathfrak{m}} G(\widehat{K}_{\mathfrak{m}}) = G(K) \prod_{\mathfrak{m}} \mathcal{G}(\widehat{R}_{\mathfrak{m}}) := \left\{ (g_K g_{\mathfrak{m}})_{\mathfrak{m}} \mid (g_K, (g_{\mathfrak{m}})_{\mathfrak{m}}) \in G(K) \times \prod_{\mathfrak{m}} \mathcal{G}(\widehat{R}_{\mathfrak{m}}) \right\}.$$

Nisnevich's decomposition problem is then the case where $\mathcal{G}$ is assumed to be reductive.

Note moreover that obtaining this decomposition also means that the *class group* (which is a priori only a pointed set) $c(\mathcal{G}) := \prod_{\mathfrak{m}} \mathcal{G}(\widehat{R}_{\mathfrak{m}}) \backslash \prod_{\mathfrak{m}} G(\widehat{K}_{\mathfrak{m}})/G(K)$ is trivial. This object was also studied by Nisnevich in his thesis (cf. [Nis82, Chapter I]).

Then, in the note [Nis84], Nisnevich brings improvements to his attempt and indicates a result of Bruhat and Tits not yet published at the time which gives the complete semisimple case.

This result (and its proof) would later be published in [BT84, 5.2.14. Proposition.], although it is not formulated in a cohomological way. We showed in [Zid] that it is indeed equivalent to the statement of the complete case, and that the reductive case can also be obtained by adjusting the proof (cf. [Zid, Proposition 5.5.]).

The case of tori was later proved by Colliot-Thélène and Sansuc in [CTS87, Theorem. 4.1.] but in a much more general context. It turns out that in our context we can provide a much simpler proof in the complete case: this is the subject of [Zid, Lemme 5.4.(2)].

Finally, Guo in [Guo22] clarifies Nisnevich's proof while this time opting for another proof of the complete case by using a reduction to the anisotropic case. He also adds the case where the ring is moreover semi-local.

We also propose in this article to obtain a simplified and new proof of this result by obtaining another proof of the decomposition problem, and combining this with the complete case that we have already treated in [Zid, Proposition 5.5.].

Our main objective is therefore to answer Question 0.2 as exhaustively as possible. The residue fields of R are then not assumed to be perfect (unless explicitly stated otherwise).

Our strategy essentially follows that of Nisnevich. We use patching techniques to split the problem into two parts: solve the complete case (which has already been explored in [Zid]) and solve a decomposition problem. However, leaving the reductive case requires using new methods (or using the known ones more cleverly).

Let us now discuss the decomposition problem. The strategy we adopt in this paper uses, for every $\mathfrak{m}$, the group $G(\widehat{K}_{\mathfrak{m}})^+$ generated by the $\widehat{K}_{\mathfrak{m}}$-points of the root subgroups of $G_{\widehat{K}_{\mathfrak{m}}}$. It relies on showing that $\prod_{\mathfrak{m}} G(\widehat{K}_{\mathfrak{m}})^+ \subset G(K) \prod_{\mathfrak{m}} \mathcal{G}(\widehat{R}_{\mathfrak{m}})$, which greatly simplifies the problem, because $\prod_{\mathfrak{m}} G(\widehat{K}_{\mathfrak{m}})^+$ is in practice large enough to conclude in a number of cases.

When G is K-isotropic, it was already known in the literature that one could approximate $\prod_{\mathfrak{m}} G(\widehat{K}_{\mathfrak{m}})^+$ with elements of $G(K)^+$ (cf. [Gil09, Lemme 5.6.]). The case where G is K-anisotropic and where there exists a $\mathfrak{m} \in \operatorname{Specm}(R)$ such that G is $\widehat{K}_{\mathfrak{m}}$-isotropic is significantly more delicate and has not been studied in the literature.

The innovative idea in this article then lies in the use of Prasad's theorem (cf. Proposition 2.1) to show that $\prod_{\mathfrak{m}} G(\widehat{K}_{\mathfrak{m}})^+ \subset G(K) \prod_{\mathfrak{m}} \mathcal{G}(\widehat{R}_{\mathfrak{m}})$, even if G is K-anisotropic: the decomposition problem is thus simplified in all cases.

The plan of this article is as follows:

(1) The first part is dedicated to patching techniques. We generalize what has already been done by Nisnevich and Guo to include the case of more general group schemes not necessarily affine (in particular those that interest us).
(2) The second part is dedicated to approximation techniques. We develop results that considerably simplify the study of the decomposition problem.
(3) The third part is dedicated to establishing crucial lemmas and the main theorems of the article.

We can already announce that in the case where the residue fields are perfect, and the group G is semisimple simply connected, Question 0.2 admits a positive answer:

Theorem 0.3. *Suppose the residue fields of R are perfect. Let G be a semisimple simply connected group. Then the facet stabilizer group schemes and the parahoric group schemes for G coincide and when $\mathcal{G}$ is one of them, the base change morphism:*

$$H^1_{\text{ét}}(R, \mathcal{G}) \to H^1_{\text{ét}}(K, G)$$

is injective.

Note also that, although we have essentially uniform proofs, what limits our results in this article (and more generally in this subject), is the fact that Bruhat-Tits theory has been poorly examined in the case of a group over a complete discretely valued field that is not quasi-split by an unramified extension.

Acknowledgments.

The author thanks Philippe Gille and Ralf Köhl for their support, their guidance and their reading of the present article. He also thanks Ofer Gabber for answering the question in Remark 1.17.

The author is also grateful to the Studienstiftung des deutschen Volkes for financially supporting this project. He was also supported by the project "Group schemes, root systems, and related representations" founded by the European Union - NextGenerationEU through Romania's National Recovery and Resilience Plan (PNRR) call no. PNRR-III-C9-2023- I8, Project CF159/31.07.2023, and coordinated by the Ministry of Research, Innovation and Digitalization (MCID) of Romania.

Notations and conventions

For any field k, the notation k^s denotes a separable closure of k.

Recall that any scheme X locally of finite presentation and separated over an integral scheme S with function field k is such that $X(S) \to X(k)$ is injective. This inclusion is implicit throughout the document (cf. [GW10, Corollary 9.9.]).

We use the definition of reductive group by Chevalley and Borel (cf. [Bor91]). In particular, they are affine, smooth and connected.

The first non-abelian set of étale and fppf cohomology considered in this article is defined by Milne in [Mil80, III.§4.] by the Čech procedure. Equivalently, they are given by the isomorphism classes of sheaf torsors, and are therefore a priori not necessarily representable by schemes (cf. [Mil80, III. Proposition 4.6.]).

In what follows, we consider R a connected semi-local Dedekind ring of dimension 1 and K its field of fractions. Everything that follows in this article trivially generalizes to the non-connected case and to the case where a component is of dimension 0.

The set $\operatorname{Specm}(R)$ denotes the maximal spectrum of R, i.e., the set of its maximal ideals. Note that in the case of R, these are also the non-zero prime ideals.

For every $\mathfrak{m} \in \operatorname{Specm}(R)$, note that $R_\mathfrak{m}$ is a discrete valuation ring and thus endows K with a discrete valuation. His residue field is denoted by $\kappa_\mathfrak{m}$. We denote by $\widehat{K}_\mathfrak{m}$ and $\widehat{R}_\mathfrak{m}$ the respective completions of K and $R_\mathfrak{m}$. Also denote by $R_\mathfrak{m}^h$, the henselization of $R_\mathfrak{m}$ and $K_\mathfrak{m}^h$ its field of fractions (cf. [Stacks, Tag 0BSK]).

Given $\mathfrak{m} \in \operatorname{Specm}(R)$, we consider a field valued by $\mathfrak{m}$, denoted $\widetilde{K}_\mathfrak{m}$, lying between K and $\widehat{K}_\mathfrak{m}$. The field K is thus dense in $\widetilde{K}_\mathfrak{m}$ for the $\mathfrak{m}$-adic topology. This field is also assumed to be henselian. We thus have: $K \subset K_\mathfrak{m}^h \subset \widetilde{K}_\mathfrak{m} \subset \widehat{K}_\mathfrak{m}$. Its ring of integers is denoted $\widetilde{R}_\mathfrak{m}$. His residue field is $\kappa_\mathfrak{m}$ as well.

It sometimes happens in the article that the hypotheses on the $\widetilde{K}_\mathfrak{m}$ are relaxed. This is then explicitly mentioned.

Also denote:

- by $\widetilde{K} := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \widetilde{K}_\mathfrak{m}$, the product of the chosen valued fields,
- by $\widetilde{R} := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \widetilde{R}_\mathfrak{m}$, the product of their ring of integers,
- by $\kappa := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \kappa_\mathfrak{m}$, the product of the residue fields,
- by $\widetilde{K}^{\mathrm{unr}} := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \widetilde{K}_\mathfrak{m}^{\mathrm{unr}}$, the product of the maximal unramified extensions,
- by $\widetilde{R}^{\mathrm{unr}} := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \widetilde{R}_\mathfrak{m}^{\mathrm{unr}}$, the product of the strict henselizations,
- by $\kappa^s := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \kappa_\mathfrak{m}^s$, the product of the separable closures,
- and finally by $I := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} I_\mathfrak{m} := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \operatorname{Gal}(\widetilde{K}_\mathfrak{m}^s / \widetilde{K}_\mathfrak{m}^{\mathrm{unr}})$, the product of inertia subgroups.

Note that $R \subset \widetilde{R}$ and $K \subset \widetilde{K}$ via the diagonal inclusion. This inclusion is implicit throughout the document.

In the case where we have $\widetilde{R}_\mathfrak{m} = \widehat{R}_\mathfrak{m}$ for every $\mathfrak{m} \in \operatorname{Specm}(R)$, note that $\widehat{R} := \widetilde{R}$ is also the completion of R by its Jacobson radical (cf. [Mat86, Theorem 8.15.]). In this case, we will also use the notation $\widehat{K} := \widetilde{K}$.

Take $\mathfrak{m} \in \operatorname{Specm}(R)$. Observe that $\widetilde{K}_\mathfrak{m}^{\mathrm{unr}} \subset \widehat{K}_\mathfrak{m}^{\mathrm{unr}} \subset \widehat{\widetilde{K}_\mathfrak{m}^{\mathrm{unr}}}$, so that $\widetilde{K}_\mathfrak{m}^{\mathrm{unr}}$ is dense in $\widehat{K}_\mathfrak{m}^{\mathrm{unr}}$. This implies that any element of $\operatorname{Gal}(\widetilde{K}_\mathfrak{m}^{\mathrm{unr}}/\widetilde{K}_\mathfrak{m})$ uniquely lifts to an element of $\operatorname{Gal}(\widehat{K}_\mathfrak{m}^{\mathrm{unr}}/\widehat{K}_\mathfrak{m})$. In fact, the induced map $\operatorname{Gal}(\widetilde{K}_\mathfrak{m}^{\mathrm{unr}}/\widetilde{K}_\mathfrak{m}) \to \operatorname{Gal}(\widehat{K}_\mathfrak{m}^{\mathrm{unr}}/\widehat{K}_\mathfrak{m})$ is bijective, and both are naturally isomorphic to $\operatorname{Gal}(\kappa_\mathfrak{m}^s/\kappa_\mathfrak{m})$.

Note that the maximal unramified extension of a complete field is not always complete. For example, the maximal unramified extension of $\kappa((t))$ is not $\kappa^s((t))$ if κ^s/κ is infinite.

Given a reductive group G over $\widetilde{K}$ (which is equivalent to giving reductive groups over the fields $\widetilde{K}_\mathfrak{m}$), we denote $\mathscr{B}(G) := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \mathscr{B}(G_{\widetilde{K}_\mathfrak{m}})$, the product of the Bruhat-Tits buildings of the $G_{\widetilde{K}_\mathfrak{m}}$ (they exist by [Zid, Proposition 1.1.]). The group $G(\widetilde{K}) = \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} G(\widetilde{K}_\mathfrak{m})$ acts naturally on $\mathscr{B}(G) := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \mathscr{B}(G_{\widetilde{K}_\mathfrak{m}})$. A facet (resp. chamber, resp. apartment) in $\mathscr{B}(G)$ is the product of facets (resp. chambers, resp. apartments) in each of the factors.

Similarly, by considering each factor separately, we generalize the notion of parahoric subgroups, stabilizer subgroups, Bruhat-Tits group schemes, etc.

Also note that $\Gamma^{\mathrm{unr}} := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} \Gamma_\mathfrak{m}^{\mathrm{unr}}$ acts naturally on $G(\widetilde{K}^{\mathrm{unr}}) := \prod_{\mathfrak{m} \in \operatorname{Specm}(R)} G(\widetilde{K}_\mathfrak{m}^{\mathrm{unr}})$.

1. Problem splitting and patching techniques

The objective of this part is to use patching techniques to separate the problem that interests us into two intermediate questions.

More precisely, we take up the idea developed by Nisnevich ([Nis82], [Nis84]) and Guo ([Guo22]). In other words, try to reduce to the case where R is local and complete (or any other more elementary situation) and understand the injectivity in that case. This therefore uses patching techniques.

Moreover, we have chosen in this section to work with algebraic spaces instead of affine schemes. Indeed, since patching techniques are not available for arbitrary schemes, working with algebraic spaces allows us to circumvent this difficulty and still obtain useful results for our problem. In a first approach, the reader can therefore consider only affine schemes.

In this part, the valued fields $\widetilde{K}_{\mathfrak{m}}$ are only assumed to contain K and to have the same residue fields as K under the $\mathfrak{m}$-adic valuations (i.e., the $\kappa_{\mathfrak{m}}$). They are therefore neither necessarily henselian, nor necessarily contained in $\widehat{K}_{\mathfrak{m}}$.

Let $\mathcal{G}$ be a group scheme over R, separated and locally of finite presentation. Also denote $G := \mathcal{G}_K$.

Question 1.1. *Consider the following commutative diagram:*

$$\begin{array}{ccccc} H^1_*(R,\mathcal{G}) & \longrightarrow & H^1_*(\widetilde{R},\mathcal{G}) & = & \prod_{\mathfrak{m}\in \mathrm{Specm}(R)} H^1_*(\widetilde{R}_{\mathfrak{m}},\mathcal{G}) \\ \downarrow & & \downarrow & & \\ H^1_*(K,G) & \longrightarrow & H^1_*(\widetilde{K},\mathcal{G}) & = & \prod_{\mathfrak{m}\in \mathrm{Specm}(R)} H^1_*(\widetilde{K}_{\mathfrak{m}},G) \end{array}$$

with $* \in \{\mathrm{fppf}, \text{ét}\}$. *What is the obstruction for this diagram to be cartesian?*

Let us recall some patching techniques:

Recall 1.2 (Patching techniques)**.** *The following functor is an equivalence of categories:*

$$\left\{\begin{array}{c}\text{Category of separated}\\ R\text{-algebraic spaces locally}\\ \text{of finite presentation}\end{array}\right\} \to \left\{\begin{array}{c}\text{Category of triples } (X',\mathfrak{X}',\tau : X'_{\widetilde{K}} \to \mathfrak{X}'_{\widetilde{K}})\\ \text{where } X' \text{ (resp. } \mathfrak{X}'\text{) is a separated algebraic space}\\ \text{loc. of fin. pres. over } K \text{ (resp. } \widetilde{R}\text{) and } \tau \text{ an isomorphism}\end{array}\right\}$$

$$\mathfrak{X} \mapsto \left(\mathfrak{X}_K, \mathfrak{X}_{\widetilde{R}}, (\mathfrak{X}_K)_{\widetilde{K}} \xrightarrow{\sim} (\mathfrak{X}_{\widetilde{R}})_{\widetilde{K}}\right).$$

Proof. This is a consequence of [MB96, Corollaire 5.6.(1)], since $\mathrm{Spec}(\widetilde{R}) \to \mathrm{Spec}(R)$ is flat by [Liu06, Corollary 1.2.14.] and induces an isomorphism at the level of closed points, since R and $\widetilde{R}$ have the same residue fields. □

By functoriality, one checks that such a functor restricts and corestricts to group algebraic spaces. By [Ana73, 4.B. Théorème], the considered group algebraic spaces are representable by schemes. We thus obtain:

Proposition 1.3. *The following functor is an equivalence of categories:*

$$\left\{\begin{array}{c}\text{Category of separated}\\ R\text{-group schemes locally}\\ \text{of finite presentation}\end{array}\right\} \to \left\{\begin{array}{c}\text{Category of triples } (G', \mathfrak{G}', \tau : G'_{\widetilde{K}} \to \mathfrak{G}'_{\widetilde{K}})\\ \text{where } G' \text{ (resp. } \mathfrak{G}'\text{) is a separated group scheme}\\ \text{loc. of fin. pres. over } K \text{ (resp. } \widetilde{R}\text{) and } \tau \text{ an isomorphism}\end{array}\right\}$$

$$\mathfrak{G} \mapsto \left(\mathfrak{G}_K, \mathfrak{G}_{\widetilde{R}}, (\mathfrak{G}_K)_{\widetilde{K}} \xrightarrow{\sim} (\mathfrak{G}_{\widetilde{R}})_{\widetilde{K}}\right).$$

What can we say now about torsors? We need a few lemmas which are moreover valid over an arbitrary base S (which is a scheme); $\mathcal{G}$ is thus assumed to be a group scheme over S not necessarily separated, nor necessarily locally of finite presentation.

Definition 1.4. *We say that $\mathfrak{X}$, an fppf sheaf on S, is a pseudo $\mathcal{G}$-torsor over S if $\mathfrak{X}$ is endowed with a free and transitive action of $\mathcal{G}$. In other words, an action such that $\mathcal{G} \times_S \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$, $(g, x) \mapsto (g.x, x)$ is an isomorphism.*

We denote by $H^1_{\text{Pseudo}}(S, \mathcal{G})$ (resp. $H^1_{\text{SLFP}}(S, \mathcal{G})$) the set of isomorphism classes of pseudo $\mathcal{G}$-torsors over S representable by algebraic spaces (resp. of pseudo $\mathcal{G}$-torsors over S representable by separated algebraic spaces locally of finite presentation).

Furthermore, we define a $\mathcal{G}$-torsor over S for the fppf (resp. étale) topology as being a sheaf endowed with an action of $\mathcal{G}$, locally isomorphic to $\mathcal{G}$ endowed with its action by translation (on the left or right depending on the convention we take).

We denote by $[\mathfrak{X}]$ the isomorphism class of $\mathfrak{X}$. Note that there is no ambiguity (neither here nor in the sequel) about the ambient category in our situation.

Remark 1.5. A pseudo $\mathcal{G}$-torsor over S is isomorphic to the trivial pseudo torsor ($\mathcal{G}$ endowed with its action by translation) if and only if it admits a section over S (cf. [Stacks, Tag 03AI]).

Lemma 1.6. *Every torsor for the fppf/étale topology is representable by an algebraic space which is a pseudo-torsor. We thus have the natural inclusions $H^1_{\text{ét}}(S, \mathcal{G}) \subset H^1_{\text{fppf}}(S, \mathcal{G}) \subset H^1_{\text{Pseudo}}(S, \mathcal{G})$. More precisely:*

(1) If $\mathcal{G}$ is flat and locally of finite presentation (resp. and also separated), the pointed set $H^1_{\text{fppf}}(S, \mathcal{G})$ is equal to the pointed set of isomorphism classes of algebraic spaces pseudo $\mathcal{G}$-torsors over S faithfully flat and locally of finite presentation (resp. and also separated).

(2) If $\mathcal{G}$ is smooth (resp. and also separated), the pointed set $H^1_{\text{ét}}(S, \mathcal{G})$ is equal to the pointed set of isomorphism classes of algebraic spaces pseudo $\mathcal{G}$-torsors over S smooth and surjective (resp. and also separated).

Proof. Note that torsors for the fppf/étale topology are representable by algebraic spaces because fppf/étale descent is always effective for them (cf. [Stacks, Tag 0ADV]).

Showing that $\mathcal{G} \times_S \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is an isomorphism can be done after fppf/étale localization. Since fppf/étale torsors are trivial fppf/étale locally, we obtain the result.

Since being flat, locally of finite presentation, smooth or separated is local for the fppf or étale topology, if $\mathcal{G}$ is so, then the fppf or étale torsors are so. Note also that $\mathcal{G}$ is always surjective over S since the morphism $\mathcal{G} \to S$ admits a section.

Conversely, let $\mathfrak{X}$ be a pseudo $\mathcal{G}$-torsor over S. Note that $\mathfrak{X} \times_S \mathfrak{X} \to \mathfrak{X}$ is a trivial $\mathcal{G}_{\mathfrak{X}}$-torsor since it possesses a section.

Now consider an étale surjective morphism $U \to \mathfrak{X}$ where U is representable by a scheme. We deduce that $(\mathfrak{X} \times_S \mathfrak{X}) \times_{\mathfrak{X}} U = \mathfrak{X} \times_S U \to U$ is also a trivial $\mathcal{G}_U$-torsor over U.

Therefore $U \to S$ is a trivializing cover for $\mathfrak{X}$. If $\mathfrak{X}$ is faithfully flat and locally of finite presentation, then U is also by composition. Therefore $\mathfrak{X}$ is trivialized by an fppf cover. Similarly, if $\mathfrak{X}$ is smooth and surjective, then U is also and therefore $\mathfrak{X}$ is trivialized by a smooth cover. Since every smooth cover can be refined to an étale cover (cf. [Stacks, Tag 055V]), we have the result. □

We thus deduce the following result:

Corollary 1.7. *If $\mathcal{G}$ is smooth, then $H^1_{\text{ét}}(S,\mathcal{G}) = H^1_{\text{fppf}}(S,\mathcal{G})$.*

Proof. An fppf torsor $\mathfrak{X}$ is fppf locally isomorphic to $\mathcal{G}$. By fppf descent, $\mathfrak{X}$ is also smooth and surjective. By the previous lemma, $[\mathfrak{X}] \in H^1_{\text{ét}}(S,\mathcal{G})$. □

Let us now return to the case where $S = \operatorname{Spec}(R)$ and $\mathcal{G}$ is separated and locally of finite presentation. We can finally state the patching techniques for torsors:

Proposition 1.8. *The following functor is an equivalence of categories:*

$$\left\{\begin{matrix}\text{Category of } R\text{-alg. spa.}\\ \text{pseudo } \mathcal{G}\text{-torsors, sep.}\\ \text{and loc. of fin. pres.}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Category of triples } (X',\mathfrak{X}',\tau : X'_{\widetilde{K}} \to \mathfrak{X}'_{\widetilde{K}})\\ \text{where } X' \text{ (resp. } \mathfrak{X}'\text{) is an alg. spa. pseudo torsor over } G \text{ (resp. } \mathcal{G}_{\widetilde{R}}\text{)}\\ \text{sep. loc. of fin. pres. over } K \text{ (resp. } \widetilde{R}\text{) and } \tau \text{ an isomorphism}\end{matrix}\right\}$$

$$\mathfrak{X} \mapsto \left(\mathfrak{X}_K, \mathfrak{X}_{\widetilde{R}}, (\mathfrak{X}_K)_{\widetilde{K}} \xrightarrow{\sim} (\mathfrak{X}_{\widetilde{R}})_{\widetilde{K}}\right).$$

If moreover $\mathcal{G}$ is flat (resp. smooth), then the previous equivalence of categories also induces one at the level of fppf torsors (resp. étale torsors).

Proof. The first result is obvious by definition of pseudo torsors and by functoriality of patching techniques (Recall 1.2): we can restrict and corestrict without difficulty.

For the second result, when $\mathcal{G}$ is flat (resp. smooth) we can also restrict and corestrict to pseudo torsors that are moreover faithfully flat (resp. smooth and surjective). Indeed, since $\operatorname{Spec}(\widetilde{R}) \to \operatorname{Spec}(R)$ is faithfully flat and quasi-compact, if a pseudo torsor is such that $\mathfrak{X}_{\widetilde{R}}$ is faithfully flat (resp. smooth and surjective), then $\mathfrak{X}$ is also by fpqc descent.

We therefore have the result by Lemma 1.6. □

Let us therefore use patching techniques to reformulate our problem. We then give a variant of [Nis84, Théorème 2.1.], or again of [Guo22, Proposition 10.]:

Theorem 1.9. *Take $* \in \{\text{SLFP}, \text{fppf}, \text{ét}\}$ (assuming moreover that $\mathcal{G}$ is flat (resp. smooth) if $* = \text{fppf}$ (resp. ét)). Denote by $\tau_g : G_{\widetilde{K}} \cong G_{\widetilde{K}}$ the isomorphism of torsors obtained by translating (to the left) by an element $g \in G(\widetilde{K})$. The map $g \mapsto (G, \mathcal{G}_{\widetilde{R}}, \tau_g)$ induces by patching the following pointed set bijection:*

$$\mathcal{G}(\widetilde{R})\backslash G(\widetilde{K})/G(K) \cong \operatorname{Ker}\left(H^1_*(R,\mathcal{G}) \to H^1_*(K,G) \times_{H^1_*(\widetilde{K},G)} H^1_*(\widetilde{R},\mathcal{G})\right).$$

Consequently, we have the natural exact sequence:

$$1 \longrightarrow \mathcal{G}(\widetilde{R})\backslash G(\widetilde{K})/G(K) \longrightarrow H^1_*(R,\mathcal{G}) \longrightarrow H^1_*(K,G) \times_{H^1_*(\widetilde{K},G)} H^1_*(\widetilde{R},\mathcal{G}) \longrightarrow 1.$$

Proof. We have a natural morphism $H^1_*(R,\mathcal{G}) \to H^1_*(K,G) \times_{H^1_*(\widetilde{K},G)} H^1_*(\widetilde{R},\mathcal{G})$ given by $[\mathfrak{X}] \to ([\mathfrak{X}_{\widetilde{R}}], [\mathfrak{X}_{\widetilde{K}}])$. This morphism is in fact surjective. Indeed, take $([X],[\mathfrak{X}'])$ in the fiber product. By definition, X and $\mathfrak{X}'$ have the same class in $H^1_*(\widetilde{K},G)$. This means that there exists an isomorphism of torsors $\tau : X_{\widetilde{K}} \to \mathfrak{X}'_{\widetilde{K}}$. We can therefore use patching techniques

(Proposition 1.8) to obtain an R-torsor $\mathfrak{X}$ over $\mathcal{G}$ that patches X and $\mathfrak{X}'$ and thus such that $[\mathfrak{X}]$ maps to $([X],[\mathfrak{X}'])$ as desired. Hence the surjectivity.

What can we say about the kernel of this morphism? We are looking for R-torsors over $\mathcal{G}$ that are trivial over $\widetilde{R}$ and $\widetilde{K}$, up to isomorphism. By patching techniques, this amounts to understanding triples of the form $(G, \mathcal{G}_{\widetilde{R}}, \tau : G_{\widetilde{K}} \xrightarrow{\sim} (\mathcal{G}_{\widetilde{R}})_{\widetilde{K}} = G_{\widetilde{K}})$ up to isomorphism. The isomorphism τ is moreover determined by the image of the neutral element which is an element of $G(\widetilde{K})$. Conversely, every element $g \in G(\widetilde{K})$ determines an isomorphism τ_g by translating by this element. The triples we are looking for are therefore exactly determined by an element of $G(\widetilde{K})$.

Let us now understand isomorphic triples. A triple $(G, \mathcal{G}_{\widetilde{R}}, \tau_{\widehat{g}})$ is isomorphic to a triple $(G, \mathcal{G}_{\widetilde{R}}, \tau_{\widehat{g}'})$ if and only if there exists $g \in G(K)$ and $p \in \mathcal{G}(\widetilde{R})$ such that the following square commutes:

$$\begin{array}{ccc} G_{\widetilde{K}} & \xrightarrow{\tau_{\widehat{g}}} & (\mathcal{G}_{\widetilde{R}})_{\widetilde{K}} \\ {\scriptstyle \tau_g}\downarrow & & \downarrow{\scriptstyle \tau_p} \\ G_{\widetilde{K}} & \xrightarrow[\tau_{\widehat{g}'}]{} & (\mathcal{G}_{\widetilde{R}})_{\widetilde{K}} \end{array}$$

In other words, $\tau_{\widehat{g}'} = \tau_p \circ \tau_{\widehat{g}} \circ \tau_g^{-1}$.

Evaluating at the neutral element, we then have $\widehat{g}' = p\,\widehat{g}\,g^{-1}$, assuming that we are manipulating left torsors. The isomorphism classes are therefore given by $\mathcal{G}(\widetilde{R})\backslash G(\widetilde{K})/G(K)$. □

Remark 1.10. Note that $\mathcal{G}(\widetilde{R})\backslash G(\widetilde{K})/G(K)$ is in bijection as pointed sets with $G(K)\backslash G(\widetilde{K})/\mathcal{G}(\widetilde{R})$ via $g \mapsto g^{-1}$. We actually obtain one or the other set by the previous calculations depending on whether we wish to work with left or right torsors. This choice has no importance.

Remark 1.11. It is interesting to note that, when $\mathcal{G}$ is affine, G is smooth, and $\widetilde{K} = \widehat{K}$, the double quotient $\mathcal{G}(\widehat{R})\backslash G(\widehat{K})/G(K)$ is isomorphic to $H^1_{\mathrm{Nis}}(R, \mathcal{G})$ by [Nis82, 2.8. Theorem]. If $\mathcal{G}$ is moreover flat, then by [Nis82, 1.3. Proposition], we have $\mathcal{G}(R^h)\backslash G(K^h)/G(K) = \mathcal{G}(\widehat{R})\backslash G(\widehat{K})/G(K)$.

Note that saying that the diagram of Question 1.1 is cartesian is equivalent to saying that we have a pointed set bijection $H^1(R,\mathcal{G}) \xrightarrow{\sim} H^1(\widetilde{R},\mathcal{G}) \times_{H^1(\widetilde{K},G)} H^1(K,G)$. In particular, $\operatorname{Ker}\left(H^1(R,\mathcal{G}) \to H^1(\widetilde{R},\mathcal{G}) \times_{H^1(\widetilde{K},G)} H^1(K,G)\right)$, hence $\mathcal{G}(\widetilde{R})\backslash G(\widetilde{K})/G(K)$, must be trivial.

However, recall that we are manipulating pointed sets and not groups a priori. Consequently, the kernel is not sufficient to understand the fibers of the morphism.

Nevertheless, the so-called twisting techniques allow us to understand its fibers. Assume from now on that $\mathcal{G}$ is flat. It can then be identified with the fppf sheaf it represents. Moreover, in what follows, we take $* \in \{\text{fppf}, \text{ét}\}$ (assuming that $\mathcal{G}$ is moreover smooth if $* = \text{ét}$). Let us recall some facts:

Take a torsor $\mathfrak{X}$ with thus $[\mathfrak{X}] \in H^1_*(R,\mathcal{G})$, and consider the twisted group of $\mathcal{G}$ by $\mathfrak{X}$ by inner automorphisms, denoted $\mathcal{G}^{\mathfrak{X}}$ (cf. [Gil15, 2.1.]). It is an fppf (or étale) form over R of $\mathcal{G}$ such that its class in $H^1_*(R, \mathrm{Aut}(\mathcal{G})))$ is given by the image of $[\mathfrak{X}]$ by the natural map $H^1_*(R,\mathcal{G}) \to H^1_*(R, \mathrm{Aut}(\mathcal{G})))$ (cf. [Gir71, Chapitre III, Corollaire 2.5.4.]). Consequently, two isomorphic torsors induce isomorphic twists.

It is also such that there exists a canonical bijection $\varphi_{\mathfrak{X}}$ from $H^1_*(R,\mathcal{G}^{\mathfrak{X}})$ to $H^1_*(R,\mathcal{G})$ that sends the class of the trivial torsor to $[\mathfrak{X}]$ (cf. [Gir71, Chapitre III, 2.6.]).

Note moreover that twisting $\mathcal{G}^{\mathfrak{X}}$ by a torsor $\mathfrak{Y}$ with thus $[\mathfrak{Y}] \in H^1_*(R, \mathcal{G}^{\mathfrak{X}})$ gives, up to isomorphism, the same group as if we twisted $\mathcal{G}$ by a torsor in the class $\varphi_{\mathfrak{X}}([\mathfrak{Y}]) \in H^1_*(R, \mathcal{G})$.

Finally, observe that thanks to [Ana73, 4.A. Théorème], an fppf/étale twist of a group scheme over R, flat, separated and locally of finite presentation (which, by descent, is an R-algebraic space in groups flat, separated and locally of finite presentation) is in fact representable by an R-group scheme. In the sequel, we can then reuse for the twists of $\mathcal{G}$ what we have already done.

From this, we deduce the following lemmas:

Lemma 1.12. *Let be* $[\mathfrak{X}] \in H^1_*(R, \mathcal{G})$. *We have:*

$$\operatorname{Ker}\left(H^1_*(R, \mathcal{G}^{\mathfrak{X}}) \to H^1_*(K, \mathcal{G}^{\mathfrak{X}})\right) \cong g^{-1}(g([\mathfrak{X}])),$$

where g *denotes* $H^1_*(R, \mathcal{G}) \to H^1_*(K, G)$.

Proof. We have the following commutative diagram with vertical arrows bijective:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \operatorname{Ker}\left(H^1_*(R, \mathcal{G}^{\mathfrak{X}}) \to H^1_*(K, \mathcal{G}^{\mathfrak{X}})\right) & \longrightarrow & H^1_*(R, \mathcal{G}^{\mathfrak{X}}) & \longrightarrow & H^1_*(K, \mathcal{G}^{\mathfrak{X}}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle 1 \mapsto [\mathfrak{X}]} & & \downarrow{\scriptstyle 1 \mapsto [\mathfrak{X}_K]} & & \\ 1 & \longrightarrow & g^{-1}(g([\mathfrak{X}])) & \longrightarrow & H^1_*(R, \mathcal{G}) & \xrightarrow{f} & H^1_*(K, G) & \longrightarrow & 1. \end{array}$$

The first line is exact. The second line is also exact by choosing $[\mathfrak{X}]$ and $[\mathfrak{X}_K]$ as neutral elements. Hence the result. □

Lemma 1.13. *Let be* $[\mathfrak{X}] \in H^1_*(R, \mathcal{G})$. *We have:*

$$(\mathcal{G}^{\mathfrak{X}})(\widetilde{R}) \backslash (\mathcal{G}^{\mathfrak{X}})(\widetilde{K}) / (\mathcal{G}^{\mathfrak{X}})(K) \cong f^{-1}(f([\mathfrak{X}])),$$

where f *denotes* $H^1_*(R, \mathcal{G}) \to H^1_*(\widetilde{R}, \mathcal{G}) \times_{H^1_*(\widetilde{K}, G)} H^1_*(K, G)$.

Proof. We have the following commutative diagram with vertical arrows bijective:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\mathcal{G}^{\mathfrak{X}})(\widetilde{R}) \backslash (\mathcal{G}^{\mathfrak{X}})(\widetilde{K}) / (\mathcal{G}^{\mathfrak{X}})(K) & \longrightarrow & H^1_*(R, \mathcal{G}^{\mathfrak{X}}) & \longrightarrow & H^1_*(\widetilde{R}, \mathcal{G}^{\mathfrak{X}}) \times_{H^1_*(\widetilde{K}, \mathcal{G}^{\mathfrak{X}})} H^1_*(K, \mathcal{G}^{\mathfrak{X}}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle 1 \mapsto [\mathfrak{X}]} & & \downarrow{\scriptstyle (1,1) \mapsto ([\mathfrak{X}_{\widetilde{R}}], [\mathfrak{X}_K])} & & \\ 1 & \longrightarrow & f^{-1}(f([\mathfrak{X}])) & \longrightarrow & H^1_*(R, \mathcal{G}) & \xrightarrow{f} & H^1_*(\widetilde{R}, \mathcal{G}) \times_{H^1_*(\widetilde{K}, G)} H^1_*(K, G) & \longrightarrow & 1. \end{array}$$

The first line is exact. The second line is also exact by choosing $[\mathfrak{X}]$ and $([\mathfrak{X}_{\widetilde{R}}], [\mathfrak{X}_K])$ as neutral elements. Hence the result. □

This allows us in particular to have results on the kernels of the arrows in the diagram of Question 1.1:

Proposition 1.14. *Denote by* $\mathfrak{C}$ *the set of* $\mathcal{G}^{\mathfrak{X}}$ *for* $[\mathfrak{X}]$ *running over the set* $\operatorname{Ker}\left(H^1_*(R, \mathcal{G}) \to H^1_*(K, G)\right)$ *(choosing only one representative for each isomorphism class). We have:*

$$\begin{array}{c} \forall \mathcal{G}' \in \mathfrak{C}, \\ \mathcal{G}'(\widetilde{R}) \backslash \mathcal{G}'(\widetilde{K}) / \mathcal{G}'(K) \\ \text{is trivial.} \end{array} \iff \begin{array}{c} \text{the kernels of } H^1_*(R, \mathcal{G}) \to H^1_*(K, G) \\ \text{and } H^1_*(\widetilde{R}, \mathcal{G}) \to H^1_*(\widetilde{K}, G) \\ \text{are in natural bijection.} \end{array}$$

Proof. Let be $[\mathfrak{X}'] \in \operatorname{Ker}\left(H^1_*(\widetilde{R}, \mathcal{G}) \to H^1_*(\widetilde{K}, \mathcal{G})\right)$. By the previous theorem, the pair $(1, [\mathfrak{X}'])$ in $H^1_*(K, G) \times_{H^1_*(\widetilde{K}, G)} H^1_*(\widetilde{R}, \mathcal{G})$ comes from a class $[\mathfrak{X}] \in H^1_*(\widetilde{R}, \mathcal{G})$. By definition, its image in $H^1_*(K, G)$ is trivial. Hence the surjectivity.

Let be an element $[\mathfrak{X}] \in \operatorname{Ker}\left(H^1_*(R,\mathcal{G}) \to H^1_*(K,\mathcal{G})\right)$. By Lemma 1.13, we have the isomorphism $(\mathcal{G}^{\mathfrak{X}})(\widetilde{R})\backslash(\mathcal{G}^{\mathfrak{X}})(\widetilde{K})/(\mathcal{G}^{\mathfrak{X}})(K) \cong f^{-1}(f([\mathfrak{X}]))$. Consequently, the double quotient is trivial if and only if $f^{-1}(f([\mathfrak{X}]))$ is trivial; that is, if and only if $[\mathfrak{X}]$ is the unique element of $H^1_*(R,\mathcal{G})$ that maps to $([\mathfrak{X}_{\widetilde{R}}],1)$ by f, or again, if and only if it is the unique element of $\operatorname{Ker}\left(H^1_*(R,\mathcal{G}) \to H^1_*(K,\mathcal{G})\right)$ having value $[\mathfrak{X}_{\widetilde{R}}]$ in $H^1_*(\widetilde{R},\mathcal{G})$. This proves the equivalence. □

We finally deduce:

Theorem 1.15. *Denote by $\mathfrak{C}$ the set of $\mathcal{G}^{\mathfrak{X}}$ for $[\mathfrak{X}]$ running over the set $H^1_*(R,\mathcal{G})$ (choosing only one representative for each isomorphism class). We have:*

$$\begin{array}{c}\forall \mathcal{G}' \in \mathfrak{C},\\ \mathcal{G}'(\widetilde{R})\backslash\mathcal{G}'(\widetilde{K})/\mathcal{G}'(K)\\ \text{is trivial.}\end{array} \Leftrightarrow \begin{array}{c}\forall \mathcal{G}' \in \mathfrak{C}, \text{the kernels of}\\ H^1_*(R,\mathcal{G}') \to H^1_*(K,\mathcal{G}')\\ \text{and } H^1_*(\widetilde{R},\mathcal{G}') \to H^1_*(\widetilde{K},\mathcal{G}')\\ \text{are in natural bijection.}\end{array} \Leftrightarrow \begin{array}{c}\text{the fibers of}\\ H^1_*(R,\mathcal{G}) \to H^1_*(K,G)\\ \text{and } H^1_*(\widetilde{R},\mathcal{G}) \to H^1_*(\widetilde{K},G)\\ \text{are in natural bijection.}\end{array} \Leftrightarrow \begin{array}{c}\text{the diagram of}\\ \text{Question 1.1}\\ \text{is cartesian.}\end{array}$$

Proof. The first equivalence is an immediate consequence of Proposition 1.14. The second equivalence comes from Lemma 1.12. Finally, the last equivalence is a classical result on cartesian diagrams. □

In summary, to answer positively Question 0.2 for a certain class of Bruhat-Tits groups, thanks to Theorem 1.15, the strategy is to establish the following three facts:

1. The class of Bruhat-Tits groups we study is stable under inner twisting (these groups are always smooth and separated);
2. The double quotient is trivial for every element of this class;
3. The triviality of the kernel is realized over $\widehat{R}$ for every element of this class.

Of course, one can envisage an analogous strategy if one is only interested in the triviality of the kernel thanks to Proposition 1.14.

Remark 1.16. As announced at the beginning of the section, the reader can avoid the notion of algebraic space by limiting themselves to affine schemes (for example if the studied group is semisimple). The proofs can then be simplified. Indeed, we use on one hand that every fpqc descent is effective for affine schemes, and on the other hand patching techniques at the level of affine schemes (cf. [MB96, Theorème 1.1]).

Remark 1.17. The viewpoint of ind-quasi-affine schemes ([Stacks, Tag 0AP5]) does not cover all the cases that interest us either, although they also satisfy fpqc descent ([Stacks, Tag 0APK]) and patching techniques (cf. below). Indeed, the Néron model $\mathcal{G}_m$ of the torus $\mathbb{G}_m$ (simple example of a non-affine Bruhat-Tits group scheme) is not ind-quasi-affine as we will establish below (proof communicated by Gabber).

Take R local with uniformizer π for simplicity. It suffices to see that the union, which we denote U, of $\pi^a\mathbb{G}_{m,R}$, $\pi^b\mathbb{G}_{m,R}$ and $\pi^c\mathbb{G}_{m,R}$ in $\mathcal{G}_m$ for a choice a,b,c of all distinct integers, is not quasi-affine. Indeed, U is quasi-compact, so the ind-quasi-affine character should imply that U is quasi-affine by definition. This would mean that $U \to \operatorname{Spec}(\mathcal{O}_U(R))$ is an open immersion (cf. (4) of [Stacks, Tag 01SM]) and thus that $\pi^b\mathbb{G}_{m,R} \to U \to \operatorname{Spec}(\mathcal{O}_U(R))$ is also.

For example, in the case where $(a,b,c) = (0,1,2)$, the ring of global functions of U is $R[X,\pi^2X^{-1}]$. Indeed, the function field of $\mathcal{G}_m$ is exactly $K(X,X^{-1})$. The functions defined on $\mathbb{G}_{m,R}$ are then $R[X,X^{-1}]$. To be also defined on $\pi\mathbb{G}_{m,R}$ and $\pi^2\mathbb{G}_{m,R}$, it is necessary to preserve $\pi R^\times$ and $\pi^2R^\times$. We then realize that the functions in question are exactly $R[X,\pi^2X^{-1}]$.

As for $\pi\mathbb{G}_{m,R}$, we realize that it is $R[\pi^{-1}X, \pi X^{-1}]$. The morphism $\pi\mathbb{G}_{m,R} \to \mathrm{Spec}(\mathcal{O}_U(R))$ is then given at the algebra level by the inclusion $R[X, \pi^2/X] \subset R[\pi^{-1}X, \pi X^{-1}]$.

A bit more formally, this gives the following morphism:

$$R[Y_1, Y_2]/(Y_1Y_2 - \pi^2) \xrightarrow{\varphi} R[Z_1, Z_2]/(Z_1Z_2 - 1)$$
$$Y_1, Y_2 \mapsto \pi Z_1, \pi Z_2$$

At the level of special fibers, we then have:

$$\kappa[Y_1, Y_2]/(Y_1Y_2) \xrightarrow{\varphi} \kappa[Z_1, Z_2]/(Z_1Z_2 - 1)$$
$$Y_1, Y_2 \mapsto 0, 0$$

In other words, we have the factorization: $(\pi\mathbb{G}_{m,R})_\kappa \to \mathrm{Spec}(\kappa) \to \mathrm{Spec}(\mathcal{O}_U(R))_\kappa$.

The morphism $\pi\mathbb{G}_{m,R} \to \mathrm{Spec}(\mathcal{O}_U(R))$ cannot therefore be an open immersion, since this is not the case over κ.

Proposition 1.18 (Patching techniques on ind-quasi-affine schemes)**.** *Denote by* INDQAFF *the fibered category of ind-quasi-affine algebraic spaces and take up the context of [MB96, 0.9] and the flatness hypothesis in [MB96, 1.0]. The functor* $\Phi_{\mathrm{INDQAFF}/S}$ *is an equivalence of categories.*

Proof. This is an immediate consequence of [MB96, Corollaire 5.4.4.] since ind-quasi-affine schemes are separated, satisfy fpqc descent ([Stacks, Tag 0APK]) and this is a local property on the base for the fpqc topology ([Stacks, Tag 0AP8]). ☐

2. Approximation techniques

In this section, K denotes an infinite field (not necessarily the field of fractions of a semi-local Dedekind ring). Let G be a reductive algebraic group over K.

We consider Σ a non-empty set (possibly infinite) of non-trivial discrete valuations of K, pairwise non-equivalent. Set $K_\Sigma := \prod_{v\in\Sigma} K_v$, where the K_v are henselian fields for the valuation v containing K. We also assume that K is dense in each K_v. Then set $G(K_\Sigma) := \prod_{v\in\Sigma} G(K_v)$. For every $v \in \Sigma$, we also view $G(K_v)$ in $G(K_\Sigma)$ by identifying it with $G(K_v) \times \prod_{w\in\Sigma\setminus\{v\}}\{1\}$.

Note also that the $G(K_v)$ are endowed with the adic topology (cf. [GGMB14, 3.1]).

Recall that the notation $G(K)^+$ denotes the subgroup of $G(K)$ generated by the K-points of the root groups of G (reduced to $\{1\}$ if there are none), or also by the K-points of the split unipotent subgroups of G, and that $RG(K)$ denotes the set of elements R-equivalent to the neutral element in $G(K)$ (cf. [CTS87, §3]). Then set $G(K_\Sigma)^+ := \prod_{v\in\Sigma} G(K_v)^+$ and $RG(K_\Sigma) := \prod_{v\in\Sigma} RG(K_v)$.

The objective of this part is to show that $G(K_\Sigma)^+ \subset \overline{G(K)}$. The underlying motivation being that $G(K_\Sigma)^+$ is an object both very manageable and sufficiently large in $G(K_\Sigma)$ to help us show the triviality of the double quotient from the previous part. We even have better. Denote by $\overline{RG(K)}$ the closure of $RG(K)$ in $G(K_\Sigma)$. We will show that $G(K_\Sigma)^+ \subset \overline{RG(K)}$.

For every $v \in \Sigma$, consider therefore the K_v-almost simple subgroups $G_{v,i}$ of $D(G)_{K_v}$, for i in a finite set I_v (cf. [Mil17, Theorem 21.51.]).

The following proposition of Prasad will play a crucial role:

Proposition 2.1 ([KP23, Proposition 2.2.14])**.** *Let L be a discretely valued henselian field and H an L-almost simple L-group. Every non-bounded open subgroup of $H(L)$ contains the subgroup $H(L)^+$.*

We will therefore show that we are indeed in the validity framework of this proposition. For this, we need to show a few lemmas.

Let's start with the following well-known lemma whose proof we recall.

Lemma 2.2. *Let H be a reductive group over an infinite field L and T a maximal torus of H. There exist $h_1, .., h_n \in H(L)$ such that* $\mathrm{Lie}(H) = \sum_{i=1}^n {}^{h_i}\mathrm{Lie}(T)$.

Proof. In the following, we use boldface to denote the underlying vector scheme of a vector space. Consider the map:

$$H \times \mathbf{Lie}(T) \xrightarrow{\varphi} \mathbf{Lie}(H)$$
$$(h, t) \mapsto \mathrm{ad}(h)(t)$$

It is dominant by the implication $(i) \implies (iv)$ of [SGA3, Exp. XIII, Théorème 5.1.] since the Cartan subgroups in a reductive group are exactly the maximal tori.

Consequently, the H-envelope of $\mathbf{Lie}(T)$, - that is the smallest vector subscheme of $\mathbf{Lie}(H)$ containing $\mathbf{Lie}(T)$ on which H acts -, is exactly $\mathbf{Lie}(H)$.

Set $E := \sum_{h \in H(L)} {}^{h}\mathrm{Lie}(T)$. This is the $H(L)$-envelope of $\mathrm{Lie}(T)$. Let us then show that $\mathbf{E}$ is H-stable. By definition, $\mathbf{E}$ is $H(L)$-stable. Since H-stability is a closed condition and $H(L)$ is dense in H (because H is unirational), we have the H-stability of $\mathbf{E}$ as desired.

Consequently, $\mathbf{E} = \mathbf{Lie}(H)$, and so $E = \mathrm{Lie}(H)$. Since $\mathrm{Lie}(H)$ is finite-dimensional, the sum defining E_0 contains a finite number of terms. This proves the result. □

Let us now show that $G_{v,i}(K_v) \cap \overline{RG(K)}$ is open for all the $G_{v,i}$.

Lemma 2.3. *Let be $v \in \Sigma$. The subgroup $\overline{RG(K)} \cap G(K_v)$ is open in $G(K_v)$ (and thus $\overline{RG(K)}$ is open in $G(K_\Sigma)$ when Σ is finite).*
In particular, for every $i \in I_v$, $G_{v,i}(K_v) \cap \overline{RG(K)}$ is an open subgroup of $G_{v,i}(K_v)$.

Proof. We use Raghunathan's trick (which comes from [Rag94, 1.2]). Consider a K-split maximal torus T of G. By Lemma 2.2, there exist $g_1, ..g_n$ such that $\mathrm{Lie}(G) = \sum_{i=1}^n {}^{g_i}\mathrm{Lie}(T)$. Take an exact sequence of tori $1 \to S \to E \xrightarrow{\pi} T \to 1$ where E is quasi-trivial (e.g., a flasque resolution of T, cf. [CTS87, Proposition 1.3.(1.3.3)]).

We can therefore consider the morphism (only of schemes!):

$$f : E^n \longrightarrow G$$
$$(x_i) \longmapsto {}^{g_1}\pi(x_1) \cdot \ldots \cdot {}^{g_n}\pi(x_n)$$

We then have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Lie}(E^n) & \xrightarrow{\mathrm{Lie}(f)} & \mathrm{Lie}(G) \\ & {\scriptstyle \mathrm{Lie}(\pi^n)} \searrow & \big\uparrow {\scriptstyle (x_i) \mapsto \sum_{i=1}^n {}^{g_i}x_i} \\ & & \mathrm{Lie}(T^n) \end{array}$$

where we know on one hand that $\mathrm{Lie}(E) \to \mathrm{Lie}(T)$ is surjective because π is smooth since S is smooth; and on the other hand $\mathrm{Lie}(T^n) \to \mathrm{Lie}(G)$ is surjective because $\mathrm{Lie}(G) = \bigoplus_{i=1}^n {}^{g_i}\mathrm{Lie}(T)$. We deduce then that $\mathrm{Lie}(f)$ is surjective. This shows that f is smooth in a neighborhood of the neutral element.

According to [GGMB14, 3.1.2 Lemme], for every $v \in \Sigma$, there exists an open set $\Omega_v \subset G(K_v)$ such that $f^{-1}(\Omega_v) \to \Omega_v$ admits a section. So $\Omega_v \subset f(E(K_v)^n)$.

Since E is quasi-trivial, it is K-rational (cf. [Mil17, Proposition 12.64.]). Consequently, by [CTG04, Proposition 2.1.], $E(K)$ is dense in $\prod_{v\in\Sigma} E(K_v)$. We deduce that $f(E(K)^n)$ is also dense in $\prod_{v\in\Sigma} f(E(K_v)^n)$. So since $f(E(K)^n) \subset RG(K)$, the group $\overline{RG(K)}$ contains $\prod_{v\in\Sigma} f(E(K_v)^n)$ and in particular $\prod_{v\in\Sigma} \Omega_v$.

Since $\overline{RG(K)} \cap G(K_v)$ contains the non-empty open set Ω_v, it is an open subgroup of $G(K_v)$. ☐

We next propose to show that $G_{v,i}(K_v) \cap \overline{RG(K)}$ is non-bounded for a potential $G_{v,i}$ isotropic over K_v. For this, we will use a lemma on tori:

Lemma 2.4. *Let T be a K-torus. We have $RT(K_\Sigma) \subset \overline{RT(K)}$.*

Proof. Take a flasque resolution $1 \to S \to E \to T \to 1$ of T. We know that $E(K)$ is dense in $E(K_\Sigma)$ by quasi-triviality. Since the image of $E(K)$ in $T(K)$ (resp. of $E(K_\Sigma)$ in $T(K_\Sigma)$) is $RT(K)$ (resp. $RT(K_\Sigma)$), we have $RT(K_\Sigma) \subset \overline{RT(K)}$ (because $E(K_\Sigma) \to T(K_\Sigma)$ is continuous for the adic topology by [GGMB14, 3.1.(ii)]). ☐

Corollary 2.5. *$G_{v,i}$ is K_v-isotropic if and only if $\overline{RG(K)} \cap G_{v,i}(K_v)$ is non-bounded.*

Proof. The reverse implication is obvious by [KP23, Theorem 2.2.9]. Let's look at the direct implication.

Take $T_i \subset G_{v,i}$ an isotropic torus of $G_{v,i}$. It is included in a maximal torus T of $D(G)_{K_v}$. Since $D(G)$ is defined over K, by weak approximation for tori (cf. e.g., the proof of [Guo22, Lemma 2.]), there exists $g \in D(G)(K_v)$ such that $T' := gTg^{-1}$ is defined over K. Set $T'_i := gT_ig^{-1}$. Since $G_{v,i}$ is normal in $D(G)_{K_v}$, T'_i is a torus of $G_{v,i}$ which is moreover isotropic since T_i is. Take a $\mathbb{G}_m$ included in T'_i. Since $\mathbb{G}_m$ is split, it is R-trivial. Consequently, we have the chain of inclusions by Lemma 2.4:

$$K_v^\times = \mathbb{G}_m(K_v) \subset RT'_i(K_v) \subset RT'_i(K_\Sigma) \subset RT'(K_\Sigma) \subset \overline{RT'(K)}.$$

Since $T'(K_\Sigma)$ is closed in $G(K_\Sigma)$, the notation $\overline{RT'(K)}$ denotes the same object, whether we place ourselves in $T'(K_\Sigma)$ or in $G(K_\Sigma)$. Since we obviously have $RT'(K) \subset RG(K)$, we have $\overline{RT'(K)} \subset \overline{RG(K)}$. But then, $K_v^\times$ belongs to $\overline{RG(K)} \cap G_{v,i}(K_v)$, we deduce that the latter is non-bounded as desired! ☐

We have therefore finally proven the lemmas necessary for our theorem:

Theorem 2.6. *Let G be a K-reductive group. We have:*

$$G(K_\Sigma)^+ = D(G)(K_\Sigma)^+ \subset \overline{RD(G)(K)} \subset \overline{RG(K)} \subset \overline{G(K)} \subset G(K_\Sigma).$$

Proof. Take $v \in \Sigma$. We have that $G_{v,i}(K_v) \cap \overline{RG(K)}$ is an open subgroup (by Lemma 2.3) non-bounded (by Corollary 2.5) of $G_{v,i}(K_v)$ for every isotropic $G_{v,i}$. This then implies that $G_{v,i}(K_v)^+ \subset \overline{RG(K)}$ by Proposition 2.1.

Observe next that, by [Mil17, Theorem 21.51.], the natural morphism $\prod_{i\in I_v} G_{v,i} \to D(G)_{K_v}$ is an isogeny. By [BT73, Corollaire 6.3.], it sends $\prod_{i\in I_v} G_{v,i}(K_v)^+$ surjectively onto $D(G)(K_v)^+$. In other words, the $G_{v,i}(K_v)^+$ generate $D(G)(K_v)^+$. Moreover, note that $D(G)(K_\Sigma)^+$ and $G(K_\Sigma)^+$ are the same groups in $G(K_\Sigma)$ thanks to [BT73, Corollaire 6.3.] applied to $D(G) \to G$.

We conclude therefore from the two previous paragraphs that $G(K_\Sigma)^+ \subset \overline{RG(K)}$. Applying what we just did for $G = D(G)$, we find $D(G)(K_\Sigma)^+ \subset \overline{RD(G)(K)}$. It then suffices to use that $RD(G)(K) \subset RG(K) \subset G(K)$ and that the inclusions pass to the closure to deduce the theorem. ☐

Remark 2.7. Obviously, $RG(K)$ and thus $\overline{RG(K)}$ is included in $\prod_{v\in\Sigma} RG(K_v)$ (an arbitrary product of closed sets is closed). Consequently, if G is semisimple simply connected, the previous theorem says that, if for every $v \in \Sigma$, G_{K_v} is strictly isotropic, then $\prod_{v\in\Sigma} RG(K_v) = G(K_\Sigma)^+ = \overline{RG(K)}$ (by [Gil09, Théorème 7.2.]). In the case where we have an anisotropic $G_{v,i}$, we do not know if $RG_{v,i}(K_v) \subset \overline{RG(K)}$; this would imply the equality $\prod_{v\in\Sigma} RG(K_v) = \overline{RG(K)}$ in full generality (since in this case, $G_{K_v} = \prod_{i\in I_v} G_{v,i}$, cf. [Mil17, Theorem 24.3.]).

There is however a case where we can conclude that the equality is indeed achieved:

For a group of type ${}^1A_n$, i.e., of the form $G := \mathrm{SL}_1(D)$, where D is a finite-dimensional division algebra over K, we have $RG(K_v) \subset \overline{RG(K)}$. Indeed, $RG(K) = [D^\times, D^\times]$ by [Vos77]. Similarly, setting $D_v := D \otimes_K K_v$, $RG(K_v) = [D_v^\times, D_v^\times]$. Consequently, the fact that $[D_v^\times, D_v^\times] \subset \overline{[D^\times, D^\times]}$ (since $D^\times$ satisfies weak approximation) gives the result.

We also have the following complementary proposition:

Proposition 2.8. *Let T be a K-torus of G. We have $RT(K_\Sigma) \subset \overline{RG(K)}$. In particular, if T is R-trivial (e.g., if T is split), then we have $T(K_\Sigma) \subset \overline{RG(K)}$.*

Moreover, for T a K_Σ-torus included in G_{K_Σ} (i.e., the data of tori in each G_{K_v}), there exists $g \in G(K_\Sigma)$ such that we have $gRT(K_\Sigma)g^{-1} \subset \overline{RG(K)}$. In particular, if T is R-trivial, then we have $gT(K_\Sigma)g^{-1} \subset \overline{RG(K)}$.

Proof. Let T be a K-torus of G. We already know by Lemma 2.4 that $RT(K_\Sigma) \subset \overline{T(K)}$. Since $T(K_\Sigma)$ is closed in $G(K_\Sigma)$, the notation $\overline{RT(K)}$ denotes the same object, whether we place ourselves in $T(K_\Sigma)$ or in $G(K_\Sigma)$. Since we obviously have $RT(K) \subset RG(K)$, we have $\overline{RT(K)} \subset \overline{RG(K)}$. Hence $RT(K_\Sigma) \subset \overline{RG(K)}$.

Now take T a K_Σ-torus of G_{K_Σ}. We write $T = \prod_{v\in\Sigma} T_v$ such that for every $v \in \Sigma$, T_v is a K_v-torus. Take $v \in \Sigma$. By weak approximation for tori (cf. e.g., the proof of [Guo22, Lemma 2.]), there exists $g_v \in G(K_v)$ such that $T'_v := g_v T_v g_v^{-1}$ is defined over K.

Observe then, by the beginning of the proof, the following inclusions:

$$g_v RT_v(K_v) g_v^{-1} = RT'_v(K_v) \subset RT'_v(K_\Sigma) \subset \overline{RG(K)}.$$

Hence $gRT(K_\Sigma)g^{-1} \subset \overline{RG(K)}$ by setting $g = (g_v)_{v\in\Sigma}$. □

Remark 2.9. It is unknown in general whether $\overline{RG(K)}$ (or $\overline{G(K)}$) is a normal subgroup of $G(K_\Sigma)$.

Let's end this part with the following general lemma.

Lemma 2.10. *Let H be a topological group, E a subset of H and U an open subgroup of H.*

(1) The set $EU := \{eu \mid (e,u) \in E \times U\}$ is open and closed in H.
(2) We have $EU = \overline{E}U$, where $\overline{E}$ is the closure of E in H.

Proof. By [Bou42, Chapitre III, §2, 5., Proposition 14.], H/U seen as a homogeneous topological space is discrete. Denote $p : H \to H/U$ the projection. In particular, $p(E)$ is open and closed in H/U. Consequently, $EU = p^{-1}(p(E))$ is open and closed in H by continuity of p.

The second point follows from the previous one. Indeed, we then have: $EU \subset \overline{E}U \subset \overline{EU} = EU$. □

As seen in part 3, this lemma allows us to bridge the gap between $\overline{G(K)}$ and the double quotient obtained by the patching methods.

3. Main Results

Let us now return to our general context, that is, R a semi-local Dedekind ring, K its field of fractions, $\widetilde{R}$ and $\widetilde{K}$, etc. Moreover, all the definitions from [Zid] generalize to $\widetilde{K} = \prod_{\mathfrak{m}\in \mathrm{Specm}(R)} \widetilde{K}_{\mathfrak{m}}$ by considering each factor separately.

Recall in particular that a subgroup H of $G(\widetilde{K})$ is said to be global if it is open and contains $G(\widetilde{K})^+$. Such a group is also said to be conformal if its action on $\mathscr{B}(G_{\widetilde{K}})$ preserves the types (cf. [Zid, Définition 2.1.]).

Let's start by gathering information related to the double quotient.

Lemma 3.1. *Let $\mathcal{G}$ be a group scheme locally of finite presentation and separated over $\widetilde{R}$. Assume that $G := \mathcal{G}_{\widetilde{K}}$ is reductive. Take H a global subgroup of $G(\widetilde{K})$, an apartment $\mathcal{A}$ of $\mathscr{B}(G_{\widetilde{K}})$ and $\mathcal{C}$, a chamber in $\mathcal{A}$. Assume that $H_{(\mathcal{A},\mathcal{C})} \subset \mathcal{G}(\widetilde{R})$.*

(1) We have $H \subset G(\widetilde{K})^+ \mathcal{G}(\widetilde{R})$.

(2) If moreover $\mathcal{G}$ is defined over R, and thus G over K, for $g \in G(\widetilde{K})$, we also have $G(K)\, g\, \mathcal{G}(\widetilde{R}) = \overline{G(K)}\, g\, \mathcal{G}(\widetilde{R})$, $g\, G(\widetilde{K})^+ \mathcal{G}(\widetilde{R}) \subset G(K)\, g\, \mathcal{G}(\widetilde{R})$, and $gH \subset G(K)\, g\, \mathcal{G}(\widetilde{R})$.

Proof.

(1) Note that $G(\widetilde{K})^+ \mathcal{G}(\widetilde{R})$ is a subgroup of $G(\widetilde{K})$ since $G(\widetilde{K})^+$ is normal in $G(\widetilde{K})$ (cf. [BT73, 6.1.]). We therefore have $H = G(\widetilde{K})^+ H_{(\mathcal{A},\mathcal{C})} \subset G(\widetilde{K})^+ \mathcal{G}(\widetilde{R})$ by [Zid, Lemme 2.8.].

(2) By Lemma 2.10, we have $G(K)\, g\, \mathcal{G}(\widetilde{R}) = \overline{G(K)g}\, \mathcal{G}(\widetilde{R}) = \overline{G(K)}\, g\, \mathcal{G}(\widetilde{R})$. Indeed, it suffices to see that $\mathcal{G}(\widetilde{R})$ is an open subgroup of $G(\widetilde{K})$. This is indeed the case because the $\mathcal{G}(\widetilde{R}_{\mathfrak{m}})$ are open in the $G(\widetilde{K}_{\mathfrak{m}})$ by [GMB23, 3.5.1 Lemme.].

This being established, we can use Theorem 2.6 which tells us that $G(\widetilde{K})^+ \subset \overline{G(K)}$. In particular,

$$g\, G(\widetilde{K})^+ \mathcal{G}(\widetilde{R}) = G(\widetilde{K})^+\, g\, \mathcal{G}(\widetilde{R}) \subset \overline{G(K)}\, g\, \mathcal{G}(\widetilde{R}) = G(K)\, g\, \mathcal{G}(\widetilde{R}).$$

Hence the result by the first part of the lemma.

□

We therefore deduce:

Proposition 3.2. *Return to the context of the previous lemma ($\mathcal{G}$ assumed defined over R). When it makes sense (for example when $G(K)\mathcal{G}(\widetilde{R})$ is a subgroup of $G(\widetilde{K})$), we denote $c'(\mathcal{G}) := G(\widetilde{K})/G(K)\, \mathcal{G}(\widetilde{R})$. Assume that $D(Z(\widetilde{K})) \subset H$, where Z is a Levi subgroup of $G_{\widetilde{K}}$.*

(1) H and $G(K)\, H$ are normal subgroups of $G(\widetilde{K})$ with abelian quotient.
We denote $c_H(G) := G(\widetilde{K})/G(K)\, H$ the quotient of $G(\widetilde{K})$ by $G(K)\, H$.

(2) If $H_{(\mathcal{A},\mathcal{C})} \subset \mathcal{G}(\widetilde{R})$ (resp. $H = G(\widetilde{K})^+ \mathcal{G}(\widetilde{R})$), then $G(K)\, \mathcal{G}(\widetilde{R})$ is a normal subgroup of $G(\widetilde{K})$ containing $G(K)\, H$ (resp. is equal to $G(K)\, H$) and with abelian quotient. Hence a surjective (resp. bijective) map $c_H(G) \to c'(\mathcal{G})$. Moreover, we have a canonical bijection:

$$c(\mathcal{G}) := G(K)\backslash G(\widetilde{K})/\mathcal{G}(\widetilde{R}) \xrightarrow{\sim} G(\widetilde{K})/G(K)\, \mathcal{G}(\widetilde{R}) =: c'(\mathcal{G}).$$

In particular, $c(\mathcal{G})$ has a natural structure of abelian group.

Proof.

(1) By [Zid, Lemme 1.5.], we have $D(G(\widetilde{K})) = G(\widetilde{K})^+ D(Z(\widetilde{K}))$. Consequently:
$$D(G(\widetilde{K})) = G(\widetilde{K})^+ D(Z(\widetilde{K})) \subset H \subset G(K)\, H.$$
Since the image of H and of $G(K)H$ in $G(\widetilde{K})^{\mathrm{ab}}$ are normal subgroups (because abelian), the same holds for H and $G(K)\, H$, and their quotients by $G(\widetilde{K})$ are of course abelian.

(2) By Lemma 3.1, we have $H \subset G(\widetilde{K})^+ \mathcal{G}(\widetilde{R})$. We then observe:
$$D(G(\widetilde{K})) \subset H \subset G(\widetilde{K})^+ \mathcal{G}(\widetilde{R}) \subset \overline{G(K)}\, \mathcal{G}(\widetilde{R}) = G(K)\, \mathcal{G}(\widetilde{R}).$$
We then conclude as before.

For the last point, it suffices to observe that, given $g \in G(\widetilde{K})$, we have:
$$\overline{G(K)} \left(g\, G(\widetilde{K})^+\, \mathcal{G}(\widetilde{R})\right) = \overline{G(K)} \left(G(\widetilde{K})^+\, g\, \mathcal{G}(\widetilde{R})\right) = \overline{G(K)}\, G(\widetilde{K})^+\, g\, \mathcal{G}(\widetilde{R}) = \overline{G(K)}\, g\, \mathcal{G}(\widetilde{R}).$$
Hence finally:
$$G(K)\, g\, \mathcal{G}(\widetilde{R}) = \overline{G(K)}\, g\, \mathcal{G}(\widetilde{R}) = \overline{G(K)}\, g \left(\mathcal{G}(\widetilde{R})\ \ G(\widetilde{K})^+\right) = \left(\mathcal{G}(\widetilde{R})\ G(\widetilde{K})^+\right) \overline{G(K)}\, g = \mathcal{G}(\widetilde{R}) \overline{G(K)}\, g = \mathcal{G}(\widetilde{R})\, G(K)\, g$$
since $G(\widetilde{K})^+ \mathcal{G}(\widetilde{R})$ is a normal subgroup of $G(\widetilde{K})$ (because it contains $D(G(\widetilde{K}))$) and $G(K)\, g\, \mathcal{G}(\widetilde{R}) = \overline{G(K)}\, g\, \mathcal{G}(\widetilde{R})$ by Lemma 3.1.

□

Lemma 3.3. *Let G be a K-reductive group, S a $\widetilde{K}$-split torus of G and $Z := Z_{G_{\widetilde{K}}}(S)$ (therefore, Z now denotes a not necessarily minimal Levi subgroup of $G_{\widetilde{K}}$). Denote $p : Z(\widetilde{K}) \to (Z/S)(\widetilde{K})$ the canonical projection. Also take H a global subgroup of $G(\widetilde{K})$ such that $D(Z(\widetilde{K})) \subset H$.*

(1) The subgroup $H \cap Z(\widetilde{K})$ is global in $Z(\widetilde{K})$.
Moreover, p is open and $p(H \cap Z(\widetilde{K}))$ is global in $(Z/S)(\widetilde{K})$.

(2) We have:
$$(Z/S)(\widetilde{K})/p(H \cap Z(\widetilde{K})) \xleftarrow{\sim} Z(\widetilde{K})/S(\widetilde{K})\,(H \cap Z(\widetilde{K})) \twoheadrightarrow c_H(G).$$

(3) If moreover S is defined over K, then Z also is, and we have:
$$c_{p(H\cap Z(\widetilde{K}))}(Z/S) = (Z/S)(\widetilde{K})/(p(H \cap Z(\widetilde{K}))\,(Z/S)(K))$$
$$\xleftarrow{\sim} Z(\widetilde{K})/((H \cap Z(\widetilde{K}))\, Z(K)\, S(\widetilde{K})) = c_{H\cap Z(\widetilde{K})}(Z) \twoheadrightarrow c_H(G).$$

Proof.

(1) Since $Z(\widetilde{K})$ and $(Z/S)(\widetilde{K})$ have no root subgroups (because they have no non-central cocharacters), we have $Z(\widetilde{K})^+$ and $(Z/S)(\widetilde{K})^+$ which are trivial. Consequently, a subgroup of $Z(\widetilde{K})$ (or $(Z/S)(\widetilde{K})$) is global if and only if it is open.

The subgroup $H \cap Z(\widetilde{K})$ is certainly open in $Z(\widetilde{K})$ since the latter is endowed with the topology induced by that of $G(\widetilde{K})$ and H is open in $G(\widetilde{K})$.

Moreover, p is open by [GGMB14, 3.1.2 Lemme] since $Z \to Z/S$ is smooth (because its kernel S is smooth). Consequently, $p(H \cap Z(\widetilde{K}))$ is open in $(Z/S)(\widetilde{K})$.

(2) Hilbert's Theorem 90 shows that $(Z/S)(\widetilde{K}) = Z(\widetilde{K})/S(\widetilde{K})$. The isomorphism is therefore a consequence of the third isomorphism theorem ([Bou70, §4, 6., Théorème 4.b)]).

Moreover, by Proposition 2.8, we have $S(\widetilde{K}) \subset \overline{G(K)}\, H = G(K)\, H$. So $(H \cap Z(\widetilde{K})\, S(\widetilde{K}) \subset G(K)\, H$. The surjectivity $Z(\widetilde{K}) \twoheadrightarrow c_H(G)$ comes from the fact that $G(\widetilde{K}) = G(\widetilde{K})^+\, Z(\widetilde{K})$ and that $G(\widetilde{K})^+ \subset H$. It therefore suffices to quotient by $(H \cap Z(\widetilde{K}))\, S(\widetilde{K})$ to obtain the desired surjective map.

(3) Again, Hilbert's Theorem 90 and the third isomorphism theorem give the isomorphism. It then suffices to see that

$$S(\widetilde{K}) \subset (H \cap Z(\widetilde{K}))\, \overline{Z(K)} = (H \cap Z(\widetilde{K}))\, Z(K)$$

by Lemma 2.10 and Proposition 2.8 to obtain the equality $Z(\widetilde{K})/((H \cap Z(\widetilde{K}))\, Z(K)\, S(\widetilde{K})) = c_{H\cap Z(\widetilde{K})}(Z)$. Finally, the surjective map is constructed as before by observing that $(H \cap Z(\widetilde{K}))\, Z(K) \subset H\, G(K)$.

□

Let us next bridge the gap between Galois cohomology and étale cohomology by the following simple lemma:

Lemma 3.4. *Let $\mathcal{G}$ be a smooth group scheme over $\widetilde{R}$. Denote $G := \mathcal{G}_{\widetilde{K}}$. Recall that $\Gamma^{\mathrm{unr}} := \prod_{\mathfrak{m}\in \mathrm{Specm}(R)} \Gamma^{\mathrm{unr}}_{\mathfrak{m}}$ acts naturally on $G(\widetilde{K}) = \prod_{\mathfrak{m}\in \mathrm{Specm}(R)} G(\widetilde{K}_{\mathfrak{m}})$. We have:*

$$\mathrm{Ker}\left(H^1_{\text{ét}}(\widetilde{R}, \mathcal{G}) \to H^1_{\text{ét}}(\widetilde{K}, G)\right) = \mathrm{Ker}\left(H^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widetilde{R}^{\mathrm{unr}})) \to H^1(\Gamma^{\mathrm{unr}}, G(\widetilde{K}^{\mathrm{unr}}))\right).$$

Proof. Obviously, the equality can be shown factor by factor. In other words, we can reduce to the case where $\widetilde{R}$ is a henselian discrete valuation ring. Observe that $H^1(\Gamma, G(\widetilde{K}^s)) = H^1_{\text{ét}}(\widetilde{K}, G)$ by [MA64, VIII, Corollaire 2.3.] (where Γ denotes the absolute Galois group of $\widetilde{K}$). Moreover, $H^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widetilde{R}^{\mathrm{unr}}))$ equals $H^1_{\text{ét}}(\widetilde{R}, \mathcal{G})$. This is a consequence of [Gil15, 2.9.2.(2)] and the fact that $H^1_{\text{ét}}(\widetilde{R}^{\mathrm{unr}}, \mathcal{G}) = 1$ since $H^1_{\text{ét}}(\widetilde{R}^{\mathrm{unr}}, \mathcal{G}) \cong H^1_{\text{ét}}(\kappa^s, \mathcal{G})$ by [SGA3, XXIV, Proposition 8.1.].

Observe next that the natural map $H^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widetilde{R}^{\mathrm{unr}})) \to H^1(\Gamma, G(\widetilde{K}^s))$ factors through $H^1(\Gamma^{\mathrm{unr}}, G(\widetilde{K}^{\mathrm{unr}}))$. By the inflation-restriction exact sequence ([Ser94, I.§5.8.a)]), we have the injection $H^1(\Gamma^{\mathrm{unr}}, G(\widetilde{K}^{\mathrm{unr}})) \to H^1(\Gamma, G(\widetilde{K}^s))$. This allows us to conclude. □

In the previous proof, we also showed that $H^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widetilde{R}^{\mathrm{unr}})) = H^1_{\text{ét}}(\widetilde{R}, \mathcal{G})$. In fact, every $\widetilde{R}$-torsor over $\mathcal{G}$ comes from a unique cocycle in $Z^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widetilde{R}^{\mathrm{unr}}))$ (cf. [Gil15, Lemme 2.2.1.] and [Gil15, 2.9. Calculs galoisiens.]). As in the end of section [Zid, 4.], we then define the twist ${}^z\mathcal{G}$ of $\mathcal{G}$ by a cocycle $z \in Z^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widetilde{R}^{\mathrm{unr}}))$ as being the twist through the torsor that z defines.

Let us now prove the main theorems of this article. In order to be faithful to Definition 0.1, we are now working with completions (i.e. $\widehat{K}$ instead of $\widetilde{K}$). We first have the following theorem:

Theorem 3.5. *Let G be a reductive group over K such that $G(\widehat{K}^{\mathrm{unr}})$ is conformal. The following map is injective:*

$$H^1_{\text{ét}}(R, \mathcal{G}) \to H^1_{\text{ét}}(K, G)$$

where $\mathcal{G}$ is a facet stabilizer R-group scheme of G.

Proof. By Theorem 1.15, it suffices to show that the class of Bruhat-Tits groups under study is stable under inner twisting, that the double quotient is trivial for every element of this class, and that the triviality of the kernel of the map is achieved over $\widehat{R}$.

Let's look at the stability. For $\mathfrak{X}$, a $\mathcal{G}$-torsor over R, the twist $\mathcal{G}^{\mathfrak{X}}$ is a group scheme with generic fiber $G^{\mathfrak{X}_K}$. The latter group is moreover isomorphic over $\widehat{K}^{\mathrm{unr}}$ to $G_{\widehat{K}^{\mathrm{unr}}}$. In particular, $(G^{\mathfrak{X}_K})(\widehat{K}^{\mathrm{unr}}) \cong G(\widehat{K}^{\mathrm{unr}})$ is conformal. Now consider $\mathfrak{X}_{\widehat{R}}$. It comes from a unique cocycle $z \in Z^1(\Gamma^{\mathrm{unr}}, \mathcal{G}(\widehat{R}^{\mathrm{unr}}))$. By [Zid, Proposition 4.14.(3)], the twist ${}^z\mathcal{G}_{\widehat{R}}$ is a facet stabilizer group scheme of $(\mathcal{G}^{\mathfrak{X}})_{\widehat{K}}$. By definition, $\mathcal{G}$ is therefore such. This concludes for the stability.

The double quotient is trivial. Indeed, take $\mathcal{C}$, a chamber of $\mathscr{B}(G_{\widehat{K}})$ containing the facet from the theorem statement. Since $G(\widehat{K}^{\mathrm{unr}})$ is conformal, $G(\widehat{K})$ is therefore $\widehat{K}$-conformal. Consequently, $G(\widehat{K})_{\mathcal{C}}$ fixes $\mathcal{C}$ and thus the facet. So $G(\widehat{K})_{\mathcal{C}} \subset \mathcal{G}(\widehat{R})$. By Lemma 3.1, $G(\widehat{K}) \subset G(K)\mathcal{G}(\widehat{R})$.

Finally, let's show that the triviality of the kernel is achieved over $\widehat{R}$. By Lemma 3.4, the statement in terms of étale cohomology is equivalent to a statement in terms of Galois cohomology over Γ^{unr}. The result then follows from [Zid, Corollaire 4.7.] since $G(\widehat{K}^{\mathrm{unr}})$ is conformal. ☐

From Theorem 3.5, we deduce:

Corollary 3.6. *Let G be a semisimple simply connected group quasi-split over $\widehat{K}^{\mathrm{unr}}$ (that is, such that, for every $\mathfrak{m} \in \mathrm{Specm}(R)$, $G_{\widehat{K}^{\mathrm{unr}}_{\mathfrak{m}}}$ is quasi-split). The following map is injective:*

$$H^1_{\text{ét}}(R, \mathcal{G}) \to H^1_{\text{ét}}(K, G)$$

where $\mathcal{G}$ is a facet stabilizer group scheme of G (or also parahoric of G, the two coincide).

Proof. By Theorem 3.5, it suffices to see that $G(\widehat{K}^{\mathrm{unr}})$ is conformal. This is a consequence of [BT84, 5.2.10.(i)]. The result [BT84, 5.2.9.] also ensures that $\mathcal{G}$ has connected fibers. ☐

This allows us to deduce Theorem 0.3:

Proof of Theorem 0.3. By the previous corollary, it suffices to show that every semisimple simply connected group over a henselian valued field with perfect residue field is quasi-split over the maximal unramified extension. This is indeed the case by [BT84, 5.1.1.]. ☐

Remark 3.7. It is reasonable to wonder if the corollary extends to semisimple simply connected groups not quasi-split over $\widehat{K}^{\mathrm{unr}}$. This is a delicate question that requires a separate study which will be carried out in a subsequent article.

This result gives in particular the semisimple simply connected case of the Nisnevich-Guo theorem. With a bit more effort, we can also prove the reductive case:

Theorem 3.8. *Let G be a reductive group over R. The following map is injective:*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G).$$

Proof. As before, thanks to Theorem 1.15, it suffices to show that the class of reductive groups over R is stable under inner twisting, that the double quotient is trivial for every element of this class, and that the triviality of the kernel of the map is achieved over $\widehat{R}$.

Stability comes from the fact that being reductive is local for the étale topology (cf. [SGA3, Exp. XIX, Définition 2.7.]).

Let us now show that the double quotient is trivial. Since G is reductive over R, we can take a maximal R-split torus $\mathcal{S}$ (with generic fiber S) and $\mathcal{Z}$ (with generic fiber Z) the associated minimal Levi subgroup (and Z is a minimal Levi subgroup of G_K for S). The quotient $\mathcal{Z}/\mathcal{S}$ is then a reductive model of Z/S and

$$(\mathcal{Z}/\mathcal{S})(\widehat{R}) = (\mathcal{Z}/\mathcal{S})(\widehat{K}) = (Z/S)(\widehat{K})$$

by [Zid, Lemme 5.2.], such that Hilbert 90 gives that $\mathcal{Z}(\widehat{R})\,S(\widehat{K}) = Z(\widehat{K})$. Now, [Zid, Lemme 5.2.] gives that $\mathcal{Z}(\widehat{R}) = Z(\widehat{K})^1$. So:

$$D(Z(\widehat{K})) \subset D(Z)(\widehat{K}) \subset Z(\widehat{K})^1 = \mathcal{Z}(\widehat{R}) \subset G(\widehat{R}) \subset G(\widehat{K})^+\,G(\widehat{R}).$$

By the point (2) of Lemma 3.3 and the obtained decomposition, we therefore have $G(\widehat{K}) = G(K)\left(G(\widehat{K})^+\,G(\widehat{R})\right)$. But Lemma 3.1 gives that $G(\widehat{K})^+\,G(\widehat{R}) \subset G(K)\,G(\widehat{R})$. Hence finally $G(\widehat{K}) = G(K)\,G(\widehat{R})$ as desired.

Finally, let's show the henselian case. By [Zid, Lemme 5.2.], we can return to the study of hyperspecial points. By Lemma 3.4, as before, we reduce to a statement in terms of Galois cohomology over Γ^{unr}. The result is then a consequence of [Zid, Proposition 5.5.]. □

The particular case where the facet is a chamber also gives a positive result:

Theorem 3.9. *Let G be a reductive group over K. Let $\mathcal{G}$ be a chamber stabilizer R-group scheme of G. Consider the following map:*

$$H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G).$$

(1) The map has trivial kernel.
(2) If moreover $G_{\widehat{K}}$ is residually quasi-split, then the map is injective.

Proof.

(1) By Proposition 1.14, we must show that twisting $\mathcal{G}$ by an element of a class in $\mathrm{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right)$ always gives a chamber stabilizer group scheme, then that the triviality of the double quotient and of the kernel over $\widehat{R}$ is achieved for these groups.

The triviality of the kernel over $\widehat{R}$ reduces again to Galois cohomology over Γ^{unr} by Lemma 3.4. We then conclude by using [Zid, Corollaire 4.8.].

From this, we also deduce stability, because being a chamber stabilizer group scheme is verified over $\widehat{R}$. Now, by triviality of the kernel over $\widehat{R}$, every twist of $\mathcal{G}$ by an element of $\mathrm{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right)$ is isomorphic over $\widehat{R}$ to $\mathcal{G}_{\widehat{R}}$.

Finally, the double quotient $\mathcal{G}(\widehat{R})\backslash G(\widehat{K})/G(K)$ is trivial by Lemma 3.1. Indeed, since $G(\widehat{K})_{(\mathcal{A},\mathcal{C})} \subset G(\widehat{K})_{\mathcal{C}} = \mathcal{G}(\widehat{R})$ point (2) gives $G(\widehat{K}) \subset G(K)\mathcal{G}(\widehat{R})$.

(2) Again, thanks to Theorem 1.15, it suffices to show that the class of groups over R that we consider is stable under inner twisting, that the double quotient is trivial for every element of this class, and that the triviality of the kernel of the map is achieved over $\widehat{R}$.

Consider $\mathcal{C}$, the $\widehat{K}$-chamber associated to $\mathcal{G}$ and the corresponding Γ^{unr}-chamber $\widetilde{\mathcal{C}}$. Take $\mathfrak{X}$, a $\mathcal{G}$-torsor over R and a cocycle $z \in Z^1(\Gamma^{\text{unr}}, G(\widehat{K}^{\text{unr}})_{\widetilde{\mathcal{C}}})$ corresponding to $\mathfrak{X}_{\widehat{R}}$. The crucial point to observe is that $\widetilde{\mathcal{C}}$ is Γ^{unr}-invariant in ${}^z\mathscr{B}(G_{\widehat{K}^{\text{unr}}})$ by [Zid, Proposition 4.13.(1)]. Since $G_{\widehat{K}}$ is residually quasi-split, $\widetilde{\mathcal{C}}$ is a $\widehat{K}^{\text{unr}}$-chamber. This therefore means that ${}^zG_{\widehat{K}}$ is also residually quasi-split, that ${}^z\mathcal{C}$ is a $\widehat{K}$-chamber of $\mathscr{B}({}^zG_{\widehat{K}})$, and that ${}^z\mathcal{G}_{\widehat{R}}$ is a group scheme stabilizer of ${}^z\mathcal{C}$ by [Zid, Proposition 4.14.]. The same therefore holds for $\mathcal{G}^{\mathfrak{X}}$.

The triviality of the double quotient and the injectivity in the case of $\widehat{R}$ is then done as before.

□

Let us finally end this article by computing exactly the kernel:

$$\operatorname{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right)$$

for the K-groups G semisimple adjoint and quasi-split over $\widehat{K}$, and where $\mathcal{G}$ is a facet stabilizer group scheme of G or a parahoric group scheme of G.

Let's start by showing that the double quotient $\mathcal{G}(\widehat{R})\backslash G(\widehat{K})/G(K)$ is trivial:

Lemma 3.10. *Let G be a semisimple adjoint group over K and quasi-split over $\widetilde{K}$. We have $G(\widetilde{K}) = \overline{RG(K)}$. In particular, $G(\widetilde{K}) = G(K)\,\mathcal{G}(\widetilde{R})$ for any group scheme $\mathcal{G}$ locally of finite presentation and separated over R such that $\mathcal{G}_K = G$.*

Proof. By Theorem 2.6 and Proposition 2.8, we have on one hand $G(\widetilde{K})^+ \subset \overline{RG(K)}$, and on the other hand the existence of a $g \in G(\widetilde{K})$ such that $g\,T(\widetilde{K})\,g^{-1} \subset \overline{RG(K)}$, where T is the centralizer of a maximal $\widetilde{K}$-split torus of $G_{\widetilde{K}}$. Indeed, since G is adjoint and quasi-split over $\widetilde{K}$, the centralizer T is an induced torus, and thus R-trivial (cf. [BT84, 4.4.16. Proposition.]). We then conclude that

$$G(\widetilde{K}) = g\,G(\widetilde{K})^+\,T(\widetilde{K})\,g^{-1} = G(\widetilde{K})^+\,g\,T(\widetilde{K})\,g^{-1} \subset \overline{RG(K)}$$

thanks to [BT73, 6.11.(i) Proposition.].

Let's next use point (2) of Lemma 3.1 to deduce $G(\widetilde{K}) = \overline{G(K)}\,\mathcal{G}(\widetilde{R}) = G(K)\,\mathcal{G}(\widetilde{R})$. □

This therefore allows us to immediately reduce to the henselian case, and thus to Galois cohomology:

Corollary 3.11. *Return to the notations of the previous lemma and assume moreover that $\mathcal{G}$ is smooth. We then have:*

$$\begin{aligned}\operatorname{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right) &= \operatorname{Ker}\left(H^1_{\text{ét}}(\widetilde{R},\mathcal{G}) \to H^1_{\text{ét}}(\widetilde{K},G)\right)\\ &= \operatorname{Ker}\left(H^1(\Gamma^{\text{unr}},\mathcal{G}(\widetilde{R}^{\text{unr}})) \to H^1(\Gamma^{\text{unr}},G(\widetilde{K}^{\text{unr}}))\right).\end{aligned}$$

Proof. Let's use Proposition 1.14. It suffices to show that the double quotient associated to a twist of $\mathcal{G}$ by an element of a class in $\operatorname{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right)$ is trivial. This is in fact obvious by the previous lemma because such a twist has a generic fiber isomorphic to G, by triviality of the element in $H^1_{\text{ét}}(K,G)$.

The second equality is then an immediate consequence of Lemma 3.4. □

We can now reuse the results [Zid, Théorème 6.8.] and [Zid, Théorème 6.15.] to obtain:

Theorem 3.12. *Let G be a semisimple adjoint group over K and quasi-split over $\widehat{K}$. We have:*

$$\operatorname{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right) = 1$$

where $\mathcal{G}$ is a parahoric group scheme of G.

Theorem 3.13. *Let G be a semisimple adjoint group over K and quasi-split over $\widehat{K}$. Also let $\mathcal{G}$ be a facet stabilizer group scheme of G. The kernel:*

$$\operatorname{Ker}\left(H^1_{\text{ét}}(R,\mathcal{G}) \to H^1_{\text{ét}}(K,G)\right)$$

has cardinality $2^{\sum_{\mathfrak{m}\in \operatorname{Specm}(R)} k_{\mathfrak{m}}}$ where, for every $\mathfrak{m} \in \operatorname{Specm}(R)$, the integer $k_{\mathfrak{m}}$ is bounded by the number of factors that are a Weil restriction of an absolutely almost simple group of type ${}^2D_n$ (for $n \geq 4$) or ${}^2A_{4n+3}$ (for $n \geq 0$) split by an unramified extension in $G_{\widehat{K}_{\mathfrak{m}}}$.

Remark 3.14. Of course, it is possible to compute this kernel explicitly by reducing to $\widehat{K}$ thanks to Corollary 3.11 and then reducing to the absolutely almost simple case thanks to the compatibility of the kernel with product and Weil restriction (cf. [Zid, Lemme 6.9.]) and by using the table [Zid, Table 2.].

References

[Ana73] Sivaramakrishna Anantharaman. Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1. In *Sur les groupes algébriques*, volume Tome 101 of *Supplément au Bull. Soc. Math. France*, pages 5–79. Soc. Math. France, Paris, 1973.

[BFF17] Eva Bayer-Fluckiger and Uriya A. First. Rationally isomorphic hermitian forms and torsors of some non-reductive groups. *Adv. Math.*, 312:150–184, 2017.

[BFFH19] E. Bayer-Fluckiger, U. A. First, and M. Huruguen. Orders that are étale-locally isomorphic. *Algebra i Analiz*, 31(4):1–15, 2019.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[Bou42] Nicolas Bourbaki. *Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie générale. Chapitres III et IV*, volume No. 916 of *Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]*. Hermann & Cie, Paris, 1942.

[Bou70] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Hermann, Paris, 1970.

[BT73] Armand Borel and Jacques Tits. Homomorphismes "abstraits" de groupes algébriques simples. *Ann. of Math. (2)*, 97:499–571, 1973.

[BT84] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984.

[BT87] François Bruhat and Jacques Tits. Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(3):671–698, 1987.

[CTG04] Jean-Louis Colliot-Thélène and Philippe Gille. Remarques sur l'approximation faible sur un corps de fonctions d'une variable. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 121–134. Birkhäuser Boston, Boston, MA, 2004.

[CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. Principal homogeneous spaces under flasque tori: applications. *J. Algebra*, 106(1):148–205, 1987.

[DG64] Michel Demazure and Alexander Grothendieck. Sga3 : Schémas en groupes. Réédition par P. Gille et P. Polo disponible sur https://webusers.imj-prg.fr/~patrick.polo/SGA3/, 1962–1964.

[GGMB14] Ofer Gabber, Philippe Gille, and Laurent Moret-Bailly. Fibrés principaux sur les corps valués henséliens. *Algebr. Geom.*, 1(5):573–612, 2014.

[Gil09] Philippe Gille. Le problème de Kneser-Tits. Number 326, pages Exp. No. 983, vii, 39–81. 2009. Séminaire Bourbaki. Vol. 2007/2008.

[Gil15] Philippe Gille. Sur la classification des schémas en groupes semi-simples. In *Autour des schémas en groupes. Vol. III*, volume 47 of *Panor. Synthèses*, pages 39–110. Soc. Math. France, Paris, 2015.

[Gir71] Jean Giraud. *Cohomologie non abélienne*, volume Band 179 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1971.

[GMB23] Philippe Gille and Laurent Moret-Bailly. Fibrés principaux et adèles. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 24(4):2403–2429, 2023.

[Guo22] Ning Guo. The Grothendieck-Serre conjecture over semilocal Dedekind rings. *Transform. Groups*, 27(3):897–917, 2022.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Hei10] Jochen Heinloth. Uniformization of $\mathcal{G}$-bundles. *Math. Ann.*, 347(3):499–528, 2010.

[KP23] Tasho Kaletha and Gopal Prasad. *Bruhat-Tits theory—a new approach*, volume 44 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2023.

[Liu06] Qing Liu. *Algebraic geometry and arithmetic curves. Transl. by Reinie Erné*, volume 6 of *Oxf. Grad. Texts Math.* Oxford: Oxford University Press, 2006.

[MA64] A. Grothendieck et J.-L. Verdier M. Artin. Sga4 : Théorie des topos et cohomologie étale des schémas. Réédition disponible sur http://fabrice.orgogozo.perso.math.cnrs.fr/SGA4/details/index.html, 1963–1964.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[MB96] Laurent Moret-Bailly. Un problème de descente. *Bull. Soc. Math. France*, 124(4):559–585, 1996.

[Mil80] James S. Milne. *Étale cohomology*, volume No. 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1980.

[Mil17] James S. Milne. *Algebraic groups. The theory of group schemes of finite type over a field*, volume 170 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2017.

[Nis82] Yevsey A. Nisnevich. *Etale Cohomology and Arithmetic of Semisimple Groups*. ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–Harvard University.

[Nis84] Yevsey A. Nisnevich. Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(1):5–8, 1984.

[Rag94] Madabusi S. Raghunathan. Principal bundles admitting a rational section. *Invent. Math.*, 116(1-3):409–423, 1994.

[Ser94] Jean-Pierre Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.

[Sta18] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

[Vos77] Viktorovich E. Voskresenskii. The reduced Whitehead group of a simple algebra. *Uspehi Mat. Nauk*, 32(6(198)):247–248, 1977.

[Zid] Anis Zidani. *Questions d'injectivité cohomologique sur les sous-groupes de Bruhat-Tits*. À paraître.

[Zid26] Anis Zidani. *Arithmetic of Bruhat-Tits Group Schemes over a Semi-Local Dedekind Ring*. PhD thesis, Kiel, 2026.

Anis Zidani, Christian-Albrechts-Universität zu Kiel, Mathematisches Seminar, Heinrich-Hecht-Platz 6, 24118 Kiel, Germany, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21 Calea Grivitei Street, 010702 Bucharest, Romania