ON A CONJECTURE I FOR UNIRATIONAL ALGEBRAIC GROUPS OVER AN IMPERFECT FIELD

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Abstract. Using the recent advancements in the structure of algebraic groups over imperfect fields, we propose a generalization of Serre's Conjecture I and of results that revolve around it. In particular, we prove that the first Galois cohomology set of any unirational algebraic group is always trivial if the cohomological dimension of the field is less or equal to 1 in Kato's sense.

Keywords: Serre's Conjecture I, Algebraic groups, Unirationality, Unipotent groups, Pseudo-reductive groups, Galois cohomology, Torsors.

MSC: 11E72, 20G15 (primary), 20G07 (secondary).

Contents

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1. Introduction

1.1. A bit of history.

In 1962, Serre asked in a conference in Brussels ([Ser62]) his famous "conjecture I" which we recall the statement:

Conjecture ([Ser62, Conjecture I]). *Let K be a perfect field such that its cohomological dimension is less than or equal to 1. Let G be a smooth connected linear algebraic group over K. Do we have: $H^1(K,G)=1$?*

Recall that a field K is said to have dimension less than or equal to 1 (in Serre's sense) if $\mathrm{Br}(L)=0$ for every algebraic extension L of K (or equivalently, for every finite separable extension L of K). Other equivalent conditions are found in [Ser94, II, §3, Prop. 5]. This is denoted by $\mathrm{scd}(K)\leqslant 1$.

This conjecture was then proven by Steinberg in 1965 in [Ste65, I.9. Thm.].

Moreover, Serre also wondered in [Ser62, 2.4. Rem. 2)] about the case where K is an imperfect field. He conjectured that the result would at least still hold for reductive groups (this has been proven later by Borel and Springer in 1968 in [BS68, 8.6], observing that Steinberg's strategy still works) and gave the following example of a smooth connected unipotent algebraic group such that its cohomology would not vanish:

Example 1.1 ([Ser62, 2.4. Rem. 2)]). Let k be a field of characteristic $p>2$. Consider the field $K=k((t))$. Let U be the K-subgroup of $\mathbb{G}_{a,K}^2$ defined by the equation $Y^p-Y=tX^p$. Then U is a smooth connected unipotent group isomorphic to $\mathbb{G}_a$ over $K((t^{\frac{1}{p}}))$ such that $H^1(K,U)\neq 0$.

Serre didn't explain why it is the case, we will then do it:

Proof. U is indeed isomorphic to $\mathbb{G}_a$ over $K':=K((t^{1/p}))$ thanks to $U_{K'}\to\mathbb{G}_{a,K'}:(x,y)\mapsto y-t^{\frac{1}{p}}x$ and $\mathbb{G}_{a,K'}\to U_{K'}:u\mapsto(t^{-\frac{1}{p}}(u^p-u),u^p)$ that are inverse of each other.

Then, observe that $\phi:(x,y)\mapsto y^p-y-tx^p$ is an additive map. We have the following exact sequence of K-groups:

$$0\to U\to\mathbb{G}_a^2\to\mathbb{G}_a\to 0.$$

This gives, when taking the cohomology:

$$0\to U(K)\to K^2\to K\to H^1(K,U)\to 0,$$

so that we have the isomorphism: $K/\{y^p-y-tx^p\mid(x,y)\in K^2\}\xrightarrow{\sim}H^1(K,U)$. Actually, since $\{y^p-y-tx^p\mid(x,y)\in K^2\}$ is an additive group, it is simply $\wp(K)+tK^p$, where $\wp:y\mapsto y^p-y$ is the Artin-Schreier map.

Let us show that at^{-1} with $a\in k^\times$ is not of the form "y^p-y-tx^p", this will give us the result. Suppose that it is the case. We then have $at^{-1}+tx^p=y^p-y$. Denote v the t-adic valuation.

Since $v(at^{-1})=-1$ and $v(tx^p)=1+pv(x)\neq -1$, we have $v(at^{-1}+tx^p)=\min(-1,v(tx^p))=-1$. Therefore $v(at^{-1}+tx^p)=v(y^p-y)<0$. Thus, $0>v(y^p-y)\geqslant\min(pv(y),v(y))$. And so $v(y^p-y)=pv(y)$. This implies $-1=v(at^{-1}+tx^p)=v(y^p-y)=pv(y)$, which is absurd. □

In fact, we have constructed a injective map from k to $H^1(K,U)$, which shows how big this set can be, despite the hypothesis of the dimension.

Serre also observed that for $p=2$, one can do something analogous (for example by taking $Y^2-Y=tX^4$ as suggested in [Ser94, III, 2.1., Exer. 3)], the proof will work similarly).

This then gives us a counter-example to Serre's conjecture I if we take k to be algebraically closed. Indeed, K would then be a (C_1) field as stated in [Ser62, 2.3] and [Ser62, 2.2.3], and would be therefore such that $\mathrm{scd}(K)\leqslant 1$ (and even such that $[K:K^p]\leqslant p$) thanks to [Ser62, 2.3].

A field verifying $\mathrm{scd}(K) \leqslant 1$ and $[K : K^p] \leqslant p$ is said to be of Kato's dimension $\leqslant 1$. This is denoted by $\dim(K) \leqslant 1$.

Actually, a similar example was already known from Rosenlicht in 1957, but he considered the group U over $k(t)$ instead and showed that it has a finite number of $k(t)$-points (cf. [Ros57, p. 46]). In particular, this shows that U doesn't contain any $k(t)$-unirational subgroup (otherwise this subgroup and hence U would have an infinite number of $k(t)$-points). This already hints at the role of unirationality in understanding a smooth connected unipotent algebraic group, as we will explain.

Indeed, recall that an algebraic group G over a field K is said to be (K-)unirational if there is a dominant rational map $\mathbb{A}^n_K \dashrightarrow G$ from some affine space. Such groups are necessarily smooth, connected, and linear. If K is infinite, $G(K)$ is dense in G since it is true for an open subset of $\mathbb{A}^n_K$.

One can thus actually wonder if Serre's example isn't too particular in the realm of smooth connected unipotent groups (and even in the realm of smooth connected linear algebraic groups) and try to see if there are other groups that would get its H^1 trivialized instead of reductive ones. This necessitates developing the theory of linear algebraic groups over imperfect fields. This actually took quite a while to get new results.

Towards this direction, between Fall 1966 and 1967, Tits gave a lecture at Yale University in which he lays the foundations of the structure of smooth connected unipotent groups over imperfect fields (this lecture is available in [Tit13a, Vol. IV, p. 657]). He notably introduced the notion of split and wound unipotent groups. The part of this lecture about the structure of smooth connected unipotent groups has then been rewritten with improved proofs in [CGP15, App. B].

And then in 1970 (resp. 1974) forms of $\mathbb{G}_a$ (resp. $\mathbb{G}_a^n$) have been classified in [Rus70] (resp. [KMT74]). After this, progress about unipotents began to slow.

Moreover, Tits gave in the Collège de France courses for the years 1991-1992 and 1992-1993 (cf. [Tit13b]) about algebraic groups over imperfect fields where he introduced the notion of pseudo-reductive group (i.e. smooth connected linear algebraic groups with trivial unipotent radical over the base field) and develop a partial theory of it.
He also gave an example of smooth connected unipotent group over $k(t)$ such that the group of $k(t)$-points is the trivial group (hence giving a more pathological example than Rosenlicht's one) by considering the equation $tX^{p^2} + Y^{p^2} + X = 0$ (even for $p = 2$).

This theory of pseudo-reductive groups has then been completed, improved and popularized thanks to the work of Conrad, Gabber and Prasad in [CGP15] and [CP16]. Note however that commutative pseudo-reductive groups are still far from being understood.

One can wonder up to this point whether Serre's conjecture I generalizes to pseudo-reductive groups, arguing that there is no unipotent to disturb things. This is in fact false as the following example shows:

Example 1.2. Consider the context of Example 1.1. By [Tot13, Cor. 9.5], there exists a commutative pseudo-reductive group G over K such that $G/T = U$ where T is the maximal torus of G. Moreover, we have $H^1(K,T) = 0$ since $\mathrm{scd}(K) \leqslant 1$ (by [Gil19, Prop. 4.6.2]), and $H^2(K,T) = 0$ (by Lemma 2.3), so $H^1(K,G) = H^1(K,U)$ which is not trivial as we saw.

After this, around 2017, Achet's work on Picard groups of unipotent groups remotivated people to study smooth connected unipotent groups. He then wrote the preprint [Ach19] in 2019 about unirational groups and then Rosengarten pushed further the general study up to this day (cf. [Ros21b], [Ros25], [Ros24] and [Ros26]).

In fact, in [Ach19], Achet proved that every commutative unirational unipotent group over a field K is the quotient of a group of the form $\mathrm{R}_{K'/K}(\mathbb{G}_m)$ where K'/K is a reduced finite K-algebra (i.e. products of finite algebraic extensions of K). As we will see in the article, this fact is crucial for cohomological computation over unirational unipotent groups.

This motivates us to understand how far a group is from being unirational. According to [BCS25, 2.1.2.(9)], any smooth algebraic group G contains a largest unirational subgroup G^{uni}. It is linear, smooth, connected and normal in G, and its construction commutes with separable extensions. Moreover, one can quotient iteratively by $(-)^{\mathrm{uni}}$ to get a natural map $G \to G'$ such that $(G')^{\mathrm{uni}} = 1$. When G is linear, smooth and connected, G' is a smooth connected wound unipotent group (since tori and split unipotent groups are unirational). It is in fact called a *strongly wound* unipotent group (cf. [Ach19, 1.3]).

In fact, one can observe that, thanks to [BCS25, Ex. 2.7.4], the group U is strongly wound! We have now the tools to understand how to generalize Serre's conjecture I.

1.2. **Results.**

Let K be a field of positive characteristic p.

In the case of Kato's dimension, as hinted in the previous part, unirationality enables us to have a Serre's Conjecture I beyond reductive groups. We are able to prove:

Theorem A (cf. Thm. 6.1)**.** *Assume* $\dim(K) \leqslant 1$. *Then for any unirational group G over K, we have* $H^1(K, G) = 1$.

Actually, this result holds for unirational solvable groups with just the assumption $\mathrm{scd}(K) \leqslant 1$, as stated in Proposition 3.3.

The proof of this result is implemented in Sections 3 to 6.

- Section 3: We see that we can reduce the problem from unirational groups to perfect groups, that is groups equal to their own derived subgroups.
- Section 4: We see that we can take the quotient of a perfect group by its unipotent K-radical (see Section 2), so that the problem reduces to pseudo-semisimple groups, that is perfect pseudo-reductive groups.
- Section 5: We review the generalized standard construction for pseudo-reductive groups, and by a case-by-case analysis, prove a refined version of Serre's Conjecture I, assuming that K has dimension $\leqslant 1$ only at prime numbers depending on the type of the group, as in the classical case for semisimple groups.
- Section 6: We combine the results from the previous sections to obtain Theorem A.

In the proof of Theorem A, we can pinpoint where hypothesis $[K : K^p] = p$ is needed. It is used in characteristic 2 and 3 to treat the H^1 sets of pseudo-semisimple groups, and it is used in Section 4 to prove that the unipotent radical of a unirational perfect group is also unirational.

Given a general smooth algebraic group G over K, under the assumption $[K : K^p] \leqslant p$, we can isolate the part concentrating all of the complexity of G regarding its Galois cohomology. As explained in Subsection 7.2, the group G has a largest unirational subgroup G^{uni}, and when $[K : K^p] \leqslant p$, the group G/G^{uni} has no non-trivial unirational subgroups. We then have:

Theorem B (cf. Thm. 7.3)**.** *Let G be a (not necessarily connected nor linear) smooth algebraic group over K. If* $\dim(K) \leqslant 1$, *then the quotient morphism* $G \to G' := G/G^{uni}$ *induces a bijection* $H^1(K, G) \xrightarrow{\sim} H^1(K, G')$. *Moreover,* $(G')^{\mathrm{uni}} = 1$, *i.e. it has no non-trivial unirational subgroup.*

Theorem B is actually a consequence of the previous theorem and of an improvement of [Ser94, III. 2.4, Thm. 3] which is only stated for *perfect* fields. In Subsection 7.1, we indeed check that we can adapt the proof of [Ser94, III. 2.4, Thm. 3], by using the notion of pseudo-Borel/parabolic subgroups from [CGP15] and [CP16] instead of Borel/parabolic subgroups to get:

Theorem C (cf. Thm. 7.1)**.** *Assume* $\mathrm{scd}(K) \leqslant 1$ *and let* G *be a (not necessarily linear) smooth algebraic group over* K*. Then every* G*-homogeneous space* X *is dominated by a* G*-torsor.*

The classical result about Serre's Conjecture I over imperfect fields implies the following: with the assumption $\mathrm{scd}(K) \leqslant 1$, every reductive group over K is quasi-split, meaning it admits a Borel subgroup over K, that is a solvable parabolic subgroup (cf. [Gil19, Thm. 5.2.5]).

In general, Pseudo-Borel subgroups, i.e. solvable pseudo-parabolic subgroups, are the natural generalization of Borel subgroups beyond reductive groups (see Subsection 6.2). A smooth connected linear group is thus said to be *quasi-split* if it has a pseudo-Borel subgroup over K. We then show:

Theorem D (cf. Thm. 6.3)**.** *Suppose that* $\mathrm{scd}(K) \leqslant 1$ *and moreover* $[K : K^p] \leqslant p$ *if* $p \in \{2, 3\}$*. Then every smooth connected linear group over* K *is quasi-split.*

It has the following consequence to Bruhat-Tits theory:

Proposition 1.3 (cf. Prop. 6.4)**.** *Let* G *be a reductive group over an henselian discretely valued field* K*. Denote by* κ *the residue field of* K*. Suppose that* $\mathrm{scd}(\kappa) \leqslant 1$ *and moreover* $[\kappa : \kappa^p] \leqslant p$ *if* $p \in \{2, 3\}$*. Then* G *is residually quasi-split.*

In any case, we still ask the following:

Question 1.4. *Are these previous results still true when supposing only that* $\mathrm{scd}(K) \leqslant 1$*?*

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2. Notation, overview of basic notions

Let us start briefly with the different notions and notations underlying the paper.

Throughout the paper, we respect the following points:

- K shall denote a field of positive characteristic p. Let us fix an algebraic closure $\overline{K}$ of K, denote by K_s the induced separable closure of K, and set $\Gamma_K := \mathrm{Gal}(K_s/K)$.
- By algebraic K-group (or algebraic group over K), we mean a group scheme over K of finite type. See [SGA3, Exp. VIA].
- For any finite field extension L/K (not necessarily separable), the Weil restriction of an algebraic L-group G is written $\mathrm{R}_{L/K}(G)$. See [CGP15, App. A.5].
- Our reductive groups are supposed to be connected (this is the convention of [Bor91]).

Dimension of a field.

For any prime $l \neq p$, we say that $\mathrm{scd}_l(K) \leqslant 1$, if for all l-primary torsion finite discrete Γ_K-modules M, we have $H^i(K, M) = 0$ for all $i \geqslant 2$.

Also, say that $\mathrm{scd}_p(K) \leqslant 1$ if $\mathrm{Br}(L)[p] = 0$ for all finite separable field extensions L/K, where $\mathrm{Br}(L)$ is the Brauer group of L, and $\mathrm{Br}(L)[p]$ is the subgroup formed by the p-torsion elements.

More generally, we say that $\mathrm{scd}(K) \leqslant 1$ if $\mathrm{scd}_l(K) \leqslant 1$ for all primes l, and say that $\dim(K) \leqslant 1$ if $\mathrm{scd}(K) \leqslant 1$ and $[K : K^p] \leqslant p$.

Remark 2.1. Let L/K be any finite field extension and assume $\mathrm{scd}_l(K) \leqslant 1$ (resp. $[K : K^p] \leqslant p$) for some prime l. Then $\mathrm{scd}_l(L) \leqslant 1$ (resp. $[L : L^p] \leqslant p$). This is [Ser94, II, §3.1, Prop. 5]. Serre's proposition is stated for $\mathrm{scd}(K)$ but holds for $\mathrm{scd}_l(K)$.

Unirational groups.

A finite type K-scheme X is called *(K-)unirational* if there is a rational schematically dominant map $\mathbb{A}^n_K \dashrightarrow X$. This means that there exists a non-empty open subscheme $U \subset \mathbb{A}^n_K$ and a regular map $U \to X$ which doesn't factor through any proper closed subscheme of X. When X is moreover irreducible, then X is unirational if and only if there exists a non-empty open subscheme $U \subset \mathbb{A}^n_K$ and a regular map $U \to X$ whose image is topologically dense in X.

If G is an unirational algebraic K-group, then G is a smooth connected linear algebraic group, as explained at the beginning of the proof of [Ros24, Thm. 7.9]. This fact enables to conveniently say "unirational algebraic group" instead of "unirational smooth connected linear algebraic group".

Pseudo-reductive groups.

A *pseudo-reductive K-group* is a smooth connected linear algebraic group G over K which does not admit any non-trivial smooth connected normal unipotent K-subgroup. These groups have been studied in [CGP15] and [CP16]. In general, for a smooth connected linear algebraic K-group G, we define its *unipotent K-radical* $\mathrm{R}_{u,K}(G)$ as its largest smooth connected normal unipotent K-subgroup. The quotient $G/\mathrm{R}_{u,K}(G)$ is always pseudo-reductive and there is a natural extension:

$$1 \to \mathrm{R}_{u,K}(G) \to G \to G/\mathrm{R}_{u,K}(G) \to 1,$$

so that the study of linear algebraic groups over imperfect fields roughly splits into the study of unipotent groups and of pseudo-reductive groups. A *pseudo-semisimple* K-group is a pseudo-reductive K-group G which is perfect, that is $G = \mathcal{D}(G)$.

Split and wound unipotent groups.

A smooth connected unipotent K-group U is called *split* if it admits a composition series whose successive quotients are isomorphic to $\mathbb{G}_{a,K}$. It is also called *(K-)wound* (cf. [CGP15, Def. B.2.1]) if any scheme map $f : \mathbb{A}^1_K \to U$ from the affine line to U is constant to some point of $U(K)$.

Thanks to [CGP15, Prop. B.3.2], we know that a smooth connected unipotent U is wound if and only if it has no central subgroup isomorphic to the additive group $\mathbb{G}_a$. In general, there exists a unique smooth connected normal split unipotent K-subgroup U_{split} of U such that U/U_{split} is wound (cf. [CP16, Thm. B.3.4]).

The largest smooth subgroup of an algebraic K-group.

Recall that, for any algebraic K-group G, there exists a largest smooth subgroup G^{sm}. It is such that $G(K') = G^{\mathrm{sm}}(K')$ for all separable extensions K'/K. It is also functorial in G and it commutes with separable base change (cf. [CGP15, Lem. C.4.1 & Rmk. C.4.2]).

Cohomology.

Given any algebraic K-group G, $H^i(K, G)$ will stand for the *fppf* i-th cohomology set of G. When G is smooth, $H^i(K, G)$ is identified with the étale, or Galois cohomology set.

For this paper, we need the following convenient lemmas:

Lemma 2.2. *Let K'/K be a finite field extension. Then for all algebraic K'-groups G', the canonical map is injective:*

$$H^1(K, \mathrm{R}_{K'/K}(G')) \to H^1(K', G').$$

It is moreover bijective when G' is smooth.

Proof. Injectivity is [SGA3, XXIV, Prop. 8.2]. Surjectivity in the smooth case is [SGA3, XXIV, Rem. 8.5]. □

Lemma 2.3. *Assume $\mathrm{scd}(K) \leqslant 1$ and let H be a commutative algebraic K-group. Then the group $H^2(K, H)$ is trivial.*

Proof. Consider the identity component $H^0 \subset H$. The group H/H^0 is étale and algebraic, and therefore finite. Hence $H^i(K, H/H^0) = 0$ since $\mathrm{scd}(K) \leqslant 1$. Moreover, thanks to [Mil17, Prop. 8.28], there exists a normal connected linear algebraic K-group $L \subset H^0$ such that H^0/L is an abelian variety. This shows that the problem reduces to the case of connected linear algebraic groups and to the case of abelian varieties.

Let us consider the case where H is a commutative linear algebraic K-group. The group H is an extension of a commutative unipotent group H/M by a group of multiplicative type M ([Mil17, Thm. 16.13]). So we just have to show $H^2(K, H/M) = 0$ and $H^2(K, M) = 0$. The fact that $H^2(K, H/M) = 0$ is [TV13, Lem. 3.3].

As for $H^2(K, M)$, choose a quasi-trivial torus such that M embeds in P, and denote by Q the quotient. The exact sequence in cohomology gives us:

$$H^1(K, Q) \to H^2(K, M) \to H^2(K, P).$$

We know by [Gil19, Prop. 4.6.2] that $H^1(K, Q)$ is trivial (since $\mathrm{scd}(K) \leqslant 1$). Moreover, the group $H^2(K, P)$ is a product of groups of the form $H^2(K', \mathbb{G}_m)$ for different algebraic extensions of K thanks to Shapiro's lemma. Since $\mathrm{scd}(K) \leqslant 1$, these groups are trivial. Hence $H^2(K, M) = 0$.

Suppose now that H is an abelian variety. For any prime l, the homomorphism $[l] : H \to H$ is surjective, and its kernel $H[l]$ is a finite group scheme (cf. [Stacks, Tag 03RP]). Let us now fix a l. The exact sequence in cohomology gives us:

$$H^2(K, H[l]) \to H^2(K, H) \to H^2(K, H).$$

But $H[l]$ is commutative and linear, so $H^2(K, H[l]) = 0$ thanks to what we did previously. This shows that $H^2(K, H)$ hasn't any l-torsion since the multiplication by l is injective. Since it is true for any l, we have $H^2(K, H) = 0$. □

3. From unirational groups to perfect groups

The objective of this section is to show that the problem of the triviality of $H^1(K, G)$ for an unirational algebraic K-group G reduces to the case where G is a perfect group (when $\mathrm{scd}(K) \leqslant 1$ of course). This is shown by proving the triviality when G is commutative.

Let us first discuss the case of unipotent groups, which only requires a restriction on the separable cohomological p-dimension:

Proposition 3.1. *Let U be a commutative unirational unipotent algebraic group over K. If $\mathrm{scd}_p(K) \leqslant 1$, then we have $H^1(K, U) = 1$.*

Proof. First of all, we reduce to the case where U is K-wound. According to [CGP15, Thm. B.3.4], there exists a unique smooth connected K-split unipotent K-subgroup $U_{\mathrm{split}} \subset U$ such that U/U_{split} is K-wound. The long exact sequence in cohomology then gives

$$H^1(K, U_{\mathrm{split}}) \to H^1(K, U) \to H^1(K, U/U_{\mathrm{split}}).$$

Since $H^1(K, U_{\mathrm{split}})$ is trivial, it suffices to prove that $H^1(K, U/U_{\mathrm{split}})$ is trivial to conclude that $H^1(K, U)$ is. Let us then suppose that U is K-wound. Since U is K-wound and unirational, according to [Ach19, Prop. 2.5], the group U is a quotient of a group W which is a product of groups of the form

$$\mathrm{R}_{L/K}\left(\mathrm{R}_{K'/L}(\mathbb{G}_{m,K'})/\mathbb{G}_{m,L}\right) \tag{3.1}$$

for a finite field extension K'/K and where L is the separable closure of K in K'. Let us denote by V the kernel of $W \to U$. Since W is unipotent, V is. The exact sequence in cohomology gives us:

$$H^1(K, W) \to H^1(K, U) \to H^2(K, V).$$

Thanks to [TV13, Lem. 3.3], we know that $H^2(K, V)$ is trivial. Let us show that $H^1(K, W)$ is trivial. It suffices to consider W of the form in (3.1). By Lemma 2.2 we have on one hand:

$$H^1(K, W) = H^1(L, \mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_m)$$

and on the other hand, we have: $H^1(L, \mathrm{R}_{K'/L}(\mathbb{G}_m)) = H^1(K', \mathbb{G}_m) = 1$ thanks to Hilbert's 90 theorem. The long exact sequence associated to the quotient $\mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_m$ gives:

$$\underbrace{H^1(L, \mathrm{R}_{K'/L}(\mathbb{G}_m))}_{0} \to H^1(L, \mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_{m,L}) \to H^2(L, \mathbb{G}_m).$$

But the p-primary component of $H^2(L, \mathbb{G}_m) = \mathrm{Br}(L)$ is trivial since $\mathrm{scd}_p(K) \leqslant 1$. This implies that $H^1(L, \mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_m)$ is trivial since it is of p-primary torsion (since $\mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_m$ is unipotent), and therefore that $H^1(K, W)$ is trivial.

Finally, we deduce from everything we have done that $H^1(K, U)$ is trivial, as desired. □

We then deduce the general commutative case:

Proposition 3.2. *Let C be a commutative unirational algebraic group over K. If $\mathrm{scd}(K) \leqslant 1$, then we have $H^1(K, C) = 1$.*

Proof. Thanks to [Mil17, Thm. 16.13], C admits a largest subgroup of multiplicative type T, and the quotient $U = C/T$ is unipotent. Since the extension splits over the algebraic closure, T is in fact smooth and connected, and therefore a torus. The long exact sequence in cohomology then gives:

$$H^1(K, T) \to H^1(K, C) \to H^1(K, U).$$

Thanks to [Gil19, Prop. 4.6.2], $H^1(K, T)$ is trivial. As for $H^1(K, U)$, it is trivial thanks to Proposition 3.1 since it is unirational as a quotient of C. Therefore, $H^1(K, C)$ is trivial. □

Let G be a smooth connected algebraic K-group. Let us remark that, since G is smooth and connected, for dimension reasons, there exists an integer $n \geqslant 0$ such that $\mathcal{D}^n(G)$ is perfect. In other words, the sequence $(\mathcal{D}^k(G))_{k \geqslant 0}$ stabilizes at some point. We denote this "limit" by G^{perf}.

We finally get the desired result:

Proposition 3.3. *Suppose that $\mathrm{scd}(K) \leqslant 1$. Let G be a unirational algebraic group over K. Then the map induced in cohomology:*

$$H^1(K, G^{\mathrm{perf}}) \to H^1(K, G)$$

is surjective. In particular, $H^1(K, G)$ is trivial when G is moreover solvable.

Proof. Consider the exact sequence:

$$1 \to \mathcal{D}(G) \to G \to G/\mathcal{D}(G) \to 1.$$

Since G is unirational, $G/\mathcal{D}(G)$ is too. Also, $\mathcal{D}(G)$ is unirational. Indeed, by [SGA3, VIB, Prop. 7.4], there is a surjective scheme map $(G \times_K G)^N \to \mathcal{D}(G)$, which is a product of the commutator map $G \times_K G \to \mathcal{D}(G)$. The short exact sequence above induces an exact sequence in cohomology:

$$H^1(K, \mathcal{D}(G)) \to H^1(K, G) \to H^1(K, G/\mathcal{D}(G)).$$

By Proposition 3.2, the group $H^1(K, G/\mathcal{D}(G))$ is trivial, thus $H^1(K, \mathcal{D}(G)) \to H^1(K, G)$ is surjective. It then suffices to iterate the process to get the result. □

Moreover, we can add the following:

Corollary 3.4. *Suppose that* $\mathrm{scd}(K) \leqslant 1$. *Let G be a unirational K-group and C be a Cartan subgroup of G. Then $H^1(K, C)$ is trivial.*

Proof. Thanks to the previous proposition, it suffices to prove that C is unirational. This is actually [Ros24, Prop. 7.12]. □

4. From the perfect to the pseudo-semisimple case

Let G be a perfect smooth connected linear algebraic K-group. We have an exact sequence:

$$1 \to \mathrm{R}_{u,K}(G) \to G \to G/\mathrm{R}_{u,K}(G) \to 1,$$

where $G/\mathrm{R}_{u,K}(G)$ is pseudo-reductive and perfect, that is pseudo-semisimple. If we know that $H^1(K, \mathrm{R}_{u,K}(G))$ and $H^1(K, G/\mathrm{R}_{u,K}(G))$ are trivial, then $H^1(K, G)$ will be trivial. But in general, we don't know whether $\mathrm{R}_{u,K}(G)$ is unirational. However, when $[K : K^p] \leqslant p$, the group $\mathrm{R}_{u,K}(G)$ is unirational by Theorem 4.9 below. So under the assumptions $\mathrm{scd}(K) \leqslant 1$ and $[K : K^p] \leqslant p$, the H^1 of a perfect group will be trivial once we know it is trivial for pseudo-semisimple groups. In the next section, we shall see it is the case.

In order to prove Theorem 4.9, we need to study the group of extensions of unipotent groups by the multiplicative group, and to use the theory of permawound groups. In Subsection 4.1, we recall some facts about extensions by the multiplicative group, and we prove that for a smooth connected unipotent K-group U, and every extension of U by $\mathbb{G}_m$, there exists a largest subgroup $V \subset U$ over which the restriction of the extension is split. Then in Subsection 4.2, we talk about permawound groups. The point is that when $[K : K^p] = p$, wound unirational unipotent groups are permawound. This enables, in Subsection 4.3, to link a wound unirational unipotent group U to the struture theory of permawound groups, so that we can say something concrete on U and on its largest subgroup over which a given extension becomes split.

4.1. About extensions by $\mathbb{G}_m$.

Recall the notions of central extensions by the multiplicative group. Let G be an algebraic K-group G. We consider *central* extensions (E) of algebraic K-groups

$$1 \to \mathbb{G}_m \to H \to G \to 1$$

and the set of isomorphism classes of such extensions denoted by $\mathrm{Ext}_c(G, \mathbb{G}_m)$. This is a group for the Baer sum. Note that when G is connected, every extension of G by $\mathbb{G}_m$ is necessarily central. If G is commutative, we can also consider the extensions (E), called *commutative*, for which the group H is commutative. The set formed by their isomorphism classes is denoted by $\mathrm{Ext}^1(G, \mathbb{G}_m)$. There is an obvious embedding

$$\mathrm{Ext}^1(G, \mathbb{G}_m) \subset \mathrm{Ext}_c(G, \mathbb{G}_m).$$

The group Ext^1 corresponds actually to the first Ext^i group in the category of *fppf* abelian sheaves as defined in [Stacks, Tag 06XP].

When G is smooth and connected, $\mathrm{Ext}_c(G, \mathbb{G}_m)$ is a subgroup of the Picard group $\mathrm{Pic}(G)$ of G. If G is moreover commutative, the groups $\mathrm{Ext}_c(G, \mathbb{G}_m)$ and $\mathrm{Ext}^1(G, \mathbb{G}_m)$ are equal because every central extension of G by $\mathbb{G}_m$ is commutative by [Ros21b, Lem. 4.2].

For any central extension (E), an extension (E) is trivial (that is isomorphic to $\mathbb{G}_m \times_K G$) if, and only if, it is split, that is there is a homomorphic section $G \to H$. In general, any homomorphism $f : G' \to G$ induces a canonical map

$$f^* : \mathrm{Ext}_c(G, \mathbb{G}_m) \to \mathrm{Ext}_c(G', \mathbb{G}_m).$$

If $i : G' \hookrightarrow G$ is an embedding, write $(E)_{|G'} = i^*(E)$. We have the same things for $\mathrm{Ext}^1(G, \mathbb{G}_m)$ for commutative G.

In this section, we are interested in the sequences of groups

$$\mathrm{Ext}(U'', \mathbb{G}_m) \xrightarrow{p^*} \mathrm{Ext}(U, \mathbb{G}_m) \xrightarrow{i^*} \mathrm{Ext}(U', \mathbb{G}_m)$$

(for $\mathrm{Ext}_c(-, \mathbb{G}_m)$ or $\mathrm{Ext}^1(-, \mathbb{G}_m)$) induced by exact sequence of unipotent groups

$$(*) \qquad 1 \xrightarrow{i} U' \to U \xrightarrow{p} U'' \to 1.$$

We know:

- The homomorphism p^* is always injective for $\mathrm{Ext}_c(-, \mathbb{G}_m)$ or $\mathrm{Ext}^1(-, \mathbb{G}_m)$.
- If U' and U are smooth and connected, $(*)$ induces an exact sequence
$$\mathrm{Ext}_c(U'', \mathbb{G}_m) \xrightarrow{p^*} \mathrm{Ext}_c(U, \mathbb{G}_m) \xrightarrow{i^*} \mathrm{Ext}_c(U', \mathbb{G}_m)$$
according to [Ros21b, Lem. 3.3]. Moreover, If U' is the subgroup $\mathcal{D}(U)$, then p^* is an isomorphism while i^* is the zero map ([Ros21a, Lem. 3.6]).
- If U' and U are commutative, [Stacks, Tag 06XP] tells us that $(*)$ induces an exact sequence
$$\mathrm{Ext}^1(U'', \mathbb{G}_m) \xrightarrow{p^*} \mathrm{Ext}^1(U, \mathbb{G}_m) \xrightarrow{i^*} \mathrm{Ext}^1(U', \mathbb{G}_m).$$

We mention the particular case:

Fact 4.1. If $[K : K^p] < \infty$, then the group $\mathrm{Ext}^1(\mathrm{R}_{K^{1/p}/K}(\alpha_p), \mathbb{G}_m)$ is trivial.

Proof. Applying the Snake Lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \alpha_p & \longrightarrow & \mathbb{G}_a & \xrightarrow{(-)^p} & \mathbb{G}_a & \longrightarrow & 1 \; , \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{R}_{K^{1/p}/K}(\alpha_p) & \longrightarrow & \mathrm{R}_{K^{1/p}/K}(\mathbb{G}_a) & \xrightarrow[(-)^p]{} & \mathbb{G}_a & \longrightarrow & 1
\end{array}$$

we obtain an isomorphism

$$\frac{\mathrm{R}_{K^{1/p}/K}(\alpha_p)}{\alpha_p} \simeq \frac{\mathrm{R}_{K_1/K}(\mathbb{G}_a)}{\mathbb{G}_a} \simeq \mathbb{G}_a^{[K:K^p]-1}.$$

Then the exact sequence of Ext^1 groups tells us that the natural homomorphism

$$\mathrm{Ext}^1(\mathrm{R}_{K^{1/p}/K}(\alpha_p), \mathbb{G}_m) \to \mathrm{Ext}^1(\alpha_p, \mathbb{G}_m)$$

is injective. So let (E) be a commutative extension

$$1 \to \mathbb{G}_m \to H \to \mathrm{R}_{K^{1/p}/K}(\alpha_p) \to 1.$$

We want to see that $(E)_{|\alpha_p}$ is split.

The group $\mathbb{G}_m$ is smooth, so we can apply [SGA3, VIIA, Prop. 8.2] to get an exact sequence of p-Lie algebras

$$0 \to \mathrm{Lie}(\mathbb{G}_m) \to \mathrm{Lie}(H) \to \mathrm{Lie}(\mathrm{R}_{K^{1/p}/K}(\alpha_p)) \to 1.$$

By [CGP15, Lem. A.7.13], the p-Lie algebra of $\mathbb{G}_m$ is Ke for a generator e, with the trivial bracket and with p-power $e^{[p]} = e$. The p-Lie algebra of $\mathrm{R}_{K^{1/p}/K}(\alpha_p)$ is $K^{1/p}f$ for a generator f, with the trivial bracket and with p-power $f^{[p]} = 0$. Since the embedding of α_p into $\mathrm{R}_{K^{1/p}/K}(\alpha_p)$ identifies $\mathrm{Lie}(\alpha_p)$ with $Kf \subset K^{1/p}f$, the extension $(E)_{|\alpha_p}$ will appear to be split if we find an element of $\mathrm{Lie}(H)$ with p-power equal to 0. But $\mathrm{Lie}(H)$ is of the form $Kx \oplus K^{1/p}y$ (where x and y are generators), and its p-power is characterized by the identities $x^{[p]} = x$ and $y^{[p]} = \lambda x$ for some $\lambda \in K$. In case $\lambda = 0$, the extension $(E)_{|\alpha_p}$ is split. In case $\lambda \neq 0$, the element $z = x + \left(\frac{-\alpha}{\lambda}\right)^{1/p} y$ is such that $z^{[p]} = 0$. It follows that $(E)_{|\alpha_p}$ is split, hence so is (E). □

Given a central extension of a smooth connected unipotent K-groups by $\mathbb{G}_m$, there exists a largest subgroup over which the extension becomes the trivial extension.

Proposition 4.2. *Let U be a smooth connected unipotent K-group, and let (E) be a group extension*

$$1 \to \mathbb{G}_m \to H \to U \to 1.$$

Then there is a largest K-sugbroup V of U for which $(E)_{|V}$ is trivial. In particular, there is a largest smooth connected subgroup over which (E) becomes trivial, which is $(V^{\mathrm{sm}})^0$.

Also, for any quotient homomorphism $p : U' \to U$ of smooth connected unipotent K-groups, the largest K-sugbroup V' of U' for which $p^*(E)_{|V'}$ is trivial is $p^{-1}(V)$. If moreover $\ker(p)$ is smooth and connected, then $(V^{\mathrm{sm}})^0 = ((V')^{\mathrm{sm}})^0/\ker(p)$.

Proof. Let us prove the existence of a largest subgroup of U over which (E) becomes trivial, when U is assumed to be commutative. It suffices to prove that if (E) is trivial when restricted to two subgroups, then it is also trivial on a subgroup containing them both. So let V_1, V_2 be K-sugbroups of U for which $(E)_{|V_1}$, $(E)_{|V_2}$ are trivial. This means there exist sections $s_1 : V_1 \to q^{-1}(V_1)$, and $s_2 : V_2 \to q^{-1}(V_2)$ of q, where q is the homomorphism $H \to U$ defining (E). Let W be the subgroup of H generated by $s_1(V_1)$, $s_2(V_2)$. The group W is unipotent, being a quotient of the direct product $V_1 \times_K V_2$. Due to the fact that the kernel of q is a torus, q maps the unipotent subgroups of H isomorphically onto their images. Thus $q(W)$ is a subgroup of U containing V_1 and V_2, and taking $s : q(W) \to H$ to be $(q_{|W})^{-1}$ yields a section of q on $q(W)$.

If U is not assumed to be commutative, as talked earlier, we have nonetheless an isomorphism

$$\mathrm{Ext}_c(U/\mathcal{D}(U), \mathbb{G}_m) \to \mathrm{Ext}_c(U, \mathbb{G}_m)$$

induced by the quotient $p : U \to U/\mathcal{D}(U)$. This enables to transport the extension (E) over $U/\mathcal{D}(U)$; call it (E'). Let V' be the largest subgroup of $U/\mathcal{D}(U)$ over which (E') is trivial, and put $V = p^{-1}(V')$. Thanks to the commutative natural diagram

$$\begin{array}{ccc} (E) \in \mathrm{Ext}_c(U/\mathcal{D}(U), \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_c(U, \mathbb{G}_m) \ni (E) \\ \downarrow & & \downarrow \\ (E')_{|V'} \in \mathrm{Ext}_c(V', \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_c(V, \mathbb{G}_m) \ni (E)_{|V} \end{array} \quad , \tag{4.1}$$

we see that $(E)_{|V}$ is trivial. Actually, V is the largest subgroup of U over which (E) is trivial, because given any other $W \subset U$ such that $(E)_{|W}$ is trivial, replacing the second row of (4.1) by

$$\mathrm{Ext}_c(p(W), \mathbb{G}_m) \hookrightarrow \mathrm{Ext}_c(W, \mathbb{G}_m),$$

we get that $(E')_{|p(W)}$ is trivial, so $p(W) \subset V'$, and $W \subset V$.

Now let $p : U' \to U$ be a quotient homomorphism between smooth connected unipotent groups. Let V, resp. V', be the largest subgroup of U, resp. U', such that $(E)_{|V}$, resp. $p^*(E)_{|V'}$, is trivial. It is clear that $p^{-1}(V) \subset V'$. Let us prove the reverse inclusion. The commutative natural square

$$\begin{array}{ccc} (E) \in \mathrm{Ext}_c(U, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}_c(U', \mathbb{G}_m) \ni p^*(E) \\ \downarrow & & \downarrow \\ (E)_{|p(V')} \in \mathrm{Ext}_c(p(V'), \mathbb{G}_m) & \hookrightarrow & \mathrm{Ext}_c(V', \mathbb{G}_m) \ni p^*(E)_{|V'} \end{array}$$

allows to assert that $(E)_{|p(V')}$ is trivial, hence $p(V') \subset V$ and $V' \subset p^{-1}(V)$. If moreover $\ker(p)$ is smooth and connected, the $p^{-1}(W)$ is smooth and connected whenever W is. The equality $(V^{\mathrm{sm}})^0 = ((V')^{\mathrm{sm}})^0 / \ker(p)$ follows at once. □

4.2. (Weakly) permawound groups.

We introduce the notions of weakly permawound groups and of permawound groups, to be used below. A *permawound* K-group ([Ros25, Def. 1.2]) is a smooth connected unipotent K-group U such that, for all algebraic K-groups E fitting into an exact sequence

$$U \to E \to \mathbb{G}_a \to 1, \tag{4.2}$$

the group E admits $\mathbb{G}_a$ as a subgroup. And a *weakly permawound* K-group ([Ros25, Def. 5.1]) is a commutative unipotent K-group U such that for all commutative algebraic K-groups E fitting into an exact sequence as above, the group E admits $\mathbb{G}_a$ as a subgroup. Note that any smooth connected commutative unipotent group is permawound if, and only if, it is weakly permawound (this is [Ros25, Prop. 6.9]). Define also a *semiwound* unipotent K-group as a (not necessarily smooth and nor necessarily connected) unipotent K-group G such that every morphism $\mathbb{A}^1_K \to G$ is the constant map to some K-rational point of G.

Notation 4.3. Set $K_n := K^{1/p^n}$ for all $n \geqslant 0$.

In particular, we are interested in the following commutative (semi)wound permawound K-groups.

Example 4.4. Assume $[K : K^p] = p$. Write

$$R_n = \mathrm{R}_{K_n/K}(\mathbb{G}_m)/\mathbb{G}_m.$$

The group R_n is permawound as stated in [Ros24, Prop. 9.5]. It is wound because of [CGP15, Ex. B.2.8]. For $0 < m \leqslant n$, there is a canonical inclusion $R_m \subset R_n$, and the p^{n-m}-power map $K_n \to K_m$ naturally induced a surjective homomorphism $f_{n/m} : R_n \to R_m$. We have in particular $f_n = f_{n/1}$, whose kernel is denoted by W_n. Note that $W_m = \ker(f_{n/(n-m+1)})$.

We will need:

Fact 4.5. Fix $m \geqslant 0$ and suppose that $[K : K^p] = p$. Consider the natural inclusions $R_m \subset W_{m+1} \subset R_{m+1}$. Then the quotient W_{m+1}/R_m admits a decomposition series whose quotients are isomorphic to $\mathrm{R}_{K_1/K}(\alpha_p)$.

Proof. To start with, let L be a field such that $[L : L^p] = p$, and take $t \in L \setminus L^p$. Write

$$G(L) = \{(x_0, \ldots, x_{p-1}) \in \mathbb{G}_a^p \,|\, x_0^p + t x_1^p + \ldots + t^{p-1} x_{p-1}^p = x_{p-1}\}$$

and P_t for the polynomial $Y_0 + tY_1 + \ldots + t^{p-1} Y_{p-1}$. From [Oes, VI.5, Prop.] (and its proof), there exist polynomials $P_0, \ldots, P_{p-1} \in \mathbb{F}_p[Y_0, \ldots, Y_{p-1}]$ such that the map $\varphi_L : \mathrm{R}_{L_1/L}(\mathbb{G}_m) \to \mathbb{G}_a^p$ defined by the expression

$$y_0 + t^{1/p} y_1 + \ldots + t^{\frac{p-1}{p}} y_{p-1} \in L_1^\times \longmapsto \left(\frac{P_i(y_0, \ldots, y_{p-1})}{P(y_0, \ldots, y_{p-1})} \right)_{i=0,\ldots,p-1}$$

is well defined, and induces a group isomorphism

$$\frac{\mathrm{R}_{L_1/L}(\mathbb{G}_m)}{\mathbb{G}_m} \to G(L).$$

The point is that the polynomials P_i do not depend on L.

Returning to W_{m+1}/R_m, we note that it is the kernel of

$$\frac{\mathrm{R}_{K_{m+1}/K}(\mathbb{G}_m)}{\mathrm{R}_{K_m/K}(\mathbb{G}_m)} \to \frac{\mathrm{R}_{K_1/K}(\mathbb{G}_m)}{\mathbb{G}_m}$$

induced by the p^m-power map. Fix $t \in K \setminus K^p$. For every $0 \leqslant n$, we use the polynomials P_i, $P_{t^{1/p^{n+1}}}$, and $P_{t^{1/p^{n+2}}}$. Because the P_i's have coefficients in $\mathbb{F}_p$, they are fixed by taking p-power, so the p-power maps $K_{n+2} \to K_{n+1}$ induces a commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}_{K_{n+1}/K}(\mathbb{G}_m) & \hookrightarrow & \mathrm{R}_{K_{n+1}/K}(\mathrm{R}_{K_{n+2}/K_{n+1}}(\mathbb{G}_m)) & \xrightarrow{\mathrm{R}_{K_{n+1}/K}(\varphi_{K_{n+1}})} & \mathrm{R}_{K_{n+1}/K}(\mathbb{G}_a^p) \\
\downarrow (-)^p & & \downarrow (-)^p & & \downarrow (-)^p \\
\mathrm{R}_{K_n/K}(\mathbb{G}_m) & \hookrightarrow & \mathrm{R}_{K_n/K}(\mathrm{R}_{K_{n+1}/K_n}(\mathbb{G}_m)) & \xrightarrow{\mathrm{R}_{K_n/K}(\varphi_{K_n})} & \mathrm{R}_{K_n/K}(\mathbb{G}_a^p)
\end{array}.$$

The groups $\mathrm{R}_{K_n/K}(\mathbb{G}_m)$ and $\mathrm{R}_{K_{n+1}/K}(\mathbb{G}_m)$ being smooth, Weil restriction behaves well by quotient by [CGP15, Cor. A.5.4(3)], so that the images of $\mathrm{R}_{K_n/K}(\varphi_{K_n})$ and $\mathrm{R}_{K_{n+1}/K}(\varphi_{K_{n+1}})$ are $\mathrm{R}_{K_{n+1}/K}(G(K_n))$, $\mathrm{R}_{K_{n+1}/K}(G(K_{n+1}))$ respectively. Then the diagram enables to read

$$(-)^p : \frac{\mathrm{R}_{K_{n+2}/K}(\mathbb{G}_m)}{\mathrm{R}_{K_{n+1}/K}(\mathbb{G}_m)} \to \frac{\mathrm{R}_{K_{n+1}/K}(\mathbb{G}_m)}{\mathrm{R}_{K_n/K}(\mathbb{G}_m)}$$

as the p-power homomorphism

$$f_n : \mathrm{R}_{K_{n+1}/K}(G(K_{n+1})) \to \mathrm{R}_{K_n/K}(G(K_n)).$$

In particular, the group W_{m+1}/R_m is isomorphic to the kernel of $f_1 \circ \cdots \circ f_m$.

It remains to see that W_{m+1}/R_m has a decomposition series with quotients $\mathrm{R}_{K_1/K}(\alpha_p)$. We slice W_{m+1}/R_m by the subgroups $N_i = \ker(f_i \circ \cdots \circ f_m)$ for $1 \leqslant i \leqslant m$. For any $1 \leqslant i \leqslant m$, the quotient N_i/N_{i+1} (with $N_{m+1} = 0$) is the kernel of f_i, which is easily seen to be isomorphic to $\mathrm{R}_{K_{i+1}/K}(\alpha_p)^{p-1}$ by the map

$$(x_0, \ldots, x_{p-2}) \mapsto (x_0, \ldots, x_{p-2}, 0).$$

But the latter group is a product of $\mathrm{R}_{K_1/K}(\alpha_p)$ by [Ros25, Prop. 7.5]. □

In the case of the wound permawound groups R_n, it is possible to identify the form of the largest subgroup over which a given central extension by $\mathbb{G}_m$ is the trivial extension.

Proposition 4.6. *Assume $[K : K^p] = p$, and consider a group of the form R_n for some $n > 0$. Then for every non trivial extension $(E) \in \mathrm{Ext}_c(R_n, \mathbb{G}_m)$, the largest K-sugbroup S of R_n for which $(E)_{|S}$ is trivial is of the form W_m for some $0 < m \leqslant n$.*

Proof. First of all, notice that R_n is smooth, connected, and commutative, so (E) is in fact commutative. This allows to consider only exact sequences of Ext^1 groups for commutative groups.

Let $0 \leqslant m \leqslant n$ be maximal such that $(E)_{|R_m}$ is trivial. Note that $m < n$ by assumption. This implies that the extension (E) is the pullback of a commutative extension (E') of R_n/R_m. We show that W_{m+1} is the largest subgroup of R_n over which (E) becomes split.

The pullback of (E') via $W_{m+1} \to W_{m+1}/R_m \subset R_n/R_m$ is $(E)_{|W_{m+1}}$, hence is trivial on R_m. But the homomorphism

$$\mathrm{Ext}^1(W_{m+1}, \mathbb{G}_m) \to \mathrm{Ext}^1(R_m, \mathbb{G}_m)$$

is injective because its kernel is $\mathrm{Ext}^1(W_{m+1}/R_m, \mathbb{G}_m)$ which is trivial since, as we saw earlier, W_{m+1}/R_m is a power of the group $\mathrm{R}_{K^{1/p}/K}(\alpha_p)$. It follows that $(E)_{|W_{m+1}}$ is trivial.

Now let us show that W_{m+1} is maximal, that is $W_{m+1} = S$, that is $\overline{S} = S/W_{m+1}$ is trivial. For this, we use the theory of (weakly) permawound groups. Assume $\overline{S} \neq 0$. Putting $R = R_{m+1}/W_{m+1}$, we show that $\overline{S} \cap R \neq 0$. Actually, we may assume that $K = K_s$. The group S has the property that R_n/S, which is also the quotient of R_n/W_{m+1} by $\overline{S}$, is wound. Indeed, assume there is an embedding $i : \mathbb{G}_a \to R_n/S$, then $i^*(\overline{E})$ is trivial, and the preimage of $i(\mathbb{G}_a)$ via the quotient $R_n \to R_n/S$ would contradict the maximality of S. Thus, $\overline{S}$ is weakly permawound by [Ros25, Prop. 5.5] (the group $R_n/W_{m+1} \simeq R_{n-m}$ being wound and unirational, it is permawound - [Ros25, Prop. 9.6]). By [Ros25, Thm. 9.5], the group $\overline{S}$ contains a subgroup S' isomorphic to $\mathrm{R}_{K_1/K}(\alpha_p)$ or to R_1. Note that, over our separably closed field K such that $[K : K^p] = p$, the group $\mathcal{V}$ of the cited theorem may be defined as the subgroup

$$\{(x_0, \ldots, x_{p-1}) \in \mathbb{G}_a^p \,|\, x_0^p + t x_1^p + \ldots + t^{p-1} x_{p-1}^p = x_{p-1}\} \subset \mathbb{G}_a^p$$

where t is any element $t \in K \setminus K^p$ (check [Ros25, Def. 7.3 & Cor. 7.2]). But this subgroup of $\mathbb{G}_a^p$ is known to be R_1 by [Oes, VI.5, Prop.].

It happens that S' cannot be $\mathrm{R}_{K_1/K}(\alpha_p)$. This is because there is no nontrivial homomorphism $\mathrm{R}_{K_1/K}(\alpha_p) \to R_n/W_{m+1}$. We know that $R_n/W_{m+1} \simeq R_{n-m}$. Given an homomorphism $\varphi : \mathrm{R}_{K_1/K}(\alpha_p) \to R_{n-m}$, since $\mathrm{Ext}^1(R_{K_1/K}(\alpha_p), \mathbb{G}_m) = 0$, the homomorphism φ must lift to an homomorphism $\phi : \mathrm{R}_{K_1/K}(\alpha_p) \to \mathrm{R}_{K_{n-m}/K}(\mathbb{G}_m)$. The functorial property characterizing Weil restriction assigns to ϕ an homomorphism $\mathrm{R}_{K_1/K}(\alpha_p)_{K_{n-m}} \to \mathbb{G}_m$ over K_{n-m}, which is trivial as the former group is unipotent. Hence φ is the zero map.

Thus we have $S' = R_1$. To see that $S' \cap R \neq 0$, we check that $S' \subset R$ (there is even equality due to dimension reason). The quotient of R_n/W_{m+1} by R is R_n/R_{m+1}, so consider any homomorphism φ from S' to R_n/R_{m+1}, that is

$$\varphi : \frac{\mathrm{R}_{K_1/K}(\mathbb{G}_m)}{\mathbb{G}_m} \longrightarrow \frac{\mathrm{R}_{K_n/K}(\mathbb{G}_m)}{\mathrm{R}_{K_{m+1}/K}(\mathbb{G}_m)} = \mathrm{R}_{K_{m+1}/K}\left(\frac{\mathrm{R}_{K_n/K_{m+1}}(\mathbb{G}_m)}{\mathbb{G}_m}\right).$$

The homomorphism corresponding via Weil restriction is an homomorphism

$$\left(\frac{\mathrm{R}_{K_1/K}(\mathbb{G}_m)}{\mathbb{G}_m}\right)_{K_{m+1}} \longrightarrow \frac{\mathrm{R}_{K_n/K_{m+1}}(\mathbb{G}_m)}{\mathbb{G}_m}$$

over K_{m+1}. However, the former group is split unipotent (see [CGP15, Ex. 1.3.2]), while the latter is wound (see [CGP15, Ex. B.2.8]). This proves that φ is the zero map, hence $S' \subset R$.

We are finally able to conclude the whole proof. If $\overline{S} \neq 0$, then $\overline{S} \supset R$, so (E) would be trivial over $R_{m+1} \subset R_n$ which is a contradiction. Hence $\overline{S} = 0$, that is $S = W_{m+1}$, as desired. □

4.3. **Unirationality of the K-unipotent radical.**

We are now able to tackle the problem of unirationality of the K-unipotent radical of a perfect smooth connected linear algebraic K-group.

Proposition 4.7. *Assume $[K : K^p] \leqslant p$. Let U be a unirational unipotent K-group, and let (E) be a group extension*

$$1 \to \mathbb{G}_m \to H \to U \to 1.$$

Then the largest smooth connected K-sugbroup V of U for which $(E)_{|V}$ is trivial is unirational.

Proof. If $[K : K^p] = 1$ then the unipotent U is split and $\mathrm{Ext}_c(U, \mathbb{G}_m) = \mathrm{Pic}(U) = 0$. So assume $[K : K^p] = p$.

The group V_{K_s} is the largest smooth connected K_s-subgroup V' of U_{K_s} for which $(E_{K_s})_{|V'}$ is trivial. Because unirationality of V_{K_s} over K_s implies unirationality of V over K by [Ros24, Thm.

1.6], we may assume $K = K_s$. Note also that we may assume that U is commutative. Indeed, we saw in the proof of Proposition 4.2 that (E) comes from a central extension (E') of $U/\mathcal{D}(G)$, and $V = p^{-1}(V')$ where $p : U \to U/\mathcal{D}(G)$ is the abelian quotient of U and V' the largest smooth connected subgroup of $U/\mathcal{D}(G)$ such that $(E')_{|V'}$ is trivial. But $\mathcal{D}(U)$ is unirational as U is, so V is unirational if V' is by [Ros26, Thm. 2.4]. Thus the case of U commutative implies the general case, an from now on, U will be commutative.

We may assume furthermore U is wound. Indeed, take a smooth connected normal K-split subgroup U_{split} of U such that U/U_{split} is wound ([CGP15, Thm. B.3.4]). Since $\mathrm{Ext}_c(U_{\text{split}}, \mathbb{G}_m) = \mathrm{Pic}(U_{\text{split}}) = 0$, the map

$$\mathrm{Ext}_c(U/U_{\text{split}}, \mathbb{G}_m) \to \mathrm{Ext}_c(U, \mathbb{G}_m)$$

is an isomorphism as seen in Subsetcion 4.1, so the extension (E) comes from an extension (E') over U/U_{split}. Then we can use Proposition 4.2 and apply it to the quotient $U \to U/U_{\text{split}}$. We find first that $U_{\text{split}} \subset V$. Secondly, we find that V/U_{split} is the largest smooth connected sugbroup of U/U_{split} on which the restriction of (E') is trivial. But V is unirational if V/U_{split} is ([Ros26, Thm. 2.4] again). Thus we may assume U is wound and commutative.

From the proof of [Ros21b, Thm. 1.3] (beginning on the bottom of page 9 at the end of section 3), or from [Ach19, Prop. 2.5], there is a quotient homomorphism

$$f : \prod_{i=1}^{r} R_{n_i} \to U$$

where $n_i > 0$. Denote by R the product of the R_{n_i,a_i}'s and call (E') the image of (E) by f^*. It is easy to see that for any two commutative algebraic K-groups G_1, G_2, the embeddings $G_i \to G_1 \times_K G_2$ induce an isomorphism

$$\mathrm{Ext}^1(G_1 \times_K G_2, \mathbb{G}_m) \to \mathrm{Ext}^1(G_1, \mathbb{G}_m) \oplus \mathrm{Ext}^1(G_2, \mathbb{G}_m)$$

whose inverse is built from the projections $G_1 \times_K G_2 \to G_i$. This implies, together with Proposition 4.2, that the largest subgroup of R on which the restriction of (E') is trivial is the product of such subgroups for each component: by Proposition 4.6, there is a partition I_1, I_2 of $\{1, \dots, r\}$, and integers $0 < m_i \leqslant n_i$ for all $i \in I_2$, such that the subgroup

$$S = \prod_{i \in I_1} R_{n_i} \times \prod_{i \in I_2} W_{m_i} \subset R$$

is the largest subgroup of R on which (E') is trivial. The following facts enable to conclude that V is unirational.

(1) The group R is (weakly) permawound. Each R_{n_i} is (weakly) permawound as said in Example 4.4, so the product R is also (weakly) permawound by [Ros25, Prop. 5.6]).
(2) The group S is weakly permawound. Indeed, we have

$$R/S = \prod_{i \in I_2} \frac{R_{n_i}}{W_{m_i}}.$$

But for any $0 < m \leqslant n$, the group $R_n/W_m \simeq R_{n-m+1}$ which is wound. Thus R_n/W_m is wound, and the product R/S itself is wound. The group R being permawound by (1), we can apply [Ros25, Prop. 5.5] to see that S is weakly permawound.
(3) The group $f(S)$ is weakly permawound as every quotient of an arbitrary weakly permawound group by [Ros25, Prop. 5.4].
(4) The group $f(S)^{\text{sm}}$ is (weakly) permawound. We may use again [Ros25, Prop. 5.5]: $f(S)$ is commutative and weakly permawound, while $f(S)/f(S)^{\text{sm}}$ is semiwound (it has only one K-rational point), thus $f(S)^{\text{sm}}$ is weakly permawound, hence permawound.

(5) The group $f(S)^{\mathrm{sm}}$ is unirational because of [Ros24, Prop. 9.7].

The group $f(S)^{\mathrm{sm}}$ being unirational, it is connected so $V = f(S)^{\mathrm{sm}}$, which conclude the proof. □

The last proposition leads us to:

Proposition 4.8. *Assume $[K : K^p] \leqslant p$ and consider an extension of algebraic K-groups*

$$1 \to T \to G \to U \to 1$$

where T is a torus, G is unirational, and U is unipotent. Then $\mathrm{R}_{u,K}(G)$ is unirational.

Proof. If $[K : K^p] = 1$ then all linear algebraic groups are unirational. So assume $[K : K^p] = p$. If $T = 1$ then the result is clear, so assume $T \neq 1$. Since $(\mathrm{R}_{u,K}(G))_{K_s} = \mathrm{R}_{u,K_s}(G_{K_s})$ by [CGP15, Prop. 1.1.9], and since unirationality descends through separable extensions by [Ros24, Thm. 1.6], we may also assume $K = K_s$, so that $T = \mathbb{G}_m^r$ for some integer $r \geqslant 1$. Since G is connected, the subgroup $T \subset G$ is necessarily central. For all $i = 0, \ldots, r$, put $G_i = G/\left(\{1\}^i \times \mathbb{G}_m^{r-i}\right)$ and call (E_i) the extension

$$1 \to \mathbb{G}_m \to G_{i+1} \to G_i \to 1$$

if $i < r$. We have $G_0 = U$, $G_r = G$. Calling p_i the quotient homomorphism $G_{i+1} \to G_i$, the homomorphism p_i maps every unipotent subgroup of G_{i+1} isomorphically onto its image, and it is a general fact for algebraic groups that a quotient homomorphism maps a normal subgroup onto a normal subgroup, so $p_i(\mathrm{R}_{u,K}(G_{i+1})) \subset \mathrm{R}_{u,K}(G_i)$. We prove by induction on $i < r$ that $\mathrm{R}_{u,K}(G_{i+1})$ is unirational.

So fix $0 \leqslant i < r$ and assume moreover that $\mathrm{R}_{u,K}(G_i)$ is unirational if $i > 1$. For every smooth connected normal unipotent subgroup V of G_i, if there is a section $s : V \to p_i^{-1}(V)$ to p_i, then $s(V)$ is normal in G_{i+1}. Indeed, by [Mil17, Cor. 1.85], it suffices to check that $s(V)$ is stable by conjugation by the K-points of G. But $p_i^{-1}(V)$ is normal in G_{i+1}, while $p_i^{-1}(V) = \mathbb{G}_m s(V)$. So, for all $g \in G(K)$, $\mathrm{int}(g)$ restricts to an isomorphism of $p_i^{-1}(V)$, which stabilizes the unipotent radical of $p_i^{-1}(V)$, that is $s(V)$. Hence $p_i(\mathrm{R}_{u,K}(G_{i+1}))$ is the maximal smooth connected subgroup V of $\mathrm{R}_{u,K}(G_i)$ such that $(E_i)_{i|V}$ is trivial. By Proposition 4.7 we get that $\mathrm{R}_{u,K}(G_{i+1})$ is unirational.

We conclude that $\mathrm{R}_{u,K}(G_r) = \mathrm{R}_{u,K}(G)$ is unirational. □

Finally, we obtain:

Theorem 4.9. *Assume $[K : K^p] \leqslant p$ and let G be an unirational algebraic K-group. Then $\mathrm{R}_{u,K}(G)$ is unirational.*

Proof. If $[K : K^p] = 1$ then all linear algebraic groups are unirational. So assume $[K : K^p] = p$. There is a unique smooth connected normal K-split subgroup V of $\mathrm{R}_{u,K}(G)$ such that $\mathrm{R}_{u,K}(G)/V$ is K-wound ([CGP15, Thm. B.3.4]). But by [Ros26, Thm. 2.6], an algebraic K-group is unirational if it is an extension of two unirational algebraic K-groups under the assumption $[K : K^p] \leqslant p$. So since V is (uni)rational, the group $\mathrm{R}_{u,K}(G)$ is unirational if $\mathrm{R}_{u,K}(G)/V$ is unirational. But V is stable by K_s-automorphisms of $\mathrm{R}_{u,K}(G)$, so is also normal in G (use [Mil17, Cor. 1.85]), and $\mathrm{R}_{u,K}(G/V) = \mathrm{R}_{u,K}(G)/V$. Thus we may assume that $\mathrm{R}_{u,K}(G)$ is K-wound.

Choose a maximal K-torus $T \subset G$. Then $\mathrm{R}_{u,K}(G) \subset \mathrm{Z}_G(T)$ because the action of T on $\mathrm{R}_{u,K}(G)$ is trivial by [CGP15, Prop. B.4.4]. By [Ros24, Prop. 7.12], since G is unirational, $\mathrm{Z}_G(T)$ is unirational. Moreover, $\mathrm{R}_{u,K}(G) = \mathrm{R}_{u,K}(\mathrm{Z}_G(T))$. Indeed, note that centralizers of tori behave well under quotients [Bor91, Cor. 2, IV.11.14], so we are reduced to the case in which G is pseudo-reductive; that is, we must show that G pseudo-reductive implies the same for $\mathrm{Z}_G(T)$, but this is [CGP15, Prop. 1.2.4]. Now, $\mathrm{Z}_G(T)/T$ is unipotent since it has no non-trivial torus. So $\mathrm{Z}_G(T)$ is unirational and an extension of a unipotent group by T. Thanks to Proposition 4.8, we have that $\mathrm{R}_{u,K}(\mathrm{Z}_G(T)) = \mathrm{R}_{u,K}(G)$ is unirational. □

4.4. **Comments.**

We don't know if Theorem 4.9 holds without the assumption $[K : K^p] \leqslant p$. It would be greatly valuable for strengthening our results. For example, if $\mathrm{char}(K) > 3$, this problem is in fact the only obstruction in getting the triviality of $H^1(K, G)$ for any arbitrary unirational K-group when $\mathrm{scd}(K) \leqslant 1$. This lets us wonder what the shape of a unipotent radical of a perfect group is.

In fact, there is an easy construction that shows that every wound commutative unipotent unirational algebraic group is the unipotent radical (and even the center) of a perfect group without any assumption on K:

Proposition 4.10. *Every wound commutative unipotent unirational algebraic group U appears as the unipotent radical and the center of a perfect smooth connected linear algebraic group.*

Proof. Recall that, thanks to [Ach19, Prop. 2.5.], we have a map

$$W := \prod_{i=1}^{n} \mathrm{R}_{L_i/K}\left(\mathrm{R}_{K_i/L_i}(\mathbb{G}_m)/\mathbb{G}_m\right) \to U$$

where $K_1, ..., K_n$ are some finite extensions of K and $L_1, ..., L_n$ are the separable closures of K in $K_1, ..., K_n$ respectively. If W is the unipotent radical and the center of a perfect smooth connected linear algebraic K-group G, then $V = \mathrm{Ker}(W \to U)$ is normal in G, so it is possible to form the quotient $G' = G/V$, and G' satisfies the statement of the proposition for U. Indeed, G' is perfect as a quotient of a perfect group. Its center is W/V by [CGP15, Prop. 2.2.12]. Also, $\mathrm{R}_{u,K}(G') = \mathrm{R}_{u,K}(G)/V = U$: We have $W/V \subset \mathrm{R}_{u,K}(G')$ and is normal in G', while the quotient

$$\frac{G/V}{W/V} = G/W = G/\mathrm{R}_{u,K}(G)$$

is pseudo-reductive. The next step is to note that the proposition holds for W if and only if it holds for each factor $\mathrm{R}_{L_i/K}\left(\mathrm{R}_{K_i/L_i}(\mathbb{G}_m)/\mathbb{G}_m\right)$ by taking products.

We first prove the case where $U = \mathrm{R}_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m$ for a finite purely inseparable extension K'/K. There exists an integer $r > 0$ such that $(K')^{p^r} \subset L$, and we have:

$$\mathrm{R}_{K'/K}(\mu_{p^r})/\mu_{p^r} \xrightarrow{\sim} \mathrm{R}_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m.$$

To explain the latter isomorphism, apply the Snake lemma to the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_{p^r} & \longrightarrow & \mathbb{G}_m & \xrightarrow{(-)^{p^r}} & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{R}_{K'/K}(\mu_{p^r}) & \longrightarrow & \mathrm{R}_{K'/K}(\mathbb{G}_m) & \xrightarrow{(-)^{p^r}} & \mathbb{G}_m & \longrightarrow & 1
\end{array}$$

whose rows are exact by [Lou22, Lem. 2.10]. This gives an isomorphism:

$$\frac{\mathrm{R}_{K'/K}(\mu_{p^r})}{\mu_{p^r}} \simeq \frac{\mathrm{R}_{K'/K}(\mathbb{G}_m)}{\mathbb{G}_m}.$$

Observe then that we have the following exact sequence:

$$1 \to \mathrm{R}_{K'/K}(\mu_{p^r})/\mu_{p^r} \to \mathrm{R}_{K'/K}(\mathrm{SL}_{p^r})/\mu_{p^r} \to \mathrm{R}_{K'/K}(\mathrm{SL}_{p^r})/\mathrm{R}_{K'/K}(\mu_{p^r}) \to 1.$$

We have the facts:

(1) The group $G = \mathrm{R}_{K'/K}(\mathrm{SL}_{p^r})/\mu_{p^r}$ is perfect, because $\mathrm{R}_{K'/K}(\mathrm{SL}_{p^r})$ is perfect by [CGP15, Cor. A.7.11].

(2) The center of $\mathrm{R}_{K'/K}(\mathrm{SL}_{p^r})$ is $\mathrm{R}_{K'/K}(\mu_{p^r})$ ([CGP15, Prop. A.5.15(1)]), so by [CGP15, Prop. 2.2.12(2)] the center of G is $\mathrm{R}_{K'/K}(\mu_{p^r})/\mu_{p^r} = U$.

(3) The group U lies in $\mathrm{R}_{u,K}$ while the quotient G/U is pseudo-reductive thanks to [CGP15, Prop. 1.3.4], so U appears to be the unipotent radical of G.

This proves that $U = \left(\mathrm{R}_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m\right)$ is the unipotent radical as well as the center of a perfect smooth connected linear algebraic group.

Returning to the case of a group of the form $U = \mathrm{R}_{L/K}\left(\mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_m\right)$ for a finite extension K'/K, where L is the separable closure of K in K', we just have to take the Weil restriction $\mathrm{R}_{L/K}(G)$, of a perfect smooth connected linear algebraic L-group G whose L-unipotent radical and center is $V = \mathrm{R}_{K'/L}(\mathbb{G}_m)/\mathbb{G}_m$. Indeed, $\mathrm{R}_{L/K}(G)$ is perfect because, extending scalars to K_s, the group $\mathrm{R}_{L/K}(G)_{K_s}$ is a product of the perfect groups G_σ which are obtained from G by extending scalars from L to K_s via the K-embeddings $L \hookrightarrow K_s$. Also the center of $\mathrm{R}_{L/K}(G)$ is the restriction of the center of G, which is V, by [CGP15, Prop. A.5.15(1)]. Finally the quotient $\mathrm{R}_{L/K}(G)/U$ is isomorphic to $\mathrm{R}_{L/K}(G/V)$ (Weil restriction respects quotients of a smooth linear algebraic groups by a smooth subgroup as stated in [CGP15, Cor. A.5.4]). But G/V being pseudo-reductive, we invoke [CGP15, Prop. 1.1.10] that ensures that $\mathrm{R}_{L/K}(G/V)$ is pseudo-reductive. This proves the proposition for U, finishing the proof. □

5. The pseudo-semisimple case

To study pseudo-semisimple groups over K, we will do a case by case analysis using the structure theory one can find in [CGP15] and [CP16]. Also, one want to express a more precise Conjecture I for pseudo-semisimple groups using the assumption $\mathrm{scd}_l(K) \leqslant 1$ for a minimal number of primes l, and for this we need to talk about the absolute root system of a pseudo-semisimple group, and use the notion of tame central universal cover.

Let G be a pseudo-semisimple K-group. By [CGP15, §3.2], there is a root system associated to G_{K_s}. It is the set, in the character group $X(T) = \mathrm{Hom}_{K_s-grp}(T, \mathbb{G}_{m,K_s})$ for any maximal torus $T \subset G_{K_s}$, of non-trivial α such that the α-weight space of $\mathrm{Lie}(G_{K_s})$ for the adjoint action by T is not zero. This is the *absolute root system* of G. We can thus talk about the *absolute type* of G. This type may be non reduced [CGP15, Thm. 2.3.10]. The possibilities for irreducible absolute root system are the reduced types A_n $(n \geqslant 1)$, B_n $(n \geqslant 2)$, C_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, E_6, E_7, E_8, F_4, G_2; and the non-reduced type BC_n $(n \geqslant 1)$.

Consider the reductive quotient $\overline{G}$ of G over $\overline{K}$. In fact, according to [CGP15, Thm. 2.3.10], the type of $\overline{G}$ is obtained from the absolute type of G by keeping the same irreducible reduced components, and by associating the type C_n to BC_n (where $C_1 = A_1$, $C_2 = B_2$).

For any irreducible type $\mathcal{T}$, we define its *set of torsion primes* as follows:

- $S(A_n) = \{$ prime divisors of $2(n+1)$ $\}$ for any $n \geqslant 1$;
- $S(B_n) = \{2\}$ for any $n \geqslant 2$;
- $S(C_n) = \{2\}$ for any $n \geqslant 3$;
- $S(D_n) = \{2\}$ for any $n \geqslant 5$;
- $S(D_4) = S(E_6) = S(E_7) = S(F_4) = \{2,3\}$;
- $S(E_8) = \{2,3,5\}$;
- $S(G_2) = \{2\}$;
- $S(BC_n) = \{2\}$ for any $n \geqslant 1$.

In general, the *set of torsion primes* of a type $\mathcal{T}$, is the union $S(\mathcal{T})$ of the sets of torsion primes of every irreducible component of $\mathcal{T}$. Write also $S(G)$ for the torsion primes of the absolute type of G. Note that this table coincides in the classical reduced case with [Gil19, Tableau 4.1].

The guiding question of this section is:

Question 5.1. *Let G be a pseudo-semisimple K-group G. Do we have $H^1(K,G) = 1$ if $\mathrm{scd}_l(K) \leqslant 1$ for all $l \in S(G)$?*

Recall from [CP16, Def. 5.1.1] that a *tame* algebraic K-group is a commutative linear algebraic K-group which does not contain any non-zero unipotent subgroup. For example, if L/K is a finite extension, and μ a L-group of multiplicative type, then $\mathrm{R}_{L/K}(\mu)$ is tame.

One interesting thing about tame groups is that central perfect smooth connected extensions of G by a tame group are classified similarly as in the semisimple case. By associating to any central extension:

$$1 \to Z \to E \to G \to 1$$

where Z is tame and E a perfect smooth connected linear algebraic K-group, the induced homomorphism:

$$E_{\overline{K}}/\mathrm{R}_{u,\overline{K}}(E_{\overline{K}}) \to G_{\overline{K}}/\mathrm{R}_{u,\overline{K}}(G_{\overline{K}}),$$

we obtain an equivalence of categories between the category of central perfect smooth connected extensions of G by a tame group, and the category of central semisimple covers of $G_{\overline{K}}/\mathrm{R}_{u,\overline{K}}(G_{\overline{K}})$ (this is [CP16, Thm. 5.1.3]).

In particular, there exists a perfect smooth connected linear algebraic K-group $\widetilde{G}$ and a surjective homomorphism $\widetilde{G} \to G$ with a central tame kernel, such that every other central perfect smooth connected tame extension is dominated by this map. It is called the *universal smooth tame central extension/cover*. The group $\widetilde{G}$ does not have any non-trivial tame central perfect smooth connected extension. Such a group is said to be *tamely simply connected*. In fact, G is tamely simply connected if and only if $G_{\overline{K}}/\mathrm{R}_{u,\overline{K}}(G_{\overline{K}})$ is semisimple simply connected. The reductive quotient over $\overline{K}$ of $\widetilde{G}$ is the simply connected semisimple cover of the reductive quotient over $\overline{K}$ of G. Therefore, in the semisimple case, $\widetilde{G}$ is just the simply connected cover of G. Note that this construction commutes with separable extensions and that, since G is pseudo-semisimple, $\widetilde{G}$ is too. Also, $\widetilde{G}$ has the same absolute type than G. For more information, check [CP16, Chap. 5].

Let us observe that the exact sequence associated to the map $\widetilde{G} \to G$ gives in cohomology:

$$H^1(K, \widetilde{G}) \to H^1(K, G) \to H^2(K, Z)$$

where Z is the kernel of $\widetilde{G} \to G$.

In fact, we can prove that $H^2(K, Z)$ vanishes:

Lemma 5.2. *Consider its universal smooth tame central extension $\widetilde{G} \to G$. Denote by Z its kernel. Assume that $\mathrm{scd}_l(K) \leqslant 1$ for all $l \in S(G)$. Then we have $H^2(K, Z) = 0$.*

Proof. Let $\mu \subset Z$ be the maximal subgroup of multiplicative type. Observe that $\mu_{\overline{K}}$ is the kernel of the simply connected cover of the reductive quotient of $G_{\overline{K}}$. Therefore, μ is of n-torsion, where n is such that its prime factors are all in $S(G)$.

Moreover, Z/μ is unipotent, so $H^2(K, Z/\mu) = 0$ by [TV13, Lem. 3.3]. But also, $H^2(K, \mu) = 0$ by [Gil19, Prop. 4.6.2] due to the cohomological dimension hypothesis, since μ has n-torsion. Hence $H^2(K, Z) = 0$. □

This lemma shows us that $H^1(K, G)$ is trivial if $H^1(K, \widetilde{G})$ is, when $\mathrm{scd}_l(K) \leq 1$ for all primes $l \in S(G)$. We will then suppose by now that G is tamely simply connected. This reduction is quite important for what will follow since such groups have a nice structure.

One very important object that comes along G is $\mathscr{C}_G$, the group defined in [CP16, 2.3.1], which is also the largest central unipotent subgroup of G by [CP16, Prop. 2.3.7.].

It is trivial if and only if G is generalized standard (cf. [CP16, Thm. 9.2.1. & Prop. 4.3.3.(ii)]). In other words, if one wants to use the classification result of [CP16], one should assume that $\mathscr{C}_G$ is trivial. As we don't find any way of tackling this issue, we will suppose by now that $\mathscr{C}_G$ is trivial (such a group is said to be of *minimal type*). It is sufficient for our applications.

Indeed, its non-triviality happens only when $\mathrm{char}(K) = 2$ and $[K : K^2] > 2$, and even only when $[K : K^2] > 8$ if the absolute type of G is reduced (cf. [CP16, 9.2]). This bound is actually optimal, even when K is separably closed (cf. [CP16, App. B.]).

Let us recall roughly how a generalized standard group (in the sense of [CP16, Def. 9.1.7.], not in the sense of [CGP15, Def. 10.1.9]) is constructed. Roughly speaking, a pseudo-reductive group is said to be *generalized standard* if it can be constructed from a *primitive pair* $(G', K'/K)$ and a commutative pseudo-reductive group through a pushforward given some glueing condition.

In fact, when the group is pseudo-semisimple, it is a central quotient of $\mathcal{D}(\mathrm{R}_{K'/K}(G'))$ (cf. [CP16, p. 175]). Moreover, thanks to [CP16, Prop. 9.1.6], the group $\mathcal{D}(\mathrm{R}_{K'/K}(G'))$ is tamely simply connected with tame center. This shows that any tamely simply connected group that is generalized standard (or equivalently, of minimal type), is of this form.

Recall that $(G', K'/K)$ is said to be a primitive pair if K'/K is a reducted finite K-algebra and G' a group over K' such that over each fiber it is a *primitive group*. Moreover, we say that an algebraic K-group is *primitive* if it is one of the following:

(1) a simply connected absolutely almost simple group;
(2) a *basic exotic* group of type B, C, F_4, G_2 (cf. [CP16, Def. 2.2.2.]);
(3) a *generalized basic exotic* group of type B, C (cf. [CP16, Def. 8.1.1 & Def. 8.2.3]) not of the previous case;
(4) a *rank-2 basic exceptional* group (cf. [CP16, Def. 8.3.6]);
(5) a *minimal type* absolutely pseudo-simple group with a non-reduced absolute root system (i.e. of type BC) of *root field* K.

In fact, the classification shows that these are exactly the absolutely pseudo-simple groups (i.e., such that their reductive quotient over an algebraic closure is simple, cf. [CGP15, Lem. 3.1.2]), tamely simply connected of minimal type of root field K (cf. [CP16, Def. 3.3.2 & Def. 9.1.2]).

Moreover, when a generalized standard group is constructed from primitive groups that are only semisimple simply connected, the group is said to be *standard* (cf. [CGP15, Def. 1.4.4]).

Our tamely simply connected pseudo-semisimple G is thus of the form $G = \mathcal{D}(\mathrm{R}_{K'/K}(G'))$ for a primitive pair $(G', K'/K)$. Write $K' = \prod_i K'_i$ where the K'_i/K are finite field extensions, then denote by L_i the separable closure of K_i in K'_i and set $G'_i := G'_{K'_i}$. Observe that separable Weil restrictions commutes with taking the derived subgroup since a separable Weil restriction is the descent of multiple copies of the same group with a Galois action permuting them. We then have:

$$G = \mathcal{D}(\mathrm{R}_{K'/K}(G')) = \mathcal{D}(\prod_i \mathrm{R}_{K'_i/K}(G'_i)) = \prod_i \mathcal{D}\left(\mathrm{R}_{L_i/K}(\mathrm{R}_{K'_i/L_i}(G'_i))\right)$$
$$= \prod_i \mathrm{R}_{L_i/K}\left(\mathcal{D}(\mathrm{R}_{K'_i/L_i}(G'_i))\right).$$

When taking the H^1, thanks to Shapiro's Lemma, we have:

$$H^1(K, G) = H^1(K, \mathcal{D}(\mathrm{R}_{K'/K}(G'))) = \prod_i H^1\left(L_i, \mathcal{D}(\mathrm{R}_{K'_i/L_i}(G'_i))\right).$$

To answer Question 5.1 for G, we are then reduced to the case where K'/K is a purely inseparable extension of fields, which means, thanks to [CP16, Prop. 9.1.6], that we can suppose G absolutely pseudo-simple.

Our strategy for computing $H^1(K, \mathcal{D}(\mathrm{R}_{K'/K}(G')))$ consists of understanding first when $\mathrm{R}_{K'/K}(G')$ is perfect. If it is the case, the computation would then reduce in computing $H^1(K', G')$, i.e. understanding the case of primitive groups thanks to Lemma 2.2.

In fact, we have the following lemma which explains that $\mathrm{R}_{K'/K}(G')$ is perfect if and only if G' is, in some sense, "simply connected" for infinitesimal central extensions:

Lemma 5.3. *Let G' be a smooth connected perfect linear algebraic K'-group and set $G := \mathrm{R}_{K'/K}(G')$, for K'/K a non-trivial finite purely inseparable extension. Then the following are equivalent:*

(1) The space of coinvariants of $\mathrm{Lie}(G')$ by the adjoint action of G' is trivial.
(2) G is perfect.
(3) The group $G'_{\overline{K}}$ has no non-trivial central perfect smooth connected extension by an infinitesimal group over the algebraic closure $\overline{K}$ of K.
(4) The Froebenius twist $G'^{(p)}$ has no non-trivial central perfect smooth connected extension by an infinitesimal group over K'.

Proof. (1) ⇒ (2): This is [CGP15, Prop. A.7.10].

(2) ⇒ (3): Suppose that there exists a non-trivial extension as in the statement:

$$1 \to \alpha \to E \to G'_{\overline{K}} \to 1$$

Set $A := K' \otimes_K \overline{K}$. Taking $\mathrm{R}_{A/\overline{K}}(\cdot) = i_*$ where i is $\mathrm{Spec}(A) \to \mathrm{Spec}(\overline{K})$, we have:

$$1 \to \mathrm{R}_{A/\overline{K}}(\alpha_A) \to \mathrm{R}_{A/\overline{K}}(E_A) \to \mathrm{R}_{A/\overline{K}}(G'_A) \to R^1 i_*(\alpha_A)$$

Suppose now that G is perfect, then $G_{\overline{K}} = \mathrm{R}_{A/\overline{K}}(G'_A)$ is. We then have that the image of $G_{\overline{K}}$ in $R^1 i_*(\alpha_A)$ is trivial since it is commutative. Therefore, we have in fact:

$$1 \to \mathrm{R}_{A/\overline{K}}(\alpha) \to \mathrm{R}_{A/\overline{K}}(E) \to \mathrm{R}_{A/\overline{K}}(G') \to 1$$

Observe moreover that we have $\dim(\mathrm{R}_{A/\overline{K}}(E)) = [A : \overline{K}] \dim(E)$ and also that $\dim(\mathrm{R}_{A/\overline{K}}(G'_A)) = [A : \overline{K}] \dim(G')$ since the equalities are true over Lie algebras and that E and G' are smooth. Moreover, since $\dim(\alpha) = 0$, we have that $\dim(E) = \dim(G')$, so that $\dim(\mathrm{R}_{K'/K}(E)) = \dim(\mathrm{R}_{K'/K}(G'))$. This implies that $\dim(\mathrm{R}_{A/\overline{K}}(\alpha)) = 0$, but this is absurd. Indeed, take $a \in (K' \cap K^{\frac{1}{p}})\backslash K$. Observe that $K(a) \otimes_K \overline{K} \cong \overline{K}[\varepsilon]/(\varepsilon^p)$ and $K[\varepsilon']/(\varepsilon'^2) \cong K[\varepsilon^{p-1}]/(\varepsilon^p) \subset K[\varepsilon]/(\varepsilon^p)$. Therefore:

$$\begin{aligned}\mathrm{R}_{\left(\overline{K}[\varepsilon']/(\varepsilon'^2)\right)/\overline{K}}(\alpha) &\cong \mathrm{R}_{\left(\overline{K}[\varepsilon^{p-1}]/(\varepsilon^p)\right)/\overline{K}}(\alpha) \\ &\subset \mathrm{R}_{\left(\overline{K}[\varepsilon]/(\varepsilon^p)\right)/\overline{K}}(\alpha) = \mathrm{R}_{K(a)\otimes_K \overline{K}/\overline{K}}(\alpha) \subset \mathrm{R}_{A/\overline{K}}(\alpha),\end{aligned}$$

but $\dim(\mathrm{R}_{\overline{K}[\varepsilon']/(\varepsilon'^2)}(\alpha)) = \dim(\alpha) + \dim(\mathrm{Lie}(\alpha)) = \dim(\mathrm{Lie}(\alpha)) > 0$. Hence the result.

(3) ⇒ (4): This is a direct consequence of the fact that $G'^{(p)}$ and G' are isomorphic over $\overline{K}$ and of the fact that a potential extension would not split over $\overline{K}$ since it would contradict its smoothness (or perfectness).

(4) ⇒ (1): Suppose the space of coinvariants $\mathrm{Lie}(G')_{G'}$ is non-trivial. Consider the map of p-Lie algebras $\mathrm{Lie}(G') \to \mathrm{Lie}(G')_{G'}$ given by taking the coinvariants. It is G'-equivariant. Thanks to Cartier's duality (cf. [Mil17, Prop. 11.37]), this map corresponds to a surjective G'-equivariant map $\phi : G'_{(p)} \to Z$ where $G'_{(p)}$ is the kernel of the Froebenius map $G' \to G'^{(p)}$ and Z is the finite group of height 1 associated to $\mathrm{Lie}(G')_{G'}$ (equipped with the trivial G'-action). We then have the following diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Ker}(\phi) & = & \mathrm{Ker}(\phi) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G'_{(p)} & \longrightarrow & G' & \longrightarrow & G'^{(p)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & Z & \longrightarrow & G'/\mathrm{Ker}(\phi) & \longrightarrow & G'^{(p)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Since ϕ is G'-equivariant, $\mathrm{Ker}(\phi)$ is normal in G' so that the diagram above make sense. In fact, the G'-equivariance of ϕ implies that that the extension of $G'^{(p)}$ by Z is central, and of course non-trivial since $Z \neq 0$. Moreover, E is perfect, smooth and connected as a quotient of G'. □

One direct consequence of the lemma is that, since any semisimple simply connected group satisfies condition (3), it thus satisfies (2). This gives an easier proof of [CGP15, Cor. A.7.11].

Also, one see from this lemma that there are primitive groups that doesn't give a perfect Weil restriction. Indeed, in [CP16, B.1-2], give examples of tamely simply connected groups of type B_n and C_n (for $n \geqslant 2$) that are not of minimal type. In fact, they are extensions of generalized exotic basic groups of type B_n (resp. C_n) by finite groups that are necessarily unipotent and infinitesimal. As for the case of type A_1, there are also counter-examples thanks to the discussions in [CGP15, 9.1.8] and [CP16, Ex. 3.1.6]. This phenomenon only happens when $[K : K^2] \geqslant 16$.

We also have in [CP16, B.4] examples of groups of type BC_n not locally of minimal type, and similarly, induces non trivial infinitesimal extensions of a group of a primitive group of type BC_n. This only happens when $[K : K^2] \geqslant 4$.

On the other hand, [CP16, Prop. B.3.4] implies that a generalized basic exotic K-group has no non-trivial central perfect smooth connected extension by an infinitesimal group over K when $[K : K^2] \leqslant 8$. One would want to apply Lemma 5.3 to conclude that such groups give perfect Weil restrictions, but we don't know if it is possible to change (4) by replacing $G'^{(p)}$ with G'.

Moreover, the propositions [CGP15, Prop. 7.3.3] and [CGP15, Prop. 9.9.4.(1)] shows us that, when $[K : K^p] \leqslant p$, the cohomology set of a basic exotic group and of a primitive group of type BC_n is in bijection with the cohomology set of a reductive group, which we know being trivial thanks to Steinberg's theorem (whose refined version is given in [Gil19, Thm. 5.2.5]). Therefore, all primitive groups have trivial H^1 when $[K : K^p] \leqslant p$. In general, we don't know how to compute such a cohomology set in the non-standard case. A separate study should be done for that.

We finally made the table 1 that summarizes what happens for every primitive group. From this table, and all we have discussed, we deduce in particular the following result:

Theorem 5.4. *Let G be a pseudo-semisimple group over K. Suppose that $\mathrm{scd}_l(K) \leqslant 1$ for all $l \in S(G)$ and that $[K : K^p] \leqslant p$ if G is not standard (which happens only when $p \in \{2, 3\}$). Then $H^1(K, G) = 1$.*

TABLE 1. Behaviour of each primitive group.

primitive group G	$H^1(K,G)$ trivial if $\forall l \in S(G), \mathrm{scd}_l(K) \leqslant 1$?	Perfect insep. Weil restriction ?	Possible absolute type	Appears when ...
semisimple simply connected	✓ [Gil19, Thm. 5.2.5]	✓ [CGP15, Cor. A.7.11]	Any reduced one	Always
basic exotic	$*$ if $[K:K^p] \leqslant p$: ✓ [CGP15, Prop. 7.3.3] $*$ if $[K:K^p] \geqslant p^2$: ?	✓ [CGP15, Prop. 8.1.2]	B_n, C_n, F_4, G_2 (for $n \geqslant 2$)	$p = 2$ for B_n, C_n, F_4 $p = 3$ for G_2
generalized basic exotic (outside the previous case)	?	$*$ if $[K:K^2] \leqslant 8$: ? $*$ if $[K:K^2] \geqslant 16$: ✗ [CP16, Ex. 3.1.6] [CP16, B.1-2] and Lem. 5.3	$A_1 = B_1, B_n, C_n$ (for $n \geqslant 2$)	$p = 2$ and $[K:K^2] \geqslant 4$
rank-2 basic exceptional	?	?	B_2, C_2	$p = 2$ and $[K:K^2] \geqslant 16$
non reduced type	$*$ if $[K:K^2] = 2$: ✓ [CGP15, Prop. 9.9.4.(1)] $*$ if $[K:K^2] \geqslant 4$: ?	$*$ if $[K:K^2] = 2$: ✓ [CGP15, Prop 10.1.4.(1)] $*$ if $[K:K^2] \geqslant 4$: ✗ [CP16, B.4] and Lem. 5.3	BC_n (for $n \geqslant 1$)	$p = 2$

6. Triviality of H^1 for unirational groups and quasi-splitness

The previous sections enable us to state the our generalization of Serre's Conjecture I for unirational algebraic K-groups. We also deduce from the previous section quasi-splitness of smooth connected linear algebraic groups in the generalized sense of [CP16, C.2].

6.1. Serre's Conjecture I for unirational groups.

We have the following result:

Theorem 6.1. *Assume* $\dim(K) \leqslant 1$. *Then for all unirational algebraic K-groups G, we have* $H^1(K, G) = 1$.

Proof. Since G is unirational, Proposition 3.3 implies that $H^1(K, G)$ is trivial as soon as $H^1(K, G^{\mathrm{perf}})$ is trivial, so we may assume that G is perfect.

Moreover, by Theorem 4.9, we know that $\mathrm{R}_{u,K}(G)$ is unirational, so $H^1(K, \mathrm{R}_{u,K}(G)) = 1$ by Proposition 3.3 again, because $\mathrm{R}_{u,K}(G)$ is solvable. Because of the cohomology sequence obtained from the short exact sequence

$$1 \to \mathrm{R}_{u,K}(G) \to G \to G/\mathrm{R}_{u,K}(G) \to 1,$$

we see that we can assume $\mathrm{R}_{u,K}(G) = 1$, in other words, that G is pseudo-semisimple.

We then conclude with Theorem 5.4 which says that the first Galois cohomology set of a pseudo-semisimple K-group is trivial. □

6.2. Quasi-splitness.

The classical result about Serre's Conjecture I over imperfect fields implies the following: with the assumption $\mathrm{scd}(K) \leqslant 1$, every reductive group over K is quasi-split, meaning it admits a Borel subgroup over K, that is a solvable parabolic subgroup, or equivalently, a minimal parabolic K_s-subgroup (cf. [Gil19, Thm. 5.2.5]). In fact, in the case of $\dim(K) \leqslant 1$, we can show that every smooth connected linear K-group is quasi-split in a generalized sense.

Let G be a smooth connected linear algebraic K-group. For any cocharacter $\lambda : \mathbb{G}_m \to G$, let $P_G(\lambda)$ be the subgroup of G whose R-points are, for all K-algebras R, the $g \in G(R)$ such that the scheme morphism $\mathbb{G}_{m,R} \to G_R$, $t \mapsto \lambda(t)g\lambda(t)^{-1}$ can be extended to $\mathbb{A}^1_R$. The groups $P_G(\lambda)$ are smooth, connected by [CGP15, Prop. 2.1.8]. We then define a *pseudo-parabolic subgroup* of G to be a subgroup of the form $\mathrm{R}_{u,K}(G)P_G(\lambda)$.

When G is reductive, pseudo-parabolic subgroups are exactly the parabolic subgroups ([CGP15, Prop. 2.2.9]). In parallel to the reductive case, a pseudo-Borel K-subgroup of a smooth connected linear K-group G is a solvable pseudo-parabolic subgroup over K (or equivalently, a minimal pseudo-parabolic K_s-subgroup). The group G is said to be *quasi-split* if it has a pseudo-borel subgroup over K. Of course, G is quasi-split if and only if its pseudo-reductive quotient is.

Moreover, thanks to [CP16, C.2], one can consider the notion of *pseudo-inner forms*, which is roughly the generalized notion of inner forms for pseudo-reductive groups. As in the semisimple case, the group functor of automorphisms $\mathrm{Aut}_{G/K}$ of a pseudo-semisimple K-group G is representable as a linear algebraic K-group (cf. [CP16, Prop. 6.2.2]). The neutral component of the maximal smooth subgroup of $\mathrm{Aut}_{G/K}$, which is denoted by $(\mathrm{Aut}^{\mathrm{sm}}_{G/K})^0$, is, in fact, the group used to define pseudo-inner forms. Contrary to the semisimple case, it is not necessarily $G/Z(G)$ (we have, however, $\mathcal{D}\left((\mathrm{Aut}^{\mathrm{sm}}_{G/K})^0\right) = G/Z(G)$; cf. [CP16, p. 216]).

In fact, we can give an more precise description of $(\mathrm{Aut}^{\mathrm{sm}}_{G/K})^0$. Fix any Cartan subgroup $C \subset G$. By [CP16, Prop. 6.2.4], we have the following isomorphism:

$$(\mathrm{Aut}^{\mathrm{sm}}_{G/K})^0 \simeq (G \rtimes Z_{G,C})/C$$

where $Z_{G,C}$ is the maximal smooth subgroup of $\mathrm{Aut}_{G,C}$, the group functor of automorphisms of G that fixes C pointwise, which is represented by a linear algebraic K-group according to [CGP15, Thm. 2.4.1]. The group $Z_{G,C}$ is moreover commutative (cf. [CGP15, Thm. 2.4.1]). In other words, $(\mathrm{Aut}^{\mathrm{sm}}_{G/K})^0$ is the pushforward of the extension given by $C \to Z_{G,C}$ by the map $C \to G/Z(G)$.

If G is this time supposed to be only pseudo-reductive, then the action of $(\mathrm{Aut}^{\mathrm{sm}}_{\mathcal{D}(G)/K})^0$ on $\mathcal{D}(G)$ extends to G (cf. [CP16, Lem. C.2.3]). We therefore define a *pseudo-inner form* of G as a group obtained by twisting G via a cocycle in the image of

$$H^1(K, (\mathrm{Aut}^{\mathrm{sm}}_{\mathcal{D}(G)/K})^0) \to H^1(K, \underline{\mathrm{Aut}}(G)),$$

where $\underline{\mathrm{Aut}}(G)$ is the functor of automorphisms of G (cf. [CP16, Def. C.2.4]).

It is not actually true that every pseudo-reductive group G admits a quasi-split pseudo-inner form (cf. [CP16, C.4]). It is, however, true in some situations (for example, if $p > 2$ or $p = 2$ and $[K : K^2] \leqslant 4$, cf. [CP16, Thm. C.2.10]), and especially when $\mathrm{scd}(K) \leqslant 1$:

Proposition 6.2. *Let G be a smooth, connected linear algebraic K-group. Suppose that $\mathrm{scd}(K) \leqslant 1$. Then G admits a quasi-split pseudo-inner form.*

Proof. The question obviously reduces to the pseudo-reductive case. Now, cf. [CP16, C.2.9] gives us that the existence of a quasi-split pseudo-inner form is equivalent to the triviality of the H^2 of a certain pseudo-reductive commutative K-group. This is of course trivial thanks to Lemma 2.3. □

Let us now prove:

Theorem 6.3. *Let G be a smooth connected linear algebraic K-group. Suppose that $\mathrm{scd}(K) \leqslant 1$ and that $[K : K^p] \leqslant p$ if $G/\mathrm{R}_u(G)$ is not standard (which happens only when $p \in \{2, 3\}$). Then G is quasi-split.*

Proof. The problem reduces to the pseudo-reductive case. Moreover, since G admits a quasi-split pseudo-inner form G^q, thanks to [CP16, Thm. C.2.10], it suffices to show that the cohomological set $H^1(K, \mathrm{Aut}^{\mathrm{sm}}_{\mathcal{D}(G^q)/K})^0)$ is trivial in order to conclude.

Note that we have the following isomorphism:

$$(\mathrm{Aut}^{\mathrm{sm}}_{\mathcal{D}(G^q)/K})^0 \simeq (\mathcal{D}(G^q) \rtimes Z_{\mathcal{D}(G^q),C})/C,$$

where C is a Cartan subgroup of $\mathcal{D}(G^q)$.

We then see that $H^1(K, \mathrm{Aut}^{\mathrm{sm}}_{\mathcal{D}(G^q)/K})^0) = 1$ as soon as $H^1(K, \mathcal{D}(G^q))$, $H^1(K, Z_{D(G^q),C})$ and $H^2(K, C)$ are trivial. Thanks to the proof of [CP16, Prop. C.2.8], we have $H^1(K, Z_{D(G^q),C}) = 1$ since $D(G^q)$ is quasi-split. Moreover, $H^2(K, C) = 1$ thanks to Lemma 2.3. Finally, $H^1(K, \mathcal{D}(G)) = 1$ thanks to Theorem 5.4. □

We then have the following consequence for Bruhat-Tits theory:

Proposition 6.4. *Let G be a reductive group over an henselian discretely valued field K. Denote by κ the residue field of K. Suppose that $\mathrm{scd}(\kappa) \leqslant 1$ and moreover $[\kappa : \kappa^p] \leqslant p$ if $p \in \{2, 3\}$. Then G is residually quasi-split.*

Proof. [Zid26, Prop. 1.6.2.] gives equivalent conditions for G being residually quasi-split. The condition (5) is actually satisfied thanks to Theorem 6.3. □

7. General reduction for the H^1 of a smooth algebraic group

Consider a smooth algebraic K-group G. We define in this part a canonical morphism $G \to G'$ such that G' has no non-trivial unirational subgroup, and we show that it induces a bijection $H^1(K,G)$ and $H^1(K,G')$ when $\dim(K) \leqslant 1$. Actually, to prove the surjectivity of such a map, we prove a surjectivity result for H^1 valid when $\mathrm{scd}(K) \leqslant 1$, and which is an improvement of [Ser94, Thm. 3, III.2.4] (stated originally for perfect fields) using the new study of linear algebraic groups over imperfect fields we can find in [CGP15] and in [CP16].

7.1. Homogeneous spaces and torsors.

Given an algebraic K-group G, recall that a *G-homogeneous space over K* is a non-empty K-scheme of finite type X endowed with a right action by G such that the morphism $G \times_K X \to X \times_K X$ given by $(g,x) \mapsto (x \cdot g, x)$ is faithfully flat. Note that when G is smooth, a non-empty G-homogeneous space is exactly a non-empty smooth K-scheme of finite type with an action of G that is transitive on the K_s-points.

Theorem 7.1. *Assume $\mathrm{scd}(K) \leqslant 1$ and let G be a (not necessarily linear) smooth algebraic K-group. Then every G-homogeneous space X is dominated by a G-torsor, that is there exists a G-torsor $\widetilde{X}$ and a G-equivariant map $\widetilde{X} \to X$.*

We roughly do the same proof of [Ser94, Thm. 3, III.2.4], but we replace Borel subgroups by pseudo-Borel subgroups and some groups by their largest smooth subgroup.

Proof. Fix a point $x_0 \in X(K_s)$ (which exists since X is smooth) and consider the pairs (H,a), called *compatible pairs*, of a smooth K_s-subgroup $H \subset G_{K_s}$ and a continuous map $a : \Gamma \to G(K_s)$ satisfying the following properties:

1. For all $h \in H(K_s)$, $x_0 \cdot h = x_0$;
2. For all $s \in \Gamma$, ${}^s x_0 = x_0 \cdot a_s$;
3. For all s, $t \in \Gamma$, $a_s {}^s a_t a_{st}^{-1} \in H(K_s)$;
4. For all $s \in \Gamma$, $a_s {}^s H a_s^{-1} = H$ as subgroups of G_{K_s}.

Compatible pairs exist: Take H to be $(\mathrm{Stab}_{G_{K_s}}(x_0))^{\mathrm{sm}}$, and take for a a continuous map such that ${}^s x_0 = x_0 \cdot a_s$ for all $s \in \Gamma$, which exists by homogeneity.

Now let (H,a) be a compatible pair with H of minimal dimension. We show that $H = 1$, thus a will be a 1-cocycle of G which corresponds to a G-torsor that dominates X.

⋄ First of all, the neutral component H^0 of H is solvable. For this, consider L, the largest smooth connected linear algebraic K_s-subgroup of H^0. It is such that the quotient H^0/L is a pseudo-abelian variety (cf. [Mil17, Prop. 8.6]). Choose also a pseudo-Borel subgroup B of L. By definition, there exists a cocharacter λ such that $B = \mathrm{R}_{u,K_s}(L) P_L(\lambda)$.

For all $s \in \Gamma$, ${}^s B = \mathrm{R}_{u,K_s}({}^s L) P_{{}^s L}({}^s \lambda)$, so $a_s {}^s B a_s^{-1}$ is also a pseudo-Borel subgroup of $a_s {}^s L a_s^{-1} = L$ (as subgroups of H). But by [CGP15, Thm. C.2.5], all pseudo-Borel subgroups of L are conjugated by $L(K_s)$, so there exists $h_s \in L(K_s)$ such that $a_s {}^s B a_s^{-1} = h_s^{-1} B h_s$. We can choose the h_s such that $h : \Gamma \to G(K_s)$, $s \mapsto h_s$ is continuous, so that we get a compatible pair $(\mathrm{N}_H(B)^{\mathrm{sm}}, ha)$. Indeed, (1), (2), and (4) are clear enough. For (3), let:

$$h_{s,t} = a_s {}^s a_t a_{st}^{-1} \text{ and } h'_{s,t} = h_s a_s {}^s(h_t a_t)(h_{st} a_{st})^{-1}$$

for all s, $t \in \Gamma$. We have $h_{s,t} \in H(K_s)$ by assumption, and

$$h'_{s,t} = h_s a_s {}^s h_t a_s^{-1} h_{s,t} h_{st}^{-1}$$

so $h_{s,t}^{-1} \in H(K_s)$ since $a_s {}^s h_t a_s^{-1} \in H(K_s)$. Moreover,

$$(h_s a_s {}^s(h_t a_t))^{-1} B h_s a_s {}^s(h_t a_t) = {}^{st} B = (h_{st} a_{st})^{-1} B h_{st} a_{st},$$

thus, we see $h'_{s,t} \in \mathrm{N}_H(B)(K_s)$, but $\mathrm{N}_H(B)(K_s) = \mathrm{N}_H(B)^{\mathrm{sm}}(K_s)$. This shows (3).

Now, since H is minimal, necessarily $H = \mathrm{N}_H(B)^{\mathrm{sm}}$, and thus $L = \mathrm{N}_L(B)^{\mathrm{sm}}$. But B is its own normalizer in L by [CGP15, Prop. 3.5.7], so $L = B$. Hence, H^0 is an extension of a pseudo-abelian variety (which is commutative by [Tot13, Thm. 2.1]) by B, which is solvable. So H^0 is solvable.

⋄ Let us see next that H itself is solvable. In view of the previous point, it is enough to show that H/H^0 is solvable. Let $S \subset (H/H^0)(K_s)$ be a l-Sylow subgroup seen as an algebraic subgroup of H/H^0. Write P for the inverse image of S in H. The group $Q = a_s {}^sPa_s^{-1}$ lies inside H and its image $\overline{Q}$ in H/H^0 is another l-Sylow subgroup: by Sylow's theorems there exists $h_s \in H(K_s)$ such that $\overline{Q} = \overline{h_s}S\overline{h_s}^{-1}$, thus $Q = h_s^{-1}Ph_s$. Of course, we can choose $h : s \mapsto h_s$ to be continuous, and $(\mathrm{N}_H(P)^{\mathrm{sm}}, ha)$ is a compatible pair, hence $H = \mathrm{N}_H(P)$ and $H/H^0 = \mathrm{N}_{H/H^0}(S)$. Every Sylow subgroup of H/H^0 being normal, we get that H/H^0 is a product of finite l-groups for several primes l, which are all solvable.

⋄ We finally show that H is actually perfect. For every $s \in \Gamma$, the s-semilinear automorphism (see [FSS98, §(1.2)]) ${}^sG \to G$, $g \mapsto a_s {}^sga_s^{-1}$ of G restricts to a scheme isomorphism $\tau_s : {}^sH \to H$. The inverse ρ_s of τ_s passes to the quotient by $\mathcal{D}(H)$ to yield an s-semilinear automorphism $\overline{\rho_s}$ of $H/\mathcal{D}(H)$. The collection of the $\overline{\rho_s}$'s defines a map $\overline{\rho} : \Gamma \to \mathrm{SAut}((H/\mathcal{D}(H))/K)$, where $\mathrm{SAut}((H/\mathcal{D}(H))/K)$ is the group of semilinear automorphism of $H/\mathcal{D}(H)$. Thanks to the properties defining a compatible pair, $\overline{\rho}$ is actually continuous (in the sense of [FSS98, Def. (1.10)]) and is a group homomorphism. Thus $\overline{\rho}$ defines a K-group structure on $H/\mathcal{D}(H)$ by [FSS98, Rem. (1.15)]. Denote by C the K-descent of $H/\mathcal{D}(H)$ defined by $\overline{\rho}$. Write, for all s, $t \in \Gamma$:

$$h_{s,t} = a_s {}^sa_t a_{st}^{-1} \in H(K_s),$$

and $\overline{h_{s,t}}$ for its image in $H/\mathcal{D}(H)(K_s)$. For all s, t, $u \in \Gamma$, we have

$$\begin{aligned} h_{s,t}^{-1}\rho_s(h_{t,u})h_{s,tu}h_{st,u}^{-1} &= \left(a_s {}^sa_t a_{st}^{-1}\right)^{-1}\left(a_s {}^sa_t {}^{st}a_u {}^sa_{tu}^{-1} a_s^{-1}\right)\left(a_s {}^sa_{tu} a_{stu}^{-1}\right)\left(a_{st} {}^{st}a_u a_{stu}^{-1}\right)^{-1} \\ &= \left(a_{st} {}^sa_t^{-1} a_s^{-1}\right)\left(a_s {}^sa_t {}^{st}a_u {}^sa_{tu}^{-1} a_s^{-1}\right)\left(a_s {}^sa_{tu} a_{stu}^{-1}\right)\left(a_{stu} {}^{st}a_u^{-1} a_{st}^{-1}\right) \\ &= 1, \end{aligned}$$

hence $\overline{\rho}_s(\overline{h_{t,u}})\overline{h_{st,u}}^{-1}\overline{h_{s,tu}}\,\overline{h_{s,t}}^{-1} = 1$ in $H/\mathcal{D}(H)(K_s)$. It follows that $(\overline{h_{s,t}})_{s,t}$ is a 2-cocycle of $H/\mathcal{D}(H)$. However, $H^2(K, C) = 1$ because of Lemma 2.3. Therefore, $(\overline{h_{s,t}})_{s,t}$ is a 2-coboundary: there exists a 1-cochain z of $H/\mathcal{D}(H)(K_s)$ such that for all s, $t \in \Gamma$, $\overline{h_{s,t}} = z_s^{-1}\overline{\rho}_s(z_t)^{-1}z_{st}$. Taking a continuous lift $y : \Gamma \to H(K_s)$ of z, we have

$$\begin{aligned} \overline{h_{s,t}} &= \overline{y_s}^{-1}\overline{a_s {}^sy_t a_s^{-1}}\,\overline{y_{st}} \\ &= \overline{y_s}^{-1}\overline{y_s}\,\overline{a_s}\,\overline{{}^sy_t}\,\overline{a_s}^{-1}\overline{y_s}^{-1}\overline{y_{st}}. \end{aligned}$$

Thus there exists $h' : \Gamma \to \mathcal{D}(H)(K_s)$, continuous, such that

$$h_{s,t} = y_s^{-1}\, y_s a_s\, {}^sy_t\, a_s^{-1}\, y_s^{-1}\, h'_{s,t}\, y_{st},\ \forall s, t \in \Gamma.$$

Now, if we write $a' = ya$, we have

$$a'_s {}^sa'_t a_{st}^{-1} = h'_{s,t} \in \mathcal{D}(H)(K_s).$$

It appears that the pair $(\mathcal{D}(H), a')$ is a compatible pair.

We conclude that H is solvable and perfect, so $H = 1$ and a is a 1-cocycle of G. □

We then deduce from Theorem 7.1:

Corollary 7.2. *Assume* $\mathrm{scd}(K) \leqslant 1$. *Let* $f : G \to G'$ *be a surjective map of smooth algebraic* K*-groups. Then the natural map*

$$H^1(K, G) \to H^1(K, G')$$

is surjective.

Proof. The proof is the same as the one from [Ser94, Cor. 2, III.2.4]. In other words, every G'-torsor can be seen as a G-homogeneous space. This homogeneous space is thus dominated by a G-torsor thanks to Theorem 7.1. This gives the result. □

7.2. **Strongly wound quotient.**

Let G be a smooth algebraic K-group. There is a largest unirational K-subgroup G^{uni} of G thanks to [BCS25, 2.1.2.(9)]. It is affine, smooth, connected and normal in G, and its construction commutes with separable extensions. If $[K : K^p] \leqslant p$, then the quotient G/G^{uni} has no non-trivial unirational subgroup. Indeed, if $H \subset G/G^{\mathrm{uni}}$ is unirational, then the inverse image of H by the quotient homomorphism $G \to G/G^{\mathrm{uni}}$ is unirational by [Ros26, Thm. 2.4], thus $H = 1$.

Theorem 7.3. *Let* G *be a smooth algebraic* K*-group. If* $\dim(K) \leqslant 1$*, then the quotient homomorphism* $G \to G' := G/G^{uni}$ *induces a bijection*

$$H^1(K, G) \xrightarrow{\sim} H^1(K, G').$$

Moreover, $(G')^{\mathrm{uni}} = 1$*, i.e. it has no non-trivial unirational subgroup.*

Proof. By Corollary 7.2, the map $H^1(K, G) \to H^1(K, G/G^{\mathrm{uni}})$ is surjective. As for the injectivity, we use the classical twisting argument, noting that every twist of G^{uni} is K-unirational because it is so over K_s, so that its H^1 is trivial as well. □

Remark 7.4. When G is a smooth algebraic K-group whose identity component G^0 is unirational, Theorem 7.3 says that $H^1(K, G) \to H^1(K, G/G^0)$ is a bijection if $\dim(K) \leqslant 1$.

More specifically, in the case of smooth connected linear algebraic K-groups, when $[K : K^p] \leqslant p$, such groups G for which G^{uni} are trivial, are called *strongly wound*, and in general the *strongly wound quotient* of G is $G \to G/G^{\mathrm{uni}}$.

Actually, strongly wound groups can be defined similarly without the assumption $[K : K^p] \leqslant p$. In this case, the quotient G/G^{uni} is not necessarily strongly wound. Instead we need to continue the process: Given G, consider the sequence of groups $G_0 = G$, and for all $i \geqslant 0$, let $G_{i+1} = G_i/G_i^{\mathrm{uni}}$. There exists a minimal index i_0 for which G_{i_0} is strongly wound, and the *strongly wound quotient* of G is $G \to G_{i_0}$. The kernel of the strongly wound quotient is then the largest *ext-unirational* subgroup of G (see [Ros26, §4]), that is a group which is obtained by taking successive extensions of unirational groups.

For an arbitrary base field K, a strongly wound K-group G is necessarily unipotent, because it can't contain a non-trivial torus. It is also wound, meaning it admits no non-constant map $\mathbb{A}^1_K \to G$. Note that G_{K_s} must also be strongly wound. Indeed, the largest unirational subgroup $(G_{K_s})^{\mathrm{uni}}$ is stable by the Galois action, hence descends over K, but unirationality over K is equivalent to unirationality over K_s by [Ros24, Thm. 7.9], so $(G_{K_s})^{\mathrm{uni}} = 1$.

Since strongly wound K-group G are unipotent, Theorem 7.3 means that the Galois cohomology of a smooth connected linear algebraic K-group is entirely determined by the cohomology of its strongly wound quotient when $\dim(K) \leqslant 1$.

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