DEGENERATION LOCUS OF $\mathbb{Q}_p$-LOCAL SYSTEMS: CONJECTURES

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Dedicated to the memory of Bas Edixhoven

Abstract. We introduce a conjecture on the arithmetic sparcity of the degeneration locus of a p-adic local system on a smooth variety over a number field and, *modulo* the Bombieri-Lang conjecture, show that it follows from a conjecture on the geometry of the level varieties attached to the local system. We present a few applications of our conjecture to classical problems in arithmetic geometry. Eventually, we give some evidences and discuss a few perspectives to attack it, in particular for p-adic local systems arising from geometry.

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Notation / conventions:

For a scheme S, let $|S|$ denote the set of closed points. A variety over a field k or a k-variety means a reduced scheme separated and of finite type over k. For a k-variety X and $x \in X$, let $k(x)$ denote the residue field of x; we often implicitly regard x as a morphism $x : \mathrm{spec}(k(x)) \to X$. For every integer $d \geq 1$, let $|X|^{\leq d} \subset |X|$ denote the set of all $x \in |X|$ with $[k(x) : k] \leq d$.

For a field k of characteristic 0, a morphism $f : Y \to X$ of k-varieties and an embedding $\infty : k \hookrightarrow \mathbb{C}$, we write $f^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ for the analytification of the base change $f_{\mathbb{C}} : Y_{\mathbb{C}} \to X_{\mathbb{C}}$ of $f : Y \to X$ along $\infty : \mathrm{spec}(\mathbb{C}) \to \mathrm{spec}(k)$. As the embedding $\infty : k \hookrightarrow \mathbb{C}$ will always be fixed, omitting the mention of it in the notation should not cause confusion. If k is a number field and $k \hookrightarrow k_v$ its completion at a finite place v of k, we write $f^{v\text{-an}} : Y^{v\text{-an}} \to X^{v\text{-an}}$ for the rigid analytification of the base change $f_{k_v} : Y_{k_v} \to X_{k_v}$ of $f : Y \to X$ along $\mathrm{spec}(k_v) \to \mathrm{spec}(k)$.

For an algebraic group G over a field Q (of characteristic 0), we write $G^\circ \subset G$ for its neutral component, $G^{\mathrm{der}} \subset G$ for its derived subgroup and $G \twoheadrightarrow G^{\mathrm{ad}}$ for its adjoint quotient. If V is a finite-dimensional Q-vector space, we write $V^\vee$ for its dual and $V^{\otimes} := \oplus_{m,n \in \mathbb{Z}_{\geq 0}} V^{\otimes m} \otimes_Q V^{\vee \otimes n}$.

1. Introduction

Let k be a number field with fixed algebraic closure $k \subset \bar{k}$. Let X be a smooth, geometrically connected variety over k, with generic point η. Fix a prime integer p and a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on X. For every $x \in X$, fix a geometric point $\overline{x}$ over x and an étale path from $\overline{x}$ to $\overline{\eta}$ inducing compatible isomorphisms $\pi_1(X, \overline{x}) \tilde{\to} \pi_1(X, \overline{\eta})$, $\mathcal{V}_{p,\overline{x}} \tilde{\to} \mathcal{V}_{p,\overline{\eta}}$. For a morphism $Z \to X$ of connected k-schemes and a geometric point $\overline{z}$ on Z mapping to $\overline{x}$ on X, write $\Pi_{Z,\mathcal{V}_p} \subset \mathrm{GL}(\mathcal{V}_{p,\overline{\eta}})$ for the image of $\pi_1(Z, \overline{z})$ acting on $\mathcal{V}_{p,\overline{\eta}}$ *via* $\pi_1(Z, \overline{z}) \to \pi_1(X, \overline{x}) \tilde{\to} \pi_1(X, \overline{\eta})$ and $G_{Z,\mathcal{V}_p} \subset \mathrm{GL}_{\mathcal{V}_{p,\overline{\eta}}}$ for its Zariski-closure. If Z is geometrically connected over k, write also $\overline{\Pi}_{Z,\mathcal{V}_p} \subset \mathrm{GL}(\mathcal{V}_{p,\overline{\eta}})$ for the image of $\pi_1(Z \times_k \overline{k}, \overline{z})$ and $\overline{G}_{Z,\mathcal{V}_p} \subset \mathrm{GL}_{\mathcal{V}_{p,\overline{\eta}}}$ for its Zariski-closure.

Define the $\mathcal{V}_p$-degeneration locus to be

$$|X|_{\mathcal{V}_p} = \{x \in |X| \mid G^\circ_{x,\mathcal{V}_p} \subsetneq G^\circ_{X,\mathcal{V}_p}\}.$$

One says that points in $|X|_{\mathcal{V}_p}$ are $\mathcal{V}_p$-exceptional and that points in $|X| \setminus |X|_{\mathcal{V}_p}$ are $\mathcal{V}_p$-generic. The aim of this note is to motivate and discuss Conjecture 1 below, which is the strongest guess we can reasonably make about the sparcity of the arithmetic truncations $|X|_{\mathcal{V}_p} \cap |X|^{\leq d}$ of $|X|_{\mathcal{V}_p}$.

One says that $\mathcal{V}_p$ is geometrically Lie perfect (GLP for short) if $\mathcal{V}_p|_{X_{\bar{k}}}$ is Lie perfect, that is to say if the Lie algebra of the p-adic analytic Lie group $\overline{\Pi}_{X,\mathcal{V}_p}$ is perfect. One says that $\mathcal{V}_p$ is geometrically curve-Lie perfect (GCLP for short) if for every non constant morphism $\phi : C \to X$ with C a smooth, connected curve over $\overline{k}$, the resulting $\mathbb{Q}_p$-local system $\phi^*\mathcal{V}_p$ is Lie perfect. One defines similarly the notion of being geometrically Lie semisimple (GLS for short) and geometrically curve-Lie semisimple (GCLS for short) by replacing the condition "perfect" by the condition "semisimple". Note that as a quotient of a perfect (resp. semisimple) Lie algebra is again perfect (resp. semisimple), the properties of being GLP, GCLP, GLS and GCLS are preserved by quotient. If $\mathcal{V}_p$ is GCLP (resp. GCLS) then, by Bertini theorem, for every smooth, geometrically connected variety Y over k and non-constant morphism $\phi : Y \to X$, $\phi^*\mathcal{V}_p$ is GLP (resp. GLS); in particular $\mathcal{V}_p$ itself is GLP (resp. GCLS).

Conjecture 1. *Let $\mathcal{V}_p$ be a GCLP $\mathbb{Q}_p$-local system on X. Then, for every integer $d \geq 1$,*
(1) *the set $|X|_{\mathcal{V}_p} \cap |X|^{\leq d}$ is not Zariski-dense in X;*
(2) $\sup\{[\Pi_{X,\mathcal{V}_p} : \Pi_{x,\mathcal{V}_p}] \mid x \in |X|^{\leq d} \setminus |X|_{\mathcal{V}_p}\} < +\infty$.

For later reference, let us single out the $d = 1$ case of of Conjecture 1.

Conjecture 2. *Let $\mathcal{V}_p$ be a GCLP $\mathbb{Q}_p$-local system on X. Then,*
(1) *the set $|X|_{\mathcal{V}_p} \cap X(k)$ is not Zariski-dense in X;*
(2) $\sup\{[\Pi_{X,\mathcal{V}_p} : \Pi_{x,\mathcal{V}_p}] \mid x \in X(k) \setminus |X|_{\mathcal{V}_p}\} < +\infty$.

One could also formulate weaker conjectures with the GCLP condition replaced by the GCLS condition; we will refer to these variants as Conjecture 1 and Conjecture 2 for GCLS $\mathbb{Q}_p$-local systems.

For the time being, the main evidence for Conjecture 1 is that it is known when X is a curve [CT13]. On the other hand, for higher-dimensional X, it remains widely open.

One says that $\mathcal{V}_p$ has positive period dimension if $\dim(\overline{G}_{X,\mathcal{V}_p}) > 0$ and that $\mathcal{V}_p$ has zero period-dimension otherwise.

Remark.
(1) Conjecture 1 and Conjecture 2 are trivial if $\mathcal{V}_p$ has zero period dimension.
(2) To prove Conjecture 1 and Conjecture 2 one may replace k with a finite field extension. Conjecture 1 is insensitive to base changes by dominant morphisms. Conjecture 2 is insensitive to base changes by open immersions with Zariski-dense image *but* it is very sensitive to base changes by *e.g.* étale covers of degree ≥ 2.
(3) One may ask for variants of Conjecture 1 and Conjecture 2 involving integral points rather than rational points. These are closely related to the Lang-Vojta conjecture (Conjecture 10). More precisely, as we will recall (Corollary 11), the Lang-Vojta conjecture predicts that, as soon as $\mathcal{V}_p$ has positive period dimension, the set of integral points (not only $\mathcal{V}_p$-exceptional ones) on X is never Zariki-dense. This, in particular, automatically implies the integral variant of Conjecture 2 (1):

    Conjecture 3. *Let $\mathcal{V}_p$ be a GCLP $\mathbb{Q}_p$-local system on X. Then, for every smooth model $\mathcal{X}$ of X over a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$ the set $|X|_{\mathcal{V}_p} \cap \mathcal{X}(U)$ is not Zariski-dense in X.*

This note is organized as follows. In Section 2, we briefly review some basic features of $\mathbb{Q}_p$-local systems arising from geometry, which are the most important examples for applications. In Section 3 we give a sample of classical consequences of Conjecture 1 in order to emphasize its significance. In Section 4, we explain how Conjecture 2 follows from the Bombieri-Lang conjecture (Conjecture 16) *modulo* a geometric conjecture - Conjecture 17 - about the geometric structure of certain level schemes naturally attached to our $\mathbb{Q}_p$-local system; this is the heuristic on which the proof of Conjecture 1 for X a curve is based. In the last and longest section 5, we give a short survey of what is known about Conjecture 17 and Conjecture 1 and discuss a few perspectives to tackle them. We insist more specifically on Conjecture 3 (and, to a lesser extent, on Conjecture 2 (1)) for $\mathbb{Q}_p$-local systems arising from geometry, for which recent progresses in the study of complex and p-adic period maps now provide an heuristic bypassing the one exposed in Section 4.

Warning: This note contains only a few and rather basic mathematical proofs. It is more intended as an introduction to Conjecture 1 (which has obsessed for many years and is still obsessing the author), its potential applications and to draw a few old and newer perspectives to attack it.

2. $\mathbb{Q}_p$-LOCAL SYSTEMS ARISING FROM GEOMETRY AND CONJECTURES

Fundamental examples of $\mathbb{Q}_p$-local systems are those arising from geometry, namely those (cut out by algebraic correspondances on $\mathbb{Q}_p$-local systems) of the form $\mathcal{V}_p := R^i_{(\mathrm{prim})} f_* \mathbb{Q}_p(j)$ for $f : Y \to X$ a smooth proper (projective) morphism and $i, j \in \mathbb{Z}$. We briefly review some basic properties of these and introduce notation / terminology that we will use throughout the paper.

So, in the remaining part of Section 2, all $\mathbb{Q}_p$-local systems are assumed to arise from geometry.

2.1. **$\mathbb{Q}_p$-local systems arising from geometry.** A $\mathbb{Q}_p$-local system of the form $\mathcal{V}_p := R^i_{(\mathrm{prim})} f_* \mathbb{Q}_p(j)$ comes with a compatible family of $\mathbb{Q}_\ell$-étale / singular realizations.

2.1.1. *Compatibility.*

$$\underline{\mathcal{V}} := \mathcal{V}_\ell := R^i_{(\mathrm{prim})} f_* \mathbb{Q}_\ell(j),\ \ell \in |\mathrm{spec}(\mathbb{Z})|, \mathcal{V}_\infty := R^i_{(\mathrm{prim})} f^{\mathrm{an}}_* \mathbb{Q}(j),\ \infty : k \hookrightarrow \mathbb{C},$$

Here the compatibility is twofold:

- Arithmetic: Let $\mathcal{O}_k$ denote the ring of integers of k. Then there exists a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$ and a smooth connected model $\mathcal{X}$ of X over U such that for every $\ell \in |\mathrm{spec}(\mathbb{Z})|$, $\mathcal{V}_\ell$ extends to a $\mathbb{Q}_\ell$-local system on $\mathcal{X}[\frac{1}{\ell}]$ and for every $x \in |\mathcal{X}|$ of residue characteristic p_x and prime $\ell \neq p_x$, the characteristic polynomial of the geometric Frobenius $\varphi_x \in \pi_1(x)$ acting on $\mathcal{V}_\ell$ is in $\mathbb{Z}[T]$, independent of $\ell \neq p_x$ and pure (of weight $i - 2j$). This follows from the smooth proper base change theorem and [D74].

- Geometric: For every $x \in X^{\mathrm{an}}$ one has the canonical Artin comparison isomorphism $\mathcal{V}_{\infty,x} \otimes_{\mathbb{Q}} \mathbb{Q}_p \tilde{\to} \mathcal{V}_{p,x}$, which is equivariant with respect to the profinite completion morphism $\pi_1(X^{\mathrm{an}}, x) \to \pi_1(X_{\mathbb{C}}, x) \tilde{\to} \pi_1(X_{\bar{k}}, x)$. In particular, if $\overline{G}_{X^{\mathrm{an}},\mathcal{V}_\infty} \subset \mathrm{GL}_{\mathcal{V}_{\infty,x}}$ denotes the Zariski-closure of the image of $\pi_1(X^{\mathrm{an}}, x)$ acting on $\mathcal{V}_{\infty,x}$,

$$(2.1.1)\quad \overline{G}_{X^{\mathrm{an}},\mathcal{V}_\infty} \times_{\mathbb{Q}} \mathbb{Q}_p = \overline{G}_{X,\mathcal{V}_p}$$

  modulo the Artin comparison isomorphism $\mathcal{V}_{\infty,x} \otimes_{\mathbb{Q}} \mathbb{Q}_p \tilde{\to} \mathcal{V}_{p,x}$. This shows $\overline{G}_{X,\mathcal{V}_p}$ admits a $\mathbb{Q}$-form which is independent of p and is a semisimple algebraic group [D71]. In particular, $\mathcal{V}_p$ is GCLS and the property that $\mathcal{V}_p$ has positive period dimension is independent of p. In the following, we simply write $\overline{G}_{X,\mathcal{V}} := \overline{G}_{X^{\mathrm{an}},\mathcal{V}_\infty}$.

  Another crucial feature of the singular realization is that $\mathcal{V}_\infty$ underlies a polarizable $\mathbb{Z}$-variation of Hodge structures ($\mathbb{Z}$-VHS for short) $\mathcal{V}_{\infty,\mathbb{Z}}$ on X^{an}.

2.1.2. *'Motivic' tensors and Tannakian groups.* For a projective variety $\mathfrak{Y}$ over $\mathbb{C}$, write $H := H^i_{(\mathrm{prim})}(\mathfrak{Y}^{\mathrm{an}}, \mathbb{Q}(j))$ and $H_p := H^i_{(\mathrm{prim})}(\mathfrak{Y}, \mathbb{Q}_p(j))$. Let

$$Z_{\mathrm{mot}}(H) \subset Z_{\mathrm{AH}}(H) \subset Z_{\mathrm{H}}(H) \subset H^{\otimes}$$

denote the subspaces of motivated tensors, absolute Hodge tensors and Hodge tensors respectively. Let $Z(H_p) \subset H_p^{\otimes}$ denote the subspace of Tate tensors, namely those $v \in H_p^{\otimes}$ fixed by $\pi_1(K)$ for some finitely generated extension K of $\mathbb{Q}$ over which $\mathfrak{Y}$ admits a model. Correspondingly, one defines

- The subspaces

$$Z_{\mathrm{mot}}(\mathcal{V}_\infty) \subset Z_{\mathrm{AH}}(\mathcal{V}_\infty) \subset Z_{\mathrm{H}}(\mathcal{V}_\infty) \subset \mathcal{V}^{\otimes}_{\infty,x}$$

  of generic motivated tensors, absolute Hodge tensors and Hodge tensors to be the set of all $v \in \mathcal{V}^{\otimes}_{\infty,x}$ which extends to a global section of $\mathcal{V}^{\otimes}_\infty$ over a connected étale cover $X' \to X_{\mathbb{C}}$ (equivalently, are fixed by a finite index subgroup $U = \pi_1(X'^{\mathrm{an}}, x) \subset \pi_1(X^{\mathrm{an}}, x)$) which specializes, at every $x' \in X'(\mathbb{C})$, to a motivated tensor, an absolute Hodge tensor and a Hodge tensor respectively.

- The subspace $Z(\mathcal{V}_p) \subset \mathcal{V}^{\otimes}_{p,x}$ of generic Tate tensors as the set of all $v \in \mathcal{V}^{\otimes}_{p,x}$ which extends to a global section of $\mathcal{V}^{\otimes}_p$ over a connected étale cover $X' \to X_{\mathbb{C}}$ (equivalently, are fixed by an open subgroup $U = \pi_1(X', x) \subset \pi_1(X_{\mathbb{C}}, x)$) which specializes to a Tate tensor at every $x' \in X'(\mathbb{C})$.

Let

$$G_{X_{\mathbb{C}},\mathcal{V}_\infty} \subset G_{X_{\mathbb{C}},\mathcal{V}_{\mathrm{AH}}} \subset G_{X_{\mathbb{C}},\mathcal{V}_{\mathrm{mot}}} \subset \mathrm{GL}_{\mathcal{V}_{\infty,x}}$$

denote the algebraic subgroups fixing all generic motivated tensors, absolute Hodge tensors and Hodge tensors on $\mathcal{V}^{\otimes}_{\infty,x}$ respectively and let $G_{X_{\mathbb{C}},\mathcal{V}_p} \subset \mathrm{GL}_{\mathcal{V}_{p,x}}$ denote the algebraic subgroup fixing all generic Tate tensors on $\mathcal{V}_{p,x}^{\otimes}$. Note that $G_{X_{\mathbb{C}},\mathcal{V}_p} = G^{\circ}_{X,\mathcal{V}_p}$, where $G_{X,\mathcal{V}_p}$ is the arithmetic monodromy group introduced in Section 1 [A23]. *modulo* the Artin comparison isomorphism (2.1.1) motivated and absolute Hodge tensors are mapped to Tate tensors so that one also has the inclusions

$$G^{\circ}_{X,\mathcal{V}_p} = G_{X_{\mathbb{C}},\mathcal{V}_p} \subset G_{X_{\mathbb{C}},\mathcal{V}_{\mathrm{AH}}} \times_{\mathbb{Q}} \mathbb{Q}_p \subset G_{X_{\mathbb{C}},\mathcal{V}_{\mathrm{mot}}} \times_{\mathbb{Q}} \mathbb{Q}_p.$$

The (generic) Mumford-Tate group $G_{X_{\mathbb{C}},\mathcal{V}_\infty}$, the (generic) absolute Mumford-Tate group $G_{X_{\mathbb{C}},\mathcal{V}_{\mathrm{AH}}}$ and the (generic) motivated motivic group $G_{X_{\mathbb{C}},\mathcal{V}_{\mathrm{mot}}}$ are connected reductive groups over $\mathbb{Q}$; the connected (generic) arithmetic monodromy group $G^{\circ}_{X,\mathcal{V}_p} = G_{X_{\mathbb{C}},\mathcal{V}_p}$ is not known to be reductive in general. For $\flat = \infty, \mathrm{AH}, \mathrm{mot}, p$ define the $\mathcal{V}_\flat$-degeneration locus as in Subsection 1, namely

$$X(\mathbb{C})_{\mathcal{V}_\flat} := \{x \in X(\mathbb{C}) \mid G^{\circ}_{x,\mathcal{V}_\flat} \subsetneq G^{\circ}_{X,\mathcal{V}_\flat}\}.$$

The relevancy of $X(\mathbb{C})_{\mathcal{V}_\flat}$ comes from the fact that it is the locus of all $x \in X(\mathbb{C})$ such that the powers Y_x^n, $n \geq 0$ of the corresponding fiber Y_x are "more simple" or "more symmetric" in the sense that they carry additional Hodge, absolute Hodge, motivated or Tate cycles than the the powers of the generic fiber.

Note that *via* the inclusion $|X| \subset X(\mathbb{C})$ induced by $\infty : k \hookrightarrow \mathbb{C}$, $|X|_{\mathcal{V}_p} = |X| \cap X(\mathbb{C})_{\mathcal{V}_p}$.

For $\flat = \infty, \mathrm{AH}, \mathrm{mot}$ the fixed part theorem implies that $G^{\mathrm{der}}_{X_{\mathbb{C}},\mathcal{V}_\flat}$ contains $\overline{G}^{\circ}_{X,\mathcal{V}}$ as a normal subgroup [A92] and that for every $x \in X(\mathbb{C})$, $G_{X_{\mathbb{C}},\mathcal{V}_\flat}$ is generated by $\overline{G}^{\circ}_{X,\mathcal{V}}$ and $G_{x,\mathcal{V}_\flat}$. Similarly, for every geometric point $\overline{x}$ over $x \in |X|$, the short exact sequence

$$1 \to \pi_1(X_{\bar{k}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \pi_1(k) \to 1$$

implies that $G^{\circ}_{X,\mathcal{V}_p}$ is generated by $\overline{G}^{\circ}_{X,\mathcal{V}_p}$ and $G^{\circ}_{x,\mathcal{V}_p}$. These observations show

$$\begin{array}{ccccc} & & X(\mathbb{C})_{\mathcal{V}_{\mathrm{mot}}} & & \\ & & \cap & & \\ X(\mathbb{C})_{\mathcal{V}_p} & \supset & X(\mathbb{C})_{\mathcal{V}_{\mathrm{AH}}} & \subset & X(\mathbb{C})_{\mathcal{V}_\infty} \end{array} \quad (2.1.2)$$

In particular, every sparcity result about $X(\mathbb{C})_{\mathcal{V}_p}$ (or $X(\mathbb{C})_{\mathcal{V}_\infty}$) automatically transfers to $X(\mathbb{C})_{\mathcal{V}_{\mathrm{mot}}}$, $X(\mathbb{C})_{\mathcal{V}_{\mathrm{AH}}}$.

2.2. **Conjectures.** With the above notation / terminology, one can state the following consequences of the Hodge and Tate conjectures, which can be seen as an arithmetic enhancement of (2.1.1).

Conjecture 4. *For every $x \in X(\mathbb{C})$,*

(Tate conjecture)	$G^{\circ}_{x,\mathcal{V}_p}(= G_{x,\mathcal{V}_{\mathrm{AH}}} \times_{\mathbb{Q}} \mathbb{Q}_p) = G_{x,\mathcal{V}_{\mathrm{mot}}} \times_{\mathbb{Q}} \mathbb{Q}_p$	*(equivalently, every Tate tensor is motivated);*
(Hodge conjecture)	$G_{x,\mathcal{V}_\infty}(= G_{x,\mathcal{V}_{\mathrm{AH}}}) = G_{x,\mathcal{V}_{\mathrm{mot}}}$	*(equivalently, every Hodge tensor is motivated).*

and their by-product,

Conjecture 5. (Mumford-Tate Conjecture) *For every $x \in X(\mathbb{C})$, $G_{x,\mathcal{V}_\infty} \times_{\mathbb{Q}} \mathbb{Q}_p = G^{\circ}_{x,\mathcal{V}_p}$ (equivalently, modulo the Artin comparison isomorphism every Tate tensor is a $\mathbb{Q}_p$-linear combinaison of Hodge tensors and conversely).*

Conjecture 4 immediately implies that for every $x \in X(\mathbb{C})$, $G^{\circ}_{x,\mathcal{V}_p}$ is reductive and admits a $\mathbb{Q}$-form which is independent of p and that the inclusions (2.1.2) are equalities:

$$X(\mathbb{C})_{\mathcal{V}_p} = X(\mathbb{C})_{\mathrm{mot}} = X(\mathbb{C})_{\mathcal{V}_{\mathrm{AH}}} = X(\mathbb{C})_{\mathcal{V}_\infty}. \quad (2.2.1)$$

In particular, the subsets $|X|_{\mathcal{V}_p}$ are independent of $p \in \mathrm{spec}(\mathbb{Z})$ and, Conjecture 4 combined with Conjecture 1 (1) yields that

Conjecture 6. *For $\flat = \infty, \mathrm{AH}, \mathrm{mot}, p$ and every integer $d \geq 1$, $X(\mathbb{C})_{\mathcal{V}_\flat} \cap |X|^{\leq d}$ is not Zariski-dense in X.*

One can upgrade (2.2.1) as follows. For $\flat = \infty, \mathrm{AH}, \mathrm{mot}, p$ say that a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is $\mathcal{V}_\flat$-special if it is maximal among all closed integral subvarieties $Z' \hookrightarrow X_{\mathbb{C}}$ with $G^{\circ}_{Z',\mathcal{V}_\flat} = G^{\circ}_{Z,\mathcal{V}_\flat}$ and let $\Sigma_{\mathcal{V}_\flat}$ denote the set of *strict* $\mathcal{V}_\flat$-special subvarieties of $X_{\mathbb{C}}$. Then Conjecture 4 implies more generally

$$\Sigma_{\mathcal{V}_p} = \Sigma_{\mathcal{V}_{\mathrm{mot}}} = \Sigma_{\mathcal{V}_{\mathrm{AH}}} = \Sigma_{\mathcal{V}_\infty}. \quad (2.2.2)$$

In particular, the subsets $\Sigma_{\mathcal{V}_p}$ are independent of $p \in \mathrm{spec}(\mathbb{Z})$ and Conjecture 6 can be reformulated as follows.

Conjecture 7. *For any of $\flat = \infty, \mathrm{AH}, \mathrm{mot}, p$ and every integer $d \geq 1$, there exists finitely many $Z_1, \dots, Z_r \in \Sigma_{\mathcal{V}_\flat}$ such that*

$$X(\mathbb{C})_{\mathcal{V}_\flat} \cap |X|^{\leq d} = \bigcup_{1 \leq i \leq r} |Z_i|^{\leq d}.$$

Lemma 8. *Conjecture 6 is equivalent to Conjecture 7.*

Proof. The implication Conjecture 7 $\Rightarrow$ Conjecture 6 is straightforward. For the implication Conjecture 6 $\Rightarrow$ Conjecture 7, let $Z_1, \dots, Z_r$ denote the irreducible components of the Zariski-closure of $X(\mathbb{C})_{\mathcal{V}_\flat} \cap |X|^{\leq d}$ in X. Then, for $i = 1, \dots, r$, $G^\circ_{Z_i, \mathcal{V}_\flat} \subsetneq G^\circ_{X, \mathcal{V}_\flat}$ since, otherwise, $X(\mathbb{C})_{\mathcal{V}_\flat} \cap |Z_i^{\mathrm{sm}}|^{\leq d} = X(\mathbb{C})_{\mathcal{V}_\flat|_{Z_i^{\mathrm{sm}}}} \cap |Z_i^{\mathrm{sm}}|^{\leq d}$, which would contradict Conjecture 6 applied to $\mathcal{V}_\flat|_{Z_i^{\mathrm{sm}}}$. As $G^\circ_{Z_i, \mathcal{V}_\flat} \subsetneq G^\circ_{X, \mathcal{V}_\flat}$, there exists $Z'_i \in \Sigma_{\mathcal{V}_\flat}$ with $Z_i \subset Z'_i$ (and $G^\circ_{Z_i, \mathcal{V}_\flat} = G^\circ_{Z'_i, \mathcal{V}_\flat}$). By definition of $X(\mathbb{C})_{\mathcal{V}_\flat}$, one has

$$X(\mathbb{C})_{\mathcal{V}_\flat} \cap |X|^{\leq d} = \bigcup_{1 \leq i \leq r} |Z_i|^{\leq d} \subset \bigcup_{1 \leq i \leq r} |Z'_i|^{\leq d} \subset X(\mathbb{C})_{\mathcal{V}_\flat} \cap |X|^{\leq d},$$

whence the assertion. $\square$

The significance of reformulating Conjecture 6 as Conjecture 7 will appear later in Subsection 5.2.2.

2.3. **Constraining the geometry of X.** The existence of $\mathbb{Q}_p$-local systems arising from geometry with positive period dimension on a X constrains the geometry of X. More precisely, recall that a variety V over k is of general type (resp. of log general type) if there exists a diagram

$$V_0 := V_{\bar{k}} \hookrightarrow V_1 \longleftarrow V_2 \hookrightarrow V_3$$

with $V_0 \hookrightarrow V_1$ an open immersion, $V_2 \to V_1$ a proper birational morphism, $V_2 \hookrightarrow V_3$ a smooth compactification (resp. a log smooth compactification) that is an open immersion with V_3 a connected variety, smooth and projective over $\bar{k}$ (resp. and $\Delta := V_3 \setminus V_2$ a normal crossing divisor) and such that the canonical divisor K_{V_3} is big (resp. $K_{V_3} + \Delta$ is big).

Theorem 9. *Assume $\mathcal{V}_p$ has positive period dimension. Then after possibly replacing X by a dense open subscheme, there always exists a dominant morphism $\alpha : X \times_k \bar{k} \to X'$ of $\bar{k}$-varieties with X' of log-general type* [1] *and a $\mathbb{Q}_p$-local system $\mathcal{V}'_p$ on X' such that $\mathcal{V}_p|_{X \times_k \bar{k}} = \alpha^* \mathcal{V}'_p$.*

Proof. Fix a log smooth compactification $j_1 : X \hookrightarrow X_1$ by a normal crossing divisor $\Delta := X_1 \setminus X$. From [Gr70, Prop.9.11 i)] the complex analytic period $\Phi^{\mathrm{an}} : X^{\mathrm{an}} \to \Gamma \setminus D$ attached to $\mathcal{V}_\infty$ (see Subsection 5.2.2 for details) extends to a proper period map $\widetilde{\Phi}^{\mathrm{an}} : \widetilde{X}^{\mathrm{an}} \to \Gamma \setminus D$ over the components of Δ around which the monodromy is finite. From [BBrT18, Thm. 1.1] (see also [Kl22, Thm. 3.20]), there exists a dominant morphism $\alpha : \widetilde{X}_{\mathbb{C}} \to X'$ of algebraic $\mathbb{C}$-varieties with X' quasi-projective and such that $\widetilde{\Phi}^{\mathrm{an}} : \widetilde{X}^{\mathrm{an}} \to \Gamma \setminus D$ factors as

$$\begin{tikzcd} \widetilde{X}^{\mathrm{an}} \arrow[r, "\widetilde{\Phi}^{\mathrm{an}}"] \arrow[d, "\alpha^{\mathrm{an}}"'] & \Gamma \setminus D \\ X'^{\,\mathrm{an}} \arrow[ur, hook, "\overline{\widetilde{\Phi}}^{\,\mathrm{an}}"'] & \end{tikzcd}$$

with $\overline{\widetilde{\Phi}}^{\,\mathrm{an}} : X'^{\,\mathrm{an}} \hookrightarrow \Gamma \setminus D$ a complex analytic closed immersion. By Hironaka, there exists a proper birational morphism $p : X'_1 \to X'$ and a log smooth compactification $j'_1 : X'_1 \hookrightarrow X'_2$ by a normal crossing divisor $\Delta := X'_2 \setminus X'_1$. The resulting period map $X'^{\,\mathrm{an}}_1 \to X'^{\,\mathrm{an}} \hookrightarrow \Gamma \setminus D$ then satisfies the assumption of Zuo's theorem [Z00, Thm. 0.1] so that $K_{X'_2} + \Delta$ is big. In other words, X' is of log general type. By the Lefschetz principle, one may assume that X' and $\alpha : \widetilde{X}_{\mathbb{C}} \to X'$ descends to $\bar{k}$. This shows that there exists a dominant morphism $\alpha : X \times_k \bar{k} \to X'$ of $\bar{k}$-varieties with X' of log-general type and a polarizable $\mathbb{Z}$-VHS $\mathcal{V}'_{\infty\mathbb{Z}}$ on $X'^{\,\mathrm{an}}$ such that $\mathcal{V}_{\infty\mathbb{Z}}|_{X^{\mathrm{an}}} = \alpha^{\mathrm{an}\,*} \mathcal{V}'_{\infty\mathbb{Z}}$. In turn, using the profinite completion morphism $\pi_1(X'^{\,\mathrm{an}}) \to \pi_1(X')$ [SGA1, XII, Thm. 5.1] $\mathcal{V}'_{\infty\mathbb{Z}}$ on $X'^{\,\mathrm{an}}$ gives rise to a $\mathbb{Z}_p$-local system $\mathcal{V}'_{\mathbb{Z}_p}$ on X' such that $\mathcal{V}'_p := \mathcal{V}'_{\mathbb{Z}_p} \otimes \mathbb{Q}_p$ has the requested property. $\square$

Remark: When X is a curve, Theorem 9 boils down to the fact that X is hyperbolic if and only if $\pi_1(X \times_k \bar{k})$ is not abelian.

Theorem 9 combined with the following conjecture of Lang-Vojta

[1] One could even choose this morphism to be defined over k - see [BBrT18, Par. after Thm.1.1].

Conjecture 10. (Lang-Vojta, [Vo86]) *Assume X is of log general type. Then for every non-empty open subset $U \subset \mathrm{spec}(\mathcal{O}_k)$ and model $\mathcal{X} \to U$ of X over U, $\mathcal{X}(U) \cap |X|$ is not Zariski-dense in X.*

has the following striking consequence on the set of integral points of X.

Corollary 11. *Assume $\mathcal{V}_p$ arises from geometry and has positive period dimension (resp. for every smooth locally closed subvariety $Y \hookrightarrow X$, $\mathcal{V}_p|_Y$ has positive period dimension). Then Conjecture 10 implies that for every non-empty open subset $U \subset \mathrm{spec}(\mathcal{O}_k)$ and model $\mathcal{X} \to U$ of X over U, $\mathcal{X}(U) \cap |X|$ is not Zariski-dense in X (resp. is finite).*

3. Arithmetic applications

In this section, we review a sample of (conjectural) arithmetic applications of Conjecture 1. Those collected in Subsections 3.1, 3.2 and 3.3 only use part (1) of Conjecture 1 while those collected in Subsection 3.4 also use part (2) of Conjecture 1. For the applications in Subsections 3.2 and 3.4 one can restricts to GCLS $\mathbb{Q}_p$-local systems but the application in Subsection 3.3 requires Conjecture 1 for GCLP $\mathbb{Q}_p$-local systems since the $\mathbb{Q}_p$-local systems involved are almost never GCLS.

3.1. **A characterization of arithmetic monodromy of subvarieties.** For a closed integral subvariety $Z \hookrightarrow X$ Conjecture 1 (1) gives an arithmetic characterization of $G^\circ_{Z,\mathcal{V}_p} \subset G^\circ_{X,\mathcal{V}_p}$ in terms of the Zariski-topology of Z, namely,

Proposition 12. *Assume Conjecture 1 (1) holds. Let $Z \hookrightarrow X$ be a closed integral subvariety, geometrically connected over k and let $H \subset G^\circ_{X,\mathcal{V}_p}$ be a connected algebraic subgroup. Then $G^\circ_{Z,\mathcal{V}_p} = H$ if and only if there exists an integer $d \geq 1$ such that the set of all $x \in |Z|^{\leq d}$ with $G_{x,\mathcal{V}_p} = H$ is Zariski-dense in Z.*

Proof. The if implication follows from the fact that, for a variety Z over k one can always find an integer $d \geq 1$ such that $|Z|^{\leq d}$ is Zariski-dense in Z and from Conjecture 1 (1) (applied to the GCLP $\mathbb{Q}_p$-local system $\mathcal{V}_p|_{Z^{\mathrm{sm}}}$, where $Z^{\mathrm{sm}} \hookrightarrow Z$ denote the smooth locus). The only if implication only requires k to be Hilbertian and is a classical consequence of the Hilbert irreducibility theorem. □

3.2. **Degeneration of motivated motivic Galois groups / Exceptional motivated cycles.** (See [C12] for details). Assume $\mathcal{V}_p$ arises from geometry (Section 2.1). The inclusions (2.1.2) immediately imply

Proposition 13. *Fix an integer $d \geq 1$ and assume Conjecture 1 (1) holds for d and $\mathcal{V}_p$. Then $X_{\mathcal{V}_{\mathrm{AH}}} \cap |X|^{\leq d}$ (hence a fortiori $X_{\mathcal{V}_{\mathrm{mot}}} \cap |X|^{\leq d}$) is not Zariski-dense in X.*

Example. Let $f : Y \to X$ be a smooth proper morphism. As $Z^2_{\mathrm{mot}}(Y_{\bar{x}}) = Z^2_{\mathrm{AH}}(Y_{\bar{x}}) = NS(Y_{\bar{x}})$ one gets, in particular,

(1) (Jumping locus of the Neron-Severi rank) Assume Conjecture 1 (1) holds for d and $\mathcal{V}_p = R^2 f_* \mathbb{Q}_p(1)$. Then the set of all $x \in |X|^{\leq d}$ such that $\mathrm{rank}(NS(Y_{\overline{\eta}})) < \mathrm{rank}(NS(Y_{\bar{x}}))$ is not Zariski-dense in X.

When $f : Y \to X$ is an abelian scheme, Example (1) in turn, implies

(2) (Jumping locus of the rank of the endomorphism ring of abelian varieties) Assume Conjecture 1 (1) holds for d and $\mathcal{V}_p = R^2 f_* \mathbb{Q}_p(1)$. Then the set of all $x \in |X|^{\leq d}$ such that $\mathrm{rank}(\mathrm{End}(Y_{\overline{\eta}})) < \mathrm{rank}(\mathrm{End}(Y_{\bar{x}}))$ is not Zariski-dense in X.

3.3. **Specialization of 1-cohomology classes.** (See [C21] for details). Let $\mathcal{V}_p$ be a GCLP $\mathbb{Q}_p$-local system on X such that $(\mathcal{V}_{p,\overline{\eta}})_{\mathrm{Lie}(\overline{\Pi}_{X,\mathcal{V}_p})} = 0$ and let $E \subset \mathrm{H}^1(\pi_1(X), \mathcal{V}_{p,\overline{\eta}})$ be a finite-dimensional $\mathbb{Q}_p$-subvector space. Consider the $\mathbb{Q}_p$-local system $\widetilde{\mathcal{V}}_p$ (which is not GCLS in general, even if $\mathcal{V}_p$ is, but which is GCLP) corresponding to the universal extension of $\pi_1(X)$-modules

$$0 \to \mathcal{V}_{p,\overline{\eta}} \to \widetilde{E} \to E \to 0$$

classifying the 1-classes in E.

Proposition 14. *Fix an integer $d \geq 1$ and assume Conjecture 1 (1) holds for d and the GCLP $\mathbb{Q}_p$-local system $\widetilde{\mathcal{V}}_p$. Then the set of all $x \in |X|^{\leq d}$ such that the restriction morphism*

$$E \subset \mathrm{H}^1(\pi_1(X), \mathcal{V}_{p,\overline{\eta}}) \stackrel{\mathrm{res}_x}{\to} \mathrm{H}^1(\pi_1(x), \mathcal{V}_{p,\bar{x}})$$

is not injective is not Zariski-dense in X.

Example. Let $f : Y \to X$ be an abelian scheme. The Kummer map and the Néron extension property for abelian schemes yield a canonical commutative diagram

$$\begin{tikzcd}
Y_\eta(k(\eta)) \otimes \mathbb{Q} \arrow[r, hook, "\kappa_p"] & \mathrm{H}^1(\pi_1(\eta), V_p(Y_{\overline{\eta}})) \; . \\
Y(X) \otimes \mathbb{Q} \arrow[r, hook] \arrow[u, "\simeq"'] \arrow[d, "\mathrm{sp}_x"] & \mathrm{H}^1(\pi_1(X), V_p(Y_{\overline{\eta}})) \arrow[u, "\simeq"', "\mathrm{inf}"] \arrow[d, "\mathrm{res}_x"] \\
Y_x(k(x))) \otimes \mathbb{Q} \arrow[r, hook, "\kappa_p"] & \mathrm{H}^1(\pi_1(x), V_p(Y_{\overline{x}}))
\end{tikzcd}$$

By the Mordell-Weil theorem, $E := Y(X) \otimes \mathbb{Q}_p \subset \mathrm{H}^1(\pi_1(X), V_p(Y_{\overline{\eta}}))$ is a finite-dimensional $\mathbb{Q}_p$-subvector space and if Y_η contains no non-zero k- isotrivial abelian subvariety, the condition $(\mathcal{V}_{p,\overline{\eta}})_{\mathrm{Lie}(\overline{\Pi}_{X,\mathcal{V}_p})} = 0$ is satisfied for $\mathcal{V}_p = (R^1 f_* \mathbb{Q}_p)^\vee$ so that one gets

(3) (Dropping locus of the rank of abelian varieties) Fix an integer $d \geq 1$ and assume Conjecture 1 (1) holds for d and the GCLP $\mathbb{Q}_p$-local system $\widetilde{\mathcal{V}}_p$. Then the set of all $x \in |X|^{\leq d}$ such that the specialization map
$$\mathrm{sp}_x : Y_\eta(k(\eta)) \otimes \mathbb{Q} \to Y_x(k(x))) \otimes \mathbb{Q}$$
is not injective is not Zariski-dense in X. In particular, the set of all $x \in |X|^{\leq d}$ such that $\mathrm{rank}(A_x(k(x)) < \mathrm{rank}(\ A_\eta(k(\eta)))$ is not Zariski-dense in X.

3.4. **Uniform boundedness of p-primary torsion.** We follow closely [CT12, Sec. 4]. For simplicity, we remove the subscript $-_p$ from the notation. Let $\mathcal{V}$ be a GCLP (or GCLS) $\mathbb{Q}_p$-local system. Write $V := \mathcal{V}_{\overline{\eta}}$. Let $T \subset V$ be a $\pi_1(X)$-stable $\mathbb{Z}_p$-lattice and set $D := V/T = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ so that one has a natural $\pi_1(X)$-equivariant isomorphism $T/p^n \tilde{\to} D[p^n]$ for each $n \geq 1$. Set $T_{(0)} := T^{\overline{G}^\circ_{X,\mathcal{V}}}$. Let $\rho : \pi_1(X) \to \mathrm{GL}(V)$ denote the representation corresponding to $\mathcal{V}$ and for every smooth, connected variety Y and morphism $\phi : Y \to X$, write $\rho_\phi : \pi_1(Y) \overset{\phi}{\to} \pi_1(X) \overset{\rho}{\to} \mathrm{GL}(V)$ denote the representation corresponding to $\phi^*\mathcal{V}$. Eventually, for a p-adic character $\chi : \pi_1(X) \to \mathbb{Z}_p^\times$, write $\chi_\phi := \chi \circ \phi : \pi_1(Y) \to \mathbb{Z}_p^\times$ and set

$$\overline{D}_\phi := \{v \in D \mid \rho_\phi(\sigma) v \in \langle v \rangle,\ \sigma \in \pi_1(Y)\},\ \ D_\phi(\chi) := \{v \in D \mid \rho_\phi(\sigma) v = \chi_\phi(\sigma) v,\ \sigma \in \pi_1(Y)\}$$
$$\overline{T}_\phi := \{v \in T \mid \rho_\phi(\sigma) v \in \langle v \rangle,\ \sigma \in \pi_1(Y)\},\ \ T_\phi(\chi) := \{v \in T \mid \rho_\phi(\sigma) v = \chi_\phi(\sigma) v,\ \sigma \in \pi_1(Y)\}$$

By definition $\overline{D}_\phi$, $\overline{T}_\phi$ (resp. D_ϕ, T_ϕ) are $\pi_1(Y)$-subsets (resp $\pi_1(Y)$-submodules) of D, T respectively. For each subset $E \subset D$ and $n \geq 0$, set $E[p^n] := E \cap D[p^n]$ and $E[p^n]^* := E \cap (D[p^n] \smallsetminus D[p^{n-1}])$, with the convention $D[p^{-1}] := \emptyset$. For each subset $E \subset T$, set $E^* := E \cap (T \smallsetminus pT)$. Then one has
$$\lim_n \overline{D}_\phi[p^n] \simeq \overline{T}_\phi,\ \ \lim \overline{D}_\phi[p^n]^* \simeq \overline{T}^*_\phi,$$
and
$$\lim_n D_\phi(\chi)[p^n] \simeq T_\phi(\chi),\ \ \lim_n D_\phi(\chi)[p^n]^* \simeq T_\phi(\chi)^*.$$

Proposition 15. *Assume Conjecture 1 holds for d and GCLS (or GCLP) $\mathbb{Q}_p$-local systems $\mathcal{V}|_Z$, for $Z \hookrightarrow X$ a locally closed smooth subvariety. Then,*

(1) *Let $\chi : \pi_1(X) \to \mathbb{Z}_p^\times$ be a p-adic character such that, for every $x \in |X|^{\leq d}$, χ_x does not appear as a subrepresentation of ρ_x. Then there exists an integer $n := n(\mathcal{V}, \chi, d) \geq 0$ such that for every $x \in |X|^{\leq d}$ the $\pi_1(x)$-module $D_x(\chi)$ is contained in $D[p^n]$;*

(2) *Assume $T_{(0)} = 0$. Then there exists an integer $n := n(\mathcal{V}, d) \geq 0$ such that for every $x \in |X|^{\leq d} \setminus X_{\mathcal{V}}$ the $\pi_1(x)$-set $\overline{D}_x$ is contained in $D[p^n]$.*

Proof. We argue as in [CT12, Cor. 4.3] with one additional induction step for (1). Write $\Pi := \Pi_{X,\mathcal{V}}$, $\overline{\Pi} := \overline{\Pi}_{X,\mathcal{V}}$ Let $\Pi(n)$ denote the kernel of the morphism $\Pi \subset \mathrm{GL}(T) \twoheadrightarrow \mathrm{GL}(T/p^n)$ induced by reduction *modulo* p^n. Recall that the $\Pi(n)$, $n \geq 1$ form a fundamental system of neighbourhoods of 1 in Π. From Conjecture 1, $|X|_{\mathcal{V}} \cap |X|^{\leq d}$ is not Zariski-dense in X and there exists an integer $N = N(\mathcal{V}, d) \geq 1$ such that for every $x \in |X|^{\leq d} \setminus |X|_{\mathcal{V}}$, $\Pi(N) \subset \Pi_x$ hence $D_x(\chi) \subset D_{\phi_N}(\chi)$ and $\overline{D}_x \subset \overline{D}_{\phi_N}$, where $\phi_N : X_{\Pi(N)} \to X$ denote the geometrically connected etale cover corresponding to the open subgroup $\Pi(N) \subset \Pi$.

Proof of (2): It is enough to show that $\overline{D}_{\phi_N}$ is finite. Each $0 \neq v \in \overline{T}_{\phi_N}$ defines a p-adic character $\chi_v : \pi_1(X_{\Pi(N)}) \to \mathbb{Z}_p^\times$ such that $\pi \cdot v = \chi_v(\pi) v$, $\pi \in \pi_1(X_{\Pi(N)})$. Since $\overline{\Pi} \cap \Pi(N)$ has finite abelianization (as

$\mathcal{V}$ is GLP), $\pi_1(X_{\Pi(N_v),\bar{k}})$ acts trivially on v for some $N_v \geq N$. This shows $\overline{T}_{\phi_N} \subset T_{(0)} = 0$. On the other hand, if $\overline{D}_{\phi_N}$ were infinite, $\overline{D}_{\phi_N}[p^n]^*$ would be nonempty for every $n \geq 0$, hence $\overline{T}^*_{\phi_N} = \lim_n \overline{D}_{\phi_N}[p^n]^*$ would be nonempty: a contradiction.

Proof of (1): Assume first χ is the trivial character $\mathbf{1}$. One argue by induction on the dimension of X. If X is 0-dimensional, the assertion is straightforward (as $D_x(\mathbf{1})$ is finite for every $x \in X(k)$ by assumption). Assume X has dimension ≥ 1. For $x \in |X|^{\leq d} \setminus |X|_{\mathcal{V}}$, $D_x(\mathbf{1}) \subset D_{\phi_N}(\mathbf{1})$. By construction, the action of $\pi_1(X)$ on $D_{\phi_N}(\mathbf{1})$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(X)/\pi_1(X_{\Pi(N)}) = \Pi/\Pi(N) = \Pi_N$. Thus, $D_x(\mathbf{1})$ coincides with the submodule of elements of $D_{\phi_N}(\mathbf{1})$ fixed by the subgroup $(\Pi_x)_N \subset \Pi_N$. As Π_N is a finite group, there are only finitely many subgroups of Π_N that coincide with $(\Pi_x)_N$ for some $x \in |X|^{\leq d} \setminus |X|_{\mathcal{V}}$. Accordingly, there are only finitely many submodules of $D_{\phi_N}(\mathbf{1})$ that coincide with $D_x(\mathbf{1})$ for some $x \in |X|^{\leq d} \setminus |X|_{\mathcal{V}}$. As $D_x(\mathbf{1})$ is finite for every $x \in |X|^{\leq d}$ (by assumption), this shows that there exists an integer $n_0 \geq 1$ such that $D_x(\mathbf{1}) \subset D[p^{n_0}]$, $x \in |X|^{\leq d} \setminus |X|_{\mathcal{V}}$. Next, let $Z := \overline{(|X|_{\mathcal{V}} \cap |X|^{\leq d})^{\mathrm{zar}}} \subsetneq X$ denote the Zariski-closure of $X_{\mathcal{V}} \cap |X|^{\leq d}$ in X. One can cover Z by finitely many locally closed smooth irreducible subvarieties - say $Z_1, \dots, Z_r$, of dimension $\leq \dim(X) - 1$. As for $i = 1, \dots, s$, $\mathcal{V}|_{Z_i}$ is again GCLS (and χ_x does not appear as a subrepresentation of ρ_x for every $x \in Z_i(k) \subset |X|^{\leq d}$), the induction hypothesis ensures that $n_i \geq 1$ such that for every $x \in |Z_i|^{\leq d}$ the $\pi_1(x) = \pi_1(k)$-module $D_x(\chi)$ is contained in $D[p^{n_i}]$. Taking $n := \max\{n_0, n_1, \dots, n_r\}$ concludes the proof. The case of a general p-adic character $\chi : \pi_1(X) \to \mathbb{Z}_p^\times$ reduces to the case $\chi = \mathbf{1}$ exactly as in the proof of [CT12, Cor. 4.3] (with "GLP replaced by "GCLS" or "GCLP"). □

Remark. The proof shows that one may replace replace $|X|^{\leq d}$ with any subset $\Xi \subset |X|^{\leq d}$ in Proposition 15.

Examples.

(4) (Uniform boundedness of p-primary torsion of abelian varieties). Let $f : Y \to X$ be an abelian scheme and consider the GCLS $\mathbb{Q}_p$-local system $\mathcal{V}_p = (R^1 f_* \mathbb{Q}_p)^\vee$. By the Mordell-Weil theorem Proposition 15 (1) applies to $T = T_p(A_{\bar{\eta}}) \subset V = V_p(A_{\bar{\eta}})$ and $\chi = \mathbf{1}$ so that one gets: Assume Conjecture 1 holds for d and GCLS (or GCLP) $\mathbb{Q}_p$-local systems $\mathcal{V}|_Z$, for $Z \hookrightarrow X$ a locally closed smooth subvariety. Then
$$\sup\{|A_x(k(x))[p^\infty]| \mid x \in |X|^{\leq d}\} < +\infty.$$
By the theory of Hilbert moduli schemes, for every integer $g \geq 1$, one can construct a variety X_g over $\mathbb{Q}$ and a principally polarized abelian scheme $Y_g \to X_g$ of relative dimension g such that for every number field k and g-dimensional principally polarized abelian variety $\mathfrak{Y}$ over k there exists $x_{\mathfrak{Y}} \in X(k)$ with $\mathfrak{Y} \simeq Y_{g,x_{\mathfrak{Y}}}$. Stratifying X_g by (finitely many) locally closed smooth irreducible subvarieties (and using that a smooth irreducible variety over k with a k-rational point is geometrically connected) and applying the above to the pullback of $Y_g^4 \times_{X_g} Y_g^{\vee\,4} \to X_g$ over each of the (finitely many) stratum, one gets the p-primary part of the torsion conjecture for abelian varieties, namely, for every integer d and prime p there exists an integer $n(d,p,g) \geq 0$ such that for every number field k with $[k : \mathbb{Q}] \leq d$ and g-dimensional abelian variety $\mathfrak{Y}$ over k one has $|\mathfrak{Y}(k)[p^\infty]| \leq p^{n(d,p,g)}$.

(5) Let $f : Y \to X$ be a smooth proper morphism and consider the GCLS $\mathbb{Q}_p$-local system $\mathcal{V}_p = R^2 f_* \mathbb{Q}_p(1)$. For a prime p, let $\Xi_p \subset X(k)$ denote the subset of all $x \in X(k)$ such that Y_x satisfies the p-adic Tate conjecture for divisors. Then Proposition 15 (1) and the above Remark with $\Xi := \Xi_p$ apply to $T = T_p(Br(Y_{\bar{\eta}})) \subset V = V_p(Br(Y_{\bar{\eta}}))$ and $\chi = \mathbf{1}$ (*e.g.* [CCh20, Prop. 2.1.1]) so that one gets: for every prime p
$$\sup\{|Br(Y_{\bar{x}})^{\pi_1(k)}[p^\infty]| \mid x \in \Xi_p\} < +\infty.$$
If $Y \to X$ is an abelian scheme or a family of K3 surfaces then $\Xi_p = X(k)$. If[2] one assumes Conjecture 1 for every integer $d \geq 1$, one obtains as in Example (1) the p-primary part of the uniform boundedness conjecture [?, Conj. 4.6], namely: for every integer $d \geq 1$ and lattice Λ there exists $n(d, \Lambda) \geq 0$ such that for every number field k with $[k : \mathbb{Q}] \leq d$ and K3-surface $\mathfrak{Y}$ over k with $NS(\mathfrak{Y}_{\bar{k}}) \simeq \Lambda$ one has $|Br(\mathfrak{Y}_{\bar{k}})^{\pi_1(k)}[p^\infty]| \leq p^{n(d,\Lambda)}$. See [CCh20, 2.2] for details.

4. The main geometric conjecture

The aim of this Section is to relate Conjecture 2 to the following celebrated Diophantine conjecture, which is the case $\Delta = \emptyset$ of Conjecture 10.

[2] If one wants to stick to the $d = 1$ case of Conjecture 1, then one should also involve a fixed level structure in the data.

Conjecture 16. (Bombieri-Lang, [L86]) *Assume X is of general type. Then $X(k)$ is not Zariski-dense in X.*

Modulo a geometric conjecture - Conjecture 17. This is achieved in Corollary 19, Proposition 20, after introducing an *ad hoc* projective system of *level schemes* (Subsection 4.1), which is the essential ingredient to reformulate the (representation-theoretic) Conjecture 2 in diophantine terms.

Before that, let us just briefly mention that Conjecture 10 and Conjecture 16 are widely open in general. The most important known cases are:

- Conjecture 16: Subvarieties of general type in abelian varieties (in particular curves - Mordell conjecture [F83], [FW84]) [F91];
- Conjecture 10: Shimura varieties of abelian type - Shafarevich conjecture for abelian varieties (in particular hyperbolic curves of genus ≤ 1 - Siegel Theorem [Si29]) [F91].

Let us also mention the following consequence of Conjecture 16 for subvarieties of general type in abelian varieties [Fr94]: Let X be proper, smooth, geometrically connected curve over k of k-gonality $\geq \gamma$. Then for every integer $0 \leq d \leq \lfloor \frac{\gamma-1}{2} \rfloor$, $|X|^{\leq d}$ is finite.

4.1. **Level schemes.** For a profinite group Π, let $\Phi(\Pi) \subset \Pi$ denote its Frattini subgroup (that is the intersection of all the maximal open subgroups of Π).

4.1.1. Let Γ be a compact p-adic Lie group. Fix a fundamental system $\Gamma(n)$, $n \geq 0$ of neighbourhoods of 1 in Γ, that is a decreasing sequence $\cdots \subset \Gamma(n+1) \subset \Gamma(n) \subset \cdots \subset \Gamma$ of open normal subgroups of Γ such that $\cap_{n\geq 0}\Gamma(n) = 1$ (or, equivalently, such that every open subgroup $U \subset \Gamma$ contains $\Gamma(n)$ for $n \gg 0$). For every $n \geq 1$, let $\mathcal{H}_n(\Gamma)$ denote the set of open subgroups $U \subset \Gamma$ such that $\Phi(\Gamma(n-1)) \subset U$ but $\Gamma(n-1) \not\subset U$ and let $\mathcal{H}_0(\Gamma) := \{\Gamma\}$. Then [CT12, Lem. 3.3, (Proof of Cor. 3.6)],

(1) $\mathcal{H}_n(\Gamma)$ is finite, $n \geq 0$.

(2) The maps $\mathcal{H}_{n+1}(\Gamma) \to \mathcal{H}_n(\Gamma)$, $U \mapsto U\Phi(\Gamma(n-1))$ (with the convention that $\Phi(\Gamma(-1)) = \Gamma$) endow the $\mathcal{H}_n(\Gamma)$, $n \geq 0$ with a canonical structure of projective system $(\mathcal{H}_{n+1}(\Gamma) \overset{\phi_n}{\to} \mathcal{H}_n(\Gamma))_{n\geq 0}$.

(3) For every $\underline{H} := (H[n])_{n\geq 0} \in \lim\limits_{n} \mathcal{H}_n(\Gamma)$,
$$H[\infty] := \lim_{n} H[n] = \bigcap_{n\geq 0} H[n] \subset \Gamma$$
is a closed subgroup of codimension ≥ 1 in Γ and $H[n] = H[\infty]\Gamma(n)$, $n \gg 0$.

(4) For every closed subgroup $H \subset \Gamma$ such that $\Gamma(n-1) \not\subset H$ there exists $U \in \mathcal{H}_n(\Gamma)$ such that $H \subset U$.

4.1.2. Let now $\mathcal{V} := \mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on X and $\rho : \pi_1(X) \to \mathrm{GL}(\mathcal{V}_{\overline{\eta}})$ the corresponding representation. Write $\Pi := \Pi_{X,\mathcal{V}}$, $\overline{\Pi} := \overline{\Pi}_{X,\mathcal{V}}$. For an open subgroup $U \subset \Pi$, let $X_U \to X$ denote the connected étale cover corresponding to $\rho^{-1}(U) \subset \pi_1(X)$. Fix a fundamental system $\Pi(n)$, $n \geq 0$ of neighbourhoods of 1 in Π and let $\mathcal{H}_n(\Pi)$, $n \geq 0$ denote the corresponding projective system of 4.1.1. From 4.1.1 (1),
$$X_n := X_n(\Pi) := \bigsqcup_{U\in\mathcal{H}_n(\Pi)} X_U \to X$$
is a (non-connected) étale cover of X and, by functoriality of étale fundamental group, the maps $\mathcal{H}_{n+1}(\Pi) \to \mathcal{H}_n(\Pi)$, $n \geq 0$ of 4.1.1 (2) endow the X_n, $n \geq 0$ with a structure of projective system
$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$
whose transition morphisms are étale covers. The connected etale cover $X_U \to X$ is defined over a finite extension k_U of k (namely the one corresponding to the open subgroup $\mathrm{im}(\pi_1(X_U) \to \pi_1(X) \to \pi_1(k)) = \pi_1(k_U) \subset \pi_1(k)$) and satisfies the following two properties:

(1) $X_U \times_{k_U} \overline{k} \to X_{\overline{k}}$ is the etale cover $X_{\overline{U}} \to X_{\overline{k}}$ corresponding to the inverse image of the open subgroup $\overline{U} := \overline{\Pi} \cap U \subset \overline{\Pi}$ in $\pi_1(X_{\overline{k}})$.

(2) For every morphism $Y \to X$, $\mathrm{im}(\pi_1(Y) \to \pi_1(X) \twoheadrightarrow \Pi) \subset U$ (up to conjugacy) if and only if $Y \to X$ factors as
$$\begin{tikzcd} & X_U \arrow[d] \\ Y \arrow[ur, dotted] \arrow[r] & X \end{tikzcd}.$$

4.2. The main geometric conjecture relating Conjecture 2 to Conjecture 16 is the following.

Conjecture 17. (Geometric version of Conjecture 2) *Assume $k = \bar{k}$ (so that $\Pi = \overline{\Pi}$) and let $\mathcal{V}$ be a (G)CLP $\mathbb{Q}_p$-local system on X. Fix a fundamental system $\Pi(n)$, $n \geq 0$ of neighbourhoods of 1 in Π. Then, for every closed subgroup $H \subset \Pi$ of positive codimension and for $n \gg 0$, $X_{H\Pi(n)}$ dominates a variety of general type.*

Remark. Conjecture 17 is independent of the choice of $\Pi(n)$, $n \geq 0$ as if $\widetilde{\Pi(n)}$, $n \geq 0$ is another fundamental system of neighbourhoods of 1 in Π, for every $n \geq 0$ there exists $N_n \geq n$ such that $\Pi(N_n) \subset \widetilde{\Pi(n)}$ so that $X_{H\Pi(n)} \to X_{H\widetilde{\Pi(n)}}$.

Consider a property P of smooth, irreducible varieties over $\bar{k}$ which is invariant by birational morphism and such that, for a generically finite morphism $V_2 \to V_1$ of smooth irreducible varieties over $\bar{k}$, V_1 satisfies P implies V_2 satisfies P.

Lemma 18. *Assume $k = \bar{k}$ (so that $\Pi = \overline{\Pi}$). If for every closed subgroup $H \subset \Pi$ of positive codimension and for $n \gg 0$, $X_{H\Pi(n)}$ satisfies* P *then for $n \gg 0$, every connected component of X_n also satisfies* P.

Proof. Let $\mathcal{H}_{n,\neg \mathrm{P}}(\Pi) \subset \mathcal{H}_n(\Pi)$ denote the subset of all $U \in \mathcal{H}_n(\Pi)$ such that X_U does not satisfy P. Then the projective system $\mathcal{H}_{n+1}(\Pi) \to \mathcal{H}_n(\Pi)$, $n \geq 0$ restricts to a projective system $\mathcal{H}_{n+1,\neg \mathrm{P}}(\Pi) \to \mathcal{H}_{n,\neg \mathrm{P}}(\Pi)$, $n \geq 0$. Assume $\mathcal{H}_{n,\neg \mathrm{P}}(\Pi) \neq \emptyset$, $n \geq 1$. By 4.1.1 (1), $\mathcal{H}_{n,\neg \mathrm{P}}(\Pi)$ is finite, $n \geq 0$ hence $\lim_n \mathcal{H}_{n,\neg \mathrm{P}}(\Pi) \neq \emptyset$. Let $\underline{H} := (H[n])_{n\geq 1} \in \lim_n \mathcal{H}_{n,\neg \mathrm{P}}(\Pi)$. By 4.1.1 (3), $H[\infty] := \bigcap_{n\geq 1} H[n] \subset \Pi$ is a closed subgroup of codimension ≥ 1 and $H[n] = H[\infty]\Pi(n)$, for $n \gg 0$, which contradicts the assumption for $H = H[\infty]$. $\square$

Lemma 18 applied to the property P $\equiv$ "dominates a variety of general type" immediately yields.

Corollary 19. *Assume $k = \bar{k}$ (so that $\Pi = \overline{\Pi}$) and let $\mathcal{V}$ be a (G)CLP $\mathbb{Q}_p$-local system on X for which Conjecture 17 holds. Then, for $n \gg 0$, every connected component of X_n dominates a variety of general type.*

Proposition 20. *Conjecture 16 and Conjecture 17 imply Conjecture 2.*

Proof. Fix a fundamental system $\Pi(n)$, $n \geq 0$ of neighbourhoods of 1 in Π and let
$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_1 \to X.$$
be the projective system of étale covers attached to it in 4.1.2. From 4.1.1 (4) and 4.1.2 (2), the exceptional locus $X_{\mathcal{V}} \subset X(k)$ is the image of
$$\lim X_n(k) \to X(k).$$
It is in particular enough to show that $X_n(k) \subset X_n$ is not Zariski-dense in X_n for $n \gg 0$. We choose $\Pi(n)$, $n \geq 0$ as follows (See Remark 17). Fix a Π-stable $\mathbb{Z}_p$-lattice $T \subset \mathcal{V}_{\bar{\eta}}$. For $n \geq 0$ let $\Pi(n) \subset \Pi$ denote the kernel of the reduction-*modulo*-p^n morphism $\Pi \subset GL(T) \twoheadrightarrow GL(T/p^n)$. With this choice of $\Pi(n)$, $n \geq 0$ there exists an integer $N_0 > 0$ such that for every $n \geq N_0$, $\Phi(\Pi(n)) = \Pi(n+1)$ and $\Phi(\overline{\Pi}(n)) = \overline{\Pi}(n+1)$ [CT12, Lem. 3.2]. From Corollary 19, there exists an integer $N_1 \geq N_0$ such that for every $n \geq N_1$ and $U \in \mathcal{H}_n(\overline{\Pi})$, X_U dominates a variety of general type. Since Π is topologically finitely generated, it contains only finitely many open subgroups $U \subset \Pi$ with $[\Pi : U] \leq 2[\overline{\Pi} : \overline{\Pi}(N_1 - 1)]$; the intersection of all such open subgroups is thus again open in Π hence contains $\Pi(N_2 - 1)$ for some integer $N_2 \geq N_1$. Let $n \geq N_2$ and $U \in \mathcal{H}_n(\Pi)$ so that $\Pi(n)(= \Phi(\Pi(n-1))) \subset U$ but $\Pi(n-1) \not\subset U$. Then, by definition, $\overline{\Pi}(n) \subset \overline{U}$. Let N be the minimal integer $\geq N_1 - 1$ such that $\overline{\Pi}(N)(= \Phi(\overline{\Pi}(N-1))) \subset \overline{U}$. If $N \geq N_1$, $\overline{\Pi}(N-1) \not\subset \overline{U}$ hence $\overline{U} \in \mathcal{H}_N(\overline{\Pi})$ and $X_{\overline{U}}$ dominates a variety of general type. If $N = N_1 - 1$, $\overline{\Pi}(N_1 - 1) \subset \overline{U}$. Since $\Pi(n-1) \not\subset U$ and $n \geq N_2$, $\Pi(N_2 - 1) \not\subset U$. Thus, by definition of N_2, $[\Pi : U] > 2[\overline{\Pi} : \overline{\Pi}(N_1 - 1)]$, hence
$$[k_U : k] = \frac{[\Pi : U]}{[\overline{\Pi} : \overline{U}]} > 2\frac{[\overline{\Pi} : \overline{\Pi}(N_1 - 1)]}{[\overline{\Pi} : \overline{U}]} \geq 2,$$
This shows that for $n \geq N_2$ and every $U \in \mathcal{H}_n(\Pi)$, either $[k_U : k] \geq 2$ - hence $X_U(k) = \emptyset$ or X_U dominates a variety of general type - hence, by Conjecture 16, $X_U(k) \subset X_U$ is not Zariski-dense in X_U. This shows that $|X|_{\mathcal{V}} \cap X(k)$ is not Zariski-dense in X. For the second part of Conjecture 1, we proceed by induction

on the dimension of X. For every $n \geq N_2$ let $X[n] \subset X$ denote the Zariski-closure of $\mathrm{im}(X_n(k) \to X(k))$ in X. If X is a curve, $X[N_2]$ is finite. For each $x \in X[N_2] \setminus |X|_{\mathcal{V}}(k)$, there exists $N_x \geq N_2$ such that x does not lift to a k-rational points on X_{N_x} so that, with $N_3 := \max\{N_x \mid x \in I\} \geq N_2$, and from 4.1.1 (4), for every $x \in X(k) \setminus |X|_{\mathcal{V}}(k)$, $\Pi(N_3) \subset \Pi_x$ hence $[\Pi : \Pi_x] \leq [\Pi : \Pi(N_3)]$. In general, as X is noetherian, there exists $N_3 \geq N_2$ such that $X[n] = X[N_3]$, $n \geq N_3$. From 4.1.1 (4), for every $x \in X \setminus X[N_3](k)$, $\Pi(N_3) \subset \Pi_x$ hence $[\Pi : \Pi_x] \leq [\Pi : \Pi(N_3)]$. Let us examine what happens for $x \in I := X[N_3](k) \setminus |X|_{\mathcal{V}}(k)$.

- If I is finite. Then, for each $x \in I$, there exists $N_x \geq N_3$ such that x does not lift to a k-rational points on X_{N_x} so that, with $N_4 := \max\{N_x \mid x \in I\} \geq N_3$, and from 4.1.1 (4), for every $x \in X(k) \setminus |X|_{\mathcal{V}}(k)$, $\Pi(N_4) \subset \Pi_x$.
- If I is not finite, consider $Z := \overline{I}^{\mathrm{zar}} \subset X[N_3]$. Then one can cover Z by finitely many locally closed smooth, irreducible subvarieties - say $Z_1, \dots, Z_r$. For each of $i = 1, \dots, r$, one of the following holds
  - $Z_i(k) = Z_i(k) \cap |X|_{\mathcal{V}}$. In that case, set $n_i = N_3$.
  - $Z_i(k) \setminus Z_i(k) \cap |X|_{\mathcal{V}}$ is finite and, as above, there exists $n_i \geq N_3$ such that for every $x \in Z_i(k) \setminus Z_i(k) \cap |X|_{\mathcal{V}}$, $\Pi(n_i) \subset \Pi_x$.
  - $Z_i(k) \setminus Z_i(k) \cap |X|_{\mathcal{V}}$ is infinite; in particular, Z_i has dimension ≥ 1 but, also, $<\dim(X)$. Let $\Pi_{Z_i} \subset \Pi$ denote the image of $\pi_1(Z_i)$ acting on $\mathcal{V}_{\overline{\eta}}$ *via* $\pi_1(X) \to \pi_1(X)$. As for $x \in Z_i(k) \setminus Z_i(k) \cap |X|_{\mathcal{V}}$, $\Pi_x \subset \Pi$ is open, $\Pi_{Z_i} \subset \Pi$ is open; in particular, $|X|_{\mathcal{V}} \cap Z_i = X_{\mathcal{V}|_{Z_i}}$. On the other hand, by definition, $\mathcal{V}|_{Z_i}$ is again GCLS so that, by induction hypothesis, there exists $n_i (\geq N_3)$ such that for every $x \in Z_i(k) \setminus Z_i(k) \cap X_{\mathcal{V}|_{Z_i}}$, $\Pi(n_i) \subset \Pi_x$.

  As a result, with $N_4 := \max\{n_1, \dots, n_r\} \geq N_3$, and from 4.1.1 (4), one gets, again, that for every $x \in X(k) \setminus |X|_{\mathcal{V}}(k)$, $\Pi(N_4) \subset \Pi_x$.

$\square$

4.3. **Remarks.**

(1) The proof of Proposition 20 shows that one "only" needs Conjecture 16 for the connected components of X_n which are of general type and for varieties of general type of dimension $\leq \dim(X) - 1$. In particular, if X is a surface, one "only" needs Conjecture 16 for the connected components of X_n which are of general type.

(2) The proof of Proposition 20 also shows that the non-Zariski density of $X_n(k)$ for $n \gg 0$ is only required for part (2) of Conjecture 2 while for part (1) of Conjecture 2, the non-Zariski density of the image of $\lim X_n(k) \to X(k)$ in X is enough, which is a much weaker statement. More generally, for every integer $d \geq 1$, one has

$$\mathrm{im}(\lim |X_n|^{\leq d} \to |X|^{\leq d}) = |X|_{\mathcal{V}} \cap |X|^{\leq d}$$

so that part (1) of Conjecture 1 for d is equivalent to the non Zariski-density of $\mathrm{im}(\lim |X_n|^{\leq d} \to |X|^{\leq d})$ in X. We feel the following conjecture (which would imply part (1) of Conjecture 2) is of independent interest.

Conjecture 21. *Let X be a smooth, geometrically connected variety over k and let*

$$X_{n+1} \to X_n \to \cdots \to X_1 \to X$$

be a projective system of geometrically connected étale covers. Let Π_n denote the Galois group of the Galois closure of $X_n \times_k \bar{k} \to X \times_k \bar{k}$, $n \geq 1$ and assume that $\Pi := \lim \Pi_n$ is a p-adic Lie group with perfect Lie algebra of dimension ≥ 1. Then $\mathrm{im}(\lim X_n(k) \to X(k))$ is not Zariski-dense in X.

(3) For higher-dimensional smooth varieties over number fields we are not aware of any geometric invariant expected to control the sparcity of points of bounded degree as the geometric gonality does for curve. Shifting the perspective, we would like to address the following (possibly too rough) question.

Question 22. *Fix integers $d_0, \delta \geq 1$. Does there exists an integer $N(d_0, \delta) \geq 1$ such that if Part (1) of Conjecture 2 holds for every GCLP $\mathbb{Q}_p$-local systems on a smooth, geometrically connected k-variety of dimension $\leq N(d_0, \delta)$ then Part (1) of Conjecture 1 holds for every integer $d \leq d_0$ and GCLP $\mathbb{Q}_p$-local systems on a smooth, geometrically connected k-variety of dimension $\leq \delta$.*

5. Results and perspectives

5.1. **Conjecture 17.** In this subsection, assume $k=\overline{k}$.

5.1.1. In full generality Conjecture 17 is known only in the two simplest cases of strongly hyperbolic Artin neighbourhoods[3], namely when X is an hyperbolic curve or the product of two hyperbolic curves. More precisely, one has:

(1) If X is an hyperbolic curve, the following enhanced version of Conjecture 17 holds ([CT13, Thm.3.3]): Assume $\mathcal{V}_p$ is (G)LP. Then, for every closed subgroup $H\subset\Pi:=\Pi_{X,\mathcal{V}_p}$ of codimension ≥ 1, the k-gonality of $X_{H\Pi(n)}$ goes to $+\infty$ with n.

(2) If $X=X_1\times X_2$ is the product of two hyperbolic curves, the following enhanced version of Conjecture 17 for (G)CLP $\mathbb{Q}_p$-local systems holds ([C22, Thm.1.1]): Assume $\mathcal{V}_p$ is (G)LP. Fix $x_i\in X_i(k)$, $i=1,2$ and set $x=(x_1,x_2)\in X(k)$. These induce closed immersions $\iota_1: X_1\tilde{\rightarrow}X_1\times x_2\hookrightarrow X$, $\iota_2: X_2\tilde{\rightarrow}x_1\times X_2\hookrightarrow X$ splitting the projections $X\rightarrow X_1$, $X\rightarrow X_2$ respectively. For $i=1,2$, write $\Pi_i:=\Pi_{X_i,\iota_i^*\mathcal{V}_p}\subset\Pi:=\Pi_{X,\mathcal{V}_p}$ and let $p_i:\Pi_1\times\Pi_2\rightarrow\Pi_i$ denote the corresponding canonical projection; write also $p:\Pi_1\times\Pi_2\twoheadrightarrow\Pi$ for the canonical product morphism. Let $\Pi(n)$, $n\geq 1$ be a fundamental system of neighbourhoods of 1 in Π. One has the following dichotomy.

   (a) Either $H\subset\Pi$ is not transverse that is, for one of $i=1,2$, $p_ip^{-1}(H)\subset\Pi_i$ is a closed subgroup of codimension ≥ 1. Then one has a projective commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X_{H\Pi(n+1)} \arrow[r] \arrow[d] & X_{H\Pi(n)} \arrow[r] \arrow[d] & \cdots \arrow[r] & X \arrow[d] \\
\cdots \arrow[r] & B_{i,H,n+1} \arrow[r] & B_{i,H,n} \arrow[r] & \cdots \arrow[r] & X_i
\end{tikzcd}$$

   with $B_{i,H,n}\rightarrow X_i$ a connected étale cover with gonality (hence, a fortiori, geometric genus) going to $+\infty$ with n;

   (b) Or $H\subset\Pi$ is transverse. Then $X_{H\Pi(n)}$ is birational to a smooth projective surface of general type and, for every integer $g\geq 0$, contains only finitely many closed integral curves with geometric genus $\leq g$ for $n\gg 0$.

   In (b), the assertion that $X_{H\Pi(n)}$ contains only finitely many closed integral curves with geometric genus $\leq g$ for $n\gg 0$ is a consequence of a celebrated theorem of Bogomolov ([Bo77]) asserting that a smooth projective surface of general type with $c_1^2-c_2>0$ contains only finitely many closed integral curves with geometric genus ≤ 1 and that for every integer $g\geq 0$, closed integral curves with geometric genus $\leq g$ form a bounded family.

5.1.2. From Theorem 9, when $\mathcal{V}_p$ arises from geometry with positive period dimension and X is *projective* then X dominates a variety of general type so that Conjecture 17 automatically holds. More generally, after possibly replacing X by a dense open subscheme, one may assume that there exists a dominant morphism $\alpha: X\rightarrow X'$ of k-varieties with X' of log general type and that $\mathcal{V}_p$ arises by base-change from a $\mathbb{Q}_p$-local system on X' so that, to tackle Conjecture 17, one may assume X is of log-general type. The strategy to achieve the results of Subsection 5.1.1 already uses this ingredient in a crucial way. Roughly, the idea is to exploit the ramification data of the covers $X_{H\Pi(n)}\rightarrow X$ around the components at infinity in the log smooth compactification $j:X\hookrightarrow\overline{X}$. There are two difficulties to overcome:

- Extract from the p-adic representation $\mathcal{V}_{p,\overline{\eta}}$ asympotic estimates for the ramification data. This step should involve subtle structural results about p-adic Lie groups and their homogeneous spaces;
- Relate the ramification data to the geometric invariants (Hodge numbers, Chern numbers, Kodaira dimensions *etc.*) one wants to control. This step is purely geometric. For instance, when X is a curve and one only wants to control the genus of $X_{H\Pi(n)}$, the Riemann-Hurwitz formula is enough. When X is the

[3]To prove Conjectures 1, 17, one may freely replace X by a non-empty open subscheme. In particular, one may assume that X is a strongly hyperbolic Artin neighbourhood that is it decomposes into a sequence

$$X=X_d\rightarrow X_{d-1}\rightarrow\cdots\rightarrow X_1\rightarrow X_0=\mathrm{spec}(k)$$

of elementary fibrations $X_n\rightarrow X_{n-1}$ into hyperbolic curves with the additionnal property that X_n embeds into a product of hyperbolic curves. Strongly hyperbolic Artin neighbourhood are anabelian in the sense that they can be reconstructed from their étale homotopy type [SSt16, Thm. 1.2]. As the level schemes are determined by X and the representation $\mathcal{V}_{p,\overline{\eta}}$ of $\pi_1(X,\overline{\eta})$, one may (at least naively) expect that they should be easier to control when X has some strong anabelian features.

product of two hyperbolic curves and one wants to control the Chern numbers / Kodaira dimension, one has to work harder and exploit specific geometric features of product-quotients surfaces.

Carrying out this rough strategy should be significantly simpler if one replaces first $X_{H\Pi(n)} \to X$ by its Galois closure $\widehat{X}_{H\Pi(n)} \to X$. A third step is then to relate the asymptotic estimates for $X_{H\Pi(n)}$ and those for $\widehat{X}_{H\Pi(n)}$. In the Galois case, a parangon of this strategy appears for instance in [M77, §4].

The basic strategy sketched above is purely algebraic and, except for the assumption that X is of log general type, might work for arbitrary GCLP $\mathbb{Q}_p$-local systems, not only for those arising from geometry. On the other hand, for a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ arising from geometry, fixing an embedding $\infty : k \hookrightarrow \mathbb{C}$, one can consider the corresponding polarizable Z-VHS $\mathcal{V}_{\infty,\mathbb{Z}}$ on $X_\mathbb{C}$ with complex period map $\Phi : X^{\mathrm{an}} \to \Gamma \setminus D$ that we may assume to be finite-to-one (Theorem 9); then $\mathcal{V}_{\infty,\mathbb{Z}} = \Phi^*\mathcal{V}^\rho_{\infty,\mathbb{Z}}$ for the polarizable $\mathbb{Z}$-VHS $\mathcal{V}^\rho_{\infty,\mathbb{Z}}$ on $\Gamma \setminus D$ defined by the tautological representation $\rho : G_{X,\mathcal{V}_\infty} \hookrightarrow \mathrm{GL}_{\mathcal{V}_{\infty,x}}$. From this viewpoint, Conjecture 17 is closely related to the hyperbolicity properties of certain covers $\Gamma_n \setminus D \to \Gamma \setminus D$. For results in this direction see [Br20a] (when D is a bounded symmetric domain), [Br20b].

5.2. **Conjecture 1.**

5.2.1. In full generality Conjecture 1 is known only when X is a curve ([CT13, Thm. 1.1]). This follows from the fact that a smooth proper curve over a number field k with $\overline{k}$-gonality γ has only finitely many points of bounded degree $d \leq \lfloor \frac{\gamma-1}{2} \rfloor$ and the enhanced form of Conjecture 17 in Subsection 5.1.1 (1).

Even assuming Conjecture 17, Conjecture 16 stands in the way to make the heuristic of Section 4 unconditional. On the other hand, as pointed out in 4.3 (2), part (1) of Conjecture 2 does not require the full strength of Conjecture 16 and should be significantly more accessible than part (2), possibly by diophantine techniques bypassing Conjecture 17. For $\mathbb{Q}_p$-local systems arising from geometry, one has at disposal a broader range of techniques due to the interplay between the various other cohomological incarnations of $\mathcal{V}_p$. In particular, one can try and control the exceptional loci $|X|_{\mathcal{V}_p} \cap X(k)$ directly *via* period maps.

For the remaining part of this section, let $f : Y \to X$ be a smooth proper morphism and set $\mathcal{V}_p = R^i_{(\mathrm{prim})} f_* \mathbb{Q}_p$, $p \in \mathrm{spec}(\mathbb{Z})$ and $\mathcal{V}_\infty = R^i_{(\mathrm{prim})} f^{\mathrm{an}}_* \mathbb{Q}$, $\infty : k \hookrightarrow \mathbb{C}$. Recall that $\mathcal{V}_p$ is then automatically GCLS and that $\mathcal{V}_\infty$ underlies a polarizable $\mathbb{Z}$-VHS $(\mathcal{V}_{\infty,\mathbb{Z}}, \mathcal{V}_{\infty,\mathbb{Z}} \otimes_\mathbb{Z} \mathcal{O}_{X^{\mathrm{an}}}, \nabla, F^\bullet)$ on X^{an}.

5.2.2. $\mathbb{C}$-*analytic period map.* As already discussed in Subsection 2.2, assuming the Mumford-Tate Conjecture (Conjecture 5), Conjecture 1 (1) immediately translates to:

Conjecture 23. *For every integer $d \geq 1$, the set $X(\mathbb{C})_{\mathcal{V}_\infty} \cap |X|^{\leq d}$ is not Zariski-dense in X.*

What makes Conjecture 23 possibly more tractable than Conjecture 1 is that the Hodge locus $X(\mathbb{C})_{\mathcal{V}_\infty}$ is controlled by the global complex analytic period map $\Phi : X^{\mathrm{an}} \to \Gamma \setminus D$, which is particularly suited for studying its geometric properties. Let us briefly review the general conjectural frame, following [Kl22] to which we refer for further details and references. Fix $o \in X^{\mathrm{an}}$, which we may assume to be $\mathcal{V}_\infty$-generic and let $u : \widetilde{X}^{\mathrm{an}} \to X^{\mathrm{an}}$ denote the corresponding universal cover of X^{an}. The polarizable $\mathbb{Z}$-VHS $\mathcal{V}_\infty$ on X^{an} gives rise to a diagram:

$$\begin{tikzcd} \widetilde{X}^{\mathrm{an}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "u"'] & D \arrow[r, hook, "\iota"] \arrow[d] & \check{D}^{\mathrm{an}} \\ X^{\mathrm{an}} \arrow[r, "\Phi"] & S := \Gamma \setminus D. & \end{tikzcd}$$

Set $f_n := \dim_\mathbb{C}(F^n_o)$, $n \in \mathbb{Z}$; write $V_{\infty,\mathbb{Z}} := \mathcal{V}_{\infty,\mathbb{Z},o}$, $V_\infty := \mathcal{V}_{\infty,o}$ and let $G_{\mathcal{V}_\infty} := G_{X_\mathbb{C},\mathcal{V}_\infty} \subset \mathrm{GL}_{V_\infty}$ denote the generic Mumford-Tate group of $\mathcal{V}_\infty$, $h_o : \mathbb{S} \to G_{\mathcal{V}_\infty,\mathbb{R}}$ the morphism of algebraic groups over $\mathbb{R}$ defining $o^*\mathcal{V}_\infty$. Let $\check{\mathbf{D}}$ be the algebraic variety over $\mathbb{Q}$ (a closed subvariety of the product of Grassmannians $\prod_n Gr(f_n, V_\infty)$) classifying the finite decreasing filtrations $F^\bullet$ on V_∞ with $\dim F^n = f_n$, $n \in \mathbb{Z}$ and such that $(F^p)^{\perp_q} = F^{n+1-p}$, where q is the polarization on $\mathcal{V}_{\infty,\mathbb{Z}}$. Let $\mathbf{D} \subset \check{\mathbf{D}}^{\mathrm{an}}$ denote the analytic open subset where the Hodge form $(x, y) \mapsto (2\pi i)^n q(x, h(i)y)$ on $V_\infty \otimes \mathbb{R}$ is positive definite. Writing $\mathbf{G} := \mathrm{GAut}(V_\infty, q)$ for the group of similitudes of (V_∞, q), $\mathbf{G}(\mathbb{C})$ acts transitively on $\check{\mathbf{D}}$, identifying $\check{\mathbf{D}}$ with the flag variety $\mathbf{G}_\mathbb{C}/\mathbf{P}_o \tilde{\to} \check{\mathbf{D}}$, where $\mathbf{P}_o \subset \mathbf{G}_\mathbb{C}$ is the parabolic subgroup fixing $F^\bullet_o$. Similarly, the neutral component $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ acts transitively on $\mathbf{D}$ and $\mathbf{M}_o := \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ \cap \mathbf{P}_o(\mathbb{C}) \subset \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ is a compact subgroup. The $\mathbb{Z}$-VHS $u^*\mathcal{V}_\infty$ is canonically trivialized as $(V_{\infty,\widetilde{X}^{\mathrm{an}}}, (V_\infty \otimes_\mathbb{C} \mathcal{O}_{\widetilde{X}^{\mathrm{an}}}, 1 \otimes d), F^\bullet, q)$ and $\widetilde{\Phi} : \widetilde{X}^{\mathrm{an}} \to \mathbf{D}$ is the map which, if $\Omega \subset \widetilde{X}^{\mathrm{an}}$

is a small enough analytic simply connected open subset sends $\tilde{x} = (\gamma, x) \in \pi_1(X^{\mathrm{an}}, o) \times \Omega = u^{-1}(\Omega)$ to $\gamma \cdot F^\bullet_x$. The map $\widetilde{\Phi} : \widetilde{X}^{\mathrm{an}} \to \mathbf{D}$ actually factors through the $G^{\mathrm{der}}_{\mathcal{V}_\infty}(\mathbb{R})^+$-orbit $D \subset \mathbf{D}$ of $F^\bullet_o$, which is an open analytic subset of the analytification of the $G_{\mathcal{V}_\infty,\mathbb{C}}$-orbit $\check{D} \subset \check{\mathbf{D}}$ of $F^\bullet_o$. Note that one can also identify D with the $G^{\mathrm{der}}_{\mathcal{V}_\infty}(\mathbb{R})^+$-orbit of $h_o : \mathbb{S} \to G_{\mathcal{V}_\infty,\mathbb{R}}$ in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathcal{V}_\infty,\mathbb{R}})$ and that, *modulo* this identification, the map $\iota : D \hookrightarrow \check{D}^{\mathrm{an}}$ is the analytic open immersion mapping $h : \mathbb{S} \to G_{\mathcal{V}_\infty,\mathbb{R}}$ to the filtration defined by the cocharacter $h_{\mathbb{C}} \circ \mu : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathcal{V}_\infty,\mathbb{C}}$, where $\mu : \mathbb{G}_{m\mathbb{C}} \to \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m\mathbb{C}} \times \mathbb{G}_{m\mathbb{C}}$, $z \mapsto (z, 1)$. By construction $\widetilde{\Phi} : \widetilde{X}^{\mathrm{an}} \to D$ is equivariant with respect to the monodromy representation $\pi_1(X^{\mathrm{an}}, o) \to \Gamma := G_{\mathcal{V}_\infty}(\mathbb{Q}) \cap \mathrm{GL}(V_{\infty,\mathbb{Z}})$ so that it factors through $\Phi : X^{\mathrm{an}} \to S := \Gamma \setminus D$. One usually refers to (G, D) as to the Hodge datum of $\mathcal{V}_\infty$ and call D (resp. $\mathbf{D}$) the Mumford-Tate domain or Hodge domain (resp. the period domain) of (G, D) and the quotient $S = \Gamma\backslash D$ of D by the arithmetic lattice $\Gamma := G_{\mathcal{V}_\infty}(\mathbb{Q}) \cap \mathrm{GL}(V_{\infty,\mathbb{Z}})$ a Hodge variety for (G, D).

More generally, a Hodge datum is a pair (G, D), where G is a connected reductive group over $\mathbb{Q}$ and $D \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})$ the $G^{\mathrm{der}}(\mathbb{R})^+$-conjugacy class of a morphism $h : \mathbb{S}_{\mathbb{R}} \to G_{\mathbb{R}}$ such that $h \circ \alpha : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$, where $\alpha : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ is the adjonction morphism; D naturally embeds as an open analytic subset into its compact dual $\check{D}$ as above. A morphism of Hodge data $(G, D) \to (G', D')$ is a morphism of algebraic groups $G \to G'$ mapping D to D'. If (G, D) is a Hodge datum, a Hodge variety for (G, D) is the quotient $S = \Gamma \setminus D$ by an arithmetic lattice $\Gamma \subset G(\mathbb{Q}) \cap G^{\mathrm{der}}(\mathbb{R})^+$ so that S is naturally endowed with the structure of a complex analytic variety which is smooth if Γ is torsion free. A morphism of Hodge data $(G, D) \to (G', D')$ mapping an arithmetic lattice $\Gamma \subset G(\mathbb{Q}) \cap G^{\mathrm{der}}(\mathbb{R})^+$ to $\Gamma' \subset G'(\mathbb{Q}) \cap G'^{\,\mathrm{der}}(\mathbb{R})^+$ induces an analytic morphism of Hodge varieties $\Gamma \setminus D \to \Gamma' \setminus D'$.

For a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$, let $\overline{G}^{\circ}_{Z,\mathcal{V}} \subset G_{Z,\mathcal{V}_\infty}$ denote the connected monodromy group and generic Mumford-Tate group of $\mathcal{V}_\infty|_Z$ respectively; recall that $\overline{G}^{\circ}_{Z,\mathcal{V}}$ is normal in $G_{Z,\mathcal{V}_\infty}$.

To these data one associates the following classes of subvarieties of $X_{\mathbb{C}}$:

- *Special subvarieties*: A special subvariety of S is the image $S' \subset S$ of the map $D' \to D \to S = \Gamma \setminus D$ induced by a morphism of Hodge data $(G', D') \to (G_{\mathcal{V}_\infty}, D)$. The inverse image $\Phi^{-1}(S') \subset X^{\mathrm{an}}$ of a special subvariety $S' \subset S$ is a finite union of closed irreducible subvarieties of $X_{\mathbb{C}}$ [CaDK95], [BKT20]. A closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be special if it is an irreducible component of the preimage $\Phi^{-1}(S_Z) \subset X^{\mathrm{an}}$ of a special subvariety $S_Z \subset S$. Equivalently, a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is special if it maximal among the closed integral subvarieties of $X_{\mathbb{C}}$ with generic Mumford-Tate group $G_{Z,\mathcal{V}_\infty}$ so that one recovers the definition of Subsection 2.2.

  Define the Hodge codimension of a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ as $Hcd(Z) := \dim(S_Z) - \dim(\Phi(Z^{\mathrm{an}}))$, where $S_Z \hookrightarrow S$ is the Hodge subvariety defined by $G_{\mathcal{V}_\infty,Z}$. A special subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be *atypical* if either $\Phi(Z^{\mathrm{an}})$ is contained in the singular locus of $\Phi(X^{\mathrm{an}})$ or if $Hcd(Z) < Hcd(X)$. It is said to be *typical* otherwise.

  CM points (that is those $x \in X(\mathbb{C})$ such that $G_{\mathcal{V}_\infty,x}$ is a torus) are always atypical unless $(G_{\mathcal{V}_\infty}, D)$ is of Shimura type and $\Phi : X^{\mathrm{an}} \to S$ is dominant.

- *Weakly special subvarieties*: A weakly special subvariety of S is either a special subvariety or a subvariety of the form $v_1 v_2^{-1}(s_2)$ for morphisms of Hodge varieties $S \stackrel{v_1}{\leftarrow} S_1 \stackrel{v_2}{\to} S_2$ and $s_2 \in S_2$. The inverse image $\Phi^{-1}(S') \subset X^{\mathrm{an}}$ of a weakly special subvariety $S' \subset S$ is a finite union of closed irreducible subvarieties of $X_{\mathbb{C}}$ [KlO21]. A closed integral subvariety $Z \hookrightarrow X$ is said to be weakly special if it is an irreducible component of the preimage $\Phi^{-1}(S_Z) \subset X^{\mathrm{an}}$ of a weakly special subvariety $S_Z \subset S$. Equivalently, a closed integral subvariety $Z \hookrightarrow X$ is weakly special if it is maximal among the closed integral subvarieties of $X_{\mathbb{C}}$ with monodromy group $\overline{G}^{\circ}_{Z,\mathcal{V}}$.

- *Bi-algebraic subvarieties*: For a closed irreducible analytic subvariety $Z \hookrightarrow \widetilde{X}^{\mathrm{an}}$, write $Z^{\mathrm{zar}} \subset \check{D}_{\mathbb{C}}$ for its algebraic model that is the Zariski closure of $\widetilde{\Phi}(Z)$ in $\check{D}_{\mathbb{C}}$. A closed irreducible analytic subvariety $Z \hookrightarrow \widetilde{X}^{\mathrm{an}}$ is said to be algebraic if it is an analytic irreducible component of $\widetilde{\Phi}^{-1}(Z^{\mathrm{zar}})$. A closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be bi-algebraic if one (equivalently every) irreducible component of $u^{-1}(Z^{\mathrm{an}})$ is an algebraic subvariety of $\widetilde{X}^{\mathrm{an}}$.

  The bi-algebraic subvarieties of $X_{\mathbb{C}}$ are exactly the weakly special subvarieties [KlO21].

- $\overline{\mathbb{Q}}$-*bi-algebraic varieties*: An algebraic subvariety $Z \hookrightarrow \widetilde{X}^{\mathrm{an}}$ is said to be $\overline{\mathbb{Q}}$-algebraic if its algebraic enveloppe $Z^{\mathrm{zar}} \hookrightarrow \check{D}_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$. A bi-algebraic subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be $\overline{\mathbb{Q}}$-bi-algebraic if it is defined over $\overline{\mathbb{Q}}$ and one (equivalently every) irreducible component of $u^{-1}(Z^{\mathrm{an}})$ is a $\overline{\mathbb{Q}}$-algebraic subvariety of $\widetilde{X}^{\mathrm{an}}$.

A closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be of positive period dimension if $\Phi(Z^{\mathrm{an}}) \subset S = \Gamma \setminus D$ has dimension > 0. If one decomposes the period domain $\Gamma \setminus D = \Gamma_1 \setminus D_1 \times \cdots \times \Gamma_r \setminus D_r$ according to the decomposition $G^{\mathrm{ad}}_{\mathcal{V}_\infty} = G_1 \times \cdots \times G_r$ of $G^{\mathrm{ad}}_{\mathcal{V}_\infty}$ into simple factors then a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be factorwise of positive period dimension if the projection of $\Phi(Z^{\mathrm{an}})$ onto each factor $\Gamma_i \setminus D_i$, $i = 1, \ldots, r$ has dimension > 0.

Remark: From [BBrT18, Thm. 1.1 and par after Thm. 1.1], one may reduce to the case where the period map $\Phi : X^{\mathrm{an}} \to S$ is finite-to-one so that, in particular, a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is of positive period dimension if and only if it is of positive dimension.

Let $\Sigma^{\mathrm{typ}}_{\mathcal{V}_\infty}$, $\Sigma^{\mathrm{atyp}}_{\mathcal{V}_\infty}$ and $\Sigma_{\mathcal{V}_\infty} = \Sigma^{\mathrm{typ}}_{\mathcal{V}_\infty} \sqcup \Sigma^{\mathrm{atyp}}_{\mathcal{V}_\infty}$ denote the set of special atypical, strict special typical and strict special subvarieties of X for $\mathcal{V}_\infty$ respectively. Let also $\Sigma^{\mathrm{f}>0}_{\mathcal{V}_\infty} \subset \Sigma^{>0}_{\mathcal{V}_\infty} \subset \Sigma_{\mathcal{V}_\infty}$ denote the subset of strict special subvarieties of factorwise positive period dimension and positive period dimension respectively. Set $\Sigma^{\mathrm{atyp},>0}_{\mathcal{V}_\infty} := \Sigma^{\mathrm{typ}}_{\mathcal{V}_\infty} \cap \Sigma^{>0}_{\mathcal{V}_\infty}$, $\Sigma^{\mathrm{atyp},\,\mathrm{f}>0}_{\mathcal{V}_\infty} :=$... and consider the following subsets of the Hodge locus $X(\mathbb{C})_{\mathcal{V}_\infty}$

$X^{>0}_{\mathcal{V}_\infty}$	$:= \bigcup_{S \in \Sigma^{>0}_{\mathcal{V}_\infty}} S$	(Hodge locus of positive period dimension);
$X^{\mathrm{f}>0}_{\mathcal{V}_\infty}$	$:= \bigcup_{S \in \Sigma^{\mathrm{f}>0}_{\mathcal{V}_\infty}} S$	(Hodge locus of factorwise positive period dimension);
$X^{\mathrm{atyp}}_{\mathcal{V}_\infty}$	$:= \bigcup_{S \in \Sigma^{\mathrm{atyp}}_{\mathcal{V}_\infty}} S$	(atypical Hodge locus);
$X^{\mathrm{typ}}_{\mathcal{V}_\infty}$	$:= X_{\mathcal{V}_\infty} \setminus X^{\mathrm{atyp}}_{\mathcal{V}_\infty}$	(typical Hodge locus);
$X^{\mathrm{atyp},>0}_{\mathcal{V}_\infty}$	$:= X^{\mathrm{atyp}}_{\mathcal{V}_\infty} \cap X^{>0}_{\mathcal{V}_\infty}$	(atypical Hodge locus of positive period dimension).
$X^{\mathrm{atyp},\mathrm{f}>0}_{\mathcal{V}_\infty}$	$:= X^{\mathrm{atyp}}_{\mathcal{V}_\infty} \cap X^{\mathrm{f}>0}_{\mathcal{V}_\infty}$	(atypical Hodge locus of factorwise positive period dimension).

Note that, when $G^{\mathrm{ad}}_{\mathcal{V}_\infty}$ is simple, $X^{\mathrm{f}>0}_{\mathcal{V}_\infty} = X^{>0}_{\mathcal{V}_\infty}$.

Eventually, let $\Sigma^{\overline{\mathbb{Q}}\text{-bizar}}_{\mathcal{V}_\infty} \subset \Sigma^{\mathrm{bizar}}_{\mathcal{V}_\infty}$ denote the set of strict $\overline{\mathbb{Q}}$-bialgebraic and strict bi-algebraic (equivalently, weakly special) subvarieties of X for $\mathcal{V}_\infty$ respectively.

Conjecture 24.

(1) (Zilber-Pink, [Kl22, Conj. 5.2]) *The subset $X^{\mathrm{atyp}}_{\mathcal{V}_\infty}$ is a strict closed algebraic subvariety of X.*

(2) ([Kl22, Conj. 5.6]) *The subset $X^{\mathrm{typ}}_{\mathcal{V}_\infty}$ is either empty or analytically dense in X^{an}.*

When $\mathcal{V}_\infty$ has level ≥ 3 (see [BaKU21, §4.6] for the definition) and $\overline{G}^{\circ}_{Z,\mathcal{V}} = G^{\mathrm{der}}_{\mathcal{V}_\infty}$, $X^{\mathrm{typ}}_{\mathcal{V}_\infty} = \emptyset$ [BaKU21, Thm. 3.3] so that, in that case, Conjecture 23 follows from (and is *a priori* strictly weaker than) Conjecture 24 (1). When $\mathcal{V}_\infty$ has level ≥ 3, Conjecture 24 (1) holds for $\Sigma^{\mathrm{atyp},\,\mathrm{f}>0}_{\mathcal{V}_\infty}$ - hence, when $G^{\mathrm{ad}}_{\mathcal{V}_\infty}$ is simple, for $\Sigma^{\mathrm{atyp},\,>0}_{\mathcal{V}_\infty}$ [BaKU21, Thm. 1.5]. In constrast to level ≥ 3, typical subvarieties are in general abundant in level ≤ 2; in particular, for Shimura varieties, $X_{\mathcal{V}_\infty} = X^{\mathrm{typ}}_{\mathcal{V}_\infty}$. In the other direction, supporting Conjecture 24 (2), if $X^{\mathrm{typ}}_{\mathcal{V}_\infty} \neq \emptyset$ then $X_{\mathcal{V}_\infty}$ is analytically dense in X^{an} [BaKU21, Thm. 3.9].

Conjecture 24 (1) and the results of [BaKU21] suggest to refine Conjecture 23 as follows:

Conjecture 25.

(1) Atypical special locus:
   - (1-1) (Positive period dimension) *The set $X^{\mathrm{atyp},>0}_{\mathcal{V}_\infty}$ is not Zariski-dense in X;*
   - (1-2) (Zero period dimension) *The set of maximal atypical special subvarieties of zero period dimension for $\mathcal{V}_\infty$ is finite;*

(2) Typical special locus: *For every integer $d \geq 1$, the set $X^{\mathrm{typ}}_{\mathcal{V}_\infty} \cap |X|^{\leq d}$ is not Zariski-dense in X.*

For Shimura varieties (level 1), that $X_{\mathcal{V}_\infty}$ is analytically dense in X^{an} follows from the classical (and *a priori* stronger) fact that the set of special=CM points is analytically dense in X^{an}. Still, in this setting, the finiteness (hence the non-Zariski density) of the set of special points of bounded degree $\leq d$ holds as a consequence of (part of) the Pila-Zannier strategy for André-Oort. Let us briefly recall the argument for a

Shimura variety of abelian type. Fix a Weil height $H : \check{D}(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$. Then one can show that there exists an integer $c \geq 1$ and constants $c_i, \alpha_i > 0$, $i = 1, 2$ (depending only on (G, D) and of the choice of a fundamental (Siegel) subset $\Sigma \subset D$ for the action of Γ) such that every CM point $x \in X$ lifts to a $\overline{\mathbb{Q}}$-point $\widetilde{x} \in \check{D}(\overline{\mathbb{Q}})$ with

$$\begin{array}{ll} (1) & [k(\widetilde{x}) : \mathbb{Q}] \leq c; \\ (2) & H(\widetilde{x}) \leq c_1 \mathrm{disc}(x)^{\alpha_1}; \\ (3) & c_2 \mathrm{disc}(x)^{\alpha_2} \leq [\mathbb{Q}(x) : \mathbb{Q}]. \end{array}$$

(where $\mathrm{disc}(x)$ the discriminant of the center of the endomorphism ring of the abelian variety corresponding to x). So, ultimately, the finiteness of the set of CM points of bounded degree $\leq d$ on a Shimura varieties of abelian type follows from the Northcott property for H. We refer to [KlUY18] and the references therein for a more detailed survey. Let us only stress that (1) and (2) are essentially (highly technical for (2)) "linear algebra" but that, for the time being, the only known proof of (3) is based on deep arithmetic inputs - in particular the so-called average Colmez conjecture [AnGHoM18], [YuZh18]. For a general Shimura variety, the argument is along the same guidelines using [PShTs21]. See also [OS18] for a proof of the finiteness of $\overline{\mathbb{Q}}$-isomorphisms CM abelian varieties of bounded dimension defined over a number field of bounded degree avoiding the machinery of Shimura varieties, and [V23] for a proof for K3 surfaces avoiding the average Colmez conjecture.

Remark. One could hope for a similar strategy to attack Conjecture 25 for special points. The first step of such strategy would be to show that special points are $\overline{\mathbb{Q}}$-bi-algebraic. However, one cannot expect this to hold for an arbitrary motive as the following example shows[4]. Let $(G = \mathrm{GSp}_g, D)$ denote the Siegel Hodge datum and $X_{\mathbb{C}} = \Gamma \setminus D$ any high enough level of the corresponding Shimura variety (parametrizing g-dimensional principally polarized abelian varieties with some level structure). By Bertini, one can construct a closed smooth integral $\overline{\mathbb{Q}}$-curve $C \hookrightarrow X_{\overline{\mathbb{Q}}}$ with $\overline{G}^{\circ}_{C,\mathcal{V}} = \overline{G}^{\circ}_{X,\mathcal{V}} (= \mathrm{Sp}_g)$ - hence, in particular $G_{\mathcal{V}_\infty, C} = G_{\mathcal{V}_\infty} (= \mathrm{GSp}_g)$. Assume C intersects strictly a strict special subvariety $S \hookrightarrow X$ of *positive* dimension. Then every point z in the intersection becomes special for $\mathcal{V}_\infty|_{C^{\mathrm{an}}}$ but is not CM. On the other hand, the natural $\overline{\mathbb{Q}}$-bi-algebraic structure attached to $\mathcal{V}_\infty|_{C^{\mathrm{an}}}$ is simply the one obtained from the $\overline{\mathbb{Q}}$-bi-algebraic structure attached to $\mathcal{V}_\infty$ on X as

$$\begin{tikzcd} \widetilde{C}^{\mathrm{an}} \arrow[r] \arrow[d] & \widetilde{X}^{\mathrm{an}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "u"'] & D \arrow[r, hook, "\iota"] \arrow[d] & \check{D}^{\mathrm{an}} \\ C^{\mathrm{an}} \arrow[r] & X^{\mathrm{an}} \arrow[r, "\Phi"] & S := \Gamma \setminus D & \end{tikzcd}$$

so that a point of C is $\overline{\mathbb{Q}}$-bi-algebraic for the natural $\overline{\mathbb{Q}}$-bi-algebraic structure attached to $\mathcal{V}_\infty|_{C^{\mathrm{an}}}$ if and only if it is $\overline{\mathbb{Q}}$-bi-algebraic for the natural weak $\overline{\mathbb{Q}}$-bi-algebraic structure attached to $\mathcal{V}_\infty$; for the latter, it is known that $\overline{\mathbb{Q}}$-bi-algebraic points are CM. To exclude this kind of pathology, one can restrict the definition of a $\overline{\mathbb{Q}}$-bi-algebraic structure by requiring that X itself be $\overline{\mathbb{Q}}$-bi-algebraic, that is for every analytic irreducible component U of $u^{-1}(X^{\mathrm{an}})$, $U^{\mathrm{zar}} \subset \check{D}_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$. In the example above, this would force C to be $\overline{\mathbb{Q}}$-bi-algebraic for the natural $\overline{\mathbb{Q}}$-bi-algebraic structure attached to $\mathcal{V}_\infty$ on X which, for Shimura varieties, is equivalent to being special. Say that such a $\overline{\mathbb{Q}}$-bi-algebraic structure is arithmetic.

For Conjecture 25 (2), very little seems to be known even for $d = 1$. Note that, for Shimura varieties, the $d = 1$ case of Conjecture 25 (2) (or, equivalently Conjecture 23 as in level 1 $X(\mathbb{C})_{\mathcal{V}_\infty} = X^{\mathrm{typ}}_{\mathcal{V}_\infty}$) appears as [U04, Conj. 4.3]. The example of the Shimura variety $X = Y(1) \times Y(1)$ over $k = \mathbb{Q}$ and the special curves $Y_0(n) \subset X$ shows that X can contain an infinite set of (positive dimensional) special typical subvarieties defined over k. The special typical curves $Y_0(n) \subset X$ are strongly special in X in the terminology of [UY14]. Actually, the results of [UY14] about lower bound for the degree of Galois orbit of special subvarieties of Shimura varieties which are not strongly special may pave the way to a partial proof of the $d = 1$ case of Conjecture 23 for Shimura varieties. More precisely, say that $x \in X(\mathbb{C})_{\mathcal{V}_\infty}$ is strongly special for $\mathcal{V}_\infty$ if the image of $G_{\mathcal{V}_\infty, x} \to G^{\mathrm{ad}}_{\mathcal{V}_\infty}$ is semisimple and let

$$X^{\mathrm{str}}_{\mathcal{V}_\infty} \subset X(\mathbb{C})_{\mathcal{V}_\infty}$$

denote the subset of strongly special points.

[4] In particular, it is unclear if one could use the Pila-Zannier strategy to recover the special case of [CT13, Thm. 1.1] when the $\mathbb{Q}_p$-local system considered arises from the generic Tate module of an abelian scheme.

Conjecture 26. *If (G, D) is a Hodge datum of Shimura type and $X := \Gamma \setminus D$ is a (connected) level of the corresponding Shimura variety defined over k then*

$$(X(\mathbb{C})_{\mathcal{V}_\infty} \setminus X^{\mathrm{str}}_{\mathcal{V}_\infty}) \cap X(k)$$

is not Zariski-dense in X.

Coming back to our initial $\mathbb{Q}_p$-local system arising from geometry, using that for a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$, $\overline{G}^{\circ}_{Z,\mathcal{V}}$ (resp. $\overline{G}^{\circ}_{Z,\mathcal{V}_p} = \overline{G}^{\circ}_{Z,\mathcal{V}} \times_{\mathbb{Q}} \mathbb{Q}_p$) is normal in $G_{Z,\mathcal{V}_\infty}$ (resp. in $G_{Z,\mathcal{V}_p}$) and that $G_{Z,\mathcal{V}_\infty}$ (resp. $G_{Z,\mathcal{V}_p}$) is generated by $\overline{G}^{\circ}_{Z,\mathcal{V}}$ and $G_{z,\mathcal{V}_\infty}$ (resp. $\overline{G}^{\circ}_{Z,\mathcal{V}_p}$ and $G_{z,\mathcal{V}_p}$), one gets [Kr21a], [Kr21b]

(1) For $\flat = \infty, \mathrm{AH}, p$, every $\mathcal{V}_\flat$-special subvariety is weakly special;
(2) Weakly non-factor special subvarieties for $\mathcal{V}_\infty$ and for $\mathcal{V}_p$ all coincide. Here a closed integral subvariety $Z \hookrightarrow X_{\mathbb{C}}$ is said to be weakly non-factor for $\mathcal{V}_\infty$ if it is not contained in a strict closed integral subvariety $Z' \hookrightarrow X$ with $\overline{G}^{\circ}_{Z,\mathcal{V}}$ a strict normal subgroup of $\overline{G}^{\circ}_{Z',\mathcal{V}}$ (in particular, $\overline{G}^{\circ}_{Z,\mathcal{V}}$ has to be non-trivial).

As a special case of (2), if $\overline{G}^{\mathrm{ad}\,\circ}_{\mathcal{V}}$ is simple, the strict maximal special subvarieties of positive period dimension for $\mathcal{V}_\infty$ and for $\mathcal{V}_p$ all coincide; in particular

$$X^{>0}_{\mathcal{V}_\infty} = X^{>0}_{\mathcal{V}_p}.$$

For instance, if $\mathcal{V}_\infty$ has level ≥ 3, this reduces Conjecture 1 (1) to showing that the set $(|X|_{\mathcal{V}_p} \cap |X|^{\leq d}) \setminus |X|^{>0}_{\mathcal{V}_p}$ of isolated (that is which are not contained in a strict special subvariety of positive period dimension) exceptional points of degree $\leq d$ is not Zariski-dense in X.

If Hodge-theoretic methods prove to be powerful to understand the geometry of positive-dimensional special subvarieties, it seems that, at least currently, they fail to control isolated special points beyond the case of motives of Shimura type. To go further, one may try to take into account not only the singular incarnations of our motive but also its crystalline incarnations. This naturally leads to consider p-adic period maps which since the main conceptual breakthrough of [K05], have been handled successfully to prove the non-Zariski density of integral points on certain moduli spaces, yielding in particular a new proof of the Mordell Conjecture [LaVe20] but also higher-dimensional results [LaVe20, Thm. 10.1], [LaS20]. Compared with Kim's approach, the main additional input of the Lawrence-Venkatesh method is that it does not only consider finite v-adic places of k but also infinite places and compare the resulting v-adic and complex period maps, which enables to exploit both arithmetic information and topological / tame geometric informations.

5.2.3. *v-adic period map.*

5.2.3.1. *The Lawrence-Venkatesh strategy*. We describe below the strategy of Lawrence-Venkatesh [LaVe20], [LaS20] as enhanced by Betts-Stix [BeS22]. More precisely, the basic construction of the v-adic period map of [LaVe20], [LaS20] relies on the crystalline - de Rham comparison theorem, which requires that the smooth proper morphism $f : Y \to X$ defining our motive has good reduction at v; using Scholze's relative p-adic Hodge theory [Sc13] and a potential horizontal semistability theorem of Shimizu [Shi20], Betts and Stix could extend the construction of the v-adic period map to places v of k where $f : Y \to X$ does not necessarily have good reduction, giving slightly more flexibility in the choice of the non-archimedean place v. See Remark 5.2.3.2 (1).

Fix a finite place $v|p$ of k and write k_v for the completion of k at v, $\mathbb{Q}_p \subset k_{v,0} := k_v \cap \mathbb{Q}_p^{\mathrm{ur}} \subset k_v$ for the maximal unramified extension of $\mathbb{Q}_p$ in k_v. Let $\sigma : \mathbb{Q}_p^{\mathrm{ur}} \tilde{\to} \mathbb{Q}_p^{\mathrm{ur}}$ denote the Frobenius automorphism.

Let $W \subset X(k_v)$ be a subset - *e.g.* $W = \mathcal{X}(U)$ for some non-empty open subset $U \subset \mathrm{spec}(\mathcal{O}_k)$, $W = |X|_{\mathcal{V}_p} \cap X(k)$, $|X|_{\mathcal{V}_p} \cap |X|^{\leq d}$ (recall that k_v has only finitely many field extensions of degree $\leq d$ so that, replacing k_v by a finite field extension K_w, one can embed $|X|^{\leq d}$ into K_w) *etc.*. The aim is to show that $W \subset X$ is not Zariski-dense. The naive starting point is to cover $X^{v\text{-an}}$ by "good" admissible open subsets U_v and, for each such U_v to show that $U_v \cap W$ is not Zariski-dense in X. Here the crucial point is the definition of a "good" admissible open subset. For this, let $\mathcal{H}^i_{\mathrm{dR}}(Y/X)$ denote the relative de Rham cohomology of $f : Y \to X$; this is a coherent locally free $\mathcal{O}_X$-module endowed with

- a canonical decreasing, separated, exhaustive filtration - the Hodge filtration $F^\bullet$, whose graded pieces are again locally free;
- a flat connection $\nabla : \mathcal{H}^i_{\mathrm{dR}}(Y/X) \to \mathcal{H}^i_{\mathrm{dR}}(Y/X) \otimes \Omega^1_{X|k}$ - the Gauss-Manin connection,

and such that for every $x\in X(k_v)$, one has a canonical isomorphism $c_x:\mathcal{H}^i_{\mathrm{dR}}(Y/X)^{v\text{-an}}_x\tilde{\to}H^i_{\mathrm{dR}}(Y_x/k_v)$. For every $x_0\in X(k_v)$ one can always find an admissible open neighbourhood $U_{v,x_0}\subset X^{v\text{-an}}$ of x_0, isomorphic to a closed polydisc and such that $(\mathcal{H}^i_{\mathrm{dR}}(Y/X)^{v\text{-an}},\nabla)|_{U_{v,x_0}}\tilde{\to}(\mathcal{O}_{U_{v,x_0}},d)^{\oplus r}$. This yields a canonical isomorphism

$$T^{\nabla}_{x_0}:(\mathcal{O}_{U_{v,x_0}}\otimes_{k_v}H^i_{\mathrm{dR}}(Y_{x_0}/k_v),d\otimes 1)\tilde{\to}(\mathcal{H}^i_{\mathrm{dR}}(Y/X)^{v\text{-an}},\nabla)|_{U_{v,x_0}}$$

characterized by the fact that $(T^{\nabla}_{x_0})_{x_0}=Id$. In particular, for every $x\in U_{v,x_0}$ one obtains a canonical isomorphism - the parallel transport

$$T^{\nabla}_{x_0,x}:=(T^{\nabla}_{x_0})_x:H^i_{\mathrm{dR}}(Y_{x_0}/k_v)\tilde{\to}H^i_{\mathrm{dR}}(Y_x/k_v).$$

So that, if $\check{\mathbf{D}}$ denotes again the flag variety parametrizing filtrations on $H^i_{\mathrm{dR}}(Y_{x_0}/k)$ with the same dimension data as the Hodge filtration $F^\bullet H^i_{\mathrm{dR}}(Y_{x_0}/k)$, one gets a (local) v-adic analytic period map

$$\Phi^v_{x_0}:U_{v,x_0}\to\check{\mathbf{D}}^{v\text{-an}},\ \ x\mapsto {T^{\nabla}_{x_0,x}}^{-1}F^\bullet H^i_{\mathrm{dR}}(Y_x/k_v).$$

Inside $\check{\mathbf{D}}\times_k k_v$ one has the Zariski-closures $\mathcal{W}_{v,x_0}\subset\mathcal{U}_{v,x_0}\subset\check{\mathbf{D}}$ of

$$\Phi^v_{x_0}(W\cap U_{v,x_0})\subset\Phi^v_{x_0}(U_{v,x_0})\subset\check{\mathbf{D}}(k_v)$$

respectively. The problem then amounts to finding conditions on a closed subvariety $\mathcal{Z}\subset\mathcal{U}_{v,x_0}$ ensuring that ${\Phi^v_{x_0}}^{-1}(\mathcal{Z})\subset U_{v,x_0}\subset X(k_v)$ is not Zariski-dense in X and satisfied by $\mathcal{W}_{v,x_0}$. For instance, if X is a curve, it is enough to show that every irreducible component of $\mathcal{W}_{v,x_0}$ is a strict closed subvariety of an irreducible component of $\mathcal{U}_{v,x_0}$. To make this rough strategy work, the idea is to combine the informations encoded in the p-adic étale and singular incarnations of $\mathcal{H}^i_{\mathrm{dR}}(Y/X)$. More precisely,

(1) Input from p-adic étale incarnation and p-adic Hodge theory: Set $G_v:=\pi_1(k_v)$. Let FM_{k_v} denote the category of filtered k_v-modules *i.e.* the category with
- Objects: $M_{\mathrm{dR}}:=(M,F^\bullet)$, where M is a finite-rank k_v-module endowed with a decreasing filtration $F^\bullet=\cdots\supset F^n\supset F^{n+1}\supset\cdots$ by k_v-submodules on M which is separated ($\cap_nF^n=0$) and exhaustive ($\cup_nF^n=M$);
- Morphisms: morphisms $f:M_1\to M_2$ of k_v-modules such that $fF^nM_1\subset F^nM_2$, $n\in\mathbb{Z}$.

For a field extension $\mathbb{Q}_p\subset k_0\subset\mathbb{Q}_p^{\mathrm{ur}}$ let $\mathrm{M}_{k_0}(\phi,N,G_v)$ denote the category of (ϕ,N,G_v)-modules over k_0 *i.e.* the category with
- Objects: $M_{\mathrm{pst}}:=(M_0,\phi,N)$, where M_0 is a finite-rank k_0-module endowed with
  – a σ-semilinear (that is satisfying: $\phi(\alpha m)=\sigma(\alpha)\phi(m)$, $\alpha\in k_0$, $m\in M_0$) G_v-equivariant automorphism $\phi:M_0\to M_0$;
  – a discrete (*i.e.* with open stabilizers) action of G_v by σ-semilinear automorphims;
  – a k_0-linear nilpotent G_v-equivariant endomorphism $N:M_0\to M_0$ satisfying $N\phi=p\phi N$;
- Morphisms: morphisms $f_0:M_{1,0}\to M_{2,0}$ of k_0-modules commuting with ϕ, N and the G_v-action,

and write $\mathrm{M}_{k_0}(\phi)\subset\mathrm{M}_{k_0}(\phi,N)\subset\mathrm{M}_{k_0}(\phi,N,G_v)$ for the full subcategories whose objects are those (Φ,N,G_v)-modules with trivial G_v-action ((ϕ,N)-modules over k_0) and those with trivial G_v-action and $N=0$ (ϕ-modules over k_0). Eventually, let $\mathrm{FM}_{k_v/k_0}(\phi,N,G_v)$ denote the category of filtered (ϕ,N,G_v)-modules over k_v/k_0 *i.e.* the category with
- Objects: $M:=(M_{\mathrm{pst}},M_{\mathrm{dR}},c)$, where
  – $M_{\mathrm{pst}}=(M_0,\phi,N)$ is a (ϕ,N,G_v) module over k_0;
  – $M_{\mathrm{dR}}=(M,F^\bullet)$ is a filtered k_v-module;
  – $c:M_0\otimes_{k_0}(k_0k_v)\tilde{\to}M\otimes_{k_v}(k_0k_v)$ is a k_0k_v-linear isomorphism, where k_0k_v denotes the compositum of k_0 and k_v in $\overline{\mathbb{Q}}_p$.
- Morphisms: pairs $(f_{\mathrm{pst}}:M_{1,\mathrm{pst}}\to M_{2,\mathrm{pst}},f_{\mathrm{dR}}:M_{1,\mathrm{dR}}\to M_{2,\mathrm{dR}})$ with $f_{\mathrm{pst}}:M_{1,\mathrm{pst}}\to M_{2,\mathrm{pst}}$ a morphism of (ϕ,N,G_v) module over k_0, $f_{\mathrm{dR}}:M_{1,\mathrm{dR}}\to M_{2,\mathrm{dR}}$ a morphism of filtered k_v-modules and the following diagram commutes

$$\begin{tikzcd}
M_{1,0}\otimes_{k_0}(k_0k_v) \arrow[r, "f_{\mathrm{pst}}\otimes Id"] \arrow[d, "c"', "\simeq"] & M_{2,0}\otimes_{k_0}(k_0k_v) \arrow[d, "\simeq"', "c"] \\
M_{1,\mathrm{dR}}\otimes_{k_v}(k_0k_v) \arrow[r, "f_{\mathrm{dR}}\otimes Id"'] & M_{2,\mathrm{dR}}\otimes_{k_v}(k_0k_v)
\end{tikzcd}$$

As above, write $\mathrm{FM}_{k_v/k_0}(\phi)\subset\mathrm{FM}_{k_v/k_0}(\phi,N)\subset\mathrm{M}_{k_v/k_0}(\phi,N,G_v)$ for the full subcategories whose objects are those filtered (Φ,N,G_v)-modules with M_{pst} an object in $\mathrm{M}_{k_0}(\phi)$ (filtered ϕ-modules over k_v/k_0) and

$\mathrm{M}_{k_0}(\phi, N)$ (filtered (ϕ, N)-modules over k_v/k_0) respectively.

The relevance of introducing the category $\mathrm{FM}_{k_v/k_0}(\phi, N, G_v)$ is that it is the category of linear objects that best approximates the category of de Rham representations. More precisely, consider Fontaine's period rings $B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}} := B_{\mathrm{dR}}(k_v)$ and recall that

- B_{dR} is a complete discrete valued field (in particular, it is endowed with the $\mathfrak{m}_{B_{\mathrm{dR}}}$-adic filtration $F^\bullet B_{\mathrm{dR}}$) with residue field $\mathbb{C}_p$, containing $\overline{\mathbb{Q}}_p\widehat{k_v^{\mathrm{ur}}}$ (but not $\mathbb{C}_p$), equipped with a canonical action of G_v which restricts to the natural one on $\overline{\mathbb{Q}}_p\widehat{k_v^{\mathrm{ur}}}$ and it has a distinguished uniformizer $t \in \mathfrak{m}_{\mathrm{dR}}$ over which G_v acts through the cyclotomic character. Futhermore $B_{dR}^{G_v} = k_v$.
- $B_{\mathrm{cris}} \subset B_{\mathrm{dR}}$ is a $\widehat{k_{v,0}^{\mathrm{ur}}}$-subalgebra stabilized by G_v with $B_{\mathrm{cris}}^{G_v} = k_{v,0}$, containing t and such that the induced canonical k_v-linear morphism $k_v \otimes_{k_{v,0}} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ is an embedding; it is equipped with a σ-semilinear Frobenius ring-automorphism $\phi : B_{\mathrm{cris}} \tilde{\to} B_{\mathrm{cris}}$ commuting with the G_v action and such that $(B_{\mathrm{cris}} \cap F^0 B_{dR})^\phi = \mathbb{Q}_p$.
- $B_{\mathrm{cris}} \subset B_{\mathrm{st}} = B_{\mathrm{cris}}[u] \subset B_{\mathrm{dR}}$ is a B_{cris}-subalgebra (generated on B_{cris} by a distinguished element u) stabilized by G_v with $B_{\mathrm{st}}^{G_v} = k_{v,0}$ and such that the induced canonical k_v-linear morphism $k_v \otimes_{k_{v,0}} B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ is an embedding; the Frobenius $\phi : B_{\mathrm{cris}} \tilde{\to} B_{\mathrm{cris}}$ extends to a G_v-equivariant automorphism $\phi : B_{\mathrm{st}} \tilde{\to} B_{\mathrm{st}}$ and B_{st} is endowed with a B_{cris}-linear G_v-equivariant nilpotent endomorphism $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$, which is the unique B_{cris}-derivation of B_{st} satisfying $Nu = -1$ (in particular $\ker(N) = B_{\mathrm{cris}}$) and satisfies $N\phi = p\phi N$.

Associated to these period rings, one has Fontaine's $\otimes$-functors

$$D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_v) \to \mathrm{FM}_{k_v},\ V \mapsto (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_v}$$

$$D_{\mathrm{st}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_v) \to \mathrm{FM}_{k_v/k_{v,0}}(\phi, N),\ V \mapsto (D_{\mathrm{st}}(V)_0 := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_v}, D_{\mathrm{dR}}(V), c_V)$$

$$D_{\mathrm{pst}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_v) \to \mathrm{FM}_{k_v/\mathbb{Q}_p^{ur}}(\phi, N, G_v),\ V \mapsto (D_{\mathrm{pst}}(V)_0 := \mathrm{colim}_{G \subset_{op} G_v} D_{\mathrm{st}}(V)^G, D_{\mathrm{dR}}(V), c_V),$$

where the comparison isomorphisms c_V are the ones induced by the inclusions $B_{\mathrm{st}} \subset B_{\mathrm{dR}}$.

For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_v)$, one always has $\dim_{\mathbb{Q}_p^{\mathrm{ur}}}(D_{\mathrm{pst}}(V)_0) \leq \dim_{\mathbb{Q}_p}(V)$ (resp. $\dim_{k_v}(D_{\mathrm{dR}}(V)) \leq \dim_{\mathbb{Q}_p}(V)$) and one says that V is potentially semistable (resp. de Rham) if $\dim_{\mathbb{Q}_p^{\mathrm{ur}}}(D_{\mathrm{pst}}(V)_0) = \dim_{\mathbb{Q}_p}(V)$ (resp. $\dim_{k_v}(D_{\mathrm{dR}}(V)) = \dim_{\mathbb{Q}_p}(V)$). *A priori*, one only has inclusion of full subcategories

$$\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}}(G_v) \subset \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_v) \subset \mathrm{Rep}_{\mathbb{Q}_p}(G_v)$$

but the p-adic monodromy theoremasserts that actually $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}}(G_v) = \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_v)$. Furthermore, the $\otimes$-functor $D_{\mathrm{pst}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_v) \to \mathrm{MF}_{k_v/\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N)$ restricts to a fully faithfull $\otimes$-functor

$$D_{\mathrm{pst}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}}(G_v) \hookrightarrow \mathrm{MF}_{k_v/\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v).$$

From [BeS22, Thm. 3.3], for every open admissible neighbourhood $U_{v,x_0} \subset X^{v\text{-an}}$ of x_0 as above and $x \in U_{v,x_0}$ there exists a unique isomorphism $T_{x_0,x} : D_{\mathrm{pst}}(H^i(Y_{\overline{x}_0}, \mathbb{Q}_p))_0 \tilde{\to} D_{\mathrm{pst}}(H^i(Y_{\overline{x}}, \mathbb{Q}_p))_0$ in $\mathrm{M}_{\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v)$ making the following diagram commute

$$\begin{tikzcd}
D_{\mathrm{pst}}(H^i(Y_{\overline{x}_0}, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} \overline{\mathbb{Q}}_p \arrow[r, "c", "\simeq"'] \arrow[d, "T_{x_0,x}\otimes Id"', "\simeq"] & D_{\mathrm{dR}}(H^i(Y_{\overline{x}_0}, \mathbb{Q}_p)) \otimes_{k_v} \overline{\mathbb{Q}}_p \arrow[r, "c_{\mathrm{dR}}", "\simeq"'] & H^i_{\mathrm{dR}}(Y_{x_0}/k_v) \otimes_{k_v} \overline{\mathbb{Q}}_p \arrow[d, "\simeq"', "T^\nabla_{x_0,x}"] \\
D_{\mathrm{pst}}(H^i(Y_{\overline{x}}, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} \overline{\mathbb{Q}}_p \arrow[r, "c", "\simeq"'] & D_{\mathrm{dR}}(H^i(Y_{\overline{x}}, \mathbb{Q}_p)) \otimes_{k_v} \overline{\mathbb{Q}}_p \arrow[r, "c_{\mathrm{dR}}", "\simeq"'] & H^i_{\mathrm{dR}}(Y_x/k_v) \otimes_{k_v} \overline{\mathbb{Q}}_p,
\end{tikzcd}$$

where c_{dR} is the p-adic étale / de Rham comparison isomorphism constructed in [Sc13, Cor. 1.8]. Let M denote the k_v-module underlying $H^i_{\mathrm{dR}}(Y_{x_0}/k_v)$ and write for simplicity $M_{\mathrm{et},x}$ for the G_v-representation $H^i(Y_{\bar{x}}, \mathbb{Q}_p)$, $M_{\mathrm{pst}} := D_{\mathrm{pst}}(H^i(Y_{\overline{x}_0}, \mathbb{Q}_p))_0$, $c := c_{M_{\mathrm{et},x_0}}$, $M_{\mathrm{dR},x} := (M, F^\bullet_x) := D_{\mathrm{dR}}(H^i(Y_{\overline{x}}, \mathbb{Q}_p))(\simeq H^i_{\mathrm{dR}}(Y_x/k_v))$.

[BeS22, Thm. 3.3] shows that the v-adic period map $\Phi^v_{x_0}: U_{v,x_0} \to \check{\mathbf{D}}^{v\text{-an}}$ fits into a commutative diagram

$$\begin{tikzcd} U_{v,x_0} \arrow[r, "\Phi^v_{x_0}"] \arrow[d, "\Psi_{v,\mathrm{et}}"'] & \check{\mathbf{D}}^{v\text{-an}} \arrow[d, "\Psi_{x_0}"] \\ \pi_0(\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_v)) \arrow[r, hook, "D_{\mathrm{pst}}"'] & \pi_0(\mathrm{FM}_{k_v/\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v)), \end{tikzcd}$$

where $\pi_0(-)$ denotes the set of isomorphism classes of objects, $\Psi_{v,\mathrm{et}}: U_{v,x_0} \to \pi_0(\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_v)$ is the map induced by the p-adic étale period map *i.e.* the map which sends $x \in U_{v,x_0} \subset X(k_v)$ to the isomorphism class of $M_{\mathrm{et},x}$ and $\Psi_{x_0}: \check{\mathbf{D}}^{v\text{-an}} \to \pi_0(\mathrm{FM}_{k_v/\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v))$ is the map which sends $F^\bullet \in \check{\mathbf{D}}^{v\text{-an}}$ to the isomorphism class of $(M_{\mathrm{pst}}, (M, F^\bullet), c)$. By definition of the category $\mathrm{FM}_{k_v/\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v)$, the fibers of Ψ_{x_0} are homogeneous spaces for the natural action of $\mathrm{Aut}(M_{\mathrm{pst}})(\mathbb{Q}_p)$ on $\check{\mathbf{D}}(k_v) = \mathrm{Res}_{k_v/\mathbb{Q}_p}(\check{\mathbf{D}})(\mathbb{Q}_p)$, where $\mathrm{Aut}(M_{\mathrm{pst}})$ is the $\mathbb{Q}_p$-linear algebraic group[5]

$$R \mapsto (R \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathrm{M}_{\mathbb{Q}_p^{\mathrm{ur}}}(\phi,N,G_v)}(M_{\mathrm{pst}}))^\times.$$

Fix a (set theoretic) section $s: \pi_0(\mathrm{FM}_{k_v/\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v) \hookrightarrow \check{\mathbf{D}}^{v\text{-an}}$ of Ψ_{x_0}. Then, for every subset $W \subset U_{v,x_0}$, the Zariski-closure $\mathcal{W} \subset \check{\mathbf{D}} \times_k k_v$ of $\Phi^v_{x_0}(W)$ is contained in

$$\mathcal{Z}_W := \bigcup_{\mu \in D_{\mathrm{pst}} \circ \Psi_{v,\mathrm{et}}(W)} \mathrm{Aut}(M_{\mathrm{pst}})_{k_v} \cdot s(\mu).$$

The strategy thus boils down to finding conditions on W ensuring $\Phi^v_{x_0}{}^{-1}(\mathcal{Z}_W)$ is not Zariski-dense in X. In the Hodge-theoretic setting, a sufficient condition is provided by the Ax-Schanuel theorem [BT19]. The crucial observation of [LaVe20] is that, using that the Gauss-Manin connection is defined over k (which can be embedded simultaneously into k_v and $\mathbb{C}$), one can transport statement such as the Ax-Schanuel theorem from the complex-analytic setting to the v-adic setting.

(2) Input from singular incarnation and complex Hodge theory: More precisely, assume $x_0 \in X(k)$, fix an embedding $\infty: k \hookrightarrow \mathbb{C}$ and an isomorphism $\mathbb{C} \tilde{\to} \overline{\mathbb{Q}}_p (\hookleftarrow k_v)$, which induces an isomorphism

$$c: \check{\mathbf{D}} \times_k \mathbb{C} \tilde{\to} \check{\mathbf{D}} \times_k \overline{\mathbb{Q}}_p (\hookleftarrow \check{\mathbf{D}} \times_k k_v).$$

Fix also a simply connected open neighbourhood $U^{\mathrm{an}}_{x_0}$ of x_0 in X^{an} so that the restriction $\Phi|_{U^{\mathrm{an}}_{x_0}}: U^{\mathrm{an}}_{x_0} \to \Gamma \setminus D$ of the complex period map $\Phi: X^{\mathrm{an}} \to \Gamma \setminus D$ lifts to $\Phi^{\mathrm{an}}_{x_0} := \tilde{\Phi}|_{U^{\mathrm{an}}_{x_0}}: U^{\mathrm{an}}_{x_0} \to \widetilde{X}^{\mathrm{an}} \to D \subset \check{\mathbf{D}}^{\mathrm{an}}$. Then the following holds:

(a) [LaS20, Lemma 7.2] *Modulo* the isomorphism $c: \check{\mathbf{D}} \times_k \mathbb{C} \tilde{\to} \check{\mathbf{D}} \times_k \overline{\mathbb{Q}}_p$, the image of the v-adic period map $\Phi^v_{x_0}: U_{v,x_0} \to \check{\mathbf{D}}^{v\text{-an}}$ lies in the image of $\overline{G}_{X,\mathcal{V}} \cdot \tilde{\Phi}^{\mathrm{an}}(x_0)$ (which is of the form $\overline{G}_{X,\mathcal{V}}/P$ where $P \subset \overline{G}_{X,\mathcal{V}}$ is the parabolic subgroup stabilizing $\tilde{\Phi}^{\mathrm{an}}(x_0)$);

(b) (v-adic Ax-Schanuel - [LaVe20, Lemma 9.3], [LaS20, Thm. 7.3]) Let $\mathcal{Z} \hookrightarrow \check{\mathbf{D}} \times_k k_v$ be a closed subvariety such that $\mathrm{codim}_{\overline{G}_{X,\mathcal{V}}/P}(\mathcal{Z}) \geq \dim(X)$. Then $\Phi^v_{x_0}{}^{-1}(\mathcal{Z})$ is not Zariski-dense in X. Actually, what Lawrence and Venkatesh shows in [LaVe20, Lemma 9.3] is that for every closed subvariety $\mathcal{Z} \hookrightarrow \check{\mathbf{D}} \times_k k_v$ if $\Phi^{-1} c^{-1} \mathcal{Z} \cap U^{\mathrm{an}}_{x_0} \subset U^{\mathrm{an}}_{x_0}$ is not Zariski-dense in X then $\Phi^v_{x_0}{}^{-1}(\mathcal{Z}) \subset U_{v,x_0}$ is not Zariski-dense in X; the inequality involving (co)dimension is then just the statement of the Ax-Schanuel theorem [BT19] for the complex period map. But the use of Ax-Schanuel provides additional informations. Let $W \subset U_{v,x_0}$ and recall the notation $\mathcal{Z}_W$ of (1). Assume $\mathrm{codim}_{\overline{G}_{X,\mathcal{V}}/P}(\mathcal{Z}_W) \geq \dim(X)$. Then,

(i) There exists finitely many strict, weakly special subvarieties $S_{W,1}, \ldots, S_{W,r} \subsetneq X_{\mathbb{C}}$ such that $u\widetilde{\Phi}^{-1} c^{-1} \mathcal{Z}_W \subset S_{W,1} \cup \cdots \cup S_{W,r}$.

(ii) As the closed subvariety $\mathcal{Z}_W \subset \check{\mathbf{D}} \times_k \overline{\mathbb{Q}}_p$ depends only on $W \subset U_{v,x_0}$ and not on $x_0 \in W$, for every $x \in W$ one has (with the same notation as above, replacing x_0 with x)

$$x \in U^{\mathrm{an}}_x \subset \Phi^{\mathrm{an}}_x{}^{-1}(c^{-1}\mathcal{Z}_W) \subset u\widetilde{\Phi}^{-1} c^{-1} \mathcal{Z} \subset S_{W,1} \cup \cdots \cup S_{W,r}.$$

[5]By construction the category $\mathrm{M}_{\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N, G_v)$ is $\mathbb{Q}_p$-linear.

In particular, when $\mathcal{V}_\infty$ has level ≥ 3 and $G^{\mathrm{ad}}_{\mathcal{V}_\infty}$ is simple, combining the above with [BaKU21] and [Kr21b], $X^{>0}_{\mathcal{V}_p} = X^{>0}_{\mathcal{V}_\infty} = X^{\mathrm{atyp},>0}_{\mathcal{V}_\infty} \subsetneq X$ is a strict closed subvariety defined over k and one obtains: Let $W \subset X(k_v)$ and let $U_v \subset X(k_v)$ be a good admissible open subset with $\mathrm{codim}_{\overline{G}_{X,\mathcal{V}}/P}(\mathcal{Z}_{W\cap U_v}) \geq \dim(X)$. Then

$$\overline{(W\cap U_v)^{\mathrm{zar}}}^{>0} \subset X^{>0}_{\mathcal{V}_p} \subsetneq X.$$

5.2.3.2. *A few comments.*

(1) In [LaVe20], [LaS20], one considers $W := \mathcal{X}(U)$ with $v \in U$, where $U \subset \mathrm{spec}(\mathcal{O}_k)$ a non-empty open subscheme over which $f: Y \to X \to \mathrm{spec}(k)$ extends to a smooth proper morphism $f: \mathcal{Y} \to \mathcal{X} \to U$ so that:
  (a) The general Tannakian picture described in 5.2.3.1 is significantly simpler: U_{v,x_0} can be taken to be the disc of all $x \in \mathcal{X}(\mathcal{O}_v)$ with $x \equiv x_0 \mathrm{mod} v$, [BeS22, Thm. 3.3] boils down to the classical crystalline - de Rham comparison isomorphism and $D_{\mathrm{pst}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_v) \to \mathrm{FM}_{k_v/\mathbb{Q}_p^{ur}}(\phi, N, G_v)$ is replaced by $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_v) \to \mathrm{FM}_{k_v/k_{v,0}}(\phi)$.
  (b) W embeds into the compact set $\mathcal{X}(\mathcal{O}_v)$ hence can be covered by *finitely many* good admissible open subsets U_v.
  (c) Faltings' finiteness lemma[6] ensures that $\Psi_{v,\mathrm{et}}(W)$ is "not too large" namely that $\Psi_{v,\mathrm{et}}(W) \subset \Psi_{v,\mathrm{et}}(W_1 \cap U_{v,x_0}) \cup \cdots \cup \Psi_{v,\mathrm{et}}(W_r \cap U_{v,x_0})$, with $W_1, \ldots, W_r$ fibers of the canonical map

$$\Psi_{\mathrm{et}} : X(k) \to \pi_0(\mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))) \to \pi_0(\mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_p}(\pi_1(k))),$$

  where the first arrow sends $x \in X(k)$ to the $\pi_1(x)$-representation $\mathcal{V}_{p,\bar{x}}$ and the second one is $\pi_1(k)$-semisimplification.

In the end, the problem is reduced to showing that the dimensions of the "generalized centralizers of Frobenius" $\mathcal{Z}_{W_i\cap U_v}$ are "small" compared with the dimension of $\overline{G}_{X,\mathcal{V}}$.

(2) The Lawrence-Vekatesh strategy is well suited to tackle Conjecture 3, which only predicts (compare with Corollary 11) the non-Zariski-density of the subset $|X|_{\mathcal{V}_p} \cap \mathcal{X}(U) \subset \mathcal{X}(U)$. One may expect that the assumption that for $x \in W$, $G^\circ_{x,\mathcal{V}_p} \subset H$ for some strict algebraic subgroup $H \subset G_{\mathcal{V}_p}$ imposes restrictions on the image of the local representation $M_{\mathrm{et},x}$, which, in turn should impose some constraints on the dimension of $\mathcal{Z}_{W\cap U_v}$ at least for certain classes of subgroups H. As a first illustration of this heursitic appears in [CS23], where one considers the set $|X|^{\mathrm{tor}}_{\mathcal{V}_p} \cap X(k)$ of all $x \in X(k)$ such that $G^\circ_{x,\mathcal{V}_p}$ is a torus (these can be regarded as the étale counterpart of CM points) and, using the above circle of ideas, shows that if
  - For two primes $\ell_1 \neq \ell_2$, $\mathcal{V}_{\ell_i}$ admits a $\mathbb{Z}_{\ell_i}$-model $\mathcal{V}^\circ_{\ell_i}$ with $\mathcal{V}^\circ_{\ell_i} \otimes \mathbb{F}_{\ell_i}$ constant, $i = 1, 2$;
  - $\mathcal{V}_\infty$ has positive period dimension,

then for[7] a set of primes p of positive density $X^{\mathrm{tor}}_{\mathcal{V}_p} \cap X(k)$ is not Zariski-dense in X (and more precisely is contained is a finite union of fibers[8] of the complex period map $\Phi : X^{\mathrm{an}} \to \Gamma \setminus D$). Note that one intermediate step in the proof of this result is to show (under the same assumptions) that $|X|^{\mathrm{tor}}_{\mathcal{V}_p} \cap \mathcal{X}(U) = |X|^{\mathrm{tor}}_{\mathcal{V}_p} \cap X(k)$ so that even if it does not appear in the statement, this is a statement about integral rather than rational points.

(3) On the other hand, there are two main obstructions to apply the Lawrence-Venkatesh strategy to tackle (Conjecture 1 and) Conjecture 2 (1). Set $W := |X|_{\mathcal{V}_p} \cap X(k)$. It is no longer true that:
  (a) W can be covered by finitely many good open admissible v-adic neighbourhoods U_v;
  (b) $\Psi_{\mathrm{et}}(W \cap U_v)$ is finite.

Overcoming any of both obstructions should require significant improvements of the original strategy. It seems that a key step would be to extend the construction and study of v-adic period maps along the boundary of a smooth compactification $X \hookrightarrow X^{\mathrm{cpt}}$.

[6]This is the only information coming from the global origin of the points in W that is used.
[7]Recall that one expects $X^{\mathrm{tor}}_{\mathcal{V}_p} \cap X(k)$ to be independent of p - see Conjectures 4 and 5.
[8]These are known to be Zariki closed in $X_{\mathbb{C}}$.

(4) In contrast to what happens for number fields, there are only finitely many extensions of a local field of bounded degree $\leq d$. So that replacing k_v by a finite field extension, the above general strategy may give hints about Conjecture 3 for *arbitrary* d provided one can control obstruction (b) of (3).

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