



A belts - The Chabauty-Kim method

k number field
 X/k smooth, complete curve of genus g
 $r \in X(k)$
 p prime (for simplicity today)

Trichotomy

$g=0$	$X(k)$ infinite
$g=1$	$X(k)$ f.g. abelian group
$g \geq 2$	$X(k)$ finite (Mordell Conj. Faltings)

Observation: this trichotomy unites a connection between the "diophantine geometry" & the "algebraic topology" on X .

Q: Can we use analogues of tools from algebraic topology to prove cases of the Mordell Conj. concretely

Answer 1 (Arithmetic): $1 \rightarrow \pi_1(X_{\bar{k}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \pi_1(k) \rightarrow 1$

\exists map $X(k) \xrightarrow{\sigma} H^1(\pi_1(k), \pi_1(X_{\bar{k}}, \bar{x}))$
 non abelian continuous Galois coh.
 gives a natural action of $\pi_1(k)$ on $\pi_1(X_{\bar{k}}, \bar{x})$

Conj (Serre Conj): σ is bijective

Widely open, not known for any X .

Answer 2 (Chabauty + reinterpretation)

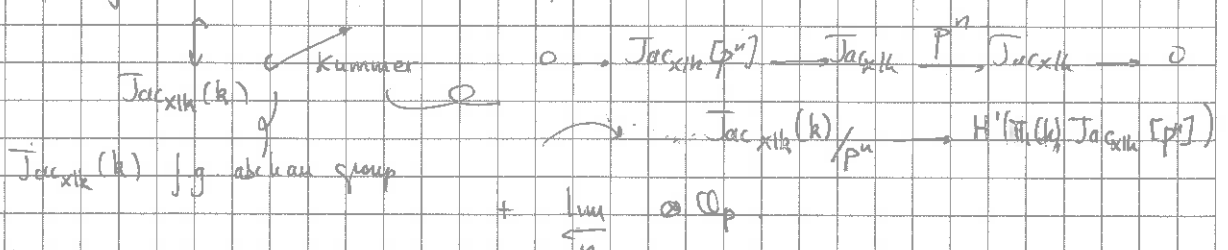
Original idea: try to combine two constraints on $X(k)$

$$\begin{matrix} X(k) \hookrightarrow X(k_p) & \text{is finite place on } k. \\ X(k) \hookrightarrow \text{Jac}_{X/k}(k) \end{matrix}$$

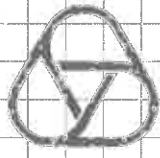
Thm (Chabauty, 1941) If rank $\text{Jac}_{X/k}(k) < g_X$ then, \forall is finite place on k
 $|X(k_p) \cap \text{Jac}_{X/k}(k)| < +\infty$

Can we replace $\text{Jac}_{X/k}(k)$ with $H_1(X_{\bar{k}}, \mathbb{Q}_p) = H^1(X_{\bar{k}}, \mathbb{Q}_p)^\vee = V_p(\text{Jac}_{X/k}) =: V^?$ with $p \neq \text{char } k$

Kummer map $f: X(k) \rightarrow H^1(\pi_1(k), V)$



Have a fundamental diagram



(D+)

$$X(k) \subset \subset H^1(\Pi_1(k), V)$$

$$\downarrow \quad \downarrow \text{loc}$$

$$X(k_v) \xrightarrow{d_v} H^1(\Pi_1(k_v), V)$$

can be identified with Dr decomposition group

Facts: 1) $\text{Im}(X(k_v)) \subset H^1_{\text{cpt}}(\Pi_1(k_v), V)$ "cuplike part"

$$\ker(H^1(\Pi_1(k_v), V) \rightarrow H^1(\Pi_1(k_v), V \otimes B_{\text{crys}}))$$

shaded by Bloch & Kato
e.g. $H^1_{\text{cpt}}(\Pi_1(k_v), V)$ is a k_v -v.s of dim g

2) $\text{Im}(X(k)) \subset H^1_{\text{cpt}}(\Pi_1(k), V)$ "uncuplike / cuplike part"

common kernel of

$$\begin{array}{ccc} H^1(\Pi_1(k), V) & \rightarrow & H^1(\Pi_1(k_v), V) & \rightarrow & H^1(\Pi_1(k_v), V \otimes B_{\text{crys}}) \\ H^1(\Pi_1(k), V) & \rightarrow & H^1(\Pi_1(k_v), V) & \rightarrow & H^1(\Pi_1(k_v), V \otimes B_{\text{crys}}) \end{array}$$

Kupp
Kupp
Merkle group

3) f_v is locally k_v -analytic & locally dense in

3.i) $X(k_v)$ can be covered by closed discs so that $f_v|_{\text{Disc}}$ is given component-wise by convergent power series

3.ii) $\text{Im}(f_v) \not\subset \text{any hyperplane in } H^1_{\text{cpt}}(\Pi_1(k_v), V)$

Thm (Chabauty revisited): $\prod_{\text{primes } p} \dim_{\mathbb{Q}_p} H^1_{\text{cpt}}(\Pi_1(k), V) < g = \dim H^1_{\text{cpt}}(\Pi_1(k), V)$ then $|X(k)| < +\infty$

Rem: $\text{rank Jac}_{X/k}(k) \leq \dim_{\mathbb{Q}_p} H^1_{\text{cpt}}(\Pi_1(k), V)$ with $=$ iff $|\text{Im Jac}_{X/k}[p^\infty]| < +\infty$

Proof: (*) $\rightarrow \text{loc}_v(H^1_{\text{cpt}}(\Pi_1(k), V))$ is contained in a proper k_v -hyperplane in $H^1_{\text{cpt}}(\Pi_1(k_v), V)$
 Choose $\alpha: H^1_{\text{cpt}}(\Pi_1(k_v), V) \rightarrow k_v$ non zero, vanishing on $\text{loc}_v(H^1_{\text{cpt}}(\Pi_1(k), V))$
 Then $\alpha \circ f_v: X(k_v) \rightarrow k_v$ vanishes on $X(k)$
 - is locally k_v -analytic
 - non vanishing on open subsets of $X(k_v)$) $\alpha \circ f_v$ has only finitely many zeros

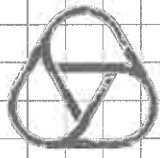
§ Non abelian Chabauty

Observation: $H_2(X_{\bar{k}}, \mathbb{Q}_p) = \Pi_1(X_{\bar{k}}, \bar{x})^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}_p$

Q: Do we get more information from Chabauty if we replace $\Pi_1(X_{\bar{k}}, \bar{x})^{\text{ab}}$ with $\Pi_1(X_{\bar{k}}, \bar{x})$?

Answer (Kim): Yes!

Step 1: Linearize $\Pi_1(X_{\bar{k}}, \bar{x})$ (on this day): Define a pro-unipotent étale fund. group $U = U^d/\mathbb{Q}_p$



- * U is a f.g. pro-unipotent group / \mathbb{Q}_p .
- * $\Pi_1(k) \curvearrowright U(\mathbb{Q}_p)$ continuously
- * " $U = \Pi_1(X_k, \bar{x}) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ " continuous Néron completion

Step 2: Non abelian Kummer maps

global $X(k) \xrightarrow{d} H^1(\Pi_1(k), U(\mathbb{Q}_p))$
 local $X(k_v) \xrightarrow{d_v} H^1(\Pi_1(k_v), U(\mathbb{Q}_p))$) getting in a fundamental commutative diagram lifting (D1)

$$(D2) \quad \begin{array}{ccc} X(k) & \xrightarrow{d} & H^1(\Pi_1(k), U(\mathbb{Q}_p)) \\ \downarrow & & \downarrow \text{loc}_v \\ X(k_v) & \xrightarrow{d_v} & H^1(\Pi_1(k_v), U(\mathbb{Q}_p)) \end{array}$$

Assume for simplicity that X has semistable reduction at v .

$X(k)^\circ := \{y \in X(k) \mid x \rightarrow y \text{ reduce onto the same component of all special fibers of a semistable model}\}$

Define $X(k_v)^\circ$ similarly

Fact: 1) $j_v(X(k_v)^\circ) \subset H^1_j(\Pi_1(k_v), U(\mathbb{Q}_p))$ "cupulline part"

$$\ker(H^1(\Pi_1(k_v), U(\mathbb{Q}_p)) \rightarrow H^1(\Pi_1(k_v), U(\mathbb{Q}_p))) \rightarrow H^1(\Pi_1(k_v), U(\mathbb{Q}_p))$$

and $H^1_j(\Pi_1(k_v), U(\mathbb{Q}_p))$ is canonically the k_v -pts of a connected affine scheme $H^1_{j,v}/k_v$

2) $j(X(k)^\circ) \subset H^1_j(\Pi_1(k), U(\mathbb{Q}_p))$

$$\begin{array}{ccccc} \text{common kernel of } & H^1(\Pi_1(k), U(\mathbb{Q}_p)) & \rightarrow & H^1(\Pi_1(k_v), U(\mathbb{Q}_p)) & \rightarrow & H^1(\Pi_1(k_v), U(\mathbb{Q}_p))_{\text{cup}} \\ & H^1(\Pi_1(k), U(\mathbb{Q}_p)) & \rightarrow & H^1(\Pi_1(k_v), U(\mathbb{Q}_p)) & \rightarrow & H^1(\Pi_1(k_v), U(\mathbb{Q}_p))_{\text{cup}} \end{array}$$

and $H^1_j(\Pi_1(k), U(\mathbb{Q}_p))$ is canonically the \mathbb{Q}_p -pts of a connected affine scheme H^1_j/\mathbb{Q}_p

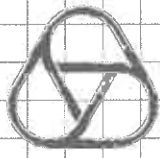
and $\text{loc}_v: H^1_j \rightarrow H^1_{j,v}$ is a map of \mathbb{Q}_p -schemes

3) $j_v: X(k_v)^\circ \rightarrow H^1_j(\Pi_1(k_v), U(\mathbb{Q}_p))$ is locally pro- k_v -analytic and locally Zariski-dense.

(locally dense means $j_v(V) \subset H^1_j \Rightarrow$ Zariski-dense $\forall \mathbb{Q}_p$ - $V \subset \text{supp } X(k_v)$)

Thm (Kim) [1] $\text{loc}_v(H^1_j(\Pi_1(k), U(\mathbb{Q}_p)))$ is not \mathbb{Z} -dense in $H^1_j(\Pi_1(k_v), U(\mathbb{Q}_p))$
Thm $X(k)^\circ$ is finite

Proof: Same as Chabauty restricted: Condition $\rightarrow \exists$ a non zero algebraic morphism

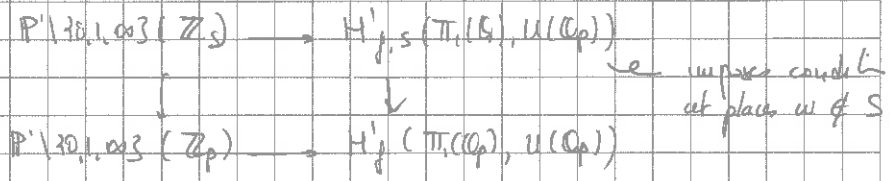


$\alpha: H_j^1(\pi_1(k_v), U(\mathbb{O}_p)) \rightarrow k_v$ at α locus vanishes on $H_j^1(\pi_1(k), U(\mathbb{O}_p))$

Then $\alpha \circ \beta_v: X(k_v)^0 \rightarrow k_v^*$ vanishes on $X(k_v)$
 is locally k_v -analytic
 non vanishing on open subsets of $X(k_v)^*$

Application: $k = \mathbb{Q}$
 S finite set of places of k } Then $P^1(0,1,\infty,3)(\mathbb{Z}_S)$ is finite
 $\mathbb{Z}[\frac{1}{S}]$
 (v \exists only finitely many $a, b \in \mathbb{Z}_S^*$ st $a+b=1$)

Proof: Choose $p=v \notin S$ and any base pt $x \in P^1(0,1,\infty,3)(\mathbb{Z}_S)$. Have
 finite commutative diagram



Then 1) U is free pro-unipotent on r generators $\Rightarrow a, b \in U(\mathbb{O}_p)$ st $\forall u \in U(\mathbb{O}_p)$
 has a unique decomposition as
 $u = a^{d_1} b^{d_2} [a,b]^{d_3} [a,[a,b]]^{d_4} [b,[a,b]]^{d_5}$ with $d_i \in \mathbb{O}_p$

Define $U_n := U / n^{th}$ terms commutators. U_n is unipotent (u of finite type / \mathbb{O}_p)
 and, more precisely, as a $\pi_1(k)$ -module, it is an extension of $(\mathbb{O}_p/\pi)^2$ by (\mathbb{O}_p/π) by $(\mathbb{O}_p/\pi)^2$ by (\mathbb{O}_p/π) by $(\mathbb{O}_p/\pi)^n$ $k_v \geq 1$

2) Dimension calculation:

$$\dim H_j^1(\pi_1(k_v), U_n(\mathbb{O}_p)) = \sum_{r=1}^n k_r \dim H_j^1(\pi_1(k_v), (\mathbb{O}_p/\pi^r)) = 1 \text{ Bloch-Kato}$$

$$\dim H_{j,S}^1(\pi_1(k_v), U_n(\mathbb{O}_p)) \leq \sum_{r=1}^n k_r \dim H_{j,S}^1(\pi_1(k_v), (\mathbb{O}_p/\pi^r)) \leq r|S| + \sum_{z \in \pi^n} k_z$$

$$H_{j,S}^1(\pi_1(\mathbb{O}), (\mathbb{O}_p/\pi)) = \mathbb{O}_p \otimes \mathbb{Z}_S^{\otimes j}$$

Soub' $\Rightarrow \dim H_{j,S}^1(\pi_1(k_v), (\mathbb{O}_p/\pi^r)) \leq \begin{cases} 1 & \text{if } r \geq 2 \text{ even} \\ 0 & \text{if } r \geq 3 \text{ odd} \end{cases}$

Conclusion: $\dim H_{j,S}^1(\pi_1(\mathbb{O}), U_n(\mathbb{O}_p)) < \dim H_j^1(\pi_1(\mathbb{O}_p), U_n(\mathbb{O}_p))$ for $n \gg 0$
 \Rightarrow loc. anal. \neq desc.
 $\Rightarrow P^1(0,1,\infty,3)(\mathbb{Z}_S)$ finite.



§ Linearizing fundamental groups.

Q: how to make (étale) fund groups into linear objects? i.e. what do we mean by " $G \otimes_{\mathbb{Z}} \mathbb{Q}$ "?

Not: k field of char 0
 Y/k smooth, connected, separated, of F^1/k
 $y \in Y(k) \implies Y$ geo connected / k .

Answer: Mal'cev completion

If Π is a f.g. discrete group, what do we mean by " $G \otimes_{\mathbb{Z}} \Pi$ "?

Ex: 1) Π abelian just take usual tensor product $G \otimes_{\mathbb{Z}} \Pi := G \otimes_{\mathbb{Z}} (\Pi / \Pi_{tors})$

2) Π 2-steps nilpotent i.e. $[a, [b, c]] = 1, a, b, c \in \Pi$



guess: $G \otimes_{\mathbb{Z}} \Pi$ should be a central extension of the divisible group $G \otimes_{\mathbb{Z}} \Pi^{ab}$ by divisible group $G \otimes_{\mathbb{Z}} [\Pi, \Pi]$

3) Π n -steps nilpotent i.e. there is a sequence of central extensions



with $\Pi_0 = 1, \Pi_1 = \Pi$

guess: $G \otimes_{\mathbb{Z}} \Pi$ should be iterated central extensions of divisible groups $G \otimes_{\mathbb{Z}} \Pi_i$

Def: (pro-unipotent groups) F char 0 field. An affine group scheme U/F is pro-unipotent if the following equiv condition hold

- 1) Any non zero F -rep of U has a non zero fixed vector
- 2) U is a (transfinite) iterated extensions of vector groups i.e. G_a^m
- 3) U is a central extensions of G_a

transfinite: $U = \varprojlim_{\lambda \in \alpha} U_\lambda$ with $1 \rightarrow \ker \rightarrow U_\lambda \rightarrow U \rightarrow 1$
 \subseteq affine group schemes / F is G_a^m
 with $U_0 = G_a^m$
 if F is a limit then $U_\beta = \varprojlim_{\lambda \in \beta} U_\lambda$

Remark: if U is f.g. then can take $\alpha = \omega \implies U_n$ finite dim

Def: (Mal'cev completion) F char 0 field, Π discrete group. The F -Mal'cev completion of Π is the pro-unipotent group $\Pi(F)/F$ representing the functor

$$\{ \text{pro-unipotent group } / F \} \rightarrow \text{Set} \\ U \rightarrow \text{Hom}_{\text{top}}(\Pi, U(F))$$



Equivalently, we have $\Pi \xrightarrow{P} \Pi_F(F)$ at $\forall U/F$ pro-unipotent group and $\Pi \xrightarrow{\varphi} U(F)$
 $\exists ! \Pi_F \xrightarrow{\varphi_P} U$ morphism of group schemes / F

$$\begin{array}{ccc} P & \nearrow & \varphi_P(F) \\ \Pi_F(F) & & \end{array}$$

Variant: Π, Π_F, Π carry topology. The continuous F -Matric completion of Π is Π_F/F representing the functor

$$\{ \text{pro-unipotent group} / F \} \rightarrow \text{Set} \\ U \mapsto \text{Hom}_{\text{cont Grp}}(\Pi, U(F))$$

Proof of existence: Special adjoint functor theorem.

Lemma: Π f.g. discrete group.
 $\Pi_{n+1} = \Pi / \langle \text{n+1 tuple commutators} \rangle$

$$1 \rightarrow \Pi_{n+1} \rightarrow \Pi_n \rightarrow \Pi_{n-1} \rightarrow 1 \text{ central ext}$$

Then Π_F is an iterated central ext of the vector groups corresponding to $\mathbb{Q} \otimes \Pi_{n-1}, n \in \mathbb{N}$

$$\Pi_F = \varprojlim_n \Pi_n, F$$

Lemma: Π f.g. discrete group, $F[\Pi]$ usual group algebra (cocommutative Hopf algebra)

$$\downarrow \varepsilon \text{ augmentation morphism - counit}$$

$$J := \ker(F[\Pi] \xrightarrow{\varepsilon} F) \text{ augmentation ideal}$$

Define the dualized J -adic completion of $F[\Pi]$ as $F[\Pi]^{J^0} = \varprojlim_n \text{Hom}_F(F[\Pi]/J^{n+1}, F)$

Then $F[\Pi]^{J^0}$ is a commutative Hopf algebra and $\Pi_F = \text{Spec}(F[\Pi]^{J^0})$

Remark: If Π top f.g. profinite group and $F = \mathbb{C}_p$ there is a similar description

$$\Pi_{\mathbb{C}_p} = \text{Spec}(\mathbb{C}_p[[\Pi]]^{J^0})$$

$$\mathbb{C}_p \otimes_{\mathbb{Z}_p} \varprojlim_{\Pi \rightarrow \Pi / \text{finite}} \mathbb{Z}_p[[\Pi]]$$

eg $\Pi = \text{free profinite group on } r \text{ generators}$
 $\mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \langle T_1, \dots, T_r \rangle$
 $\mathbb{C}_p \langle T_1, \dots, T_r \rangle = \mathbb{C}_p[[\Pi]]^{J^0}$

Remark: By universal property of Matric completion, if Π is a f.g. discrete group then

$$\Pi_{\mathbb{C}_p} = \widehat{\Pi}_{\mathbb{C}_p}$$

usual (discrete) \mathbb{C}_p -Matric compl of Π continuous \mathbb{C}_p -Matric completion of $\widehat{\Pi}$



Def: 1) Fix $K \subset \sigma \subset \mathbb{C}$ $U^B := U^B(Y, y) = \pi_1^{top}(Y(\sigma), y) \otimes \mathbb{Q}$
 \mathbb{Q}_2 Natür completion of $\pi_1^{top}(Y(\sigma), y)$

2) $U^{et} := U^{et}(Y, y) = \pi_1^{et}(Y, y) \otimes \mathbb{Q}$
 \mathbb{Q}_p continuous Natür completion of $\pi_1^{et}(Y, y)$.

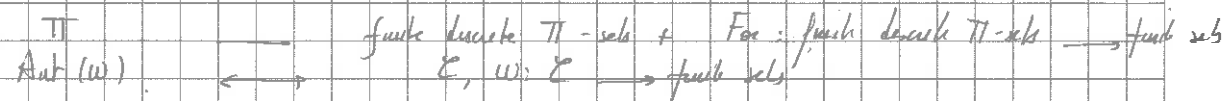
Prop: 1) $\pi_1(K) \hookrightarrow \pi_1^{et}(Y, y)$ continuously $\longrightarrow \pi_1(K) \hookrightarrow U^{et}(\mathbb{Q}_p)$ continuously.

2) Given $K \subset \sigma \subset \mathbb{C}$ $\pi_1^{et}(Y, y) \sim \widehat{\pi_1^{top}(Y(\sigma), y)}$ hence $U^{et} = U^B \otimes_{\mathbb{Q}} \mathbb{Q}_p$

§ Tannakian formalism:

Grothendieck's Galois theory: have an equivalence of categories

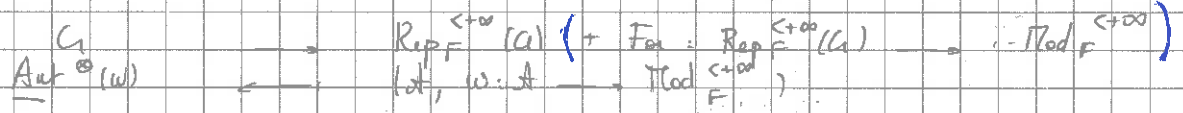
{ profinite groups } \longleftrightarrow "categories which behave like the finite étale covers of a connected scheme + fiber functor"



Idea: linearize categories of covers \hookrightarrow local systems.

Thm (Tannaka) F char 0 field, there is an equivalence of categories

{ affine group schemes / F } \longleftrightarrow "neutral categories which behave like the F -local systems on a connected scheme (+ fiber functor)"



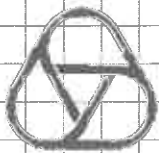
Def (sketch) a rigid F -linear abelian \otimes -category A is a F -linear category A equipped with a \otimes -operation $\otimes: A \times A \longrightarrow A$ satisfying the usual conditions.

- a \otimes -unit $1 \in A$
 a dual operation $A \xrightarrow{v} A^{op}$
 $A \xrightarrow{v} A^{op}$

A fiber functor $\omega: A \longrightarrow \text{Mod}_F^{<+\infty}$ is an exact, conservative, F -linear functor which is compatible with \otimes -structure (in part $\omega(V \otimes W) = \omega(V) \otimes \omega(W)$)
 $\omega(1) = 1$
 $\omega(W^v) = \omega(W)^v$

A neutral Tannakian category A is a rigid F -linear abelian \otimes -category for which $\exists \geq 1$ fiber functors

Def: A neutral Tannakian category A / F
 $\omega, \omega': A \longrightarrow \text{Mod}_F^{<+\infty}$ fiber functors
 $\Gamma(A)$ is an F -algebra, define $\text{Iso}^\otimes(\omega, \omega')(A)$
 $\text{Isom}^\otimes(A, \text{Mod}_F^{<+\infty}) \cong \text{Isom}^\otimes(A, \text{Mod}_F^{<+\infty}) \cong \text{Isom}^\otimes(A, \text{Mod}_F^{<+\infty})$



Fact: $\underline{\text{Iso}}^\circ(w, w')(\Lambda)$ is functorial in Λ and representable by a non-empty affine scheme $\underline{\text{Iso}}^\circ(w, w')/F$

The schemes carry operations:

composition $\underline{\text{Iso}}^\circ(w, w') \times_F \underline{\text{Iso}}^\circ(w', w'') \rightarrow \underline{\text{Iso}}^\circ(w, w'')$

identity $1_w \in \underline{\text{Iso}}^\circ(w, w)(F)$

inversion $\underline{\text{Iso}}^\circ(w, w') \rightarrow \underline{\text{Iso}}^\circ(w', w)$

In part $\underline{\text{Aut}}^\circ(w) := \underline{\text{Iso}}^\circ(w, w)$ is an affine group scheme

$\underline{\text{Iso}}^\circ(w, w')$ is a right-bisor under $\underline{\text{Aut}}^\circ(w)$

Remark: In Tannakian correspondence $k/F \leftarrow A = \text{Rep}_F^{<+\infty}(C_1)$

C_1/F pro-unipotent \iff every object in $\text{Rep}_F^{<+\infty}(C_1)$ is a finite iterated extension of 1

Ex: $\text{Uni}_{\mathcal{O}_p}^{\text{ét}}(Y) =$ category of loc \mathcal{O}_p -sheaves on $Y_{\mathbb{K}}$ which are finite iterated extensions of \mathcal{O}_p

$y \in Y(k)$ gives a functor $\omega_y: \text{Uni}_{\mathcal{O}_p}^{\text{ét}}(Y) \rightarrow \text{Mod}_{\mathcal{O}_p}^{<+\infty}$

Fact: $\text{Uni}_{\mathcal{O}_p}^{\text{ét}}(Y)$ is a neutral Tannakian category and ω_y is a fiber functor and $\underline{\text{Aut}}^\circ(\omega_y)$ is pro-unipotent.

Exercise: $\underline{\text{Aut}}^\circ(\omega_y) \simeq U^{\text{ét}}$ so, in practice, $\text{Uni}_{\mathcal{O}_p}^{\text{ét}}(Y) \simeq \text{Rep}_{\mathcal{O}_p}^{<+\infty}(U^{\text{ét}})$
unipotent part of \mathcal{O}_p -rep of $\Pi_1^{\text{ét}}(Y_{\mathbb{K}}, y)$

Fact: If $y, z \in Y(k) \rightsquigarrow \omega_y, \omega_z: \text{Uni}_{\mathcal{O}_p}^{\text{ét}}(Y) \rightarrow \text{Mod}_{\mathcal{O}_p}^{<+\infty}$

then $\Pi_1(k) \simeq \underline{\text{Iso}}^\circ(\omega_y, \omega_z) =: y P_z^{\text{ét}}$ continuous and compatible with the action of $U^{\text{ét}} = \underline{\text{Aut}}^\circ(\omega_y)$ \iff $\varphi \in \Pi_1(k) \rightsquigarrow \underline{\text{Iso}}^\circ(\omega_y, \omega_z)(\mathcal{O}_p)$ continuous

There is a bijection $\left\{ \begin{array}{l} \text{fibers under } U^{\text{ét}} \\ + \text{ compatible continuous } \Pi_1(k)\text{-action} \end{array} \right\} \longleftrightarrow H^1(\Pi_1(k), U^{\text{ét}}(\mathcal{O}_p))$

Cor: We have a map: $Y(k) \xrightarrow{\omega_y} H^1(\Pi_1(k), U^{\text{ét}}(\mathcal{O}_p))$ This is the non-abelian Kummer map.
 $\downarrow \omega_z$ $\downarrow \omega_y$ $\downarrow \omega_y$
 $\xrightarrow{\omega_z} [z P_y^{\text{ét}}]$

$$\begin{array}{ccccc}
 Y(k) & \xrightarrow{\omega_y} & H^1(\Pi_1(k), \Pi_1^{\text{ét}}(Y_{\mathbb{K}}, y)^{(p)}) & \xrightarrow{\omega_y} & H^1(\Pi_1(k), \Pi_1^{\text{ét}}(Y_{\mathbb{K}}, y)^{(p), \text{ab}}) \\
 & \searrow \omega_z & \downarrow \omega_z & & \downarrow \omega_z \\
 & & H^1(\Pi_1(k), U^{\text{ét}}(\mathcal{O}_p)) & \xrightarrow{\omega_z} & H^1(\Pi_1(k), U^{\text{ét}}(\mathcal{O}_p))
 \end{array}$$



The non-abelian Kummer map.

Recap: k/\mathbb{Q}_p number field
 X/k smooth, projective curve of genus g
 $x \in X(k)$
 v finite place of k v.p.
 $U/\mathbb{Q}_p = \Pi(X_{\bar{k}}, \bar{x})_{\mathbb{Q}_p}$
 \mathbb{Q}_p pro-unipotent étale fund. group

and $X(k) \xrightarrow{\psi} H^1(\Pi_v(k), U^{ét}(\mathbb{Q}_p))$
 \downarrow
 $[x \ P^{\text{ét}} \ y]$
 \uparrow
 tensor of paths $\text{Iso}^{\otimes}(\omega_x \omega_y)(\mathbb{Q}_p)$

The above construction works actually for arbitrary base field k and is functorial. So that

$X(k) \xrightarrow{\psi} H^1(\Pi_v(k), U^{ét}(\mathbb{Q}_p))$
 $\downarrow \text{loc}_v$
 $X(k_v) \xrightarrow{\psi_v} H^1(\Pi_v(k_v), U^{ét}(\mathbb{Q}_p))$

Hint: understand j_v is a v -adic neighbourhood of x . More precisely,

For a sufficiently small $V \ni x$ $j_v(V) \subset H^1(\Pi_v(k_v), U^{ét}(\mathbb{Q}_p))$
 \uparrow
 k_v -pts of an affine scheme

$j_v: V \rightarrow H^1(\Pi_v(k_v), U^{ét}(\mathbb{Q}_p))$ is pro- k_v -analytic

$j_v(V) \subset H^1(\Pi_v(k_v), U^{ét}(\mathbb{Q}_p))$ is Zariski dense.
 $\downarrow \alpha_v$ algebraic
 $k_v \uparrow$
 using $\dim > \dim \text{ind-loc}_v$

§ Representations on pro-unipotent groups

let U/\mathbb{Q}_p be a pro-unipotent finitely generated group scheme
 let $\Pi_v(k_v) \curvearrowright U(\mathbb{Q}_p)$ continuous $(\hookrightarrow \Pi_v(k_v) \curvearrowright \mathcal{O}(U) \text{ continuous})$
 $\hookrightarrow \Pi_v(k_v) \curvearrowright \text{Lie}(U) \text{ --- })$

Def: Λ k_v -algebra $\text{D}_{\text{DR}}(U/\Lambda) = U(\text{BDR}_{k_v} \otimes_{k_v} \Lambda)^{\Pi_v(k_v)}$

Fact: The functor $\text{D}_{\text{DR}}: k_v\text{-algebras} \rightarrow \text{pro-unipotent groups}$ is representable by a pro-unipotent group scheme $U_{\text{DR}}/\text{BDR}_{k_v}$

Def: One says a pro-unipotent f.g. group scheme U equipped with $\Pi_v(k_v) \curvearrowright U(\mathbb{Q}_p)$ continuous is de Rham if the following equivalent conditions hold

- i) $\text{D}_{\text{DR}}(U)_{\text{BDR}} \hookrightarrow U_{\text{BDR}}$ is an isomorphism of group schemes / BDR_{k_v}
- ii) $\mathcal{O}(U)$ is ind-de Rham
- iii) $\text{Lie } U$ is pro-de Rham



Fact: U^{et} is de Rham

Define $D_{DR}^+(U)$ similarly with $B_{dR} \leftarrow B_{dR}^+$
 $D_{cis}^{q=1}(U) \leftarrow B_{cis}^{q=1}$

Def: Assume $U = \Pi_1(kr) \rightarrow U(\mathcal{O}_p)$ is de Rham. Then define

$$H_e^1(\Pi_1(kr), U(\mathcal{O}_p)) := \ker(H^1(\Pi_1(kr), U(\mathcal{O}_p)) \rightarrow H^1(\Pi_1(kr), U(B_{cis}^{q=1})))$$

induced by $U(\mathcal{O}_p) \hookrightarrow U(B_{cis}^{q=1})$

$$\downarrow$$

$$H_f^1(\Pi_1(kr), U(\mathcal{O}_p)) := \ker(\dots \rightarrow B_{cis})$$

$$\downarrow$$

$$H_g^1(\Pi_1(kr), U(\mathcal{O}_p)) := \ker(\dots \rightarrow B_{dR})$$

$\approx B_{st}$

$$\downarrow$$

$$H^1(\Pi_1(kr), U(\mathcal{O}_p))$$

non abelian Bloch-Kato exponential!

Prop: $H_e^1(\Pi_1(kr), U(\mathcal{O}_p)) \cong \frac{D_{DR}(U(\mathcal{O}_p))}{D_{cis}^{q=1}(U)(\mathcal{O}_p)} / \frac{D_{DR}(U)(kr)}{D_{DR}^+(U)(kr)}$

kr -pts of an affine scheme / kr

If $D_{cis}^{q=1}(U) = 1$ then $H_f^1 = H_e^1 = \frac{D_{DR}(U)(kr)}{D_{DR}^+(U)(kr)}$

Fact: $D_{cis}^{q=1}(U^{et}) = 1$

$$\mathcal{O}_p \otimes A(kr) \cong H_f^1(\Pi_1(kr), \mathcal{V}_p(A))$$

Lemma: $\{ U$ -torsors + compatible continuous $\Pi_1(kr)$ -action $\} \hookrightarrow H^1(\Pi_1(kr), U(\mathcal{O}_p))$

because U pro unipotent

$$\downarrow \cong \frac{H_e^1 = \frac{D_{DR}(U)(kr)}{D_{cis}^{q=1}(U)(\mathcal{O}_p)}}{D_{DR}^+(U)(kr)}$$

ii $\Pi_1(kr) \rightarrow \mathcal{G}(P)$ continuous $(\Rightarrow \Pi_1(kr) \rightarrow \mathcal{G}(P)$ continuous)

let P be a U -torsor + compatible continuous $\Pi_1(kr)$ action. Then

$$[P] \in H_f^1 \iff P(B_{cis}^{q=1})^{\Pi_1(kr)} \neq \emptyset$$

In that case, $P(B_{dR}^+)^{\Pi_1(kr)} \neq \emptyset$ and $[P] =$ double class containing $g^{-1}r \in U(B_{dR}^+)^{\Pi_1(kr)}$
 \parallel
 $D_{DR}(U)$

$$\text{where } g \in P(B_{cis}^{q=1})^{\Pi_1(kr)} \subset P(B_{dR}^+)^{\Pi_1(kr)}$$

$$r \in P(B_{dR}^+)^{\Pi_1(kr)} \subset P(B_{dR}^+)^{\Pi_1(kr)}$$

\mathcal{G} de Rham fundamental group



Aim: for $y \in X(k_0)$ sufficiently close to x describe the
 $\left(\begin{matrix} \pi_1(k_0) \\ \text{pt } y \in \mathbb{P}_y^{\text{ét}}(B_{\text{ét}}^{(d=1)}) \\ \text{pt } y \in \mathbb{P}_y^{\text{ét}}(B_{\text{ét}}^+) \end{matrix} \right) \subset \mathbb{P}_y^{\text{ét}}(B_{\text{ét}}) \pi_1(k_0)$

These should vary "nicely" with y

Df: (de Rham fund. group)

$\text{Uni}_{k_0}^{\text{dR}} = \left\{ \begin{matrix} \text{algebraic vector bundles } V \text{ on } X_{k_0} \\ + \text{ integral connection } \nabla: V \rightarrow V \otimes_{\mathcal{O}_X} \Omega_{X/k_0}^1 \end{matrix} \right\}$ of $\mathcal{H} = (\mathcal{O}_{X_{k_0}}, d)$ (iterated extensions?)

$x \in X(k_0)$ determines a fibre functor $\omega_x: \text{Uni}_{k_0}^{\text{dR}} \rightarrow \text{Mod}_{k_0}^{(+\infty)}$ which makes

$\text{Uni}_{k_0}^{\text{dR}}$ a neutral Tannakian category / k_0 .

$\hookrightarrow U^{\text{dR}} := \text{Aut}^{\otimes}(\omega_x) / k_0 = \text{pro-unipotent de Rham fund. group}$

\hookrightarrow de Rham path torsors $\pi \mathbb{P}_y^{\text{dR}} = \text{Iso}^{\otimes}(W_x, W_y)$.

Remark: $(\mathcal{O}_X \otimes \mathbb{P}_y^{\text{dR}})$ carries a Hodge filtration

Big Thm! (Andreatta - Tavarita - Kim, Bhatt - Torow - Scholze)

$\exists \pi_1(k_0)$ -equivariant, filtration preserving isomorphism

$$B_{\text{ét}} \otimes_{\text{ét}} \mathcal{O}(U^{\text{ét}}) \sim B_{\text{ét}} \otimes_{k_0} \mathcal{O}(U^{\text{dR}})$$

$$B_{\text{ét}} \otimes_{\text{ét}} \mathcal{O}(\pi \mathbb{P}_y^{\text{ét}}) \sim B_{\text{ét}} \otimes_{k_0} (\mathcal{O}(\pi \mathbb{P}_y^{\text{dR}})) \quad \forall y \in X(k_0)$$

Cor: $\pi \mathbb{P}_y^{\text{ét}}(B_{\text{ét}}) \pi_1(k_0) \sim \pi \mathbb{P}_y^{\text{dR}}(B_{\text{ét}})$

As a result, want to package up $\pi \mathbb{P}_y^{\text{dR}}$ as y varies

Lemma: $\omega_x: \text{Uni}_{k_0}^{\text{dR}} \rightarrow \text{Mod}_{k_0}^{(+\infty)}$ is representable by $E^{\text{dR}} \in \text{pro-Uni}_{k_0}^{\text{dR}}$

and $e \in E_x^{\text{dR}} = \omega_x(E^{\text{dR}})$ E^{dR} is a "coalgebra":

$$\begin{array}{ccccc} \Delta: E^{\text{dR}} & \xrightarrow{\quad} & E^{\text{dR}} \otimes E^{\text{dR}} & , & \epsilon: E^{\text{dR}} \xrightarrow{\quad} \mathbb{1} = (\mathcal{O}_{X_{k_0}}, d) \\ \parallel & & \parallel & & \parallel \\ \lim_{\leftarrow} E_i & & \lim_{\leftarrow} E_i \otimes E_i & & \uparrow \\ e & \xrightarrow{\quad} & e \hat{\otimes} e & & \uparrow \\ \uparrow & & \uparrow & & \uparrow \\ E_x^{\text{dR}} & & E_x^{\text{dR}} \otimes E_x^{\text{dR}} & & \uparrow \\ & & & & \uparrow \\ & & & & \mathcal{O}_{X_{k_0}} - k_0 \end{array}$$

$\forall y \in X(k_0)$ then $\mathcal{O}(\pi \mathbb{P}_y^{\text{dR}})^{\times} \sim E_y^{\text{dR}}$ isomorphism of coalgebras

$$\pi \mathbb{P}_y^{\text{dR}}(k_0) \sim (E_y^{\text{dR}})^{\times} \text{ group-like}$$

$$\left\{ \gamma \in E_y^{\text{dR}} \mid \Delta(\gamma) = \gamma \otimes \gamma \right\} \\ \epsilon(\gamma) = 1$$



$\leftrightarrow X$ curve

"group-like sections" of E^{DR} should give compatible families of elements $y \in P_y^{DR}(k(x))$

Big Theorem 2 (Andreatta-Fantuzzi-Kim)

On any Zariski open affine $Y \subset X \ni$ a group-like algebraic section r of E^{DR} at $\forall y \in Y(k(x))$
 $r(y) \in {}_x P_y^{DR}(k(x)) (= {}_x P_y^{DR}(B_{DR}^+) \pi_1(k(x)))$, lands in ${}_x P_y^{DR}(B_{DR}^+) \pi_1(k(x))$

On a v -order neighbourhood V of x st $\exists!$ $k(x)$ -analytic section g of E^{DR} st that
 $g(x) = e \in E_x^{DR}$ and $\nabla g = 0$.

Furthermore, g is group-like and $g(y) \in {}_x P_y^{DR}(B_{DR}^+) \pi_1(k(x))$

\downarrow $g(y) \in H_c^1 \subset H_g^1$
 \downarrow $g|_V$ is pro- $k(x)$ -analytic since on V
 $g(y) = g(y)^{-1} r(y)$

Local denseness: $g(V) \subset H_g^1$ is Zariski-dense

Does not seem to follow from abstract theory \rightarrow really need to carry out computations here!

Pass to an Zariski open affine curve $Y \subset G \subset X$

for 2 reasons: algebraic section r exists globally
 all fund. groups of Y are free on $\chi = \chi(Y)$ generators

Prop (Kim) let $k(x) \ll \langle z_1, \dots, z_g \rangle$ be non-commutative power series

$k(x) \ll \langle z_1, \dots, z_g \rangle \otimes \mathcal{O}_{Y, k(x)}$ vector bundle on $Y_{k(x)}$ endowed with connection ∇ :

$$\nabla \left(\sum_w \frac{[w]}{w} f_w \right) = \sum_w \frac{[w]}{w} df_w - \sum_{i=1}^g [z_i w] f_w (z_i)$$

where we choose (z_i) , (w_j) algebraic differentials on $Y_{k(x)}$ whose classes are a basis for $H_{DR}^1(Y_{k(x)})$

Then $E^{DR} = (k(x) \ll \langle z \rangle \otimes \mathcal{O}_{Y, k(x)}, \nabla)$ st $e \in E_x^{DR} \leftrightarrow 1 \in k(x) \ll \langle z \rangle$

Key claim: (Choose local parameter t at x) $\exists!$ $g \in k(x) \ll \langle z \rangle \otimes \mathcal{O}_{Y, k(x)}(k[[t]])$

$dg = 0 \rightarrow g_w(t) = 1$ everywhere formal

$df_{z_i} = \frac{[z_i w]}{w} dt$ integrate formally

st $g(0) = 1$ \wedge $g_w(0) = 1$
 $g_w(t) = 0$ $w \neq 1$

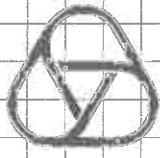
and $\nabla g = 0$ (flat section)

$$(k(x) \ll \langle z \rangle \otimes \mathcal{O}_{Y, k(x)})(k[[t]])$$

\parallel $k(x) \ll \langle z \rangle \otimes k[[t]]$

$$\sum_w \frac{[w]}{w} g_w(t)$$

then the $g_w \in k[[t]]$ are linearly independent over $k((t))$



Why local dimer? $\mathcal{O}(U^{DR}) = k[x] \langle\langle z \rangle\rangle^{\circ} = \varinjlim \left(k[x] \langle\langle z \rangle\rangle_{j,m} \right)^{\vee} \left(\neq \left(\varprojlim k[x] \langle\langle z \rangle\rangle_{j,m} \right)^{\vee} \right)$

a basis of $\mathcal{O}(U^{DR})$ is given by "duals of words" is w° st

$$w^{\circ}(w') = \begin{cases} 1 & \text{if } w' = w \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} X(k[x]) &\rightarrow H^1(\mathbb{P}^1(k[x]), U^{DR}(k[x])) \\ H^1 &= U^{DR}(k[x]) / F^{\circ} U^{DR}(k[x]) \\ &\uparrow \\ &U^{DR}(k[x]) \end{aligned}$$

Our description of E^{DR} gives an identification

$$\begin{aligned} z \mathbb{P}_y^{DR}(k[x]) &\simeq U^{DR}(k[x]) \\ \downarrow &\quad \downarrow \\ (q(y), r(y)) &\rightarrow (q'(y), r'(y)) \end{aligned}$$

and $j_0(y) = q'(y) - r'(y)$ with $q' : V \rightarrow U^{DR}(k[x])$ $k[x]$ -analytic
 $r' : Y(k[x]) \rightarrow U^{DR}(k[x])$ $k[x]$ -algebraic

then $j_0(V)$ is not \mathbb{Z} -dense iff $\exists \sum_w \lambda_w w^{\circ} \neq 0$ in $\mathcal{O}(U^{DR})$ st
 $(\sum_w \lambda_w w^{\circ})(\circ q'^{-1}(\cdot) r'^{-1}) = 0$

the point is if $r' = 1$ then $w^{\circ} \circ q' = q_w$ hence $\sum_w \lambda_w q_w = 0 \nexists$ linear indep
if $r' \neq 1$ then $w^{\circ} \circ (q'^{-1} r')$ and $w^{\circ} \circ q'$ are related by an invertible matrix with algebraic coefficients \nexists linear indep.