A. Betts. The Chabauty-Kim method

§ k number field
X/k smooth, complete curve of genus g
$x \in X(k)$
p prime — for simplicity today

Trichotomy
- $g = 0$ — $X(k)$ infinite
- $g = 1$ — $X(k)$ f.g. abelian group
- $g \geq 2$ — $X(k)$ finite (Mordell Conj. — Faltings)

Observation: this trichotomy asserts a connection between the "diophantine geometry" & the "algebraic topology" on X!

Q: Can we use analogues of tools from algebraic topology to prove cases of the Mordell Conj concretely

Answer 1 (Grothendieck): $1 \to \pi_1(X_{\bar{k}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \pi_1(k) \to 1$ — x gives a natural action of $\pi_1(k)$ on $\pi_1(X_{\bar{k}}, \bar{x})$

$\exists$ map $X(k) \xrightarrow{\sigma} \underbrace{H^1(\pi_1(k), \pi_1(X_{\bar{k}}, \bar{x}))}_{}$

non abelian continuous Galois coh.

Conj (Section Conj): σ is bijective

Widely open, not known for any X.

Answer 2 (Chabauty + reinterpretation)

Original idea: try to combine two constraints on $X(k)$

$X(k) \hookrightarrow X(k_v)$ v finite place on k.
$X(k) \hookrightarrow \mathrm{Jac}_{X/k}(k)$

Thm (Chabauty, 1941) If rank $\mathrm{Jac}_{X/k}(k) < g_X$ then, $\forall v$ finite place on k

$$|X(k_v) \cap \mathrm{Jac}_{X/k}(k)| < +\infty$$

Can we replace $\mathrm{Jac}_{X/k}(k)$ with $H_1(X_{\bar{k}}, \mathbb{Q}_p) = H^1(X_{\bar{k}}, \mathbb{Q}_p)^\vee = V_p(\mathrm{Jac}_{X/k}) =: V$? with $p = \mathrm{char}$

Kummer map $j: X(k) \to H^1(\pi_1(k), V)$

$\mathrm{Jac}_{X/k}(k) \xrightarrow{\text{Kummer}} H^1(\pi_1(k), V)$

$\mathrm{Jac}_{X/k}(k)$ f.g. abelian group

$0 \to \mathrm{Jac}_{X/k}[p^n] \to \mathrm{Jac}_{X/k} \xrightarrow{p^n} \mathrm{Jac}_{X/k} \to 0$

$\mathrm{Jac}_{X/k}(k)/p^n \to H^1(\pi_1(k), \mathrm{Jac}_{X/k}[p^n])$

$+ \varprojlim_n \otimes \mathbb{Q}_p$

Have a fundamental diagram

$$\begin{tikzcd} X(k) \arrow[r, hook, "j"] \arrow[d, hook] & H^1(\pi_1(k), V) \arrow[d, "\mathrm{loc}_v"] \\ X(k_v) \arrow[r, "j_v"] & H^1(\underbrace{\pi_1(k_v)}, V) \end{tikzcd} \qquad (D+)$$

can be identified with D_v decomposition group

Facts: 1) $j_v(X(k_v)) \subset H^1_f(\pi_1(k_v), V)$ "crystalline part"

$$= \ker\left(H^1(\pi_1(k_v), V) \longrightarrow H^1(\pi_1(k_v), V \otimes B_{cris})\right)$$

studied by Bloch & Kato

e.g. $H^1_f(\pi_1(k_v), V)$ is a k_v v-s of dim g

2) $j(X(k)) \subset H^1_f(\pi_1(k), V)$ "unramified / crystalline part"

$=$ common kernel of

$H^1(\pi_1(k), V) \longrightarrow H^1(\pi_1(k_v), V) \longrightarrow H^1(\pi_1(k_v), V \otimes B_{cris})$ $\forall v \mid p$

$H^1(\pi_1(k), V) \longrightarrow H^1(\pi_1(k_v), V) \longrightarrow H^1(I_v, V \otimes B_{cris})$ $\forall v \nmid p$

I_v inertia group.

3) j_v is locally k_v-analytic & locally dense i.e.

3.i) $X(k_v)$ can be covered by closed discs so that $j_v|_{Disc}$ is given component-wise by convergent power series.

3.ii) If $\emptyset \neq U \subset_{op} X(k_v)$ then $j_v(U)$ is not contained in any hyperplane in $H^1_f(\pi_1(k_v), V)$.

Thm (Chabauty revisited): If $\dim_{\mathbb{Q}_p} H^1_f(\pi_1(k), V) \overset{(*)}{<} g = \dim H^1_f(\pi_1(k_v), V)$ then

$$|X(k)| < +\infty$$

Rem: $\operatorname{rank} \mathrm{Jac}_{X/k}(k) \leq \dim_{\mathbb{Q}_p} H^1_f(\pi_1(k), V)$ with $=$ iff $|Ш_{\mathrm{Jac}_{X/k}}[p^\infty]| < +\infty$

Proof: $(*) \Longrightarrow \mathrm{loc}_v(H^1_f(\pi_1(k), V))$ is contained in a proper k_v hyperplane in $H^1_f(\pi_1(k_v), V)$

Choose $\alpha : H^1_f(\pi_1(k_v), V) \longrightarrow k_v$ non zero, vanishing on $\mathrm{loc}_v(H^1_f(\pi_1(k), V))$.

Then $\alpha \circ j_v : X(k_v) \longrightarrow k_v$
- vanishes on $X(k)$
- is locally k_v-analytic
- non vanishing on open subsets of $X(k_v)$

$\Big\}$ $\alpha \circ j_v$ has only ∞ly many zeros

§ Non abelian Chabauty

Observation: $H_1(X_{\bar{k}}, \mathbb{Q}_p) = \pi_1(X_{\bar{k}}, \bar{x})^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

Q: Do we get more information from Chabauty if we replace $\pi_1(X_{\bar{k}}, x)^{ab}$ with $\pi_1(X_{\bar{k}}, x)$?

Answer (Kim): Yes!

Step 1: Linearize $\pi_1(X_{\bar{k}}, \bar{x})$ (on this day): Define a pro-unipotent étale fund. group $U = U^{et}/\mathbb{Q}_p$

- U is a f.g. pro-unipotent group $/\mathbb{Q}_p$
- $\pi_1(k) \curvearrowright U(\mathbb{Q}_p)$ continuously
- " $U = \pi_1(X_{\bar{k}}, \bar{x}) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p$ " continuous Malcev completion

Step 2: Non abelian Kummer maps

global $X(k) \xrightarrow{j} H^1(\pi_1(k), U(\mathbb{Q}_p))$

local $X(k_v) \xrightarrow{j_v} H^1(\pi_1(k_v), U(\mathbb{Q}_p))$

fitting in a fundamental commutative diagram (D1)

(D2)
$$\begin{array}{ccc} X(k) & \overset{j}{\hookrightarrow} & H^1(\pi_1(k), U(\mathbb{Q}_p)) \\ \downarrow & & \downarrow \mathrm{loc}_v \\ X(k_v) & \xrightarrow{j_v} & H^1(\pi_1(k), U(\mathbb{Q}_p)) \end{array}$$

Assume for simplicity that X has semistable reduction at v.

$X(k)^0 := \{ y \in X(k) \mid x$ & y reduce onto the same component of all special fibers of a semistable model $\}$.

Define $X(k_v)^0$ similarly

Facts: 1) $j_v(X(k_v)^0) \subset H^1_f(\pi_1(k_v), U(\mathbb{Q}_p))$ "crystalline part"

$$\ker\big(H^1(\pi_1(k_v), U(\mathbb{Q}_p)) \longrightarrow H^1(\pi_1(k_v), U(B_{cris}))\big)$$

and $H^1_f(\pi_1(k_v), U(\mathbb{Q}_p))$ is canonically the k_v-pts of a connected affine scheme $H_{f,v}/k_v$

2) $j(X(k)^0) \subset H^1_f(\pi_1(k), U(\mathbb{Q}_p))$

= common kernel of $H^1(\pi_1(k), U(\mathbb{Q}_p)) \to H^1(\pi_1(k_v), U(\mathbb{Q}_p)) \to H^1(\pi_1(k_v), U(B_{cris}))\ \forall v$
& $H^1(\pi_1(k), U(\mathbb{Q}_p)) \to H^1(\pi_1(k_w), U(\mathbb{Q}_p)) \to H^1(I_w, U(B_{cris}))\ \forall w$

and $H^1_f(\pi_1(k), U(\mathbb{Q}_p))$ is canonically the $\mathbb{Q}_p$-pts of a connected affine scheme $H^1_f/\mathbb{Q}_p$

and $\mathrm{loc}_v : H^1_f \longrightarrow H^1_{f,v}$ is a map of $\mathbb{Q}_p$-schemes

3) $j_v : X(k_v)^0 \longrightarrow H^1_f(\pi_1(k_v), U(\mathbb{Q}_p))$ is locally pro k_v-analytic and locally Zariski-dense

(locall density means $j_v(V) \subset H^1_f$ is Zariski-dense $\forall \emptyset \neq V \subset_{op} X(k_v)$)

Thm (Kim) If $\mathrm{loc}_v(H^1_f(\pi_1(k), U(\mathbb{Q}_p)))$ is not Z-dense in $H^1_f(\pi_1(k_v), U(\mathbb{Q}_p))$
Then $X(k)^0$ is finite

Proof: Same as Chabauty revisited: Condition $\longrightarrow$ $\exists$ a non zero algebraic morphism

$\alpha : H^1_g(\pi_1(k_v), U(\mathbb{Q}_p)) \longrightarrow k_v$ st $\alpha \circ \log_v$ vanishes on $H^1_g(\pi_1(k), U(\mathbb{Q}_p))$

Then $\alpha \circ j_v : X(k_v)^\circ \longrightarrow k_v^\times$
- vanishes on $X(k)$
- is locally k_v-analytic
- non vanishing on open nbhds of $X(k_v)^\circ$.

Application: $k = \mathbb{Q}$, S finite set of places of k. Then $\mathbb{P}^1 \setminus \{0,1,\infty\}(\mathbb{Z}_S)$ is finite, $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{S}]$

(ie $\exists$ only finitely many $a, b \in \mathbb{Z}_S^\times$ st $a + b = 1$)

Proof: Choose $p = v \notin S$ and any base pt $x \in \mathbb{P}^1 \setminus \{0,1,\infty\}(\mathbb{Z}_S)$. Have functorial commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \setminus \{0,1,\infty\}(\mathbb{Z}_S) & \longrightarrow & H^1_{g,S}(\pi_1(\mathbb{Q}), U(\mathbb{Q}_p)) \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \setminus \{0,1,\infty\}(\mathbb{Z}_p) & \longrightarrow & H^1_g(\pi_1(\mathbb{Q}_p), U(\mathbb{Q}_p)) \end{array}$$

$\leftarrow$ imposes conditions at places $w \notin S$

Then 1) U is free pro-unipotent on 2 generators ie $\exists\, a, b \in U(\mathbb{Q}_p)$ st $\forall u \in U(\mathbb{Q}_p)$ has a unique decomposition as

$$u = a^{\alpha_1} b^{\alpha_2} [a,b]^{\alpha_3} [a,[a,b]]^{\alpha_4} [b,[a,b]]^{\alpha_5} \dots \quad \text{with } \alpha_i \in \mathbb{Q}_p$$

Define $U_n := U /_{n+1 \text{ tuple commutators}}$; U_n is unipotent (ie of finite type $/\mathbb{Q}_p$) and, more precisely, as a $\pi_1(k)$-module, it is an extension of $\mathbb{Q}_p(1)^2$ by $\mathbb{Q}_p(2)$ by $\mathbb{Q}_p(3)^2$ by ... by $\mathbb{Q}_p(r)^{k_r}$ by ... by $\mathbb{Q}_p(n)^{k_n}$ $\quad k_r \geq 1$

2) Dimension calculation:

- $\dim H^1_g(\pi_1(k_v), U_n(\mathbb{Q}_p)) = \sum_{r=1}^{n} k_r \underbrace{\dim H^1_g(\pi_1(k_v), \mathbb{Q}_p(r))}_{=1 \text{ Bloch-Kato}}$

$$= \sum_{r=1}^{n} k_r$$

- $\dim H^1_{g,S}(\pi_1(k_v), U_n(\mathbb{Q}_p)) \leq \sum_{r=1}^{n} k_r \dim H^1_{g,S}(\pi_1(k_v), \mathbb{Q}_p(r))$

$$\leq 2|S| + \sum_{\substack{2 \leq r \leq n \\ 2 \mid r}} k_r$$

$H^1_{g,S}(\pi_1(\mathbb{Q}), \mathbb{Q}_p(1)) = \mathbb{Q}_p \otimes_{\mathbb{Z}} \mathbb{Z}_S^\times$

Soulé $\Longrightarrow$ $\dim H^1_{g,S}(\pi_1(k_v), \mathbb{Q}_p(r)) \leq \begin{cases} 1 & \text{if } r \geq 2 \text{ even} \\ 0 & \phantom{\text{if }} r \geq 3 \text{ odd} \end{cases}$

Conclusion: $\dim H^1_{g,S}(\pi_1(\mathbb{Q}), U_n(\mathbb{Q}_p)) < \dim H^1_g(\pi_1(\mathbb{Q}_p), U_n(\mathbb{Q}_p))$ for $n \gg 0$.

$\Longrightarrow$ $\ker_v$ not Z-dense

$\Longrightarrow$ $\mathbb{P}^1 \setminus \{0,1,\infty\}(\mathbb{Z}_S)$ finite

§ Linearizing fundamental groups.

Q: how to make (étale) fund groups into linear objects? i.e. what do we mean by "$\mathbb{Q}_p \otimes_{\mathbb{Z}} \pi_1^{et}$"?

Not: k field of char 0.
Y/k smooth, connected, separated, of FT$/k$
$y \in Y(k)$ ($\Longrightarrow$ Y geo. connected $/k$).

Answer: Mal'čev completion

If π is a f.g. discrete group, what do we mean by "$\mathbb{Q} \otimes_{\mathbb{Z}} \pi$"?

Ex: 1) π abelian just take usual tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \pi := \mathbb{Q} \otimes_{\mathbb{Z}} (\pi/\pi_{tors})$

2) π 2-steps nilpotent i.e. $[a,[b,c]] = 1$, $a,b,c \in \pi$

$$1 \to [\pi,\pi] \to \pi \to \pi^{ab} \to 1 \quad \text{central extension}$$

guess: $\mathbb{Q} \otimes_{\mathbb{Z}} \pi$ should be a central extension of the divisible group $\mathbb{Q} \otimes_{\mathbb{Z}} \pi^{ab}$ by divisible group $\mathbb{Q} \otimes_{\mathbb{Z}} [\pi,\pi]$

3) π n-steps nilpotent i.e. there is a sequence of central extensions

$$1 \to \pi_{[r]} \to \pi_{r+1} \to \pi_r \to 1$$

with $\pi_0 = 1$, $\pi_1 = \pi$

guess: $\mathbb{Q} \otimes_{\mathbb{Z}} \pi$ should be iterated central extensions of divisible groups $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{[r]}$

Déf: (pro-unipotent groups) F char 0 field. An affine group scheme $U/_F$ is pro-unipotent if the following equiv. condition hold

1) Any non zero fin. dim F-rep of U has a non zero fixed vector
2) U is a (transfinite) iterated extensions of vectors groups i.e. $\mathbb{G}_a^m$
3) U ————— central extensions of $\mathbb{G}_a$

transfinite: $U = \varprojlim_{\lambda < \alpha} U_\lambda$ with $1 \to \ker \to U_{\lambda+1} \to U_\lambda \to 1$
affine group schemes $/F$ $\quad \ker \cong \mathbb{G}_{aF}^m$

with $U_0 \simeq \mathbb{G}_{aF}^m$
if β is a limit then $U_\beta = \varprojlim_{\lambda<\beta} U_\lambda$

Rmk: if U is f.g. then can take $\alpha = \omega$ & U_n finite dim.

Déf: (Mal'čev completion) F char 0 field, π discrete group. The F-Mal'čev completion of π is the pro-unipotent group $\pi^F/_F$ representing the functor

$$\{\text{pro-unipotent groups}/F\} \longrightarrow Set$$
$$U \longmapsto \mathrm{Hom}_{Grp}(\pi, U(F))$$

Equivalently, we have $\pi \xrightarrow{\mu} \pi_F(F)$ st $\forall$ $U/_F$ pro-unipotent group and $\pi \xrightarrow{\varphi} U(F)$, $\exists !$ $\pi_F \xrightarrow{\varphi_U} U$ morphism of group schemes$/_F$ st

$$\begin{tikzcd} \pi \arrow[r, "\varphi"] \arrow[d, "\mu"'] & U(F) \\ \pi_F(F) \arrow[ur, "\varphi_U(F)"'] & \end{tikzcd}$$

Variant: If F, π carry topologies the continuous F-Malcev completion of π is π_F/F representing the functor

$$\{\text{pro-unipotent groups}/F\} \longrightarrow \text{Set}$$
$$U \longmapsto \mathrm{Hom}_{\text{cont Grp}}(\pi, U(F))$$

Proof of existence: Special adjoint functor thm.

Lemma: π f.g. discrete group.
$\pi_n := \pi/_{n+1 \text{ tuple commutators}}$

so that $1 \longrightarrow \pi_{[n]} \longrightarrow \pi_{n+1} \longrightarrow \pi_n \longrightarrow 1$ central ext

Then $\pi_{n,F}$ is an iterated central ext of the vector groups corresponding to $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{[r]}$, $r \leq n$

$$\pi_F = \varprojlim_n \pi_{n,F}.$$

Lemma: π f.g. discrete group, $F[\pi]$ usual group algebra (cocommutative Hopf algebra)

$$F[\pi] \xrightarrow{\varepsilon} F \quad \text{augmentation morphism - counit}$$

$J := \ker(F[\pi] \xrightarrow{\varepsilon} F)$ augmentation ideal
$\lhd\, F[\pi]$

Define the dualized J-adic completion of $F[\pi]$ as $F[\pi]^{\wedge o} = \varinjlim_n \mathrm{Hom}_F(F[\pi]/_{J^{n+1}}, F)$

Then $F[\pi]^{\wedge o}$ is a commutative Hopf algebra and $\pi_F = \mathrm{Spec}(F[\pi]^{\wedge o})$

Rmk: If π top f.g. profinite group and $F = \mathbb{Q}_p$ there is a similar description

$$\pi_{\mathbb{Q}_p} = \mathrm{Spec}(\mathbb{Q}_p[[\pi]]^{\wedge o}) \qquad \text{eg } \pi = \text{free profinite group on } r \text{ generators}$$

$$\mathbb{Q}_p[[\pi]] = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_{\pi \twoheadrightarrow \bar\pi \text{ finite}} \mathbb{Z}_p[\bar\pi] \qquad \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\langle T_1, \dots, T_r \rangle$$

$$\mathbb{Q}_p\langle T_1, \dots, T_r\rangle = \mathbb{Q}_p[[\pi]]^{\wedge o}.$$

Rmk: By universal property of Malcev completion, if π is a f.g. discrete group then

$$\underbrace{\pi_{\mathbb{Q}_p}}_{\text{usual (discrete) } \mathbb{Q}_p\text{-Malcev compl of } \pi} = \underbrace{\hat{\pi}_{\mathbb{Q}_p}}_{\text{continuous } \mathbb{Q}_p \text{ Malcev completion of } \hat\pi}$$

Def: 1) Fix $K \overset{\sigma}{\hookrightarrow} \mathbb{C}$ $\quad U^B := U^B(Y,y) = \widehat{\pi_1^{top}(Y(\mathbb{C}),y)}_{\mathbb{Q}}$

$\mathbb{Q}$ Mal'cev completion of $\pi_1^{top}(Y(\mathbb{C}),y)$

2) $U^{et} := U^{et}(Y,y) = \widehat{\pi_1^{et}(Y_{\bar{k}},\bar{y})}_{\mathbb{Q}_p}$

$\mathbb{Q}_p$ continuous Mal'cev completion of $\pi_1^{et}(Y_{\bar{k}},\bar{y})$.

Rmk: 1) $\pi_1(k) \curvearrowright \pi_1^{et}(Y_{\bar{k}},y)$ continuously $\longrightarrow$ $\pi(k) \curvearrowright U^{et}(\mathbb{Q}_p)$ continuously.

2) Given $K \overset{\sigma}{\hookrightarrow} \mathbb{C}$ $\quad \pi_1^{et}(Y_{\bar{k}},\bar{y}) \simeq \widehat{\pi_1^{top}(Y(\mathbb{C}),\bar{y})}$ hence $U^{et} = U^B \otimes_{\mathbb{Q}} \mathbb{Q}_p$

§ Tannakian formalism:

Grothendieck's Galois theory: have an equivalence of categories

$\{$profinite groups$\}$ $\longleftrightarrow$ "categories which behave like the finite étale covers of a connected scheme + fibre functor"

$\pi \longrightarrow$ finite discrete π-sets + For: finite discrete π-sets $\longrightarrow$ finite sets

$\mathrm{Aut}(\omega) \longleftarrow \mathcal{C}$, $\omega: \mathcal{C} \longrightarrow$ finite sets

Idea: linearize category of covers $\curvearrowright$ local system.

Thm (Tannaka) F char 0 field, there is an equivalence of categories

$\{$affine group scheme $/F\}$ $\longrightarrow$ "neutral categories which behave like the F-local systems on a connected scheme $/F$ (+ fiber functor)"

$G \longrightarrow \mathrm{Rep}_F^{<+\infty}(G)$ (+ For: $\mathrm{Rep}_F^{<+\infty}(G) \longrightarrow \mathrm{Mod}_F^{<+\infty}$)

$\mathrm{Aut}^{\otimes}(\omega) \longleftarrow (\mathcal{A}, \omega: \mathcal{A} \longrightarrow \mathrm{Mod}_F^{<+\infty})$

Def (sketch) a rigid F-linear abelian $\otimes$-category $\mathcal{A}$ is a F-linear category $\mathcal{A}$ carrying

- a $\otimes$-operation $\otimes: \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$, $(V,W) \longmapsto V\otimes W$
- a $\otimes$-unit $\mathbb{1} \in \mathcal{A}$
- a dual operation $\mathcal{A} \xrightarrow{\vee} \mathcal{A}^{op}$, $V \longmapsto V^\vee$

satisfying the usual conditions

A fiber functor $\omega: \mathcal{A} \longrightarrow \mathrm{Mod}_F^{<+\infty}$ is an exact, conservative, F-linear functor which is compatible with $\otimes$-structure (in part $\omega(V\otimes W) = \omega(V)\otimes\omega(W)$, $\omega(\mathbb{1}) = \mathbb{1}$, $\omega(V^\vee) = \omega(V)^\vee$)

A neutral Tannakian category $\mathcal{A}$ is a rigid F-linear abelian $\otimes$-category for which $\exists \geq 1$ fibre functor

Def: $\mathcal{A}$ neutral Tannakian category $/F$

$\omega, \omega': \mathcal{A} \longrightarrow \mathrm{Mod}_F^{<+\infty}$ 2 fibre functors

If Λ is an F-algebra, define $\mathrm{Iso}^{\otimes}(\omega,\omega')(\Lambda) =$

$$\mathrm{Isom}^{\otimes}_{\mathrm{Fct}(\mathcal{A},\mathrm{Mod}_\Lambda)}\left(\mathcal{A} \xrightarrow{\omega} \mathrm{Mod}_F^{<+\infty} \xrightarrow{\otimes\Lambda} \mathrm{Mod}_\Lambda^{<+\infty},\ \mathcal{A} \xrightarrow{\omega'} \mathrm{Mod}_F^{<+\infty} \xrightarrow{\otimes\Lambda} \mathrm{Mod}_\Lambda^{<+\infty}\right)$$

Fact: $\underline{Iso}^{\otimes}(\omega,\omega')(\Lambda)$ is functorial in Λ and representable by a non empty affine scheme $\underline{Iso}^{\otimes}(\omega,\omega')/F$

The schemes carry operations:

- composition $\underline{Iso}^{\otimes}(\omega,\omega') \times_F \underline{Iso}^{\otimes}(\omega',\omega'') \longrightarrow \underline{Iso}^{\otimes}(\omega,\omega'')$
- identity $1_\omega \in \underline{Iso}^{\otimes}(\omega,\omega)(F)$
- inversion $\underline{Iso}^{\otimes}(\omega,\omega') \longrightarrow \underline{Iso}^{\otimes}(\omega',\omega)$

In part. $\underline{Aut}^{\otimes}(\omega) := \underline{Iso}^{\otimes}(\omega,\omega)$ is an affine group scheme

$\underline{Iso}^{\otimes}(\omega,\omega')$ is a right-torsor under $\underline{Aut}^{\otimes}(\omega)$

Rmk: In Tannakian correspondance $G/F \longleftarrow \mathcal{A} = Rep_F^{<+\infty}(G)$

G/F pro-unipotent $\Longleftrightarrow$ every object in $Rep_F^{<+\infty}(G)$ is a finite iterated extension of 1.

Ex: $Uni^{et}_{\mathbb{Q}_p}(Y)$ = category of lcc $\mathbb{Q}_p$-sheaves on $Y_{\bar{k}}$ which are finite iterated extensions of $\mathbb{Q}_p$

$y \in Y(k)$ gives a functor $\omega_y : Uni^{et}_{\mathbb{Q}_p}(Y) \longrightarrow Mod^{<+\infty}_{\mathbb{Q}_p}$

Fact: $Uni^{et}_{\mathbb{Q}_p}(Y)$ is a neutral Tannakian category and ω_y is a fiber functor and $\underline{Aut}^{\otimes}(\omega_y)$ is pro-unipotent.

Exercise: $\underline{Aut}^{\otimes}(\omega_y) \simeq U^{et}$ so, in practice, $Uni^{et}_{\mathbb{Q}_p}(Y) \simeq Rep^{<+\infty}_{\mathbb{Q}_p}(U^{et})$

$\simeq$ unipotent cts $\mathbb{Q}_p$-rep of $\pi_1^{et}(Y_{\bar{k}}, \bar{y})$

Fact: If $y,z \in Y(k)$ $\rightsquigarrow$ $\omega_y, \omega_z : Uni^{et}_{\mathbb{Q}_p}(Y) \longrightarrow Mod^{<+\infty}_{\mathbb{Q}_p}$

then $\pi_1(k) \curvearrowright \underline{Iso}^{\otimes}(\omega_y,\omega_z) =: {}_yP^{et}_z$ continuous and compatible with the action of $U^{et} = \underline{Aut}^{\otimes}(\omega_y)$

i.e. $\pi_1(k) \curvearrowright \underline{Iso}^{\otimes}(\omega_y,\omega_z)(\mathbb{Q}_p)$ continuous

There is a bijection $\left\{ \begin{array}{c} \text{torsors under } U^{et} \\ + \text{ compatible continuous } \pi_1(k)\text{-action} \end{array} \right\} \longleftrightarrow H^1(\pi_1(k), U^{et}(\mathbb{Q}_p))$

Cor: We have a map: $\begin{array}{ccc} Y(k) & \longrightarrow & H^1(\pi_1(k), U^{et}(\mathbb{Q}_p)) \\ y & \longmapsto & [{}_xP^{et}_y] \end{array}$ this is the non abelian Kummer map

$$\begin{tikzcd} Y(k) \arrow[r] \arrow[dr] & H^1(\pi_1(k), \pi_1^{et}(Y_{\bar{k}}, y)^{(p)}) \arrow[r] \arrow[d] & H^1(\pi_1(k), \pi_1^{et}(Y_{\bar{k}}, y)^{(p),ab}) \arrow[d] \\ & H^1(\pi_1(k), U^{et}(\mathbb{Q}_p)) \arrow[r] & H^1(\pi_1(k), U^{et,ab}(\mathbb{Q}_p)) \end{tikzcd}$$

The non abelian Kummer map.

Récap: $k/\mathbb{Q}$ number field
X/k smooth, projective curve of genus g
$x \in X(k)$
v finite place of k $v|p$

$\rightsquigarrow$ $U^{ét}/\mathbb{Q}_p = \pi_1(X_{\bar{k}}, \bar{x})_{\mathbb{Q}_p}$

$\mathbb{Q}_p$ pro-unipotent étale fund. group

and $X(k) \longrightarrow H^1(\pi_1(k), U^{ét}(\mathbb{Q}_p))$
$y \longmapsto [{}_xP^{ét}_y]$

torsor of paths $\mathrm{Iso}^{\otimes}(\omega_x, \omega_y)(\mathbb{Q}_p)$

The above construction works actually for arbitrary base field k and is functorial. So that

$$\begin{array}{ccc} X(k) & \longrightarrow & H^1(\pi_1(k), U^{ét}(\mathbb{Q}_p)) \\ \downarrow & & \downarrow \mathrm{loc}_v \\ X(k_v) & \xrightarrow{j_v} & H^1(\pi_1(k_v), U^{ét}(\mathbb{Q}_p)) \end{array}$$

Aim: understand j_v in a v-adic neighbourhood of x. More precisely,

For a sufficiently small $V \ni x$
- $j_v(V) \subset H^1_g(\pi_1(k_v), U^{ét}(\mathbb{Q}_p))$
  k_v-pts of an affine scheme
- $j_v : V \longrightarrow H^1_g(\pi_1(k_v), U^{ét}(\mathbb{Q}_p))$ is pro-k_v-analytic
- $j_v(V) \subset H^1_g(\pi_1(k_v), U^{ét}(\mathbb{Q}_p))$ is Zariski dense.
  $\downarrow \alpha$ algebraic
  k_v φ
  using dim > dim im loc_v

§ Representations on pro-unipotent groups

Let $U/\mathbb{Q}_p$ be a pro-unipotent finitely generated group scheme
Let $\pi_1(k_v) \curvearrowright U(\mathbb{Q}_p)$ continuous ($\Longleftrightarrow \pi_1(k_v) \curvearrowright \mathcal{O}(U)$ continuous
$\Longleftrightarrow \pi_1(k_v) \curvearrowright \mathrm{Lie}(U)$ ——)

Déf: Λ k_v-algebra $D_{dR}(U)(\Lambda) = U(B_{dR} \otimes_{k_v} \Lambda)^{\pi_1(k_v)}$

Fact: The functor D_{dR}: k_v-algebras $\to$ pro-unipotent groups is representable by a pro-unipotent group scheme U_{dR}/k_v

Déf: One says a pro unipotent f.g. group scheme U equipped with $\pi_1(k_v) \curvearrowright U(\mathbb{Q}_p)$ continuous is de Rham if the following equivalent conditions hold.

i) $D_{dR}(U)_{B_{dR}} \longrightarrow U_{B_{dR}}$ is an isomorphism of group schemes $/B_{dR}$

ii) $\mathcal{O}(U)$ is ind-de Rham

iii) Lie U is pro-de Rham.

Fact: $U^{ét}$ is de Rham.

Define $D_{dR}^+(U)$ similarly with $B_{dR} \longleftarrow B_{dR}^+$

$D_{cris}^{\varphi=1}(U)$ $\qquad B_{cris} \longleftarrow B_{cris}^{\varphi=1}$

Def: Assume $U + \pi_1(k_v) \curvearrowright U(\mathbb{Q}_p)$ is de Rham. Then define

$$H^1_e(\pi_1(k_v), U(\mathbb{Q}_p)) := \ker\left(H^1(\pi_1(k_v), U(\mathbb{Q}_p)) \longrightarrow H^1(\pi_1(k_v), U(B_{cris}^{\varphi=1}))\right)$$

induced by $U(\mathbb{Q}_p) \hookrightarrow U(B_{cris}^{\varphi=1})$

$$\downarrow$$

$$H^1_f(\pi_1(k_v), U(\mathbb{Q}_p)) := \ker(\text{————————} \; B_{cris}))$$

$$\downarrow$$

$$H^1_g(\pi_1(k_v), U(\mathbb{Q}_p)) := \ker(\qquad B_{dR}))$$

or B_{st}

$$\downarrow$$

$$H^1(\pi_1(k_v), U(\mathbb{Q}_p))$$

non abelian Bloch-Kato exponential

Prop: $H^1_e(\pi_1(k_v), U(\mathbb{Q}_p)) \simeq D_{cris}^{\varphi=1}(U)(\mathbb{Q}_p) \backslash D_{dR}(U(\mathbb{Q}_p)) / D_{dR}^+(U)(k_v)$

If $D_{cris}^{\varphi=1}(U) = 1$ then $H^1_f = H^1_e = D_{dR}(U)(k_v) / D_{dR}^+(U)(k_v)$

k_v-pts of an affine scheme / k_v

Fact: $D_{cris}^{\varphi=1}(U^{ét}) = 1$

$$\mathbb{Q}_p \otimes A(k_v) \simeq H^1_f(\pi_1(k_v), V_p(A))$$

Lemma: $\left\{ \begin{array}{l} U\text{-torsors} \\ + \text{ compatible continuous } \pi_1(k_v)\text{-action} \end{array} \right\} / \simeq \;\longleftrightarrow\; H^1(\pi_1(k_v), U(\mathbb{Q}_p))$

because U pro unipotent: $\emptyset_x$

ie $\pi_1(k_v) \curvearrowright P(\mathbb{Q}_p)$ continuous ($\Longleftrightarrow \pi_1(k_v) \curvearrowright \mathcal{O}(P)$ continuous)

$$\cup$$

$$H^1_e = D_{cris}^{\varphi=1}(U)(\mathbb{Q}_p) \backslash D_{dR}(U)(k_v) / D_{dR}^+(U)(k_v)$$

let P be a U-torsor + compatible continuous $\pi_1(k_v)$ action. Then

- $[P] \in H^1_f \Longleftrightarrow P(B_{cris}^{\varphi=1})^{\pi_1(k_v)} \neq \emptyset$
- In that case, $P(B_{dR}^+)^{\pi_1(k_v)} \neq \emptyset$ and $[P] =$ double class containing $q^{-1}r \in U(B_{dR})^{\pi_1(k_v)}$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad D_{dR}(U)$$

where $q \in P(B_{cris}^{\varphi=1})^{\pi_1(k_v)} \subset P(B_{dR})^{\pi_1(k_v)}$

$\qquad r \in P(B_{dR}^+)^{\pi_1(k_v)} \subset P(B_{dR})^{\pi_1(k_v)}$

§ de Rham fundamental groups

<u>Aim</u>: for $y \in X(k_v)$ sufficiently close to x describe pts.

$$\left.\begin{array}{l} q(y) \in {}_xP^{et}_y(B_{cris}^{\varphi=1})^{\pi_1(k_v)} \\ r(y) \in {}_xP^{et}_y(B_{dR}^+)^{\pi_1(k_v)} \end{array}\right) \subset {}_xP^{et}_y(B_{dR})^{\pi_1(k_v)}$$

These should vary "nicely" with y

<u>Def</u>: (de Rham fund. group)

$$\mathrm{Uni}^{dR}_{k_v} = \left\{ \begin{array}{l} \text{algebraic vector bundle } V \text{ on } X_{k_v} \\ + \text{ integral connection } \nabla : V \longrightarrow V \otimes_{\mathcal{O}_X} \Omega^1_{X_{k_v}/k_v} \end{array} \right\} \text{ st } \begin{array}{l} (V, \nabla) \text{ iterated extensions} \\ \text{of } \mathbb{1} = (\mathcal{O}_{X_{k_v}}, d) \end{array} \Big)$$

$x \in X(k_v)$ determines a fibre functor $\omega_x : \mathrm{Uni}^{dR}_{k_v} \longrightarrow \mathrm{Mod}/_{k_v}^{<+\infty}$ which makes $\mathrm{Uni}^{dR}_{k_v}$ a neutral Tannakian category $/ k_v$.

$\curvearrowright$ $U^{dR} := \underline{\mathrm{Aut}}^{\otimes}(\omega_x) / k_v$ = pro unipotent de Rham fund group.

$\curvearrowright$ de Rham path torsor ${}_xP^{dR}_y = \underline{\mathrm{Iso}}^{\otimes}(\omega_x, \omega_y)$.

<u>Rmk</u>: $\mathcal{O}({}_xP^{dR}_y)$ carries a Hodge filtration.

<u>Big Thm!</u> (Andreatta - Iovita - Kim, Bhatt - Morrow - Scholze)

$\exists$ $\pi_1(k_v)$ - equivariant, filtration preserving isomorphisms

$$B_{dR} \otimes_{\mathbb{Q}_p} \mathcal{O}(U^{et}) \xrightarrow{\sim} B_{dR} \otimes_{k_v} \mathcal{O}(U^{dR}).$$

$$B_{dR} \otimes_{\mathbb{Q}_p} \mathcal{O}({}_xP^{et}_y) \xrightarrow{\sim} B_{dR} \otimes_{k_v} \mathcal{O}({}_xP^{dR}_y) \quad \forall y \in X(k_v)$$

<u>Cor</u>: ${}_xP^{et}_y(B_{dR})^{\pi_1(k_v)} = {}_xP^{dR}_y(k_v)$.

As a result, want to package up ${}_xP^{dR}_y$ as y varies

<u>Lemma</u>: $\omega_x : \mathrm{Uni}^{dR}_{k_v} \longrightarrow \mathrm{Mod}^{<+\infty}_{k_v}$ is representable by $\mathcal{E}^{dR} \in \text{pro-}\mathrm{Uni}^{dR}_{k_v}$ and $e \in \mathcal{E}^{dR}_x = \omega_x(\mathcal{E}^{dR})$. $\mathcal{E}^{dR}$ is a "coalgebra":

$$\Delta : \mathcal{E}^{dR} \longrightarrow \mathcal{E}^{dR} \otimes \mathcal{E}^{dR}, \qquad \varepsilon : \mathcal{E}^{dR} \longrightarrow \mathbb{1} = (\mathcal{O}_{X_{k_v}}, d)$$

$$\begin{array}{ccccccc} \varprojlim_i \mathcal{E}_i & & \varprojlim_i \mathcal{E}_i \otimes \mathcal{E}_i & \qquad & e & \longrightarrow & 1 \\ & & & & \cap & & \cap \\ e & \longrightarrow & e \hat{\otimes} e & & \mathcal{E}^{dR}_x & & \mathcal{O}_{X,x} = k_v \\ \cap & & \cap & & & & \\ \mathcal{E}^{dR}_x & & \mathcal{E}^{dR}_x \otimes \mathcal{E}^{dR}_x & & & & \end{array}$$

<u>If</u> $y \in X(k_v)$ then $\mathcal{O}({}_xP^{dR}_y)^* \xrightarrow{\sim} \mathcal{E}^{dR}_y$ isomorphism of coalgebras

$${}_xP^{dR}_y(k_v) \xrightarrow{\sim} (\mathcal{E}^{dR}_y)^{\text{group-like}}$$

$$\| $$

$$\left\{ \gamma \in \mathcal{E}^{dR}_y \;\middle|\; \begin{array}{l} \Delta(\gamma) = \gamma \otimes \gamma \\ \varepsilon(\gamma) = 1 \end{array} \right\}$$

... X curve

"group-like sections" of $\mathcal{E}^{dR}$ should give compatible families of elements $y \in {}_xP_y^{dR}(k_v)$

Big thm 2 (Andreatta-Iovita-Kim)

- On any Zariski open affine $Y \subset X$ $\exists$ a group like algebraic section r of $\mathcal{E}^{dR}$ st $\forall y \in Y(k_v)$ $r(y) \in {}_xP_y^{dR}(k_v)$ $\left(= {}_xP_y^{et}(B_{dR})^{\pi_1(k_v)}\right)$ lands in ${}_xP_y^{et}(B_{dR}^+)^{\pi_1(k_v)}$
- On a v-adic neighbourhood V of x st $\exists !$ k_v-analytic section g of $\mathcal{E}^{dR}$ st that $g(x) = e \in \mathcal{E}_x^{dR}$ and $\nabla g = 0$.

Furthermore, g is group-like and $g(y) \in {}_xP_y^{et}(B_{cris}^{\varphi=1})^{\pi_1(k_v)}$ $\Rightarrow$ $j_v(y) \in H^1_e \subset H^1_g$

$\Rightarrow$ $j_v|_V$ is pro-k_v-analytic since on V $j_v(y) = g(y)^{-1} r(y)$

§ Local denseness: $j_v(V) \subset H^1_g$ is Zariski-dense

Does not seem to follow from abstract theory $\curvearrowright$ really need to carry out computations here!

- Pass to an Zariski open affine curve $Y \subset X$, $Y \ni x$

for 2 reasons: - algebraic section r exists globally
- all fund group of Y are free on $\chi = \chi(Y)$ generators.

- Prop (Kim) let $k_v\langle\langle z_1, \dots, z_\chi\rangle\rangle$ be non-commutative power series.

$k_v\langle\langle z_1, \dots, z_\chi\rangle\rangle \otimes \mathcal{O}_{Y_{k_v}}$ vector bundle on Y_{k_v} endowed with connection ∇:

$$\nabla\Big(\sum_{\substack{w \\ \text{words in } z_1,\dots,z_\chi}} [w] f_w\Big) = \sum_w [w]\, df_w - \sum_{i=1}^{\chi} [z_i w] f_w\, \omega_i$$

where we chose $\omega_1, \dots, \omega_\chi$ algebraic differentials on Y_{k_v} whose classes are a basis for $H^1_{dR}(Y_{k_v})$

Then $\mathcal{E}^{dR} = (k_v\langle\langle z\rangle\rangle \otimes \mathcal{O}_{Y_{k_v}}, \nabla)$ st $e \in \mathcal{E}_x^{dR} \longleftrightarrow 1 \in k_v\langle\langle z\rangle\rangle$

- Key claim: Choose local parameter t at x. $\exists !$ $g \in k_v\langle\langle z\rangle\rangle \otimes \mathcal{O}_{Y_{k_v}}(k_v[[t]])$

$dg_\emptyset = 0 \to g_\emptyset(w) = 1$ everywhere formal

$dg_{z_i w} = g_w \omega_i$ $\Big\}$ integrate formally in $k[[t]]\,dt$ etc.

$= \sum_w [w] g_w(t)$ $\quad$ $\| \quad k\langle\langle z\rangle\rangle \otimes k_v[[t]]$

st $g(0) = 1$ i.e. $g_\emptyset(0) = 1$, $g_w(0) = 0$ $w \neq \emptyset$

Then the $g_w \in k_v[[t]]$ are linearly independent over $k_v(Y_{k_v})$

and $\nabla g = 0$ (flat section)

$\in (k\langle\langle z\rangle\rangle \otimes \Omega_{Y_{k_v}/k_v})(k_v[[t]])$

Why local denseness? $\mathcal{O}(U^{dR}) = k_v\langle\langle z \rangle\rangle^\circ = \varinjlim \left(k_v\langle z\rangle / J^{n+1} \right)^\vee$ $\left(\neq \left(\varprojlim k_v\langle z\rangle / J^{n+1} \right)^\vee \right)$

a basis of $\mathcal{O}(U^{dR})$ is given by "duals of words" i.e. w^* st

$$w^*(w') = \begin{cases} 1 & \text{if } w' = w \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{ccc} X(k_v) & \longrightarrow & H^1(\pi_1(k_v), U^{et}(\mathbb{Q}_p)) \\ & & \cup \\ & & H^1_f = U^{dR}(k_v) / F^0 U^{dR}(k_v) \\ & & \uparrow \\ & & U^{dR}(k_v) \end{array}$$

Our description of $\mathcal{E}^{dR}$ gives an identification

$$\begin{array}{ccc} {}_xP_y^{dR}(k_v) & \xrightarrow{\sim} & U^{dR}(k_v) \\ \psi & & \psi \\ (q(y), r(y)) & \longmapsto & (q'(y), r'(y)) \end{array}$$

and $j_v(y) = q'(y)^{-1} r'(y)$ with $q' : V \longrightarrow U^{dR}(k_v)$ k_v-analytic

$r' : Y(k_v) \longrightarrow U^{dR}(k_v)$ k_v-algebraic

then $j_v(V)$ is not Z-dense iff $\exists \sum_w \lambda_w w^* \neq 0$ in $\mathcal{O}(U^{dR})$ st

$$\left(\sum_w \lambda_w w^* \right) \left(q'^{-1}(\cdot)\, r'(\cdot) \right) = 0$$

the point is if $r' = 1$ then $w^* \circ q' = q_w$ hence $\sum_w \lambda_w q_w = 0$ ↯ linear indep

if $r' \neq 1$ then $w^* \circ (q'^{-1} r')$ and $w^* \circ q'$ are related by an invertible matrix with algebraic coefficients ↯ linear indep.