

# ANSWER TO A DECOMPOSITION QUESTION ON TORI RAISED BY COLLIOT-THÉLÈNE AND SANSUC

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ABSTRACT. The aim of this note is to present a simple strategy to answer negatively a decomposition question on tori posed by Colliot-Thélène and Sansuc in the article *Principal Homogeneous Spaces under Flasque Tori: Applications* of 1987. We then deduce a torus  $T$  over  $\mathbb{Q}$  and a prime number  $p$  such that  $T(\mathbb{Z}_p)T(\mathbb{Q}) \neq T(\mathbb{Q}_p)$ , where  $T(\mathbb{Z}_p)$  denotes the maximal compact subgroup of  $T(\mathbb{Q}_p)$ .

**Keywords:** Algebraic tori, Weak approximation, Kottwitz morphism.

**MSC:** 11E72, 14G25, 20G30.

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## INTRODUCTION

The purpose of this note is to answer the following question:

**Question 1** ([4, Remark 8.3]). *Let  $K$  be a global field,  $\widehat{K}$  a completion of  $K$  with respect to a discrete valuation, and  $\widehat{R}$  the associated discrete valuation ring. Consider a torus  $T$  over  $K$ . Let  $T(\widehat{R})$  denote the maximal compact subgroup of  $T(\widehat{K})$ . Is the equality*

$$T(\widehat{R})T(K) = T(\widehat{K})$$

*satisfied?*

**2. Context.** Question 1 first appeared in 1987 in [4], where Colliot-Thélène and Sansuc record it following a suggestion by Bruhat and Tits. There, they prove the case of an unramified  $K$ -torus  $T$  using the theory of flasque resolutions (cf. [4, Proposition 8.1.(i)]).

Later, in 2007 in [5], Colliot-Thélène and Suresh studied variants of the problem: the case where  $R$  is a semi-local ring (thus involving several valuations to be considered simultaneously), and the case of a decomposition “ $RT(\widehat{K})T(\widehat{R}) = T(\widehat{K})$ ”, where  $RT(\widehat{K})$  is the subgroup of  $R$ -trivial elements of  $T(\widehat{K})$  (see [7, 7.1. Définition]). They succeeded in finding counterexamples for both situations.

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They also provided a list of cases where the decomposition “ $\mathrm{RT}(\widehat{K})T(\widehat{R}) = T(\widehat{K})$ ” holds (see [5, Proposition 2.1]). However, even after this work, Question 1 remained open.

In this note, we provide a negative answer to Question 1 (see Propositions 14 and 16). More generally, we develop a systematic strategy to construct tori that do not satisfy the required decomposition. The crucial ingredient for this is the Kottwitz morphism, which had not been used until now for this type of problem.

**3. The Kottwitz morphism.** Suppose this time that  $\widehat{K}$  denotes a discretely valued Henselian field with valuation  $v$  and whose residue field is perfect. The Kottwitz morphism can be viewed as a functorial generalization of the valuation morphism

$$\begin{aligned} \kappa_{\mathbb{G}_m} : \widehat{K}^\times &\longrightarrow \mathbb{Z} \\ k &\longmapsto v(k) \end{aligned}$$

to any reductive group  $G$  over  $\widehat{K}$ . It was defined by Kottwitz in [11, Section 7].

For example, the Kottwitz morphism for  $\mathrm{GL}_n$  over  $\widehat{K}$  is:

$$\begin{aligned} \kappa_{\mathrm{GL}_n} : \mathrm{GL}_n(\widehat{K}) &\longrightarrow \mathbb{Z} \\ M &\longmapsto v(\det(M)), \end{aligned}$$

while for  $\mathrm{PGL}_n$  over  $\widehat{K}$ , we have:

$$\begin{aligned} \kappa_{\mathrm{PGL}_n} : \mathrm{PGL}_n(\widehat{K}) &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ \overline{M} &\longmapsto v(\det(\overline{M})) \pmod{n}. \end{aligned}$$

The morphism is defined using the algebraic fundamental group (see [10, Definition 11.3.2]), denoted by  $\pi_1(G)$ . It is a  $\mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K})$ -module (where  $\widehat{K}^{\mathrm{sep}}$  is the separable closure of  $\widehat{K}$ ). For a  $\widehat{K}$ -torus  $T$ , we have  $\pi_1(T) = T_*$ , the Galois module of cocharacters of  $T$ .

Consider  $\widehat{K}^{\mathrm{unr}}$ , the maximal unramified extension of  $\widehat{K}$ . Let  $\Gamma^{\mathrm{unr}} := \mathrm{Gal}(\widehat{K}^{\mathrm{unr}}/\widehat{K})$  and  $I := \mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K}^{\mathrm{unr}})$ . Consider a reductive group  $G$  over  $\widehat{K}$ . It turns out that the Kottwitz morphism of  $G$  over  $\widehat{K}^{\mathrm{unr}}$  is a morphism of  $\Gamma^{\mathrm{unr}}$ -modules:  $\kappa_{G_{\widehat{K}^{\mathrm{unr}}}} : G(\widehat{K}^{\mathrm{unr}}) \rightarrow \pi_1(G)_I$ .

The Kottwitz morphism of  $G$  over  $\widehat{K}$  is then the morphism  $\kappa_G : G(\widehat{K}) \rightarrow (\pi_1(G)_I)^{\Gamma^{\mathrm{unr}}}$  obtained from  $\kappa_{G_{\widehat{K}^{\mathrm{unr}}}}$  by taking the  $\Gamma^{\mathrm{unr}}$ -invariants. It enjoys many properties:

- It is surjective when  $\widehat{K}$  is a local field (see [10, Corollary 11.7.6]).
- It is trivial for any simply connected semisimple group (since its  $\pi_1$  is trivial).
- Its kernel is  $G(\widehat{K})^0$ , the open subgroup generated by the parahoric subgroups of  $G$  (see [10, Proposition 11.5.4]).
- The preimage of the torsion part of  $(\pi_1(G)_I)^{\Gamma^{\mathrm{unr}}}$  is  $G(\widehat{K})^1$ , the intersection of the kernels of  $g \mapsto v(\chi(g))$  for  $\chi$  running through the characters of  $G$  (see [10, Lemma 11.5.2]).

Furthermore, for a  $\widehat{K}$ -torus  $T$ , it turns out that  $T(\widehat{K})^0$  is the subgroup of points of the connected Néron model of  $T$  (see [10, Corollary B.8.7]), and  $T(\widehat{K})^1$  is the maximal bounded subgroup of  $T(\widehat{K})$ .

**4. Strategy and outline of the paper.** Our strategy to answer Question 1 consists of two steps. In Section 1, we prove that for a reductive group  $G$  over  $\widehat{K}$  (which is now a local field) split by an extension whose wild inertia subgroup is cyclic, the map  $G(\widehat{K})/\overline{G(K)} \rightarrow G(\widehat{K})/G(K) G(\widehat{K})^0$  is an isomorphism (see Proposition 9). This is proven using the Kottwitz morphism. Next, in Section 2, we find a torus  $T$  over  $\widehat{K}$  satisfying the previous hypothesis (in fact, split by a tamely ramified extension) such that  $T(\widehat{K})^1 = T(\widehat{K})^0$  and  $\overline{T(K)} \neq T(\widehat{K})$  (meaning that  $T$  does not satisfy weak approximation). Weak approximation over number fields is very well understood thanks to the work of Sansuc (see [13]). In [1], Borovoi provides a version including function fields. We then deduce in Propositions 14 and 16 a negative answer to Question 1 (and even a negative answer to an analogous

question in the case of a semisimple group). It is worth noting that this strategy also works for studying the semi-local variant of the problem.

In addition, we discuss some further problems surrounding the Kottwitz morphism in Sections 3 and 4 of this article. On the one hand, in Section 3, we give two examples of tori over  $\mathbb{R}((t))$  whose Kottwitz morphisms are not surjective. Thus, surjectivity is only achieved for specific fields. On the other hand, we show in Section 4 that, for a  $K$ -torus  $T$ , the groups  $\text{coker}(\kappa_{T_{\widehat{K}}})$ ,  $\text{coker}(T(K) \rightarrow ((T_*)_I)^{\Gamma^{\text{unr}}})$ , and  $T(\widehat{K})/T(K)T(\widehat{K})^0$  are stably birational invariants of  $T$ .

## NOTATIONS

Let  $K$  be a field. We denote:

- $\Omega_K$  the set of places of  $K$ ,  $\Sigma$  a finite subset of  $\Omega_K$ , and  $\overline{\Sigma} := \Omega_K \setminus \Sigma$ ,
- $K_v$  the completion of  $K$  at a place  $v \in \Sigma$  and  $K_\Sigma := \prod_{v \in \Sigma} K_v$  (with  $K$  diagonally embedded),
- $K_v^{\text{unr}}$  the maximal unramified extension of  $K_v$  and  $K_\Sigma^{\text{unr}} := \prod_v K_v^{\text{unr}}$ ,
- $\Gamma_v^{\text{unr}} := \text{Gal}(K_v^{\text{unr}}/K_v)$ ,  $I_v$  the inertia subgroup of  $K_v$ ,  $\Gamma^{\text{unr}} := \prod_v \Gamma_v^{\text{unr}}$ , and  $I := \prod_v I_v$ .

We assume that, for every  $v \in \Sigma$ , the residue field of  $K_v$  is *perfect*. The notations, properties, and definitions introduced in this note for a complete field generalize to the case of  $K_\Sigma$  by working factor by factor. This is implicit throughout the document.

We also adopt the notations  $(-)^{\text{sep}}$ ,  $\pi_1(-)$ ,  $(-)_*$ ,  $\kappa(-)$ ,  $(-)^0$ , and  $(-)^1$  from §3.

In Sections 1 and 2, the field  $K$  is assumed to be *global*.

## 1. A FEW LEMMAS ON THE KOTTWITZ MORPHISM

The purpose of this section is to establish several results crucial for developing our answer to Question 1, which involve the Kottwitz morphism. In particular, we show that under certain conditions, the problem reduces to a question of weak approximation.

**5. Cyclic wild inertia.** A reductive group over  $K_\Sigma$  is said to have *cyclic wild inertia* if for every  $v \in \Sigma$ , there exists a Galois splitting extension of  $K_v$  such that its wild inertia subgroup is cyclic (this is in particular the case if the extension is tamely ramified). We observe that this is equivalent to requiring the inertia subgroup to be a metacyclic group (that is, one whose Sylow subgroups are cyclic).

**Lemma 6.** *Let  $T$  be a torus over  $K_\Sigma$  which is a direct factor of a quasi-trivial torus over  $K_\Sigma^{\text{unr}}$ . Then  $\pi_1(T)_I$  is free and  $T(K_\Sigma)^0 = T(K_\Sigma)^1$ .*

*Proof.* The construction of the functor  $\pi_1(-)_I$  behaves well with respect to direct products, so that  $\pi_1(T)_I$  is a direct factor of the “ $\pi_1(-)_I$ ” of a quasi-trivial torus, which is free. Consequently,  $\pi_1(T)_I$  is free. By [10, Corollary 11.1.6], we then have  $T(K_\Sigma)^0 = T(K_\Sigma)^1$ .  $\square$

**Lemma 7.** *Let  $T$  be a  $K_\Sigma$ -torus with cyclic wild inertia that is either flasque or coflasque. Then  $T$  is a direct factor of a quasi-trivial torus over  $K_\Sigma^{\text{unr}}$ , the group  $\pi_1(T)_I$  is free, and  $T(K_\Sigma)^0 = T(K_\Sigma)^1$ .*

*Proof.* The theorem of Endo and Miyata (see [4, p. 0.5]) shows that a flasque or coflasque torus is a direct factor of a quasi-trivial torus when this torus is split by a metacyclic extension. Therefore,  $T$  is a direct factor of a quasi-trivial torus over  $K_\Sigma^{\text{unr}}$ . The rest then follows from Lemma 6.  $\square$

**Lemma 8.** *Let  $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$  be an exact sequence of reductive  $K_\Sigma$ -groups. Assume that  $A$  is a torus and that the group  $\pi_1(A)_I$  is free. Then we have the following exact sequences:*

$$1 \longrightarrow \pi_1(A)_I \longrightarrow \pi_1(B)_I \longrightarrow \pi_1(C)_I \longrightarrow 1$$

and

$$1 \longrightarrow A(K_\Sigma)^0 \longrightarrow B(K_\Sigma)^0 \longrightarrow C(K_\Sigma)^0 \longrightarrow 1.$$

*Proof.* Recall that the functor  $\pi_1(-)$  is exact (see [2, Theorem 3.8]). Taking coinvariants under  $I$ , we obtain:

$$H_1(I, \pi_1(A)) \longrightarrow \pi_1(A)_I \longrightarrow \pi_1(B)_I \longrightarrow \pi_1(C)_I \longrightarrow 1.$$

Since  $\pi_1(A)_I$  is free by Lemma 6, the image of  $H_1(I, \pi_1(A))$  (which is a torsion group) in  $\pi_1(A)_I$  is trivial. Hence the first exact sequence.

As for the second exact sequence, since the residue field is perfect, Steinberg's theorem (see [10, Theorem 2.3.3.(1)]) gives  $H^1(I, A(K_\Sigma^{\text{sep}})) = 1$ . We then have:

$$1 \longrightarrow A(K_\Sigma^{\text{unr}}) \longrightarrow B(K_\Sigma^{\text{unr}}) \longrightarrow C(K_\Sigma^{\text{unr}}) \longrightarrow H^1(I, A(K_\Sigma^{\text{sep}})) = 1.$$

Combining the two previous exact sequences via the Kottwitz morphism and using the Snake Lemma, we obtain:

$$1 \longrightarrow A(K_\Sigma^{\text{unr}})^0 \longrightarrow B(K_\Sigma^{\text{unr}})^0 \longrightarrow C(K_\Sigma^{\text{unr}})^0 \longrightarrow 1.$$

Taking invariants under  $\Gamma^{\text{unr}}$  and using the fact that  $H^1(\Gamma^{\text{unr}}, A(K_\Sigma^{\text{unr}})^0) = 1$  by [10, Lemma 11.7.1], we then obtain:

$$1 \longrightarrow A(K_\Sigma)^0 \longrightarrow B(K_\Sigma)^0 \longrightarrow C(K_\Sigma)^0 \longrightarrow H^1(\Gamma^{\text{unr}}, A(K_\Sigma^{\text{unr}})^0) = 1.$$

Hence the desired exact sequence.  $\square$

**Proposition 9.** *Let  $G$  be a reductive group over  $K$ . If the group  $G_{K_\Sigma}$  has cyclic wild inertia, then the following natural map is an isomorphism:*

$$G(K_\Sigma)/\overline{G(K)} \xrightarrow{\sim} G(K_\Sigma)/G(K)G(K_\Sigma)^0.$$

*Proof.* Since  $G(K_\Sigma)^0$  is an open subgroup of  $G(K_\Sigma)$ , we have the inclusion  $\overline{G(K)} \subset G(K)G(K_\Sigma)^0$ . It is therefore sufficient to prove that  $G(K_\Sigma)^0 \subset \overline{G(K)}$ . Note that since the group  $G_{K_\Sigma}$  has cyclic wild inertia, we must have a flasque resolution (see [3, Proposition-Définition 3.1]) over  $K_\Sigma^{\text{unr}}$  given by  $1 \rightarrow F' \rightarrow H' \rightarrow G_{K_\Sigma^{\text{unr}}} \rightarrow 1$  where  $F'$  also has cyclic wild inertia. Consequently, it is a direct factor of a quasi-trivial torus by Lemma 7.

Now consider a flasque resolution of  $G$  over  $K$  given by  $1 \rightarrow F \rightarrow H \rightarrow G \rightarrow 1$ . Since  $F$  is unique up to a quasi-trivial torus (see [3, Proposition 3.2.(iii)]), the previous discussion shows that  $F$  is in fact a direct factor of a quasi-trivial torus over  $K_\Sigma^{\text{unr}}$ . Furthermore, [1, Proposition 2.6] indicates that  $H$  satisfies weak approximation, in other words:  $\overline{H(K)} = H(K_\Sigma)$ . This implies that the subgroup  $H(K_\Sigma)^0 \subset H(K_\Sigma)$  maps into  $\overline{G(K)}$ . But Lemma 8 tells us that  $H(K_\Sigma)^0 \rightarrow G(K_\Sigma)^0$  is surjective, so that  $G(K_\Sigma)^0 \subset \overline{G(K)}$  as desired.  $\square$

**Remark 10.** The result no longer holds if the group does not have cyclic wild inertia. Consider the extension  $L/K := \mathbb{Q}(\zeta_8)/\mathbb{Q} = \mathbb{Q}(\sqrt{-1}, \sqrt{2})/\mathbb{Q}$  (where  $\zeta_8 = e^{\frac{2\pi i}{8}}$ ) and consider the norm torus  $T := R_{L/K}^{(1)}(\mathbb{G}_m)$ . According to [15, 11.6. Example 3.2)],  $\mathbb{Q}_2(\zeta_8)/\mathbb{Q}_2$  is a totally ramified biquadratic extension such that  $T(\mathbb{Q}_2)/\overline{T(\mathbb{Q})} = \mathbb{Z}/2\mathbb{Z} \neq 0$ . However,  $\text{coker}(T(\mathbb{Q}) \rightarrow \text{Im}(\kappa_{T_{\mathbb{Q}_2}})) = 0$  (i.e.,  $T(\mathbb{Q}_2)/T(\mathbb{Q})T(\mathbb{Q}_2)^0 = 0$ ). Indeed, by adapting Section 3.1 to our context and using its notation, we observe that  $N(a(1 - \zeta_8)) \in T(\mathbb{Q})$  and  $N(b(1 - \zeta_8)) \in T(\mathbb{Q})$  map respectively to  $\bar{a}$  and  $\bar{b}$  under  $\kappa_{T_{\mathbb{Q}_2}}$ , and thus generate  $\text{Im}(\kappa_{T_{\mathbb{Q}_2}})$ .

## 2. WEAK APPROXIMATION AND CONSTRUCTION OF EXAMPLES

In this section, we compute defects of weak approximation. Combined with Proposition 9, we provide a negative answer to Question 1 (and to its semisimple variant), when  $K$  is a number field (see Proposition 14) and when  $K$  is a function field (see Proposition 16).

**Lemma 11.** *Let  $\mu := R_{K'/K}(\mu_n)/\mu_n$  with  $n \in \mathbb{N}^\times$  such that  $\text{char}(K) \nmid n$ . We have:*

$$\text{coker}(H^1(K, \mu) \rightarrow H^1(K_\Sigma, \mu)) \xrightarrow{\sim} \text{coker}(\text{Br}(K'/K)[n] \rightarrow \text{Br}(K'_{\Sigma'}/K_\Sigma)[n]) \xrightarrow{\sim} \mathbb{Z}/\frac{\gcd(d_\Sigma, n)}{\gcd(d_\Sigma, d_{\Sigma'}, n)}\mathbb{Z},$$

where for a set of places  $S$  of  $K$ , we denote  $d_S := \text{lcm}_{v \in S}[K'_v : K_v]$ , and where for every  $v \in \Omega_K$ ,  $v'$  denotes a place of  $K'$  lying over  $v$ .

*Proof.* Let  $T := R_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m$ . Observe that  $\mu = \ker(T \xrightarrow{\times n} T)$ . Using the long exact sequence in cohomology over  $K$  and  $K_\Sigma$ , we obtain the following diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T(K)/nT(K) & \longrightarrow & H^1(K, \mu) & \longrightarrow & \text{Br}(K'/K)[n] & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T(K_\Sigma)/nT(K_\Sigma) & \longrightarrow & H^1(K_\Sigma, \mu) & \longrightarrow & \text{Br}(K'_{\Sigma'}/K_\Sigma)[n] & \longrightarrow & 1. \end{array}$$

Since  $T$  admits by definition a flasque resolution  $1 \rightarrow \mathbb{G}_m \rightarrow R_{K'/K}(\mathbb{G}_m) \rightarrow T \rightarrow 1$ , it is retract  $K$ -rational (see [4, Proposition 7.4]). Consequently,  $T(K)$  is dense in  $T(K_\Sigma)$  (see [4, Proposition 8.1.(iii)]). Furthermore, since  $\text{char}(K) \nmid n$ , multiplication by  $n$  in  $T$  is smooth, so that  $nT(K_\Sigma)$  is an open subset of  $T(K_\Sigma)$  (see [6, Lemma 3.1.2]). We therefore have  $T(K) \cdot nT(K_\Sigma) = T(K_\Sigma)$ . This implies that  $T(K)/nT(K) \rightarrow T(K_\Sigma)/nT(K_\Sigma)$  is surjective.

The Snake Lemma applied to the previous diagram then yields the isomorphism:

$$\text{coker}(H^1(K, \mu) \rightarrow H^1(K_\Sigma, \mu)) \xrightarrow{\sim} \text{coker}(\text{Br}(K'/K)[n] \rightarrow \text{Br}(K'_{\Sigma'}/K_\Sigma)[n]).$$

Let us now compute the right-hand side. The Brauer-Hasse-Noether theorem (see [9, Theorem 14.11]) restricted to  $\text{Br}(K'/K)[n]$  yields the following exact sequence:

$$1 \longrightarrow \text{Br}(K'/K)[n] \longrightarrow \bigoplus_{v \in \Omega_K} \text{Br}(K' \otimes_K K_v/K_v)[n] \xrightarrow{\sum \text{inv}_v} \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Consider the partial maps  $\phi_\Sigma := \sum_{v \in \Sigma} \text{inv}_v$  and  $\phi_{\Sigma'} := \sum_{v \in \Sigma'} \text{inv}_v$ . We thus have  $\phi_\Sigma(x) = -\phi_{\Sigma'}(x) = \phi_{\Sigma'}(-x)$  for  $x \in \text{Br}(K'/K)[n]$ . In fact,  $\text{Im}((\phi_\Sigma)|_{\text{Br}(K'/K)[n]}) = \text{Im}(\phi_\Sigma) \cap \text{Im}(\phi_{\Sigma'})$ . This finally allows us to deduce the following exact sequence:

$$1 \longrightarrow \text{Br}(K'/K)[n] \longrightarrow \text{Br}(K'_{\Sigma'}/K_\Sigma)[n] \xrightarrow{\phi_\Sigma} \text{Im}(\phi_\Sigma)/(\text{Im}(\phi_\Sigma) \cap \text{Im}(\phi_{\Sigma'})) \longrightarrow 1.$$

To conclude, we compute the order of the quotient (which is cyclic). Let  $v$  be a place of  $K$ . Observe that  $\text{Br}(K'_v/K_v) = \text{Br}(K' \otimes_K K_v/K_v)$ . Also,  $d_v := [K'_v : K_v] = |\text{Br}(K'_v/K_v)|$  and the image of  $\text{Br}(K'_v/K_v)$  in  $\mathbb{Q}/\mathbb{Z}$  is  $(\frac{1}{d_v}\mathbb{Z})/\mathbb{Z}$  (see [9, Corollary 8.10]). Thus, the image of  $\text{Br}(K' \otimes_K K_v/K_v)[n]$  is  $(\frac{1}{\gcd(d_v, n)}\mathbb{Z})/\mathbb{Z}$ . Consequently,  $\text{Im}(\phi_\Sigma) = \frac{1}{\gcd(d_\Sigma, n)}\mathbb{Z}/\mathbb{Z}$  and  $\text{Im}(\phi_{\Sigma'}) = \frac{1}{\gcd(d_{\Sigma'}, n)}\mathbb{Z}/\mathbb{Z}$ . We then observe that  $\text{Im}(\phi_{\Sigma'}) \cap \text{Im}(\phi_\Sigma) = \frac{1}{\gcd(|\text{Im}(\phi_\Sigma)|, |\text{Im}(\phi_{\Sigma'})|)}\mathbb{Z}/\mathbb{Z} = \frac{1}{\gcd(d_\Sigma, d_{\Sigma'}, n)}\mathbb{Z}/\mathbb{Z}$ . We finally conclude that the cokernel has order  $\frac{\gcd(d_\Sigma, n)}{\gcd(d_\Sigma, d_{\Sigma'}, n)}$  as desired.  $\square$

**Proposition 12.** *Let  $K'/K$  be a finite Galois extension, and let  $\mu$  be a finite étale  $K$ -group of multiplicative type split over  $K'$ . Set  $\Sigma' := \{w \in \Omega_{K'} \mid \exists v \in \Sigma, w \mid v\}$ . We then have  $K'_{\Sigma'} = K' \otimes_K K_\Sigma$ . Assume that the wild inertia subgroup of  $K'_{\Sigma'}/K_\Sigma$  is cyclic. Then, there exist a semisimple  $K$ -group  $G$  and a  $K$ -torus  $T$  with  $T(K_\Sigma)^0 = T(K_\Sigma)^1$ , both split over  $K'$ , such that:*

$$\begin{aligned} G(K_\Sigma)/G(K)G(K_\Sigma)^0 &\xleftarrow{\sim} G(K_\Sigma)/\overline{G(K)} \xrightarrow{\sim} \text{coker}(H^1(K, \mu) \rightarrow H^1(K_\Sigma, \mu)) \quad \text{and} \\ T(K_\Sigma)/T(K)T(K_\Sigma)^0 &\xleftarrow{\sim} T(K_\Sigma)/\overline{T(K)} \xrightarrow{\sim} \text{coker}(H^1(K, \mu) \rightarrow H^1(K_\Sigma, \mu)). \end{aligned}$$

*Proof.* The left-hand isomorphisms are a consequence of Proposition 9. Indeed,  $G$  and  $T$  have cyclic wild inertia since they are split over  $K'$ , and thus over  $K'_{\Sigma'}$ .

There exists a finite family of integers  $(n_i)$  such that  $\mu_{K'} \cong \prod \mu_{n_i}$ . We then embed  $\mu$  into the center of  $G' := R_{K'/K}(\prod \mathrm{SL}_{n_i})$ . Setting  $G := G'/\mu$ , we obtain a central isogeny with kernel  $\mu$ , where  $G$  is a semisimple group (this was already known to Serre, see [14, p.158]). On the other hand, according to [4, Proposition 1.3], one can find an exact sequence  $1 \rightarrow \mu \rightarrow P \rightarrow T \rightarrow 1$  such that  $P$  is quasi-trivial and  $T$  is coflasque. We can also observe that, by construction,  $P$  and  $T$  can be chosen to be split over  $K'$ . [1, Theorem 2.12] then gives the right-hand isomorphisms.

Finally, Lemma 7 shows that  $T(K_{\Sigma})^0 = T(K_{\Sigma})^1$ .  $\square$

### 2.1. An example in the case of number fields

**13. Study of a number field.** Let  $K := \mathbb{Q}$ ,  $K' := \mathbb{Q}(\sqrt{3}, \sqrt{11})$ , and  $\Sigma := \{3\}$ . We thus have  $K_{\Sigma} = \mathbb{Q}_3$ . As for  $K'_{\Sigma'}$ , we have:

$$K'_{\Sigma'} := K' \otimes_{\mathbb{Q}} \mathbb{Q}_3 \cong \mathbb{Q}[X, Y]/(X^2 - 3, Y^2 - 11) \otimes_{\mathbb{Q}} \mathbb{Q}_3 \cong \mathbb{Q}_3[X, Y]/(X^2 - 3, Y^2 - 11).$$

Furthermore,  $X^2 - 3$  is irreducible over  $\mathbb{Q}_3$  since 3 is a uniformizer. Moreover,  $X^2 - 11$  has no root in  $\mathbb{Z}/3\mathbb{Z}$ , and therefore none in  $\mathbb{Z}_3(\sqrt{3})$ . Since  $X^2 - 11 \in \mathbb{Z}_3(\sqrt{3})[X]$ , it also has no root in  $\mathbb{Q}_3(\sqrt{3})$ . Consequently,  $\mathbb{Q}_3[X, Y]/(X^2 - 3, Y^2 - 11) = \mathbb{Q}_3(\sqrt{3}, \sqrt{11})$  is a field. As a result,  $\Sigma' = \{\sqrt{3}\}$  and  $K'_{\Sigma'} \cong \mathbb{Q}_3(\sqrt{3}, \sqrt{11})$ , which is of dimension 4 and, of course, has Galois group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Therefore,  $K'_{\Sigma'}/K_{\Sigma}$  is tamely ramified.

Let us find the prime numbers for which  $\mathbb{Q}(\sqrt{3}, \sqrt{11})/\mathbb{Q}$  is ramified. Of course, all decomposition groups are cyclic in the unramified case. According to [12, Chapter 2, Exercise 42.(f)], the discriminant of this extension is  $16 \times 3^2 \times 11^2$ . The prime numbers dividing the discriminant, that is, the ramified primes, are then 2, 3, and 11.

- At  $p = 2$ , we note that  $P := X^2 - 33$  has 1 as a root modulo 8 (and  $P'(1) = 2 \pmod{4}$ ). This means that  $|P(1)|_2 \leq \frac{1}{8} < |P'(1)|_2^2 = (\frac{1}{2})^2$ . Newton's method then yields a root in  $\mathbb{Q}_2$ . We also note that  $X^2 - 3$  has no root in  $\mathbb{Z}/4\mathbb{Z}$ , hence no root in  $\mathbb{Z}_2$ , and consequently no root in  $\mathbb{Q}_2$  since  $X^2 - 3 \in \mathbb{Z}_2[X]$ . Thus,  $X^2 - 3$  is irreducible over  $\mathbb{Q}_2$ . We then obtain:

$$K' \otimes_{\mathbb{Q}} \mathbb{Q}_2 \cong \mathbb{Q}_2[X, Y]/(X^2 - 3, Y^2 - 11) \cong \left( \mathbb{Q}_2[\sqrt{3}] \right) [Y]/(Y^2 - (\frac{\sqrt{33}}{\sqrt{3}})^2) \cong \left( \mathbb{Q}_2(\sqrt{3}) \right)^2.$$

The two local fields of  $K'$  lying over 2 are therefore cyclic extensions of degree 2.

- At  $p = 11$ ,  $P := X^2 - 3$  has 5 as a root modulo 11 (and  $P'(5) \not\equiv 0 \pmod{11}$ ). By Hensel's lemma,  $P := X^2 - 3$  has a root in  $\mathbb{Q}_{11}$ . Furthermore,  $X^2 - 11$  is irreducible over  $\mathbb{Q}_{11}$  since 11 is a uniformizer. We then have:

$$K' \otimes_{\mathbb{Q}} \mathbb{Q}_{11} \cong \mathbb{Q}_{11}[X, Y]/(X^2 - 3, Y^2 - 11) \cong \left( \mathbb{Q}_{11}(\sqrt{11}) \right) [X]/(X^2 - (\sqrt{3})^2) \cong \left( \mathbb{Q}_{11}(\sqrt{11}) \right)^2.$$

Thus, the two local fields of  $K'$  lying over 11 are cyclic extensions of degree 2.

**Proposition 14.** *There exist a semisimple  $\mathbb{Q}$ -group  $G$  and a  $\mathbb{Q}$ -torus  $T$  with  $T(\mathbb{Q}_3)^0 = T(\mathbb{Q}_3)^1$ , both split over  $\mathbb{Q}(\sqrt{3}, \sqrt{11})$ , such that:*

$$0 \neq \mathbb{Z}/2\mathbb{Z} \xleftarrow{\sim} G(\mathbb{Q}_3)/\overline{G(\mathbb{Q})} \xrightarrow{\sim} G(\mathbb{Q}_3)/G(\mathbb{Q}) G(\mathbb{Q}_3)^0 \text{ and}$$

$$0 \neq \mathbb{Z}/2\mathbb{Z} \xleftarrow{\sim} T(\mathbb{Q}_3)/\overline{T(\mathbb{Q})} \xrightarrow{\sim} T(\mathbb{Q}_3)/T(\mathbb{Q}) T(\mathbb{Q}_3)^0.$$

*Proof.* Let us resume the context of §13. Under the notation of Lemma 11, we have  $d_\Sigma = 4$  and  $d_{\overline{\Sigma}} = 2$ . Set  $\mu := R_{K'/K}(\mu_4)/\mu_4$ . This same lemma then yields

$$\text{coker} \left( H^1(K, \mu) \rightarrow H^1(K_\Sigma, \mu) \right) \xrightarrow{\sim} \mathbb{Z} / \frac{\gcd(d_\Sigma, 4)}{\gcd(d_\Sigma, d_{\overline{\Sigma}}, 4)} \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}.$$

The result then follows from Proposition 12.  $\square$

## 2.2. An example in the case of function fields

**15. Study of a function field.** Let  $p$  be a prime such that  $p \equiv 3 \pmod{4}$ . Let  $K := \mathbb{F}_p(t)$ ,  $K' := \mathbb{F}_p(\sqrt{t}, \sqrt{1-t})$ , and  $\Sigma := \{t^{-1}\}$ . We thus have  $K_\Sigma = \mathbb{F}_p((t^{-1}))$ . As for  $K'_{\Sigma'}$ , we have:

$$K'_{\Sigma'} := K' \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p((t^{-1})) \cong \mathbb{F}_p((t^{-1}))[X, Y] / (X^2 - t, Y^2 - (1-t)).$$

Note that  $X^2 - t$  is irreducible over  $\mathbb{F}_p((t^{-1}))$  since, otherwise, this would imply the existence of an element with valuation  $-\frac{1}{2}$ , which is absurd. Next, observe that  $1-t = (-1)t(1-t^{-1})$ . We know that  $t$  is a square in  $\mathbb{F}_p((t^{-\frac{1}{2}}))$ . Furthermore,  $1-t^{-1}$  is also a square in  $\mathbb{F}_p((t^{-\frac{1}{2}}))$  by Hensel's lemma, since it is a square over  $\mathbb{F}_p$  when reduced modulo  $t^{-\frac{1}{2}}$ . Consequently,  $X^2 - (1-t)$  is irreducible over  $\mathbb{F}_p((t^{-\frac{1}{2}}))$  if and only if  $-1$  is not a square. This is indeed the case, since the condition  $p \equiv 3 \pmod{4}$  implies that  $-1$  is not a square in  $\mathbb{F}_p$ . We thus deduce that  $\Sigma' = \{t^{-\frac{1}{2}}\}$  and that  $K'_{\Sigma'}$  is a field isomorphic to  $\mathbb{F}_{p^2}((t^{-\frac{1}{2}}))$ . It is a tamely ramified extension of  $\mathbb{F}_p((t^{-1}))$  with Galois group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

Note that  $K$  can potentially only ramify at  $t$ ,  $1-t$ , and  $t^{-1}$ . The other places are therefore unramified, hence have a cyclic local Galois group, and thus have degree at most 2. Indeed, at any other place,  $t$  and  $1-t$  are units, so their reductions over  $\mathbb{F}_p$  are non-trivial. Consequently, the reductions in  $\mathbb{F}_p[X]$  of the polynomials  $X^2 - t$  and  $X^2 - (1-t)$  are separable ( $p \neq 2$ ). Thus, they each admit a root in  $\mathbb{F}_p^{\text{sep}}$ , and therefore over the maximal unramified extension by Hensel's lemma. Hence, the local extensions are unramified.

- At  $t$ ,  $X^2 - t$  is of course irreducible since  $t$  is a uniformizer. On the other hand,  $X^2 - (1-t)$  admits a root over  $\mathbb{F}_p$  when reduced modulo  $t$ . Hensel's lemma then ensures that it has a root in  $\mathbb{F}_p((t))$ . We then obtain:

$$K' \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p((t)) \cong \mathbb{F}_p((t))[X, Y] / (X^2 - t, Y^2 - (1-t)) \cong \mathbb{F}_p((t^{\frac{1}{2}}))[X] / (X^2 - (\sqrt{1-t})^2) \cong \left( \mathbb{F}_p((t^{\frac{1}{2}})) \right)^2.$$

The two local fields of  $K'$  lying over  $t$  are therefore cyclic extensions of degree 2.

- At  $1-t$ , the polynomial  $X^2 - (1-t)$  is irreducible since  $1-t$  is a uniformizer. Also, since  $t = 1 - (1-t)$ , the polynomial  $X^2 - t$  admits a root over  $\mathbb{F}_p$  when reduced modulo  $1-t$ . Hensel's lemma then yields one in  $\mathbb{F}_p((1-t))$ . We then have:

$$K' \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p((1-t)) \cong \mathbb{F}_p((1-t))[X, Y] / (X^2 - t, Y^2 - (1-t)) \cong \mathbb{F}_p \left( \left( (1-t)^{\frac{1}{2}} \right) \right) [X] / (X^2 - (\sqrt{t})^2) \cong \mathbb{F}_p \left( \left( (1-t)^{\frac{1}{2}} \right) \right)^2.$$

Thus, the two local fields of  $K'$  lying over  $1-t$  are cyclic extensions of degree 2.

**Proposition 16.** *Let  $p$  be a prime such that  $p \equiv 3 \pmod{4}$ . There exist a semisimple  $\mathbb{F}_p(t)$ -group  $G$  and an  $\mathbb{F}_p(t)$ -torus  $T$  with  $T(\mathbb{F}_p((t^{-1})))^0 = T(\mathbb{F}_p((t^{-1})))^1$ , both split over  $\mathbb{F}_p(\sqrt{t}, \sqrt{1-t})$ , such that:*

$$0 \neq \mathbb{Z}/2\mathbb{Z} \xleftarrow{\sim} G(\mathbb{F}_p((t^{-1}))) / \overline{G(\mathbb{F}_p(t))} \xrightarrow{\sim} G(\mathbb{F}_p((t^{-1}))) / G(\mathbb{F}_p(t)) G(\mathbb{F}_p((t^{-1})))^0 \quad \text{and}$$

$$0 \neq \mathbb{Z}/2\mathbb{Z} \xleftarrow{\sim} T(\mathbb{F}_p((t^{-1}))) / \overline{T(\mathbb{F}_p(t))} \xrightarrow{\sim} T(\mathbb{F}_p((t^{-1}))) / T(\mathbb{F}_p(t)) T(\mathbb{F}_p((t^{-1})))^0.$$

*Proof.* The proof is identical to that of Proposition 14.  $\square$

### 3. COUNTEREXAMPLES TO THE SURJECTIVITY OF THE KOTTWITZ MORPHISM

Here, we consider  $K = \mathbb{R}(t)$  and  $\Sigma = \{t\}$ , so that  $K_\Sigma = \mathbb{R}(\!(t)\!)$ , which is denoted hereafter by  $\widehat{K}$ . In this section, we construct two examples of tori  $T$  over  $K$  that both satisfy the equality  $T(\widehat{K}) = T(K)T(\widehat{K})^0$ , but for which the Kottwitz morphism  $\kappa_{T_{\widehat{K}}}$  is not surjective (see Propositions 19 and 21).

**17. Context of the counterexamples.** Let  $L := \mathbb{C}(u)$ ,  $\widehat{L} := \mathbb{C}(\!(u)\!)$ ,  $\mathcal{O}_{\widehat{L}} := \mathbb{C}[[u]]$ , and  $\mathcal{O}_{\widehat{K}} := \mathbb{R}[[t]]$ , where  $u$  is such that  $u^2 = t$ . Denote by  $v_{\widehat{L}}$  the valuation of  $\widehat{L}$ . Consider  $\Gamma := \text{Gal}(L/K) \cong \text{Gal}(\widehat{L}/\widehat{K}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  generated by  $\sigma : i \mapsto -i$  and  $\tau : u \mapsto -u$ . Note that  $\widehat{K}^{\text{unr}} = \mathbb{C}(\!(t)\!)$ . In what follows, it is harmless to assume that  $I$  denotes  $\text{Gal}(\widehat{L}/\widehat{K}^{\text{unr}})$ .

Let  $T$  be a  $\widehat{K}$ -torus split over  $\widehat{L}$ . Recall that we have an identification of  $\Gamma$ -modules  $X \otimes L^\times \xrightarrow{\sim} T(\widehat{L})$  given by  $\chi \otimes l \mapsto \chi(l)$ . Similarly,  $X \otimes \mathcal{O}_{\widehat{L}}^\times \xrightarrow{\sim} T(\widehat{L})^1$ . Note also that the decomposition  $\mathcal{O}_{\widehat{L}}^\times \cong \mathbb{C}^\times \times U^1$  (where  $U^1$  is the subgroup of  $\mathcal{O}_{\widehat{L}}^\times$  with constant term 1) is  $\Gamma$ -invariant and thus induces the  $\Gamma$ -invariant decomposition  $X \otimes \mathcal{O}_{\widehat{L}}^\times \xrightarrow{\sim} (X \otimes \mathbb{C}^\times) \oplus (X \otimes U^1)$ .

Recall from [10, Proposition 11.1.1] that we have the following diagram:

$$\begin{array}{ccc} T(\widehat{L}) & \xrightarrow{\text{id} \otimes v_{\widehat{L}}} & X \\ N_{\widehat{L}/\widehat{K}^{\text{unr}}} \downarrow & & \downarrow \cdot \\ T(\widehat{K}^{\text{unr}}) & \xrightarrow{\kappa_{T_{\widehat{K}^{\text{unr}}}}} & X_I \end{array} \quad (1)$$

where  $N$  is the norm and  $\cdot := X \rightarrow X_I$  is the canonical projection. Consequently,  $N(T(\widehat{L})^1) = T(\widehat{K}^{\text{unr}})^0$ . This yields the decomposition  $T(\widehat{K}^{\text{unr}})^0 = N(X \otimes \mathbb{C}^\times) \oplus N(X \otimes U^1)$ . Furthermore, since  $\mathbb{C}^\times$  is a divisible  $\tau$ -invariant group, we have  $N(X \otimes \mathbb{C}^\times) = N(X) \otimes \mathbb{C}^\times = X^I \otimes \mathbb{C}^\times$ . Finally, since  $U^1$  has a structure of a  $\mathbb{Q}$ -vector space,  $\widehat{H}^0(I, X \otimes U^1) = 0$  and thus  $N(X \otimes U^1) = (X \otimes U^1)^I$ . Whence finally:

$$T(\widehat{K}^{\text{unr}})^0 \cong (X^I \otimes \mathbb{C}^\times) \oplus (X \otimes U^1)^I \quad (2)$$

Note also that this decomposition is  $\Gamma^{\text{unr}}$ -invariant.

#### 3.1. First counterexample

**18. The Galois module of the norm torus.** Consider the norm torus  $T := R_{L/K}^{(1)}(\mathbb{G}_m)$ . Its cocharacter module  $X$  is the zero-trace sublattice of  $\mathbb{Z}[\Gamma]$ . In other words, if we denote by  $(e_1, e_\tau, e_\sigma, e_{\sigma\tau})$  the natural basis of  $\mathbb{Z}[\Gamma]$ , we have:

$$X := \left\{ \sum_{\alpha \in \Gamma} a_\alpha e_\alpha \in \mathbb{Z}[\Gamma] \mid \sum_{\alpha \in \Gamma} a_\alpha = 0 \right\}.$$

One checks that a basis of  $X$  is given by  $(a, b, c) := (e_1 - e_\tau, e_\tau - e_\sigma, e_\sigma - e_{\sigma\tau})$ . The actions of  $\sigma$  and  $\tau$  on the basis are then given by the following matrices:

$$\text{for } \sigma : \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \text{ and for } \tau : \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Let us then compute  $X_I$ . By definition, we have:

$$\begin{aligned} \bar{a} &= \tau(\bar{a}) = \overline{\tau(a)} = -\bar{a} \\ \bar{b} &= \tau(\bar{b}) = \overline{\tau(b)} = \overline{a + b + c} \\ \bar{c} &= \tau(\bar{c}) = \overline{\tau(c)} = -\bar{c}. \end{aligned}$$

Whence the relations  $2\bar{a} = 0$  and  $\bar{a} = \bar{c}$ . Consequently,  $X_I \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$  with basis  $(\bar{a}, \bar{b})$ . Furthermore, the action of  $\sigma$  on  $X_I$  is then given by the relations:

$$\begin{aligned}\sigma(\bar{a}) &= \overline{\sigma(a)} = \bar{c} = \bar{a} \\ \sigma(\bar{b}) &= \overline{\sigma(b)} = \overline{-(a+b+c)} = -\bar{b}.\end{aligned}$$

We then deduce that  $(X_I)^{\Gamma^{\text{unr}}} \cong \mathbb{Z}/2\mathbb{Z}$  with generator  $\bar{a}$ .

This time, let us compute  $X^I$ . Take  $n_1a + n_2b + n_3c \in X^I$ . We have:

$$n_1a + n_2b + n_3c = \tau(n_1a + n_2b + n_3c) = -n_1a + n_2(a+b+c) - n_3c = (n_2 - n_1)a + n_2b + (n_2 - n_3)c.$$

We deduce that  $2n_1 = 2n_3 = n_2$ . Thus  $X^I \cong \mathbb{Z}$  and a generator is given by  $E := a + 2b + c$ . Moreover,  $\sigma(E) = \sigma(a) + 2\sigma(b) + \sigma(c) = c - 2(a+b+c) + a = -E$ . Whence the action of  $\sigma$  on  $X^I$ .

**Proposition 19.** *The norm  $K$ -torus  $T := R_{L/K}^{(1)}(\mathbb{G}_m)$  is such that*

$$0 \cong \text{Im}(\kappa_{T_{\widehat{K}}}) \subsetneq ((T_*)_I)^{\Gamma^{\text{unr}}} \cong \mathbb{Z}/2\mathbb{Z}.$$

In particular,  $T(\widehat{K}) = T(\widehat{K})^0 = T(K)T(\widehat{K})^0$ .

*Proof.* Let us resume the context of §18. Set  $l = a(u)$  and compute  $N(l)$ . We find:

$$N(l) = l\tau(l) = a(u)(\tau(a)(\tau(u))) = a(u)a^{-1}(-u) = a(u)a((-u)^{-1}) = a(-1).$$

Furthermore, the commutativity of diagram (1) shows that  $\kappa_{T_{\widehat{K}}^{\text{unr}}}(a(-1)) = \bar{a} \in (X_I)^{\Gamma^{\text{unr}}}$ , since  $v_{\widehat{L}}(u) = 1$ . However,  $\sigma(a(-1)) = \sigma(a)(\sigma(-1)) = c(-1)$ . Consequently,  $a(-1) \notin T(\widehat{K})$ . Let us show that  $\bar{a}$  is not in the image of  $\kappa_{T_{\widehat{K}}}$ .

Consider the cocycle  $z : \alpha \mapsto a(-1)\alpha(a(-1))^{-1}$  with values in  $T(\widehat{K}^{\text{unr}})^0$  (since  $\sigma(\bar{a}) = \bar{a}$ ). This cocycle is a coboundary if and only if  $\bar{a} \in \text{Im}(\kappa_{T_{\widehat{K}}})$ . Note also that:

$$a(-1)\sigma(a(-1))^{-1} = a(-1)c(-1)^{-1} = a(-1)c(-1) = a(-1)b(-1)^2c(-1) = E(-1) \in X^I \otimes \mathbb{C}^\times.$$

For the sake of contradiction, suppose that  $z$  is a coboundary. It can therefore be written as  $\alpha \mapsto y\alpha(y)^{-1}$  with  $y \in T(\widehat{K}^{\text{unr}})^0$ . Let us use the decomposition (2) to write  $y = y_0u$ . We then have  $y\sigma(y)^{-1} = (y_0\sigma(y_0)^{-1})(u\sigma(u)^{-1})$ .

By identifying the terms in the decomposition, we obtain  $E(-1) = y_0\sigma(y_0)^{-1}$ . Furthermore, since  $X^I = \langle E \rangle$ , we can write  $y_0 = E(x)$  with  $x \in \mathbb{C}^\times$ . Whence finally:

$$E(-1) = y_0\sigma(y_0)^{-1} = E(x)\sigma(E)(\sigma(x)^{-1}) = E(x)E(\sigma(x)) = E(|x|^2).$$

We then deduce that  $|x|^2 = -1$ . This is a contradiction! Therefore,  $z$  is not a coboundary and  $\bar{a} \notin \text{Im}(\kappa_{T_{\widehat{K}}})$ .

Since  $\mathbb{Z}/2\mathbb{Z} \cong (X_I)^{\Gamma^{\text{unr}}} = \langle \bar{a} \rangle$ , we deduce that  $\text{Im}(\kappa_{T_{\widehat{K}}}) = 0$ , and thus  $T(\widehat{K}) = T(\widehat{K})^0$ . The decomposition  $T(\widehat{K}) = T(K)T(\widehat{K})^0$  then follows trivially.  $\square$

### 3.2. Second counterexample

**20. The Galois module of an induced torus over  $\mathbb{C}(t)$ .** Let us now consider the lattice  $X = \mathbb{Z}^4$  whose standard basis is denoted by  $(e, e', f, f')$ . We associate with  $\sigma$  and  $\tau$  the following matrices:

$$\text{for } \sigma : \begin{pmatrix} A & 0 \\ I_2 - A & -A \end{pmatrix} \text{ and for } \tau : \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \text{ where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An immediate block computation shows that this defines a  $\Gamma$ -module structure on  $X$ . We then deduce a unique  $K$ -torus  $T$  such that  $T_* = X$ . By construction,  $T_{\mathbb{C}(t)}$  is induced.

Let us compute  $X_I$ . Since  $\tau(e) = e'$  and  $\tau(f) = f'$ , we find that  $X_I$  is defined by the relations  $\bar{e} = \bar{e}'$  and  $\bar{f} = \bar{f}'$ , so that  $X_I \cong \mathbb{Z}^2$  with basis  $(\bar{e}, \bar{f})$ . Its  $\Gamma^{\text{unr}}$ -action is given by:

$$\begin{aligned}\sigma(\bar{e}) &= \overline{\sigma(e)} = \overline{e' + f - f'} = \bar{e} \\ \sigma(\bar{f}) &= \overline{\sigma(f)} = \overline{-f'} = -\bar{f}.\end{aligned}$$

We then deduce that  $(X_I)^{\Gamma^{\text{unr}}} \cong \mathbb{Z}$  with generator  $\bar{e}$ .

Now let us compute  $X^I$ . Take  $ne + n'e' + mf + m'f' \in X^I$ . We have:

$$ne + n'e' + mf + m'f' = \tau(ne + n'e' + mf + m'f') = n'e + ne' + m'f + mf'.$$

Whence  $n = n'$  and  $m = m'$ . Thus  $ne + n'e' + mf + m'f' = n(e + e') + m(f + f')$ . We then deduce that  $X^I \cong \mathbb{Z}^2$  and admits a basis given by  $(e + e', f + f')$ . The action of  $\Gamma^{\text{unr}}$  on  $X^I$  is given by:

$$\begin{aligned} \sigma(e + e') &= \sigma(e) + \sigma(e') = (e' + f - f') + (e - f + f') = e + e' \\ \sigma(f + f') &= \sigma(f) + \sigma(f') = -f' - f = -(f + f'). \end{aligned}$$

**Proposition 21.** *The  $K$ -torus  $T$  induced over  $\mathbb{C}(t)$  introduced in §20 is such that*

$$2\mathbb{Z} \cong \text{Im}(\kappa_{T_{\widehat{K}}}) \subsetneq ((T_*)_I)^{\Gamma^{\text{unr}}} \cong \mathbb{Z}.$$

Moreover,  $T(\widehat{K}) = T(K)T(\widehat{K})^0$  and  $T(\widehat{K}^{\text{unr}})^0 = T(\widehat{K}^{\text{unr}})^1$  (i.e.,  $(T_*)_I$  is free).

*Proof.* Let us resume the context of §20. Set  $l = e(iu)$  and compute  $N(l)$ . We find:

$$N(l) = l\tau(l) = e(iu)(\tau(e)(\tau(iu))) = e(iu)e'(-iu) = (e + e')(iu)e'(-1).$$

Furthermore, the commutativity of diagram (1) shows that  $\kappa_{T_{\widehat{K}^{\text{unr}}}}(N(l)) = \bar{e} \in (X_I)^{\Gamma^{\text{unr}}}$ , since  $v_{\widehat{L}}(iu) = 1$ . Let us show that  $\bar{e}$  is not in the image of  $\kappa_{T_{\widehat{K}}}$ .

Consider the cocycle  $z : \alpha \mapsto N(l)\alpha(N(l))^{-1}$  with values in  $T(\widehat{K}^{\text{unr}})^0$  (since  $\sigma(\bar{e}) = \bar{e}$ ). This cocycle is a coboundary if and only if  $\bar{e} \in \text{Im}(\kappa_{T_{\widehat{K}}})$ . Note also that:

$$\sigma(N(l)) = \sigma((e + e')(iu)e'(-1)) = \sigma(e + e')(-iu)\sigma(e')(-1) = (e + e')(-iu)(e - f + f')(-1).$$

Consequently:

$$\begin{aligned} N(l)\sigma(N(l))^{-1} &= (e + e')(iu)e'(-1)((e + e')(-iu)(e - f + f')(-1))^{-1} \\ &= e'(-1)(e + e')(-1)(e - f + f')(-1) = (2e + 2e' - f + f')(-1) = (f + f')(-1) \in X^I \otimes \mathbb{C}^\times. \end{aligned}$$

For the sake of contradiction, suppose that  $z$  is a coboundary. It can therefore be written as  $\alpha \mapsto y\alpha(y)^{-1}$  with  $y \in T(\widehat{K}^{\text{unr}})^0$ . Let us use the decomposition (2) to write  $y = y_0u$ . We then have  $y\sigma(y)^{-1} = (y_0\sigma(y_0)^{-1})(u\sigma(u)^{-1})$ .

By identifying the terms in the decomposition, we obtain  $(f + f')(-1) = y_0\sigma(y_0)^{-1}$ . Furthermore, since  $\mathbb{Z}(e + e')$  and  $\mathbb{Z}(f + f')$  are stable under  $\sigma$ , and since  $(e + e', f + f')$  is a basis of  $X^I$ , we have  $y_0 \in \mathbb{Z}(f + f') \otimes \mathbb{C}^\times$ . We can thus write  $y_0 = (f + f')(x)$  for  $x \in \mathbb{C}^\times$ . Whence finally:

$$(f + f')(-1) = y_0\sigma(y_0)^{-1} = (f + f')(x)\sigma(f + f')(\sigma(x)^{-1}) = (f + f')(x)(f + f')(\sigma(x)) = (f + f')(|x|^2).$$

We then deduce that  $|x|^2 = -1$ . This is a contradiction! Therefore,  $z$  is not a coboundary and  $\bar{e} \notin \text{Im}(\kappa_{T_{\widehat{K}}})$ .

However,  $2\bar{e} \in \text{Im}(T(K) \rightarrow (X_I)^{\Gamma^{\text{unr}}})$ . Indeed,  $N(e(t)) = (e + e')(t) \in T(K)$  and  $v_{\widehat{L}}(t) = 2$ . It follows that  $\text{Im}(T(K) \rightarrow (X_I)^{\Gamma^{\text{unr}}}) = \text{Im}(\kappa_{T_{\widehat{K}}})$ , that is to say  $T(\widehat{K}) = T(K)T(\widehat{K})^0$ .  $\square$

**Remark 22.** Let  $T$  be a  $\widehat{K}$ -torus and  $\mathcal{T}$  its Néron model. We have:

$$\begin{aligned} \text{coker}(\kappa_{\mathcal{T}}) &\xrightarrow{\sim} \ker \left( H^1(\Gamma^{\text{unr}}, T(\widehat{K}^{\text{unr}})^0) \rightarrow H^1(\Gamma^{\text{unr}}, T(\widehat{K}^{\text{unr}})) \right) \\ &\xrightarrow{\sim} \ker \left( H^1(\Gamma^{\text{unr}}, T(\widehat{K}^{\text{unr}})^0) \rightarrow H^1(\text{Gal}(\widehat{K}^{\text{sep}}/\widehat{K}), T(\widehat{K}^{\text{sep}})) \right) \xrightarrow{\sim} \ker \left( H^1(\mathcal{O}_{\widehat{K}}, \mathcal{T}^0) \rightarrow H^1(\widehat{K}, T) \right). \end{aligned}$$

The first isomorphism arises from the exact sequence in cohomology induced by  $\kappa_{T_{\widehat{K}^{\text{unr}}}}$ , the second comes from the inflation-restriction exact sequence [14, I.§5.8.a)], and the last one is obtained via [8, 2.9.2.(2)] and [SGA3, XXIV, Proposition 8.1.(i)].

Consequently, our two previous counterexamples also induce a counterexample to a Grothendieck–Serre type problem for Néron models of tori.

## 4. BIRATIONAL PROPERTIES

In this section,  $K$  is no longer necessarily a global field. We prove the following theorem:

**Theorem 23.** *Let  $T$  be a  $K$ -torus, and let  $1 \rightarrow S \rightarrow P \rightarrow T \rightarrow 1$  be a flasque resolution of  $T$ . Let  $(S_*)_{I, \text{free}}$  denote the quotient of  $(S_*)_I$  by its torsion part. We have the following natural isomorphisms:*

$$\begin{aligned} \text{coker}(\kappa_{T_{K_\Sigma}}) &\xrightarrow{\sim} \text{coker}(H^1(K_\Sigma, S) \rightarrow H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}})), \\ \text{coker}(T(K) \rightarrow ((T_*)_I)^{\Gamma^{\text{unr}}}) &\xrightarrow{\sim} \text{coker}(H^1(K, S) \rightarrow H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}})), \text{ and} \\ T(K_\Sigma)/T(K)T(K_\Sigma)^0 &\xrightarrow{\sim} \text{coker}(H^1(K, S) \rightarrow \text{Im}(H^1(K_\Sigma, S) \rightarrow H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}}))). \end{aligned}$$

*In particular, these three quantities are stable birational invariants of  $T$ .*

*Proof.* Observe that we have the exact sequence:

$$H_1(I, T_*) \longrightarrow (S_*)_I \longrightarrow (P_*)_I \longrightarrow (T_*)_I \longrightarrow 1.$$

Since  $(P_*)_I$  is free and  $H_1(I, T_*)$  is torsion, it immediately follows:

$$1 \longrightarrow (S_*)_{I, \text{free}} \longrightarrow (P_*)_I \longrightarrow (T_*)_I \longrightarrow 1.$$

Note that  $(P_*)_I$  is a permutation module since  $P_*$  is. Furthermore,  $\kappa_{P_{K_\Sigma}}$  is surjective, as this holds for  $\mathbb{G}_m$ . We then obtain the following diagram with exact rows:

$$\begin{array}{ccccccc} \dots & \longrightarrow & P(K_\Sigma) & \longrightarrow & T(K_\Sigma) & \longrightarrow & H^1(K_\Sigma, S) \longrightarrow H^1(K_\Sigma, P) = 1 \\ & & \downarrow \kappa_{P_{K_\Sigma}} & & \downarrow \kappa_{T_{K_\Sigma}} & & \downarrow \\ \dots & \longrightarrow & ((P_*)_I)^{\Gamma^{\text{unr}}} & \longrightarrow & ((T_*)_I)^{\Gamma^{\text{unr}}} & \longrightarrow & H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}}) \longrightarrow H^1(\Gamma^{\text{unr}}, (P_*)_I) = 1 \end{array}$$

A diagram chase then yields the first isomorphism.

Moreover,  $P(K) \rightarrow (P_*)_I$  is also surjective because  $P(K)$  is dense in  $P(K_\Sigma)$ . Replacing  $K_\Sigma$  by  $K$  in the first row of the previous diagram, the same diagram chase then yields the second isomorphism.

The last isomorphism is obtained by observing that

$$\begin{aligned} T(K_\Sigma)/T(K)T(K_\Sigma)^0 &\xrightarrow{\sim} (T(K_\Sigma)/T(K_\Sigma)^0)/(T(K)/T(K) \cap T(K_\Sigma)^0) \\ &\xrightarrow{\sim} \text{Im}(\kappa_{T_{K_\Sigma}})/\text{Im}(T(K) \rightarrow (T_*)_I) \\ &\xrightarrow{\sim} \text{coker}(T(K) \rightarrow (T_*)_I)/\text{coker}(\kappa_{T_{K_\Sigma}}) \\ &\xrightarrow{\sim} \text{coker}(H^1(K, S) \rightarrow H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}}))/\text{coker}(H^1(K_\Sigma, S) \rightarrow H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}})) \\ &\xrightarrow{\sim} \text{coker}(H^1(K, S) \rightarrow \text{Im}(H^1(K_\Sigma, S) \rightarrow H^1(\Gamma^{\text{unr}}, (S_*)_{I, \text{free}}))). \end{aligned}$$

Finally, observe that these three quantities are invariant if we replace  $S$  by  $S \times P_0$ , where  $P_0$  is a quasi-trivial torus. We then conclude that these quantities are stable birational invariants of  $T$  according to [15, §4].  $\square$

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