VARIATIONS ON A TANNAKIAN CEBOTAREV DENSITY THEOREM

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Abstract. Let X be a normal variety of positive dimension over a finite field and let $\mathcal{C}$ be either a locally constant constructible Weil $\overline{\mathbb{Q}}_\ell$-sheaf, a locally constant constructible quasi-tame $\overline{\mathbb{Q}}_{\mathfrak{u}}$-sheaf or an overconvergent $\overline{\mathbb{Q}}_p$-F-isocrystal on X. We prove the following Tannakian Cebotarev density theorem: let S be a set of closed points of X of upper Dirichlet density 1 (resp. >0) and Φ the union of conjugacy classes of Frobenius elements corresponding to S in the Tannakian group $G(\mathcal{C})$ of $\mathcal{C}$. Then Φ is Zariski-dense in (resp. contains at least one connected component of) $G(\mathcal{C})$. We use the theory of companions and its by-product, the existence of a weight filtration, to reduce the general statement to the case of locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaves, where the assertion is an easy consequence of the (classical) Cebotarev density theorem. The reduction step relies on group-theoretic arguments which might be of independent interest. When X is smooth, our strategy can be adapted to reduce the Tannakian Cebotarev density theorem for convergent $\overline{\mathbb{Q}}_p$-F-isocrystals satisfying a weak form of the parabolicity conjecture of Crew (resp. for overconvergent $\overline{\mathbb{Q}}_p$-F-isocrystals) to the case of direct sums of isoclinic $\overline{\mathbb{Q}}_p$-F-isocrystals, which is due to Hartl and Pál. Since the parabolicity conjecture is known for convergent $\overline{\mathbb{Q}}_p$-F-isocrystals admitting an overconvergent extension by recent work of D'Addezio and is straightforward for convergent $\overline{\mathbb{Q}}_p$-F-isocrystals admitting a slope filtration, this in particular proves unconditionally the Tannakian Cebotarev density theorem in those two cases. Let us point out that this variant of our strategy for convergent (resp. overconvergent) $\overline{\mathbb{Q}}_p$-F-isocrystals is purely p-adic and "elementary" in the sense that it does not resort to automorphic techniques *via* the companion conjecture (nor to the "à la Weil II" formalism of Frobenius weights).

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A variety over a field k means a reduced scheme separated and of finite type over k. For a variety X over k, let $|X|$ denote the set of its closed points and, if X is integral, let η denote its generic point. For a morphism $f : Y \to X$ of varieties over k, write $\overline{f} : \overline{Y} \to \overline{X}$ for the base-change of $f : Y \to X$ along $\mathrm{spec}(\overline{k}) \to \mathrm{spec}(k)$, where $k \hookrightarrow \overline{k}$ is an algebraic closure.

Let k be a finite field of characteristic $p > 0$, $k \hookrightarrow \overline{k}$ an algebraic closure, $\pi_1(k) := Gal(\overline{k}|k)$ the corresponding absolute Galois group and $\varphi \in \pi_1(k)$ the geometric Frobenius.

1. Introduction

Let X be a normal, connected variety of positive dimension over k. The classical Cebotarev density theorem asserts that for every finite continuous quotient $\pi_1(X) \twoheadrightarrow \Pi$ and union of conjugacy classes $\Delta \subset \Pi$, the set $S_\Delta \subset |X|$ of closed points $x \in |X|$ such that the corresponding Frobenius elements $\varphi_x \in \pi_1(X)$ map to Δ has (Dirichlet) density $\delta(S_\Delta) = |\Delta|/|\Pi|$. Passing to the limit, one gets that for every $S \subset |X|$ with upper (Dirichlet) density $\delta^u(S) = 1$ the union of the $\pi_1(X)$-conjugacy classes of the $\varphi_x \in \pi_1(X)$, $x \in S$ is dense in $\pi_1(X)$ for the profinite topology. This implies that for a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{C}$ on X ($\ell \neq p$), if $G(\mathcal{C}^{an})$ denotes the image of the corresponding representation of $\pi_1(X)$ on $\mathcal{C}^{an} := \mathcal{C}_{\overline{\eta}}$ and if, for $x \in |X|$, $\Phi_x^{\mathcal{C}^{an}} \subset G(\mathcal{C}^{an})$ denotes the $G(\mathcal{C}^{an})$-conjugacy class of the image of φ_x in $G(\mathcal{C}^{an})$ then the union of the $\Phi_x^{\mathcal{C}^{an}}$, $x \in S$ is ℓ-adically dense in $G(\mathcal{C}^{an})$. In particular, for every $S \subset |X|$ with $\delta^u(S) = 1$ (resp. >0)

A) If $G(\mathcal{C})$ denotes the Zariski closure of $G(\mathcal{C}^{an})$ in $\mathrm{GL}_{\mathcal{C}_{\overline{\eta}}}$ and if, for $x \in |X|$, $\Phi_x^{\mathcal{C}} \subset G(\mathcal{C})$ denotes the conjugacy class generated by $\Phi_x^{\mathcal{C}^{an}}$ in $G(\mathcal{C})$ then the union $\Phi^{\mathcal{C}}$ of the $\Phi_x^{\mathcal{C}}$, $x \in S$ is Zariski-dense in $G(\mathcal{C})$ (resp. the Zariski-closure of $\Phi^{\mathcal{C}}$ contains a connected component of $G(\mathcal{C})$);
B) If $\mathcal{C}'$ is another $\overline{\mathbb{Q}}_\ell$-local system on X such that $(x^*\mathcal{C})^{ss} \simeq (x^*\mathcal{C}')^{ss}$, $x \in S$ then $\mathcal{C}^{ss} \simeq \mathcal{C}'^{\,ss}$ (where $(-)^{ss}$ stands for semisimplification).

The Cebotarev density theorem plays a fundamental part in arithmetic geometry in that it often enables to reduce problems about $\overline{\mathbb{Q}}_\ell$-local systems on X to problems about semisimple $\overline{\mathbb{Q}}_\ell$-local systems on points (that is the datum of a vector space with a semisimple automorphism). This prompts the question of similar statements for local systems with other coefficients such as $\overline{\mathbb{Q}}_p$-local systems (*i.e.* convergent or overconvergent $\overline{\mathbb{Q}}_p$-F-isocrystals) or $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems (*i.e.* quasi-tame ultraproduct local systems) - see Subsection

1.1. Such coefficients share with $\overline{\mathbb{Q}}_\ell$-local systems the property that they form a Tannakian category with good functorial properties with respect to morphisms of varieties. In particular, one can attach to every such local system $\mathcal{C}$ a Tannakian group $G(\mathcal{C})$ and a collection of $G(\mathcal{C})$-conjugacy classes $\Phi_x^{\mathcal{C}}$, $x \in |X|$. But, except in the unit-root case, the Tannakian structure on $\overline{\mathbb{Q}}_p$-local systems does not upgrade to a category of finite-dimensional continuous $\overline{\mathbb{Q}}_p$-representations of $\pi_1(X)$ while, though the Tannakian structure on $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems does upgrade to a category of finite-dimensional continuous $\overline{\mathbb{Q}}_{\mathfrak{u}}$-representations of $\pi_1(X)$, the ultraproduct topology is not Hausdorff so that the classical Cebotarev density theorem is useless. Still, for such coefficients, the weaker Statements A), B) make sense and are already of significant importance (*e.g.* Statement B) for pure local systems is a key ingredient in the proof of the Langlands' correspondance).

Of course, Statement A) implies Statement B) and, as observed by Tsuzuki and Abe [A18b, Prop. A.4.1], one can prove Statement B) for $S = |X|$ by a simple L-function argument using weights provided a suitable "à la Weil II" formalism of Frobenius weights is available, which is the case for overconvergent $\overline{\mathbb{Q}}_p$- and $\overline{\mathbb{Q}}_{\mathfrak{u}}$-local systems (and at the cost of invoking the companion conjecture, the same argument proves Statement B) for arbitrary $S \subset |X|$ with $\delta^u(S) = 1$). Statement B) for overconvergent $\overline{\mathbb{Q}}_p$-local systems automatically implies Statement B) for convergent $\overline{\mathbb{Q}}_p$-local systems admitting an overconvergent extension that is those lying in the essential image of the natural functor $\mathcal{C}^\dagger \mapsto \mathcal{C}$ from overconvergent to convergent $\overline{\mathbb{Q}}_p$-local systems (since over a point, this functor is an equivalence of categories). Statement B) also holds for convergent $\overline{\mathbb{Q}}_p$-local systems admitting a slope filtration hence, in particular, it always holds over a dense open subscheme (see Corollary 1.3.2.2 below). In this work, we focus on the upgraded Statement A).

In the remaining part of this introduction, we settle the notation and recall the basic common Tannakian features of the various categories of local systems we consider, state our results, describe our strategy and compare our work with the one of Hartl-Pál [HP18] for $\overline{\mathbb{Q}}_p$-local systems, to which we owe a lot.

1.1. **(Motivic) Q-coefficients.** Fix an infinite set $\mathcal{L}$ of primes $\neq p$ and let $\mathcal{U}$ denote the set of non-principal ultrafilters on $\mathcal{L}$.

- For $\mathfrak{l} \in \mathcal{L} \cup \{p\}$ let $\overline{\mathbb{Q}}_{\mathfrak{l}}$ denote the algebraic closure of the completion of $\mathbb{Q}$ at $\mathfrak{l}$;
- For $\mathfrak{u} \in \mathcal{U}$, let $\overline{\mathbb{Q}}_{\mathfrak{u}}$ denote the quotient of $\prod_{\ell\in\mathcal{L}} \overline{\mathbb{F}}_\ell$ by the maximal ideal generated by the characteristic functions of $\mathcal{L} \setminus S$, $S \in \mathfrak{u}$.

Let X be a normal variety over k. Let Q be either $\overline{\mathbb{Q}}_\ell$ for $\ell \in \mathcal{L}$, $\overline{\mathbb{Q}}_{\mathfrak{u}}$ for $\mathfrak{u} \in \mathcal{U}$ or $\overline{\mathbb{Q}}_p$. In the following a motivic Q-local system or Q-coefficient $\mathcal{C}$ on X means either:

- A locally constant constructible Weil $\overline{\mathbb{Q}}_\ell$-sheaf on X [D80, (1.1)];
- A locally constant constructible quasi-tame Weil $\overline{\mathbb{Q}}_{\mathfrak{u}}$-sheaf on X [C19a, §3.6.2];
- An overconvergent $\overline{\mathbb{Q}}_p$-F-isocrystal on X [Cr92, §1] (and *e.g.* [AM15, §7.3] or [A18b, §4.1] for scalar extension to $\overline{\mathbb{Q}}_p$).

A motivic $\overline{\mathbb{Q}}_\ell$-coefficient (resp. $\overline{\mathbb{Q}}_{\mathfrak{u}}$-coefficient) on X is said to be étale if it arises from a locally constant constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf on X (resp. from a locally constant constructible quasi-tame $\overline{\mathbb{Q}}_{\mathfrak{u}}$-sheaf on X).

Let $\mathcal{C}^\dagger(X, Q)$ denote the category of motivic Q-coefficients on X and, for $Q = \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{Q}}_{\mathfrak{u}}$, let $\mathcal{C}^{\dagger,et}(X, Q) \subset \mathcal{C}^\dagger(X, Q)$ denote the full subcategory of étale motivic Q-coefficients. For $Q = \overline{\mathbb{Q}}_p$, we will also consider the category $\mathcal{C}(X, \overline{\mathbb{Q}}_p)$ of convergent $\overline{\mathbb{Q}}_p$-F-isocrystals on X [Cr92, §1] (and *e.g.* [AM15, §7.3] for scalar extension to $\overline{\mathbb{Q}}_p$). There is a natural functor $\alpha : \mathcal{C}^\dagger(X, \overline{\mathbb{Q}}_p) \to \mathcal{C}(X, \overline{\mathbb{Q}}_p)$, which is fully faithful if X is smooth over k [Ked04, Thm. 1.1]; an object in its essential image is said to be $\dagger$-extendable or to admit an overconvergent extension.

To unify the presentation, we will use the terminology Q-coefficient on X for either a motivic Q-coefficient or, when $Q = \overline{\mathbb{Q}}_p$, a convergent $\overline{\mathbb{Q}}_p$-coefficient, that is a convergent $\overline{\mathbb{Q}}_p$-F-isocrystal on X, and we will write $\mathcal{C}^{(\dagger)}(X, Q)$ for the corresponding category (so $\mathcal{C}^{(\dagger)}(X, Q) = \mathcal{C}^\dagger(X, Q)$ for $Q = \overline{\mathbb{Q}}_\ell$ or $Q = \overline{\mathbb{Q}}_{\mathfrak{u}}$ and $\mathcal{C}^{(\dagger)}(X, \overline{\mathbb{Q}}_p) = \mathcal{C}(X, \overline{\mathbb{Q}}_p)$ or $\mathcal{C}^\dagger(X, \overline{\mathbb{Q}}_p)$).

For $Q = \overline{\mathbb{Q}}_\ell$ (resp. $Q = \overline{\mathbb{Q}}_{\mathfrak{u}}$), let $\mathcal{C}^{geom,\dagger}(X, Q)$ denote the category of locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaves (resp. locally constant constructible quasi-tame $\overline{\mathbb{Q}}_{\mathfrak{u}}$-sheaves) on $\overline{X}$. For $Q = \overline{\mathbb{Q}}_p$, let $\mathcal{C}^{geom}(X, Q)$

(resp. $\mathcal{C}^{geom,\dagger}(X,Q)$) denote the category of convergent (resp. overconvergent) isocrystals on X. The category $\mathcal{C}^{(\dagger)}(X,Q)$ (resp. $\mathcal{C}^{geom,(\dagger)}(X,Q)$) is a Q-linear abelian rigid $\otimes$-category and if X is connected (resp. geometrically connected over k), it is (neutral) Tannakian. More precisely, every geometric point $\overline{x}$ with value in $\overline{k}$ over a closed point $x \in |X|$ induces an exact $\otimes$-functor $(-)_{\overline{x}} : \mathcal{C}^{(\dagger)}(X,Q) \to \mathrm{Vect}_Q$ (resp. $(-)_{\overline{x}} : \mathcal{C}^{geom,(\dagger)}(X,Q) \to \mathrm{Vect}_Q$), which, if X is connected (resp. geometrically connected over k), is a fiber functor. Let $F_X : X \to X$ denote the $|k|$-th power Frobenius endomorphism on X. (By abuse of notation, we denote $F_X \times Id_{\mathrm{spec}(\overline{k})} : \overline{X} \to \overline{X}$ again by F_X.) Then the pull-back functor $F_X^* : \mathcal{C}^{geom,(\dagger)}(X,Q) \tilde{\to} \mathcal{C}^{geom,(\dagger)}(X,Q)$ is a Q-linear $\otimes$-autoequivalence and $\mathcal{C}^{(\dagger)}(X,Q)$ identifies with the category of pairs $\mathcal{C} = (\overline{\mathcal{C}}, \Phi)$, with $\overline{\mathcal{C}} \in \mathcal{C}^{geom,(\dagger)}(X,Q)$ and $\Phi : F_X^*\overline{\mathcal{C}} \tilde{\to} \overline{\mathcal{C}}$ an isomorphism in $\mathcal{C}^{geom,(\dagger)}(X,Q)$. (Thus, if X is geomterically connected over k, $\mathcal{C}^{(\dagger)}(X,Q)$ gives rise to a (neutral) Tannakian category with Frobenius. See [D'A20a, App.].) The natural functor $\overline{(-)} : \mathcal{C}^{(\dagger)}(X,Q) \to \mathcal{C}^{geom,(\dagger)}(X,Q)$, $\mathcal{C} = (\overline{\mathcal{C}}, \Phi) \mapsto \overline{\mathcal{C}}$ corresponding to forgetting the Frobenius-structure Φ is an exact $\otimes$-functor and commutes with the fiber functors $(-)_{\overline{x}}$. Let $\overline{\mathcal{C}}^{(\dagger)}(X,Q) \subset \mathcal{C}^{geom,(\dagger)}(X,Q)$ denote the smallest full subcategory of $\mathcal{C}^{geom,(\dagger)}(X,Q)$ containing the essential image of $\overline{(-)} : \mathcal{C}^{(\dagger)}(X,Q) \to \mathcal{C}^{geom,(\dagger)}(X,Q)$ and stable under tensor products, duals and subquotients (thus, if X is geometrically connected over k, this is the smallest Tannakian subcategory of $\mathcal{C}^{geom,(\dagger)}(X,Q)$ containing the essential image of $\overline{(-)} : \mathcal{C}^{(\dagger)}(X,Q) \to \mathcal{C}^{geom,(\dagger)}(X,Q)$). Every morphism $f : Y \to X$ of normal varieties over k induces a commutative diagram of exact $\otimes$-functors

$$\begin{array}{ccc} \mathcal{C}^{(\dagger)}(X,Q) & \xrightarrow{f^*} & \mathcal{C}^{(\dagger)}(Y,Q) \\ {\scriptstyle\overline{(-)}}\downarrow & & \downarrow{\scriptstyle\overline{(-)}} \\ \overline{\mathcal{C}}^{(\dagger)}(X,Q) & \xrightarrow[f^*]{} & \overline{\mathcal{C}}^{(\dagger)}(Y,Q). \end{array}$$

For $Q = \overline{\mathbb{Q}}_\ell$ or $Q = \overline{\mathbb{Q}}_{\mathfrak{u}}$, one has the following Galois-theoretic description of $\mathcal{C}^{\dagger,(et)}(X,Q)$ and $\overline{\mathcal{C}}^{\dagger}(X,Q)$. If X is connected, let $\pi_1(X,\overline{x})$ denote the étale fundamental group of X and $W(X,\overline{x}) = \pi_1(X,\overline{x}) \times_{\pi_1(k)} \varphi^{\mathbb{Z}}$ the Weil group of X. Then the fiber functor $(-)_{\overline{x}} : \mathcal{C}^{\dagger}(X,Q) \to \mathrm{Vect}_Q$ factors through the category $Rep(W(X,\overline{x}),Q)$ of finite-dimensional Q-representations of $W(X,\overline{x})$ and its restriction to the full subcategory $\mathcal{C}^{\dagger,et}(X,Q) \subset \mathcal{C}^{\dagger}(X,Q)$ factors through $Rep(\pi_1(X,\overline{x}),Q)$; the resulting functors $(-)_{\overline{x}} : \mathcal{C}^{\dagger}(X,Q) \to Rep(W(X,\overline{x}),Q)$, $(-)_{\overline{x}} : \mathcal{C}^{\dagger,et}(X,Q) \to Rep(\pi_1(X,\overline{x}),Q)$ are fully faithfull hence induce equivalences of categories onto their essential images, which we denote by $Rep^{\dagger}(W(X,x),Q)$ and $Rep^{\dagger}(\pi_1(X,x),Q)$ respectively. If X is geometrically connected over k, $(-)_{\overline{x}} : \overline{\mathcal{C}}^{\dagger}(X,Q) \to \mathrm{Vect}_Q$ factors through $Rep(\pi_1(\overline{X},\overline{x}),Q)$ and induces an equivalence of categories onto its essential image $Rep^{\dagger}(\pi_1(\overline{X},\overline{x}),Q)$. To sum it up, one has

$$\begin{array}{ccccc} \mathcal{C}^{\dagger,et}(X,Q) & \hookrightarrow & \mathcal{C}^{\dagger}(X,Q) & \xrightarrow{\overline{(-)}} & \overline{\mathcal{C}}^{\dagger}(X,Q) \\ {\scriptstyle (-)_{\overline{x}}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (-)_{\overline{x}}} & & \simeq\downarrow{\scriptstyle (-)_{\overline{x}}} \\ Rep^{\dagger}(\pi_1(X,\overline{x}),Q) & \underset{|_{W(X,\overline{x})}}{\hookrightarrow} & Rep^{\dagger}(W(X,\overline{x}),Q) & \xrightarrow[|_{\pi_1(\overline{X},\overline{x})}]{} & Rep^{\dagger}(\pi_1(\overline{X},\overline{x}),Q) \end{array}$$

Let Π denote either $\pi_1(X,\overline{x})$ or $W(X,\overline{x})$ (resp. $\pi_1(\overline{X},\overline{x})$, if X is geometrically connected over k). For[1] $Q = \overline{\mathbb{Q}}_\ell$, $Rep^{\dagger}(\Pi,\overline{\mathbb{Q}}_\ell)$ is the (resp. a certain) category of finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of Π of the form $V_\ell \otimes_{Q_\ell} \overline{\mathbb{Q}}_\ell$ with V_ℓ a continuous Q_ℓ-representation of Π and Q_ℓ a finite extension of $\mathbb{Q}_\ell$.

Fix a geometric point $\overline{x}$ on X. Assume X is connected. For $\mathcal{C} \in \mathcal{C}^{(\dagger)}(X,Q)$, let $\langle\mathcal{C}\rangle \subset \mathcal{C}^{(\dagger)}(X,Q)$ denote the Tannakian subcategory generated by $\mathcal{C}$ in $\mathcal{C}^{(\dagger)}(X,Q)$ and let $G(\mathcal{C},\overline{x})$ denote the attached Tannakian group that is the group of $\otimes$-automorphisms of the restriction to $\langle\mathcal{C}\rangle$ of the fiber functor $(-)_{\overline{x}} : \mathcal{C}^{(\dagger)}(X,Q) \to \mathrm{Vect}_Q$. When X is geometrically connected over k, define similarly $\langle\overline{\mathcal{C}}\rangle$, $G(\overline{\mathcal{C}},\overline{x})$.

[1]For $Q = \overline{\mathbb{Q}}_{\mathfrak{u}}$, one has a similar "explicit" description (but we will not use it in the following), namely $Rep^{\dagger}(\Pi,\overline{\mathbb{Q}}_{\mathfrak{u}})$ is the (resp. a certain) category of finite-dimensional $\overline{\mathbb{Q}}_{\mathfrak{u}}$-representations of Π of the form $\underline{V} \otimes_{\underline{F}} \overline{\mathbb{Q}}_{\mathfrak{u}}$ where $\underline{F} = \prod_{\ell\in\mathcal{L}} F_\ell$ with F_ℓ a finite field extension of $\mathbb{F}_\ell$, $\underline{V} = \prod_{\ell\in\mathcal{L}} V_\ell$ with V_ℓ a continuous F_ℓ-representation of Π and the Π-representation $\underline{V}$ corresponds to a quasi-tame locally constant constructible $\underline{F}$-sheaf of finite $\mathfrak{u}$-rank on X (resp. $\overline{X}$).

If $X = \mathrm{spec}(k)$, the fiber functor $(-)_{\overline{x}} : \mathcal{C}^{(\dagger)}(X, Q) \to \mathrm{Vect}_Q$ induces an equivalence of categories onto the category of finite-dimensional Q-vector spaces endowed with an action of φ (and $G(\mathcal{C}, \overline{x}) \subset GL(\mathcal{C}_{\overline{x}})$ is the Zariski-closure of the abstract group generated by the image of φ acting on $\mathcal{C}_{\overline{x}}$) while $(-)_{\overline{x}} : \overline{\mathcal{C}}^{(\dagger)}(X, Q) \to \mathrm{Vect}_Q$ is an equivalence of categories (and $G(\overline{\mathcal{C}}, \overline{x}) = 1$).

More generally, if X is geometrically connected over k, one has natural closed immersions $G(\overline{\mathcal{C}}, \overline{x}) \subset G(\mathcal{C}, \overline{x}) \subset GL(\mathcal{C}_{\overline{x}})$ and it follows from the description of Tannakian categories with Frobenius that $G(\overline{\mathcal{C}}, \overline{x})$ is a closed normal subgroup of $G(\mathcal{C}, \overline{x})$ and that $G(\mathcal{C}, \overline{x})/G(\overline{\mathcal{C}}, \overline{x})$ is naturally a quotient of the pro-algebraic completion of $\varphi^{\mathbb{Z}}$ classifying the "constant objects" in $\langle \mathcal{C} \rangle$ [D'A20a, Thm. A.2.2]. For $Q = \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{Q}}_{\mathrm{u}}$, $G(\mathcal{C}, \overline{x}) \subset GL(\mathcal{C}_{\overline{x}})$ (resp. $G(\overline{\mathcal{C}}, \overline{x}) \subset GL(\mathcal{C}_{\overline{x}})$) is the Zariski-closure of the image of $W(X, \overline{x})$ (resp. $\pi_1(\overline{X}, \overline{x})$) acting on $\mathcal{C}_{\overline{x}}$.

Every geometric point $\overline{x}$ over a closed point $x \in |X|$, regarded as a morphism $x : \mathrm{spec}(k(x)) \to X$, induces a closed embedding $G(x^*\mathcal{C}, \overline{x}) \to G(\mathcal{C}, \overline{x})$ of algebraic groups over Q. Write $\varphi_x \in G(x^*\mathcal{C}, \overline{x})$ for the image of the (geometric) Frobenius acting on $\mathcal{C}_{\overline{x}}$. Fixing a geometric point $\times$ on X with value in $\overline{k}$, any isomorphism of fiber functors $\gamma : (-)_{\overline{x}} \tilde{\to} (-)_{\times}$ induces an isomorphism of algebraic groups $G(\mathcal{C}, \overline{x}) \tilde{\to} G(\mathcal{C}, \times)$, which is independent of the choice of γ up to $G(\mathcal{C}, \times)$-conjugacy so that the $G(\mathcal{C}, \times)$-conjugacy class $\Phi_x^{\mathcal{C}}$ of the image of φ_x *via* $G(x^*\mathcal{C}, \overline{x}) \to G(\mathcal{C}, \overline{x}) \tilde{\to} G(\mathcal{C}, \times)$ is independent of the choice of γ. In the following we will omit base-points from the notation unless necessary.

1.2. For a subset $S \subset |X|$, the series $F_S(t) = \sum_{s \in S} |k(s)|^{-t}$ converges absolutely and locally uniformly for $Re(t) > d := \dim(X)$. Write

$$\sigma_S(t) := \sup \left\{ \frac{F_S(t')}{F_{|X|}(t')} \mid d < t' < t \right\},\ t \in \mathbb{R}_{>d}$$

and let $\delta^u(S) := \lim_{t \to d,\, t \in \mathbb{R}_{>d}} \sigma_S(t)$ denote the upper Dirichlet density of S (*e.g.* [P97, Appendix B]). By definition $0 \leq \delta^u(S) \leq 1$.

1.3. Let $\mathcal{C}$ be a Q-coefficient on X. For a subset $S \subset |X|$ of closed points of X, write $\Phi_S^{\mathcal{C}} := \cup_{x \in S} \Phi_x^{\mathcal{C}}$ and for every $G(\mathcal{C})$-invariant subset $\Delta \subset G(\mathcal{C})$, write $S_\Delta^{\mathcal{C}} := \{x \in S \mid \Phi_x^{\mathcal{C}} \subset \Delta\}$. In the following, we will also omit the superscript $(-)^{\mathcal{C}}$ from the notation unless necessary.

With the above notation and definitions, one can formulate the following unified Tannakian version of the Cebotarev density theorem, which was originally stated for $\overline{\mathbb{Q}}_p$-coefficients by Hartl and Pál as [HP18, Conj. 1.4].

Conjecture. *Let X be a normal connected variety of positive dimension over k. Let $\mathcal{C}$ be a Q-coefficient on X and let $S \subset |X|$ be a subset of closed points of X. Assume S has upper Dirichlet density $\delta^u(S) > 0$ (resp. $\delta^u(S) = 1$). Then the Zariski-closure of Φ_S contains at least one connected component of $G(\mathcal{C})$ (resp. Φ_S is Zariski-dense in $G(\mathcal{C})$).*

The assumptions and conclusions of Conjecture 1.3 remain unchanged if one replaces k with the algebraic closure of k in the function field of X so that to prove Conjecture 1.3 we may and will assume that X is geometrically connected over k. Conjecture 1.3 can be reduced to the (classical) Cebotarev density theorem in the following cases.

1.3.1. **Étale $\overline{\mathbb{Q}}_\ell$-coefficients.** This essentially amounts to comparing the Haar density and the Zariski density in the sense of [Se12, §5.2.1] - see Subsection 5.1 for details.

1.3.2. **Unit-root convergent $\overline{\mathbb{Q}}_p$-F-isocrystals.** This immediately follows from:

1.3.2.1.**Fact.** (Katz, Crew [Cr87, Thm. 2.1]) *The full subcategory of unit-root convergent $\overline{\mathbb{Q}}_p$-F-isocrystals on X is equivalent to the category of finite-dimensional continuous $\overline{\mathbb{Q}}_p^{\sigma}$-representations of $\pi_1(X)$, where σ is a lift of Frobenius.*

1.3.2.2. **Corollary.** *Assume $\delta^u(S)=1$. Let $\mathcal{C}$, $\mathcal{C}'$ be convergent $\overline{\mathbb{Q}}_p$-coefficients admitting a slope filtration (see Subsection 9.1) on X and such that $(x^*\mathcal{C})^{ss}\simeq(x^*\mathcal{C}')^{ss}$, $x\in S$ then $\mathcal{C}^{ss}\simeq\mathcal{C}'^{\,ss}$. In particular, for every pair of convergent $\overline{\mathbb{Q}}_p$-coefficients $\mathcal{C}$, $\mathcal{C}'$ on X such that $(x^*\mathcal{C})^{ss}\simeq(x^*\mathcal{C}')^{ss}$, $x\in S$ there exists a dense open subscheme $U\subset X$ such that $(\mathcal{C}|_U)^{ss}\simeq(\mathcal{C}'|_U)^{ss}$.*

Proof. For the first assertion, one may assume that $\mathcal{C}$, $\mathcal{C}'$ are semisimple. Then, as $\mathcal{C}$, $\mathcal{C}'$ admit a slope filtration on X, they can be written as $\mathcal{C}=\oplus_{1\leq i\leq s}\mathcal{I}_i$, with $\mathcal{I}_i\neq 0$ (semisimple) isoclinic of slope σ_i, $i=1,\dots,s$ and $\sigma_1>\sigma_2>\cdots>\sigma_s$ and $\mathcal{C}'=\oplus_{1\leq i\leq s'}\mathcal{I}'_i$, with $\mathcal{I}'_i\neq 0$ (semisimple) isoclinic of slope σ'_i, $i=1,\dots,s'$ and $\sigma'_1>\sigma'_2>\cdots>\sigma'_{s'}$. The condition $(x^*\mathcal{C})^{ss}\simeq(x^*\mathcal{C})^{ss}$ for one $x\in|X|$ is already enough to ensure that $\{\sigma_1,\dots,\sigma_s\}=\{\sigma'_1,\dots,\sigma'_{s'}\}$ and that $(x^*\mathcal{I}_i)^{ss}\simeq(x^*\mathcal{I}'_i)^{ss}$, $i=1,\dots,s$. For each $i=1,\dots,s$, fix a geometrically constant rank-one convergent $\overline{\mathbb{Q}}_p$-coefficient of slope σ_i. Then $\mathcal{I}_i\otimes\mathcal{L}_i^\vee$, $\mathcal{I}'_i\otimes\mathcal{L}_i^\vee$ are both unit-root and still satisfy $(x^*(\mathcal{I}_i\otimes\mathcal{L}_i^\vee))^{ss}\simeq(x^*(\mathcal{I}'_i\otimes\mathcal{L}_i^\vee))^{ss}$, $x\in|S|$. From Fact 1.3.2.1 and the (classical) Cebotarev density theorem, this in turn implies $\mathcal{I}_i\otimes\mathcal{L}_i^\vee\simeq\mathcal{I}'_i\otimes\mathcal{L}_i^\vee$ hence $\mathcal{I}_i\simeq\mathcal{I}'_i$, $i=1,\dots,s$. The second assertion follows from the first, together with the fact that there exists a dense open suscheme $U\subset X$ such that $\mathcal{C}|_U$, $\mathcal{C}'|_U$ both admit a slope filtration ([K79, Thm. 2.3.1, Cor. 2.6.3], see Fact 9.1.1) and that $\delta^u(S\cap U)=\delta^u(S)=1$. □

1.3.3. **Direct sum of isoclinic convergent $\overline{\mathbb{Q}}_p$-F-isocrystals.** As in the proof of Corollary 1.3.2.2 the idea is to use that an isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficient is, up to twist, unit-root in order to reduce to the (classical) Cebotarev density theorem *via* Fact 1.3.2.1. However, the reduction is quite tricky.

Fact. (Hartl-Pál[2] [HP18, Thm. 1.8]) *Conjecture 1.3 holds for direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients. In particular, for every convergent $\overline{\mathbb{Q}}_p$-coefficient $\mathcal{C}$ on X there exists a dense open subscheme $U\subset X$ such that Conjecture 1.3 holds for $(\mathcal{C}|_U)^{ss}$.*

The argument of Hartl and Pál goes as follows. From the above, it is enough to prove Conjecture 1.3 for a convergent $\overline{\mathbb{Q}}_p$-coefficient of the form $\mathcal{C}=\mathcal{U}\oplus\mathcal{A}$ with $\mathcal{U}$ unit-root and $\mathcal{A}$ a direct sum of rank-one geometrically constant convergent $\overline{\mathbb{Q}}_p$-coefficients. In particular, $G(\mathcal{C})=G(\mathcal{U})\times_{G(\langle\mathcal{U}\rangle\cap\langle\mathcal{A}\rangle)}G(\mathcal{A})\subset G(\mathcal{U})\times G(\mathcal{A})$ (*cf.* [HP18, Prop. 3.6(c)]). Conjecture 1.3 for $\mathcal{U}$ is Fact 1.3.2.1. Conjecture 1.3 for $\mathcal{A}$ boils down to a purely group-theoretic statement [HP18, Thm. 7.4]: If A is a commutative linear algebraic group over Q with connected component $A^\circ\simeq\mathbb{G}_{m,Q}^r\times\mathbb{G}_{a,Q}^\epsilon$, where $\epsilon=0,1$ and $a\in A$ is such that $a^{\mathbb{Z}}\subset A$ is Zariski-dense in A then, for every infinite subset $S\subset\mathbb{Z}$, the Zariski-closure of $a^S\subset A$ contains a connected component of A. The mixed case is quite subtle since, *a priori* the Zariski-density of $\Phi_S^{\mathcal{U}}\subset G(\mathcal{U})$ and $\Phi_S^{\mathcal{A}}\subset G(\mathcal{A})$ does not necessarily imply the Zariski-density of $\Phi_S^{\mathcal{C}}\subset G(\mathcal{C})$. Hartl and Pál however prove this is indeed the case using that the Zariski-density of $\Phi_S^{\mathcal{U}}\subset G(\mathcal{U})$ arises from the analytic density of $\Phi_S^{\mathcal{U}^{an}}\subset G(\mathcal{U}^{an})$ (where $\mathcal{U}^{an}$, $G(\mathcal{U}^{an})$ and $\Phi_S^{\mathcal{U}^{an}}$ are defined similarly to the case of étale $\overline{\mathbb{Q}}_\ell$-coefficients, *cf.* Section 1) *via* an effective version of the (classical) Cebotarev density theorem [HP18, Thm. 3.16], which enables them to apply an *ad-hoc* Zariski-density criterion [HP18, Thm. 7.10]. The latter is a statement in p-adic analytic geometry involving counting estimates and relying, in particular, on a uniform version of Osterlé's estimates for the number of points on the reduction modulo-p^n of a p-adic analytic hypersurface of $\mathbb{Z}_p^N$ [HP18, Prop. 7.8].

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Remarks.

(1) We point out that, though subtle, the proof of Fact 1.3.3 is "purely p-adic" and "elementary" in the sense that it does not resort to automorphic techniques *via* the companion conjecture (or to the formalism of Frobenius weights).
(2) As every irreducible motivic $\overline{\mathbb{Q}}_\ell$-coefficient is a twist of an étale motivic $\overline{\mathbb{Q}}_\ell$-coefficient by a geometrically constant motivic $\overline{\mathbb{Q}}_\ell$-coefficient, the proof of [HP18, Thm. 1.8] works similarly to show that Conjecture 1.3 holds for direct sums of irreducible motivic $\overline{\mathbb{Q}}_\ell$-coefficients.

1.4. **Motivic Q-coefficients.** A Q-coefficient $\mathcal{C}$ and a Q'-coefficient $\mathcal{C}'$ are said to be compatible or companions (with respect to a fixed isomorphism $Q\simeq Q'$) if for every $x\in|X|$ the characteristic polynomials of φ_x acting on $\mathcal{C}_{\overline{x}}$, $\mathcal{C}'_{\overline{x}}$ coincide (see Subsection 5.2). Let $\mathcal{C}$ be a semisimple motivic Q-coefficient. The conjectural formalism of pure motives predicts that there should exist a reductive group $G(\mathcal{C}^{mot})$ over $\overline{\mathbb{Q}}$ together with a faithful finite-dimensional $\overline{\mathbb{Q}}$-representation $\mathcal{C}^{mot}$ and (semisimple) conjugacy classes $\Phi_x^{\mathcal{C}^{mot}}\subset G(\mathcal{C}^{mot})$,

[2]Though Hartl and Pál assume X is a curve, their proof of [HP18, Thm. 1.8] makes no use of this assumption.

$x \in |X|$ such that for every semisimple motivic Q'-coefficient $\mathcal{C}'$ which is compatible with $\mathcal{C}$ and $x \in |X|$, the Tannakian group $G(\mathcal{C}')$ and the conjugacy class $\Phi_x^{\mathcal{C}'} \subset G(\mathcal{C}')$ arise from $G(\mathcal{C}^{mot})$ and $\Phi_x^{\mathcal{C}^{mot}} \subset G(\mathcal{C}^{mot})$ by base-change from $\overline{\mathbb{Q}}$ to Q'. In particular, if $\mathcal{C}$ admits a $\overline{\mathbb{Q}}_\ell$-companion $\mathcal{C}_\ell$ which is étale, then Conjecture 1.3 for $\mathcal{C}$ should follow from Conjecture 1.3 for $\mathcal{C}_\ell$. Though the existence of $G(\mathcal{C}^{mot})$ and $\Phi_x^{\mathcal{C}^{mot}} \subset G(\mathcal{C}^{mot})$, $x \in |X|$ is still completely conjectural, one now knows that, provided X is smooth over k, $\mathcal{C}$ always admits a $\overline{\mathbb{Q}}_\ell$-companion $\mathcal{C}_\ell$ which is étale for $\ell \gg 0$ (Corollary 5.2.3); this is a consequence of the companion conjecture of Deligne (see Fact 5.2.1 for references). The first issue is thus to show that the validity of Conjecture 1.3 for $\mathcal{C}_\ell$ transfers "automatically" to $\mathcal{C}$ only by means of the compatibility property. For this, we reformulate Conjecture 1.3 in terms of the image of the characteristic polynomial map attached to $\mathcal{C}$ (Proposition 3.4.1). This reformulation reduces Conjecture 1.3 for motivic semisimple Q-coefficients to showing that the image of the characteristic polynomial map is independent of the companions, which we deduce from Kazhdan-Larsen-Varshavsky's reconstruction theorem for connected reductive groups [KaLV14, Thm. 1.2] (Section 6). Using the weight filtration on $\mathcal{C}$ (another by-product of the companion conjecture), we then reduce Conjecture 1.3 for arbitrary motivic Q-coefficients to motivic Q-coefficients $\mathcal{C}$ which are direct sums of pure motivic Q-coefficients (Section 7). Such a $\mathcal{C}$ is not semisimple in general but $\overline{\mathcal{C}}$ is, which forces $G(\mathcal{C}) \simeq \mathbb{G}_{a,Q}^{\epsilon} \times G(\mathcal{C}^{ss})$ with $\epsilon = 0, 1$ and $\mathcal{C}^{ss}$ the semisimplification of $\mathcal{C}$. Conjecture 1.3 for such a $\mathcal{C}$ then easily follows from Conjecture 1.3 for $\mathcal{C}^{ss}$ (Section 8). This yields our first main result.

Theorem. *Conjecture 1.3 holds for motivic Q-coefficients.*

1.5. $\overline{\mathbb{Q}}_p$**-coefficients.** From the motivic point of view, the proof of Theorem 1.4 is "the" natural one but it does not cover Conjecture 1.3 for convergent $\overline{\mathbb{Q}}_p$-coefficients. Also, considering the deepness of the theory of companions, one may ask for alternative more "elementary" proofs of Conjecture 1.3, even for motivic Q-coefficients. For motivic $\overline{\mathbb{Q}}_p$-coefficients and convergent $\overline{\mathbb{Q}}_p$-coefficients satisfying a weak form of the (generalized) parabolicity conjecture of Crew (in particular, †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficients) one can provide such an elementary proof by adjusting the Tannakian arguments of the proof of Theorem 1.4 to reduce Conjecture 1.3 for such $\overline{\mathbb{Q}}_p$-coefficients to Conjecture 1.3 for direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-F-isocrystals (Fact 1.3.3 above). This approach was suggested to us by Ambrosi; it roughly consists in replacing the weight filtration by the slope filtration. More precisely, as already mentioned in the proof of Corollary 1.3.2.2, for every convergent $\overline{\mathbb{Q}}_p$-coefficient $\mathcal{C}$ on X, there exists a dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration on U (Fact 9.1.1) and as the slope filtration behaves like the weight filtration with respect to Frobenii, one can apply the same group-theoretic arguments as in the proof of Theorem 1.4 to reduce Conjecture 1.3 for $\mathcal{C}|_U$ to Conjecture 1.3 for direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients on U. But Conjecture 1.3 for $\mathcal{C}|_U$ does not automatically imply Conjecture 1.3 for $\mathcal{C}$ since the closed immersion $G(\mathcal{C}|_U) \subset G(\mathcal{C})$ is not an isomorphism in general. However, a generalization of the parabolicity conjecture of Crew (Conjecture 9.2.1) predicts that $G(\mathcal{C}|_U)$ should be the stabilizer in $G(\mathcal{C})$ of the slope filtration on $\mathcal{C}|_U$. We show that a weaker form of this conjecture (Conjecture 9.2.4) is enough to deduce Conjecture 1.3 for $\mathcal{C}$ from Conjecture 1.3 for $\mathcal{C}|_U$. As the parabolicity conjecture is now known for †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficients by recent works of D'Addezio - see Fact 9.2.2, this yields our second main result:

Theorem. *Assume X is smooth over k. Then Conjecture 1.3 holds for convergent $\overline{\mathbb{Q}}_p$-coefficients on X satisfying the weak (generalized) parabolicity conjecture 9.2.4. In particular, it holds (uncondionally) for:*
- *convergent $\overline{\mathbb{Q}}_p$-coefficients admitting a slope filtration on X;*
- *†-extendable convergent $\overline{\mathbb{Q}}_p$-coefficients on X.*

1.5.1. **Remark.** Theorem 1.5 also holds for a direct sum of a convergent $\overline{\mathbb{Q}}_p$-coefficient $\mathcal{I}$ admitting a slope filtration and a †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficient $\mathcal{C}$ provided that for some (equivalently every) dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration the canonical morphism $G(\mathcal{C}|_U, \mathcal{I}|_U) \to G(\mathcal{C}, \mathcal{I})$ is a closed immersion, where we write $G(\mathcal{C}, \mathcal{I}) := G(\langle \mathcal{C} \rangle \cap \langle \mathcal{I} \rangle)$ - See Subsection 9.5.

Our arguments to prove Theorem 1.5 also provide an alternative purely p-adic and "elementary" (in the sense above) proof of Theorem 1.4 for motivic $\overline{\mathbb{Q}}_p$-coefficients (hence in particular a proof of Statement B) by reduction to Fact 1.3.3. See Section 9 for details.

1.5.2. **Remark.** Though $\overline{\mathbb{Q}}_u$-sheaves are built from étale torsion sheaves (just as $\overline{\mathbb{Q}}_\ell$-sheaves), quite surprisingly we are not aware for the time being of an "elementary" argument avoiding the companion conjecture to prove Conjecture 1.3 for $\overline{\mathbb{Q}}_u$-coefficients. Also, Remark 1.3.3 (2) provides an "elementary" proof of Conjecture 1.3 for semisimple $\overline{\mathbb{Q}}_\ell$-coefficients but we do not know how to treat the case of arbitrary (non-étale) $\overline{\mathbb{Q}}_\ell$-coefficients without resorting to the theory of companions.

1.6. **Tannakian arguments.** Aside from the arithmetico-geometric inputs (the companion conjecture, existence of slope and weight filtrations *etc.*) and the reconstruction theorem [KaLV14, Thm. 1.2], the main technical difficulties are in the Tannakian reduction steps, for which we need a series of lemmas (of possibly independent interest) to transfer Zariski-density properties of conjugacy invariant subsets. The general situation is the following. Fix an algebraic group $\widehat{G}$ over an algebraically closed field Q of characteristic 0. Assume $\widehat{G}$ is given with a faithful, r-dimensional Q-representation V ($r<\infty$) and that V is endowed with a filtration $S_\bullet V$: $V=S_1V\supsetneq\cdots\supsetneq S_sV\supsetneq S_{s+1}V=0$ (defined by a cocharacter $\omega:\mathbb{G}_{m,Q}\to\widehat{G}\subset\mathrm{GL}_V$). Let $G\subset\widehat{G}$ denote a subgroup of the stabilizer of $S_\bullet V$ in $\widehat{G}$ containing the centralizer of the image of ω in $\widehat{G}$ and consider the G-representation $\widetilde{V}:=\oplus_i S_iV/S_{i+1}V$. Let $\widetilde{G}$ denote the image of G acting on $\widetilde{V}$ and $R:=\ker(G\twoheadrightarrow\widetilde{G})$. Fix also a union $\Phi\subset G$ of G-conjugacy classes, let $\widetilde{\Phi}\subset\widetilde{G}$ (resp. $\widehat{\Phi}\subset\widehat{G}$) denote its image in $\widetilde{G}$ (resp. the conjugacy-invariant subset generated by Φ in $\widehat{G}$). Eventually, let $\chi:GL_{\widetilde{V}}\to\mathbb{P}_{r,Q}:=\mathbb{G}_{m,Q}\times\mathbb{A}^{r-1}_Q$ denote the characteristic polynomial map. Fix a $g\in\Phi$ with image $\widetilde{g}\in\widetilde{G}$. We have:

(*) (Proposition 3.4.1) If $\widetilde{G}=R_u(\widetilde{G})\times\widetilde{G}^{red}$ and $\Psi:=Id\times\chi:\widetilde{G}=R_u(\widetilde{G})\times\widetilde{G}^{red}\to R_u(\widetilde{G})\times\mathbb{P}_{r,Q}$ then $\overline{\widetilde{\Phi}}^{zar}\supset\widetilde{G}^\circ\widetilde{g}$ iff $\overline{\Psi(\widetilde{\Phi})}^{zar}\supset\Psi(\widetilde{G}^\circ\widetilde{g})$.

(**) (Lemma 3.5.2) If for every $\gamma\in\Phi$, $R^{\gamma^{ss}}=1$ then $\overline{\Phi}^{zar}\supset G^\circ g$ iff $\overline{\widetilde{\Phi}}^{zar}\supset\widetilde{G}^\circ\widetilde{g}$.

(***) (Lemma 3.6) If $\overline{\Phi}^{zar}\supset G^\circ g$ then $\overline{\widehat{\Phi}}^{zar}\supset\widehat{G}^\circ g$.

Here $R_u(G)\subset G$ denotes the unipotent radical and $G\twoheadrightarrow G^{red}:=G/R_u(G)$ the maximal reductive quotient and for an element $\gamma\in G$, $\gamma=\gamma^u\gamma^{ss}=\gamma^{ss}\gamma^u$ denotes the multiplicative Jordan decomposition of γ in G.

The proofs of the above assertions are significantly simpler if G is connected. To treat the non-connected case, we introduce the notion of quasi-Cartan subgroup, which is a well-behaved generalization for non-connected algebraic groups of the classical notion of Cartan subgroup. For a reductive G the theory of quasi-Cartan subgroups (then called maximal quasi-tori) is due to Hartl-Pál; our theory of quasi-Cartan subgroups for arbitrary G is a mild enhancement of theirs.

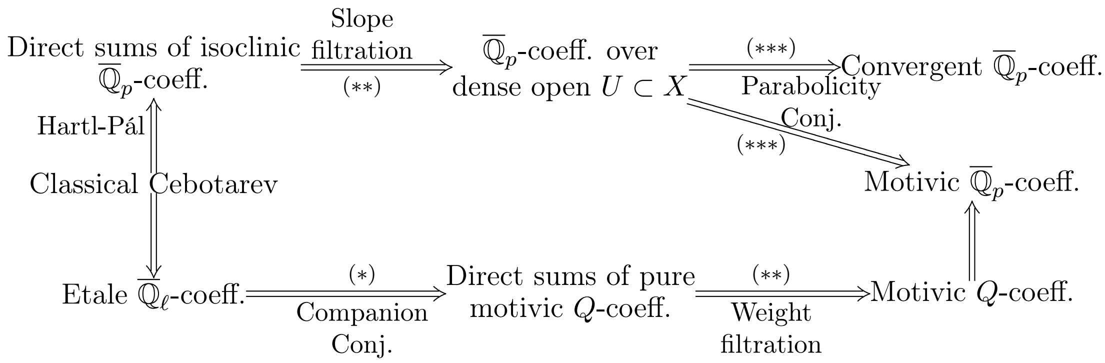
1.7. **Comparison with [HP18].** As mentioned, our work owes a lot to [HP18]. In particular, they treat the core case of direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients (Fact 1.3.3) and they develop the theory of maximal quasi-tori ([HP18, Sections 8, 9]), which lead us to the one of quasi-Cartan subgroups. Aside from Conjecture 1.3 for direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients, they also provide in the following cases:

- ([HP18, Thm. 1.12]) a proof - *via* equidistribution - of Conjecture 1.3 for semisimple motivic $\overline{\mathbb{Q}}_p$-coefficients . Their argument relies heavily on the Frobenius weight formalism and also uses indirectly the companion conjecture for motivic $\overline{\mathbb{Q}}_p$-coefficients through the fact that an irreducible motivic $\overline{\mathbb{Q}}_p$-coefficient with finite determinant is pure of weight 0. It goes as follows. As in the proof of Fact 1.3.3, it is enough to prove Conjecture 1.3 for $\mathcal{C}^\dagger=\mathcal{D}^\dagger\oplus\mathcal{A}^\dagger$ with $\mathcal{D}^\dagger$ a direct sum of irreducible motivic $\overline{\mathbb{Q}}_p$-coefficients with finite determinant and $\mathcal{A}^\dagger$ a direct sum of rank-one geometrically constant motivic $\overline{\mathbb{Q}}_p$-coefficients. Conjecture 1.3 for $\mathcal{A}^\dagger$ is again by [HP18, Thm. 7.4]. Conjecture 1.3 for $\mathcal{D}^\dagger$ follows from Deligne's equidistribution theorem for motivic $\overline{\mathbb{Q}}_p$-coefficients - a formal consequence of the "à la Weil 2" Frobenius weights formalism, *via* the unitarian trick. It is to make the unitarian trick work, that one needs that an irreducible motivic $\overline{\mathbb{Q}}_p$-coefficient with finite determinant is pure of weight 0. The mixed case is again more delicate and requires an *ad-hoc* Zariski-density criterion using a bit of measure theory (after choosing suitable real structures) and a beatifully simple compactness argument ([HP18, Thm. B.10]). See [HP18, Section 12] for details.
- ([HP18, Thm. 1.11]) a proof - *via* reduction to the overconvergent case - of Conjecture 1.3 for convergent $\overline{\mathbb{Q}}_p$-coefficients of the form $\mathcal{C}=\mathcal{E}\oplus\mathcal{I}$ with $\mathcal{I}$ a direct sum of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients and $\mathcal{E}$ a semisimple "locally weakly firm" †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficient. Here, "locally weakly firm" means there exists a non-empty open subscheme $U\subset X$ such that $\mathcal{F}|_U$ has a slope filtration and the maximal

quasi-torus of $G(\mathcal{F}|_U)$ is abelian. This latter condition is restrictive but holds *e.g* when $G(\mathcal{F}|_U)$ is connected.

To sum it up, for the time being, the status of Conjecture 1.3 seems to be as follows:
- $\overline{\mathbb{Q}}_u$-coefficients: proof *via* companion conjecture;
- Motivic $\overline{\mathbb{Q}}_p$-coefficients:
  – proof *via* companion conjecture (semisimple case) and weight filtration;
  – proof *via* equidistribution, using the purity part of companion conjecture (semisimple case) and weight filtration;
  – (purely p-adic and "elementary") proof by reduction to the case of direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients *via* slope filtration and parabolicity conjecture for †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficients;
- Convergent $\overline{\mathbb{Q}}_p$-coefficients: *assuming the weak (generalized) parabolicity conjecture*, (purely p-adic and "elementary") proof by reduction to the case of direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients *via* slope filtration.

1.8. The following diagram provides a synthetical overview of the architecture of our proofs.



The paper is organized as follows. Section 2 lists the basic group-theoretical notation used in the paper. Sections 3-4 contain preliminaries. Section 3 gathers the results from the theory of algebraic groups required to perform steps (*), (**), (***) of the proofs (namely the corresponding statements (*), (**), (***) of Subsection 1.6); it can be read independently of the rest of the paper or be skipped and used as a toolbox by the reader mostly interested in the global structure and arithmetico-geometric inputs of the proofs. In Section 4, we reduce Theorem 1.4 to the case where X is smooth and explain why the resp. part of Conjecture 1.3 follows from the non-resp. part. Sections 5-8 are devoted to the proof of Theorem 1.4. In Section 5, we recall why Theorem 1.4 for étale $\overline{\mathbb{Q}}_\ell$-coefficients follows from the classical Cebotarev density theorem and review the basic features of the theory of companions; in particular we show there that every motivic Q-coefficient admits an étale $\overline{\mathbb{Q}}_\ell$-companion for $\ell \gg 0$. In Section 6, we prove Theorem 1.4 for semisimple motivic Q-coefficients. In Section 7, we review the basic features of the weight filtration, describe the structure of the Tannakian group of a direct sum of motivic pure Q-coefficients and combine both ingredients to reduce Theorem 1.4 to the case of direct sum of motivic pure Q-coefficients (Step (**) for motivic Q-coefficients in the above diagram). Eventually, in Section 8, we conclude the proof of Theorem 1.4 for arbitrary motivic coefficients (Step (*) in the above diagram). Section 9 is devoted to the proof of Theorem 1.5. In Subsection 9.1 we review the basic features of the theory of slopes and in Subsections 9.2 and 9.3 we discuss (some variants of) the parabolicity conjecture. In Subsections 9.4 and 9.5 we reduce Conjecture 1.3 for $\mathcal{C}|_U$ to Conjecture 1.3 for direct sums of isoclinic convergent $\overline{\mathbb{Q}}_p$-F-isocrystals (Step (**) for convergent $\overline{\mathbb{Q}}_p$-coefficients in the above diagram) and show that if $\mathcal{C}$ satisfies the parabolicity conjecture then Conjecture 1.3 for $\mathcal{C}|_U$ implies Conjecture 1.3 for $\mathcal{C}$ (Step (***) in the above diagram).

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2. Notation

For an algebraic group G over a field Q, write
- $G^\circ \subset G$ for its neutral component and $p_{G^\circ}: G \twoheadrightarrow \pi_0(G) := G/G^\circ$ for its group of connected components;
- $R_u(G) \subset G$ for its unipotent radical and $p_{R_u(G)}: G \twoheadrightarrow G^{red} := G/R_u(G)$ for its maximal reductive quotient.

As $R_u(G) \subset G^\circ$, one has a canonical commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{p_{R_u(G)}} & G^{red} \\ {\scriptstyle p_{G^\circ}}\downarrow & & \downarrow{\scriptstyle p_{G^{red\circ}}} \\ \pi_0(G) & \xrightarrow{\simeq} & \pi_0(G^{red}). \end{array}$$

For subgroups $H, K \subset G$, write $Z_K(H) \subset N_K(H) \subset G$ for the subgroups of K centralizing and normalizing H respectively. For $g \in G$, write $Z_K(g)$ for $Z_K(\overline{\langle g\rangle}^{zar})$.

When Q is of characteristic 0, let $Q^+[G]$ denote the Grothendieck semiring of the category of semisimple Q-rational representations of G. Namely, $Q^+[G]$ is the set of isomorphism classes of finite-dimensional Q-rational semisimple representations of G endowed with $[V_1] + [V_2] = [V_1 \oplus V_2]$, $[V_1] \cdot [V_2] = [V_1 \otimes V_2]$.

For an arbitrary (abstract) group Γ, let $\Gamma//\Gamma$ denote the set of conjugacy classes of Γ.

3. Preliminaries on algebraic groups

Let Q be an algebraically closed field of characteristic 0. Given an algebraic group G over Q and $g \in G$, let $g = g^u g^{ss}$ denote its multiplicative Jordan decomposition in G and let Φ_g denote the G-conjugacy class of g. For a subset $\Phi \subset G$, write $\Phi^? := \{g^? \mid g \in \Phi\}$, $? = u, ss$.

Let G be an (a not necessarily connected) algebraic group over Q.

3.1. We will use the following elementary observation in several places below.

Lemma. *Let $R, L \subset G$ be closed subgroups such that $G = R \rtimes L$. Let $R' \subset R$ and $S, S' \subset L$ be closed subgroups. Assume $S' \subset N_G(R')$ and write $G' := R' \rtimes S'$. Then $Z_{S'}(S)$ normalizes $Z_{R'}(S)$ and $Z_{G'}(S) = Z_{R'}(S) \rtimes Z_{S'}(S)$*

Proof. Clearly $Z_{S'}(S)$ normalizes $Z_{R'}(S)$ and $Z_{R'}(S) \rtimes Z_{S'}(S) \subset Z_{G'}(S)$. Conversely, let $\rho\lambda \in Z_{G'}(S)$ with $\rho \in R'$, $\lambda \in S'$. Then for every $s \in S$, $\rho\lambda s = s\rho\lambda$ if and only if $\rho(\lambda s\lambda^{-1})\rho^{-1}(\lambda s\lambda^{-1})^{-1} = s(\lambda s\lambda^{-1})^{-1} \in R \cap L = 1$ hence if and only if $\lambda s\lambda^{-1} = s$ and $\rho s\rho^{-1} = s$. As this holds for every $s \in S$, this means $\rho \in Z_{R'}(S)$, $\lambda \in Z_{S'}(S)$ as desired. □

For instance, if $L \subset G$ is a Levi subgroup, for every $g \in L$, $Z_G(g) = Z_{R_u(G)}(g) \rtimes Z_L(g)$.

3.2. **Quasi-Cartan.** A Cartan in G is the centralizer $Z_G(T^\circ)$ of a maximal torus $T^\circ \subset G^\circ$. A quasi-Cartan in G is a closed algebraic subgroup $C \subset G$ of the form

$$C = N_G(B^\circ) \cap N_G(T^\circ)$$

for some Borel $B^\circ \subset G^\circ$ and maximal torus $T^\circ \subset B^\circ$. In particular [B91, IV, 11.19, Prop. (b)],

$$C \cap G^\circ = Z_{G^\circ}(T^\circ) \subset G^\circ$$

hence $C \cap G^\circ \subset G^\circ$ is a Cartan in G° and $C^\circ = C \cap G^\circ$ [B91, IV, 12.1, Thm. (e)]. One also has $Z_G(T^\circ)^\circ = Z_{G^\circ}(T^\circ)$ so that the neutral component of the quasi-Cartan of G attached to $T^\circ \subset B^\circ$ depends only on T° and coincides with the neutral component of the Cartan of G attached to T°. A big quasi-Cartan in G is a closed algebraic subgroup $\widetilde{C} \subset G$ of the form $\widetilde{C} = p_{R_u(G)}^{-1}(T)$ for $T \subset G^{red}$ a quasi-Cartan.

If G is connected, quasi-Cartan are Cartan. If $G = R_u(G) \times G^{red}$ (in particular if G is reductive), big quasi-Cartan are quasi-Cartan.

3.2.1. Since all pairs $T^\circ \subset B^\circ$ of a maximal torus contained in a Borel subgroup of G° are conjugate under G° [B91, IV, 11.19, Prop. (c)], all quasi-Cartan in G are conjugate under G°. In particular, all big quasi-Cartan in G are conjugate under G°.

3.2.2. Let $C \subset G$ be a quasi-Cartan, then the canonical morphism $\pi_0(C) \to \pi_0(G)$ is an isomorphism.

Indeed, the surjectivity follows from [B91, IV, 11.19, Prop. (c)] and the injectivity follows from $C^\circ = C \cap G^\circ$.

3.2.3. Since a surjective morphism of algebraic groups $G \twoheadrightarrow G'$ maps Borel subgroups (resp. maximal tori, resp. the unipotent radical) of G onto Borel subgroups (resp. maximal tori, resp. the unipotent radical) of G' [B91, IV, 11.14, Prop. (1); 14.11, Cor.], it also maps quasi-Cartan (resp. big quasi-Cartan) of G onto quasi-Cartan (resp. big quasi-Cartan) of G'. (Use 3.2.2 for π_0 and [B91, IV, 12.4, Prop. (1)] for neutral components.) In particular, every quasi-Cartan is contained in a big quasi-Cartan and the set of quasi-Cartan of G coincides with the set of quasi-Cartan of big quasi-Cartan of G.

3.2.4. Let $C \subset G$ be a quasi-Cartan. For every $g \in G$ the following are equivalent:

- (3.2.4.1) $g \in G^{ss}$;
- (3.2.4.2) there exists $\gamma \in G$ such that $\gamma g \gamma^{-1} \in C^{ss}$ (*i.e.* $\Phi_g \cap C^{ss} \neq \emptyset$);
- (3.2.4.3) there exists $\gamma^\circ \in G^\circ$ such that $\gamma^\circ g (\gamma^\circ)^{-1} \in C^{ss}$.

Indeed, since $G^\circ \subset G$ and $G^{ss} \cap C = C^{ss}$, (3.2.4.3) $\Rightarrow$ (3.2.4.2) $\Rightarrow$ (3.2.4.1) is straightforward and (3.2.4.1) $\Rightarrow$ (3.2.4.2) follows from [St68, Thm. 7.5], which asserts that every semisimple element is contained in a quasi-Cartan. Eventually, assume (3.2.4.2). From 3.2.2, there exists $\widetilde{\gamma} \in C$ such that $\gamma^\circ := \widetilde{\gamma}^{-1}\gamma \in G^\circ$ hence $\gamma^\circ g (\gamma^\circ)^{-1} = \widetilde{\gamma}^{-1}\gamma g \gamma^{-1}\widetilde{\gamma} \in \widetilde{\gamma}^{-1} C^{ss} \widetilde{\gamma} = C^{ss}$. Thus, (3.2.4.3) holds.

Remark. If G is reductive, $C^\circ \subset G^\circ$ is a maximal torus [B91, IV, 13.17, Cor. 2 (c)] hence $C = C^{ss}$.

3.2.5. Let $\widetilde{C} \subset G$ be a big quasi-Cartan (resp. let $C \subset G$ be a quasi-Cartan). For every $g \in \widetilde{C}^{ss}$, $Z_{\widetilde{C}^\circ}(g)^\circ \subset Z_{G^\circ}(g)^\circ$ is a big quasi-Cartan (resp. for every $g \in C^{ss}$, $Z_{C^\circ}(g)^\circ \subset Z_{G^\circ}(g)^\circ$ is contained in a Cartan).

Indeed, fix a Levi subgroup $L \subset G$ [Ho81, VIII, Thm. 4.3] and a quasi-Cartan $T \subset L$. Then $R_u(G) \rtimes T \subset R_u(G) \rtimes L = G$ is a big quasi-Cartan in G. In particular, from 3.2.1 and [Ho81, VIII, Thm. 4.3], up to replacing L (and T) by a conjugate, one may assume $\widetilde{C} = R_u(G) \rtimes T$ and $g \in T$ hence $\widetilde{C}^\circ = R_u(G) \rtimes T^\circ$ and (see Lemma 3.1)

$$Z_{\widetilde{C}^\circ}(g) = Z_{R_u(G)}(g) \rtimes Z_{T^\circ}(g), \; Z_{\widetilde{C}^\circ}(g)^\circ = Z_{R_u(G)}(g) \rtimes Z_{T^\circ}(g)^\circ$$

From Fact 3.2.6 below, $Z_{L^\circ}(g)$ is reductive and $Z_{T^\circ}(g)^\circ \subset Z_{L^\circ}(g)^\circ$ is a Cartan (equivalently, a maximal torus) while from $Z_{G^\circ}(g) = Z_{R_u(G)}(g) \rtimes Z_{L^\circ}(g)$, one gets

$$Z_{G^\circ}(g)^\circ = Z_{R_u(G)}(g) \rtimes Z_{L^\circ}(g)^\circ.$$

The proof for quasi-Cartan is similar. More precisely, one may assume $C = Z_{R_u(G)}(T^\circ) \rtimes T$ and $g \in T$. Then $Z_{C^\circ}(g)^\circ = Z_{Z_{R_u(G)}(g)}(T^\circ) \rtimes Z_{T^\circ}(g)^\circ$ is contained in the Cartan $Z_{Z_{R_u(G)}(g)}(Z_{T^\circ}(g)^\circ) \rtimes Z_{T^\circ}(g)^\circ \subset Z_{G^\circ}(g)^\circ$.

3.2.6. **Fact.** ([St68, Thm. 8.1], [HP18, Thm. 8.2]) *Let G be a reductive group, $T \subset G$ a quasi-Cartan and $g \in T$. Then $Z_{G^\circ}(g)$ is reductive and $Z_{T^\circ}(g)^\circ \subset Z_{G^\circ}(g)$ is a maximal torus.*

For general G, let $L \subset G$ be a Levi subgroup, $B^\circ \subset L^\circ$ a Borel subgroup and $T^\circ \subset B^\circ$ a maximal torus. Write $T := N_L(B^\circ) \cap N_L(T^\circ) \subset L$ for the corresponding quasi-Cartan of L. Then, for every $g \in T$, $Z_L(g)$ is reductive, $T_g := Z_T(g)^\circ \subset Z_L(g)$ is a maximal torus of $Z_L(g)$ hence of $Z_G(g) = Z_{R_u(G)}(g) \rtimes Z_L(g)$ (see Lemma 3.1) and the neutral component of the corresponding Cartan subgroup of $Z_G(g)$ is

$$C_g := Z_{Z_G(g)^\circ}(T_g)^\circ = Z_{Z_{R_u(G)}(g)}(T_g) \times T_g.$$

3.3. **A criterion for Zariski-density.** The following is a slight enhancement of the argument in [HP18, Thm. 8.9 (c)].

3.3.1. **Lemma.** *Let $g \in G^{ss}$ and write $\iota_g : G \to G$ for left conjugation by g (i.e. $\iota_g(\gamma) = g\gamma g^{-1}$). The morphism*

$$\begin{array}{ccc} G^\circ \times Z_{G^\circ}(g)^\circ & \to & G^\circ \\ (\gamma, \omega) & \to & \gamma\omega\iota_g(\gamma)^{-1} \end{array}$$

is dominant. If G° is commutative, it is surjective.

Proof. Since G° is connected, it is enough to show that the tangent map at $(1, 1) \in G^\circ \times Z_{G^\circ}(g)^\circ$ is surjective. One can compute it explicitly as

$$\begin{array}{ccc} \mathfrak{g} \oplus \ker(Ad(g) - 1) & \to & \mathfrak{g} \\ (\Gamma, \Omega) & \to & (1 - Ad(g))(\Gamma) + \Omega, \end{array}$$

where $\mathfrak{g} := Lie(G^\circ)$, $Ad(g) : \mathfrak{g} \to \mathfrak{g}$, $\Omega \to g\Omega g^{-1}$ hence $Lie(Z_G(g)^\circ) = \ker(Ad(g) - 1)$. As g is semisimple, $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ is semisimple as well hence $\mathfrak{g} = \mathrm{im}(Ad(g) - 1) \oplus \ker(Ad(g) - 1)$. If G° is commutative, $G^\circ \times Z_{G^\circ}(g)^\circ \to G^\circ$ is a morphism of algebraic groups hence its image is closed [B91, I, 1.4, Cor. (a)]. □

3.3.2. **Corollary.** *Let $g \in G^{ss}$ and let $C_g \subset Z_{G^\circ}(g)^\circ (\subset G^\circ)$ be a Cartan subgroup of $Z_{G^\circ}(g)^\circ$.*

- (3.3.2.1) *The map*

$$\begin{array}{cccc} p\colon & G^\circ \times C_g g & \to & G^\circ g \\ & (\gamma, cg) & \to & \gamma c g \gamma^{-1} \end{array}$$

  is dominant hence $p(G^\circ \times C_g g)$ contains a dense open subset of $G^\circ g$. If G° is commutative, it is surjective.
- (3.3.2.2) *Let $\Phi \subset G$ be a union of G°-conjugacy classes. Then the following are equivalent*
  - (3.3.2.2.1) $\overline{\Phi}^{zar} \supset C_g g$;
  - (3.3.2.2.2) $\overline{(\Phi \cap C_g g)}^{zar} = C_g g$;
  - (3.3.2.2.3) $\overline{(\Phi \cap G^\circ g)}^{zar} = G^\circ g$;
  - (3.3.2.2.4) $\overline{\Phi}^{zar} \supset G^\circ g$.

Proof. (3.3.2.1) is equivalent to showing that the morphism

$$\begin{array}{ccc} G^\circ \times C_g & \to & G^\circ \\ (\gamma, c) & \to & \gamma c \iota_g(\gamma)^{-1} \end{array}$$

is dominant (or surjective if G° is commutative) or, equivalently, that the composite of the morphisms

$$\begin{array}{ccccc} G^\circ \times Z_{G^\circ}(g)^\circ \times C_g & \to & G^\circ \times Z_{G^\circ}(g)^\circ & \to & G^\circ \\ (\gamma, (\omega, c)) & \to & (\gamma, \omega c \iota_g(\omega)^{-1}) & & \\ & & (\gamma, \omega') & \to & \gamma\omega'\iota_g(\gamma)^{-1} \end{array}$$

is dominant (or surjective if G° is commutative). This follows from Lemma 3.3.1 and the fact that the morphism

$$\begin{array}{ccc} Z_{G^\circ}(g)^\circ \times C_g & \to & Z_{G^\circ}(g)^\circ \\ (\omega, c) & \to & \omega c \omega^{-1} = \omega c \iota_g(\omega)^{-1} \end{array}$$

is dominant (or surjective if G° is commutative) since $C_g \subset Z_{G^\circ}(g)^\circ$ is a Cartan subgroup [B91, IV, 12.1, Thm. (b)].

For (3.3.2.2), we show (3.3.2.2.4) ⇒ (3.3.2.2.3) ⇒(3.3.2.2.2) ⇒ (3.3.2.2.1) ⇒ (3.3.2.2.4). The implication (3.3.2.2.2) ⇒ (3.3.2.2.1) is straightforward and the implication (3.3.2.2.4) ⇒ (3.3.2.2.3) follows from the fact that $G^\circ g$ is open in G. Assume (3.3.2.2.1). Then $p(G^\circ \times C_g g) \subset \overline{\Phi}^{zar}$ and (3.3.2.2.1) ⇒ (3.3.2.2.4) follows from (3.3.2.1). Assume (3.3.2.2.3). From (3.3.2.1) and Chevalley's constructibility theorem, for every non-empty open subset $U \subset C_g g$, $p(G^\circ \times U)$ contains again a dense open subset of $G^\circ g$. Hence $\Phi \cap p(G^\circ \times U) \neq \emptyset$. This shows (3.3.2.2.3) ⇒ (3.3.2.2.2). □

3.3.3. **Corollary.** *Assume G is reductive. Let $T \subset G$ be a quasi-Cartan and $g \in T$. Let $\Phi \subset G$ be a union of G°-conjugacy classes. The following are equivalent*

- (3.3.3.1) $\overline{\Phi}^{zar} \supset Z_{T^\circ}(g)^\circ g$;
- (3.3.3.2) $\overline{(\Phi \cap Z_{T^\circ}(g)^\circ g)}^{zar} = Z_{T^\circ}(g)^\circ g$;
- (3.3.3.3) $\overline{(\Phi \cap G^\circ g)}^{zar} = G^\circ g$;
- (3.3.3.4) $\overline{\Phi}^{zar} \supset G^\circ g$.

Proof. Since $g \in T$, g is semisimple by the remark after 3.2.4 hence one can apply (3.3.2.2). But from Fact 3.2.6, $Z_{G^\circ}(g)$ is reductive and $Z_{T^\circ}(g)^\circ \subset Z_{G^\circ}(g)^\circ$ is a maximal torus so that one can take $C_g = Z_{T^\circ}(g)^\circ$ in (3.3.2.2). □

3.4. Assume $G = R_u(G) \times G^{red}$ and write $(-)_u : G \to R_u(G)$, $(-)_{red} : G \to G^{red}$ for the first and second projections respectively. Let $T \subset G^{red}$ be a quasi-Cartan. Then for every $g \in T$, $C_g := R_u(G) \times Z_{T^\circ}(g)^\circ \subset Z_{G^\circ}(g)^\circ$ is a Cartan. Fix a faithful Q-linear representation V of G^{red} and an isomorphism $V \tilde{\to} Q^r$, identifying $\mathrm{GL}(V) \tilde{\to} \mathrm{GL}_{r,Q}$. Write

$$\begin{array}{cccc} \chi : & \mathrm{GL}_{r,Q} & \to & \mathbb{P}_{r,Q} := \mathbb{G}_{m,Q} \times \mathbb{A}_Q^{r-1} \\ & g & \to & \det(Id - Tg) \end{array}$$

for the characteristic polynomial map and $\psi := Id \times \chi : R_u(G) \times G^{red} \to R_u(G) \times \mathbb{P}_{r,Q}$. For a conjugacy class $\overline{\Delta} \subset \pi_0(G)$ with inverse image $\Delta := p_{G^\circ}^{-1}(\overline{\Delta}) = R_u(G) \times \Delta_{red} \subset R_u(G) \times G^{red} = G$, write $\chi(G)_{\overline{\Delta}} := \chi(\Delta_{red})$ and $\psi(G)_{\overline{\Delta}} := \psi(\Delta) = R_u(G) \times \chi(G)_{\overline{\Delta}}$.

3.4.1. **Proposition.**
- (3.4.1.1) *For every conjugacy class $\overline{\Delta} \subset \pi_0(G)$ and $g \in T$ lifting an element $\overline{g} \in \overline{\Delta}$ (cf. 3.2.2),*

$$\psi(G)_{\overline{\Delta}} = \psi(C_g g) = R_u(G) \times \chi(G)_{\overline{\Delta}} = R_u(G) \times \chi(Z_{T^\circ}(g)^\circ g)$$

and the restriction $\psi : C_g g \to R_u(G) \times \mathbb{P}_{r,Q}$ is a finite morphism. In particular, $\psi(G)_{\overline{\Delta}} \subset R_u(G) \times \mathbb{P}_{r,Q}$ is a closed irreducible subvariety of dimension $\dim(C_g)$ and the irreducible components of $\psi(G)$ are the maximal elements among the $\psi(G)_{\overline{\Delta}}$ for $\overline{\Delta} \subset \pi_0(G)$ varying among all conjugacy classes of $\pi_0(G)$;
- (3.4.1.2) *Let $\overline{\Delta} \subset \pi_0(G)$ be a conjugacy class with inverse image $\Delta := p_{G^\circ}^{-1}(\overline{\Delta}) \subset G$. Let $\Phi \subset \Delta$ be a union of conjugacy classes. Then*

$$\overline{\Phi}^{zar} = \Delta \Leftrightarrow \overline{\psi(\Phi)}^{zar} = \psi(G)_{\overline{\Delta}}.$$

Proof. For (3.4.1.1), it is enough to prove it when $G = G^{red}$ hence $C_g = Z_{T^\circ}(g)^\circ$, which we assume from now on. Consider a conjugacy class $\overline{\Delta} \subset \pi_0(G)$ with inverse images $\Delta := p_{G^\circ}^{-1}(\overline{\Delta}) \subset G$ and $\Delta_T := p_{T^\circ}^{-1}(\overline{\Delta})(= \Delta \cap T) \subset T$. Then $\chi(\Delta) = \chi(\Delta^{ss})$ and from 3.2.4, $\chi(\Delta^{ss}) = \chi(\Delta_T) = \cup_{\overline{g} \in \overline{\Delta}} \chi(T^\circ g)$ while, from (3.3.2.1) applied to T, $\chi(T^\circ g) = \chi(Z_{T^\circ}(g)^\circ g)$. This shows that

$$\chi(G)_{\overline{\Delta}} = \cup_{\overline{g} \in \overline{\Delta}} \chi(Z_{T^\circ}(g)^\circ g).$$

But since $Z_{T^\circ}(g)^\circ$ and g commute, one may assume that the algebraic subgroup $\overline{\langle Z_{T^\circ}(g)^\circ, g\rangle}^{zar} \hookrightarrow G$ they generate in G is contained in the diagonal torus $\mathbb{G}_{m,Q}^r \subset GL_{r,Q}$. Then $\chi(Z_{T^\circ}(g)^\circ g)$ is the image of the morphism

$$Z_{T^\circ}(g)^\circ g \hookrightarrow \overline{\langle Z_{T^\circ}(g)^\circ, g\rangle}^{zar} \subset \mathbb{G}_{m,Q}^r \stackrel{\chi}{\to} \mathbb{P}_{r,Q},$$

which is finite as the composite of closed immersions with the finite morphism $\chi : \mathbb{G}_{m,Q}^r \to \mathbb{P}_{r,Q}$. This shows that $\chi(Z_{T^\circ}(g)^\circ g)$ is a closed irreducible subvariety of dimension $\dim(Z_{T^\circ}(g)^\circ)$. Hence $\chi(G)_{\overline{\Delta}}$ is a closed subvariety as well. But $\chi(G)_{\overline{\Delta}}$ is also irreducible since for every $g \in G$ lifting an element in $\overline{\Delta}$, $\chi(G)_{\overline{\Delta}} = \chi(G^\circ g)$. On the other hand, since for every $g, h \in T$, $Z_{T^\circ}(hgh^{-1})^\circ = hZ_{T^\circ}(g)^\circ h^{-1}$, the $\chi(Z_{T^\circ}(g)^\circ g)$, $\overline{g} \in \overline{\Delta}$ all have the same dimension. This forces $\chi(G)_{\overline{\Delta}} = \chi(Z_{T^\circ}(g)^\circ g)$ for every $\overline{g} \in \overline{\Delta}$. The last part of (3.4.1.1) follows from the tautological equality $\chi(G) = \cup_{\overline{\Delta}} \chi(G)_{\overline{\Delta}}$. We now prove (3.4.1.2). For every $h \in \Phi$, $\Phi_h = (\Phi_h)_u \times (\Phi_h)_{red}$ with $(\Phi_h)_u = \Phi_{h_u} \subset R_u(G)$ and $(\Phi_h)_{red} = \Phi_{h_{red}} \subset G^{red}$. Note that $h^u = h_u(h_{red})^u$, $h^{ss} = (h_{red})^{ss}$. Set $\Phi' := \{h_u \cdot h^{ss} \mid h \in \Phi\}$. We show the following implications:

$$\overline{\Phi}^{zar} = \Delta \stackrel{(1)}{\Rightarrow} \overline{\psi(\Phi)}^{zar} = \psi(G)_{\overline{\Delta}} \stackrel{(2)}{\Leftrightarrow} \overline{\psi(\Phi')}^{zar} = \psi(G)_{\overline{\Delta}} \stackrel{(3)}{\Leftrightarrow} \overline{\Phi'}^{zar} = \Delta \stackrel{(4)}{\Rightarrow} \overline{\Phi}^{zar} = \Delta.$$

Implication (1) and the $\Leftarrow$ part of equivalence (3) follow from the fact that $\psi(G)_{\overline{\Delta}}$ is closed (3.4.1.1). Equivalence (2) follows from $\psi(\Phi) = \psi(\Phi')$. For the $\Rightarrow$ part of equivalence (3), let $g \in T$ lifting a representative of $\overline{\Delta} \subset \pi_0(G)$. From 3.2.4 and Lemma 3.3.1 (applied to T), $\psi(\Phi') = \psi(\Phi' \cap C_g g)$. Since the restriction $\psi : C_g g \to R_u(G) \times \mathbb{P}_{r,Q}$ is (finite hence) closed, $\psi(\overline{(\Phi' \cap C_g g)}^{zar}) = \overline{\psi(\Phi' \cap C_g g)}^{zar}$. Since the restriction $\psi : C_g g \to R_u(G) \times \mathbb{P}_{r,Q}$ is finite, $C_g g$ is irreducible and $\overline{(\Phi' \cap C_g g)}^{zar} \subset C_g g$ is a closed subset,

$$\psi(\overline{(\Phi' \cap C_g g)}^{zar}) = \psi(C_g g) \Leftrightarrow \dim(\overline{(\Phi' \cap C_g g)}^{zar}) = \dim(C_g g) \Leftrightarrow \overline{(\Phi' \cap C_g g)}^{zar} = C_g g,$$

which, from (3.3.2.2), is also equivalent to $\overline{\Phi'}^{zar} \supset G^\circ g$ hence $\overline{\Phi'}^{zar} \supset \Delta$. Implication (4) follows from $\overline{\Phi'}^{zar} \subset \overline{\Phi}^{zar}$. Indeed, for every $h \in \Phi$, $\{h_u\} \times \Phi_{h_{red}} \subset \Phi$ hence $\{h_u\} \times \overline{\Phi_{h_{red}}}^{zar} \subset \overline{\Phi}^{zar}$. But from [St74, Lemma on p. 92] (whose proof works for arbitrary (not necessarily connected) reductive groups), $\overline{\Phi_{h_{red}}}^{zar}$ contains h^{ss}. □

3.4.2. **Remark.** Though we will not use this in the following, let us point out that one can always choose V in such a way that it separates the conjugacy classes of $\pi_0(G)$. Namely, there exists a finite-dimensional faithful Q-representation V of G^{red} such that, with the above notation, the connected components of $\psi(G)$ are irreducible and the map sending $\overline{\Delta} \in \pi_0(G)//\pi_0(G)$ to $\psi(G)_{\overline{\Delta}} \in \pi_0(\psi(G))$ induces a bijection $\pi_0(G)//\pi_0(G)\tilde{\to}Irr(\psi(G))(= \pi_0(\psi(G)))$.

Again, it is enough to prove the assertion when $G = G^{red}$, which we assume. We proceed in two steps.

- (3.4.2.1) Let Γ be a finite group (*e.g.* $\Gamma = \pi_0(G)$). There exists a faithful finite-dimensional Q-representation W of Γ whose character $\tau_W : \Gamma \to Q$, $\gamma \mapsto \tau_W(\overline{g}) := Tr(\gamma|W)$ induces an injective map $\Gamma//\Gamma \hookrightarrow Q$.

  Let N denote the exponent of Γ and let $\zeta_N \in Q$ denote a primitive Nth root of unity. Then any character $\tau : \Gamma \to Q$ takes its values in $\mathbb{Z}[\zeta_N] \subset Q$. Fix a non-trivial character $\tau_1 : \Gamma \to Q$. If τ_1 separates the conjugacy classes of Γ, we are done. Otherwise, there exists $\overline{\gamma}_1 \neq \overline{\gamma}'_1 \in \Gamma//\Gamma$ such that $\tau_1(\gamma_1) = \tau_1(\gamma'_1)$. Fix an integer $n_1 \geq 1$ such that $\mathrm{im}(\tau_1)$ injects into $\mathbb{Z}[\zeta_N]/n_1$ and a character $\tau'_2 : \Gamma \to \mathbb{Z}[\zeta_N] \subset Q$ such that $\tau'_2(\gamma_1) \neq \tau'_2(\gamma'_1)$. Set $\tau_2 := \tau_1 + n_1\tau'_2$. By construction, $|\mathrm{im}(\tau_1)| < |\mathrm{im}(\tau_2)|$. And one iterates the construction.

- (3.4.2.2) From (3.4.2.1), there exists a faithful finite-dimensional Q-representation W of $\pi_0(G)$ whose character $\tau_W : \pi_0(G) \to Q$ separates the conjugacy classes of $\pi_0(G)$. Fix an arbitrary faithful finite-dimensional Q-representation V' of G of Q-dimension say N. Then $V := V' \oplus W^{\oplus N+1}$ has the required property since one can recover χ_W from χ_V by the following recipe. For $g \in G$ write $\chi_V(g) = \prod_{i\in I}(T-t_i)^{a_i}$ with $t_i \neq t_j$ for $i \neq j$ and $a_i = (N+1)q_i + r_i$ with $0 \leq r_i < N+1$, $i \in I$. Then $\chi_W(g) = \prod_{i\in I}(T-t_i)^{q_i}$.

3.5. Assume G fits into a short exact sequence of algebraic groups over Q

$$1 \to R \to G \xrightarrow{p} \widetilde{G} \to 1$$

with $R \subset R_u(G)$.

3.5.1. **Lemma.** *Let $S \subset G$ be a reductive (not necessarily connected) subgroup such that $Z_R(S) = 1$; set $\widetilde{S} := p(S) \subset \widetilde{G}$. Then the morphism $p : Z_G(S) \to Z_{\widetilde{G}}(\widetilde{S})$ is an isomorphism.*

Proof. Fix a Levi subgroup $L \subset G$ containing S and write $\widetilde{L} := p(L) \subset \widetilde{G}$; this is a Levi subgroup of $\widetilde{G}$ containing $\widetilde{S}$. From Lemma 3.1, $Z_G(S) = Z_{R_u(G)}(S) \rtimes Z_L(S)$ and $Z_{\widetilde{G}}(\widetilde{S}) = Z_{R_u(\widetilde{G})}(\widetilde{S}) \rtimes Z_{\widetilde{L}}(\widetilde{S})$. Since p induces an isomorphism $p : Z_L(S)\tilde{\to}Z_{\widetilde{L}}(\widetilde{S})$, it is enough to show that it also induces an isomorphism $p : Z_{R_u(G)}(S)\tilde{\to}Z_{R_u(\widetilde{G})}(\widetilde{S})$. As the Lie correspondence gives an equivalence of categories between the unipotent algebraic groups and the nilpotent (finite-dimensional) Lie algebras [DG70, Chap. IV, §2, 4.5, Cor. b)], it is enough to show that p induces an isomorphism from $Lie(R_u(G))^S$ to $Lie(R_u(\widetilde{G}))^{\widetilde{S}}$. This isomorphism follows from the (S-equivariant) exact sequence $0 \to Lie(R) \to Lie(R_u(G)) \to Lie(R_u(\widetilde{G})) \to 0$, together with the the fact that $Lie(R)^S = 0$ by assumption and the reductivity of S. □

In particular, for every $g \in G^{ss}$ with image $\widetilde{g} := p(g) \in \widetilde{G}$ and such that $R^g = 1$, the injective morphism $Z_G(g) \hookrightarrow Z_{\widetilde{G}}(\widetilde{g})$ induces isomorphisms $Z_G(g)\tilde{\to}Z_{\widetilde{G}}(\widetilde{g})$, $Z_{G^\circ}(g)^\circ\tilde{\to}Z_{\widetilde{G}^\circ}(\widetilde{g})^\circ$.

3.5.2. Let $\Phi \subset G$ be a union of conjugacy classes. Write $\widetilde{\Phi} := p(\Phi)$.

Lemma. *Assume $R^g = 1$ for every $g \in \Phi^{ss}$. Let $g \in \Phi^{ss}$ with image $\widetilde{g} := p(g) \in \widetilde{G}$. Then $\overline{\Phi^{zar}} \supset G^\circ g$ if and only if $\overline{\widetilde{\Phi}^{zar}} \supset \widetilde{G}^\circ\widetilde{g}$.*

Proof. Fix a Levi subgroup $L \subset G$ containing g, a maximal torus and a Borel subgroup $T^\circ \subset B^\circ \subset L$; write $\widetilde{T}^\circ := p(T^\circ) \subset \widetilde{B}^\circ := p(B^\circ) \subset \widetilde{L} := p(L) \subset \widetilde{G}$. Set $T := N_L(B^\circ) \cap N_L(T^\circ) \subset L$ for the corresponding quasi-Cartan of L and $\widetilde{T} := p(T) = N_{\widetilde{L}}(\widetilde{B}^\circ) \cap N_{\widetilde{L}}(\widetilde{T}^\circ) \subset \widetilde{L}$. From 3.2.4, up to replacing g by an L°-conjugate, one may assume $g \in T$ hence $\widetilde{g} \in \widetilde{T}$. From Fact 3.2.6, $Z_G(g) = Z_{R_u(G)}(g) \rtimes Z_L(g)$, $Z_L(g)$ is reductive, and $T_g := Z_{T^\circ}(g)^\circ \subset Z_G(g)$ is a maximal torus with corresponding Cartan subgroup $C_g = Z_{Z_G(g)^\circ}(T_g)^\circ = Z_{Z_{R_u(G)}(g)}(T_g) \times T_g \subset Z_G(g)$. Similarly $Z_{\widetilde{G}}(\widetilde{g}) = Z_{R_u(\widetilde{G})}(\widetilde{g}) \rtimes Z_{\widetilde{L}}(\widetilde{g})$, $Z_{\widetilde{L}}(\widetilde{g})$ is reductive, and $T_{\widetilde{g}} := Z_{\widetilde{T}^\circ}(\widetilde{g})^\circ \subset Z_{\widetilde{G}}(\widetilde{g})$ is a maximal torus with corresponding Cartan subgroup $C_{\widetilde{g}} = Z_{Z_{\widetilde{G}}(\widetilde{g})^\circ}(T_{\widetilde{g}})^\circ = Z_{Z_{R_u(\widetilde{G})}(\widetilde{g})}(T_{\widetilde{g}}) \times T_{\widetilde{g}} \subset Z_{\widetilde{G}}(\widetilde{g})$. Since $R^g = 1$ and $g \in G^{ss}$, it follows from 3.5.1 that the injective morphism $p : Z_G(g) \hookrightarrow Z_{\widetilde{G}}(\widetilde{g})$ induces isomorphisms $p : Z_G(g)\tilde{\to}Z_{\widetilde{G}}(\widetilde{g})$, $p : Z_{G^\circ}(g)^\circ\tilde{\to}Z_{\widetilde{G}^\circ}(\widetilde{g})^\circ$, $p : T_g\tilde{\to}T_{\widetilde{g}}$ and

$p : C_g \tilde{\to} C_{\widetilde{g}}$. From (3.3.2.2) applied to $\Phi \subset G$, $\overline{\Phi^{zar}} \supset G^\circ g$ if and only if $\overline{(\Phi \cap C_g g)^{zar}} = C_g g$. Similarly, $\overline{\widetilde{\Phi}^{zar}} \supset \widetilde{G}^\circ \widetilde{g}$ if and only if $\overline{(\widetilde{\Phi} \cap C_{\widetilde{g}}\widetilde{g})^{zar}} = C_{\widetilde{g}}\widetilde{g}$. From the isomorphism $p : Z_G(g) \tilde{\to} Z_{\widetilde{G}}(\widetilde{g})$ (which induces a homeomorphism $p : C_g g \tilde{\to} C_{\widetilde{g}}\widetilde{g}$), it is thus enough to show that $p(\Phi \cap C_g g) = \widetilde{\Phi} \cap C_{\widetilde{g}}\widetilde{g}$. The inclusion $p(\Phi \cap C_g g) \subset \widetilde{\Phi} \cap C_{\widetilde{g}}\widetilde{g}$ is straightforward. To prove the converse inclusion it is enough to show that for every $\widetilde{\phi} \in \widetilde{\Phi} \cap C_{\widetilde{g}}\widetilde{g}$ there exists $\phi \in \Phi \cap Z_G(g)$ such that $p(\phi) = \widetilde{\phi}$. Let $\phi \in \Phi$ such that $p(\phi) = \widetilde{\phi}$.

- We first show that up to replacing ϕ by an R-conjugate, one may assume $\phi^{ss} \in L$.

  Write $\phi = \phi^R \phi^L$ with $\phi^R \in R_u(G)$, $\phi^L \in L$ and $\widetilde{\phi} = \widetilde{\phi}^R \widetilde{\phi}^L$ with $\widetilde{\phi}^R \in R_u(\widetilde{G})$, $\widetilde{\phi}^L \in \widetilde{L}$. As $\widetilde{\phi} \in C_{\widetilde{g}}\widetilde{g} = Z_{Z_{R_u(\widetilde{G})}(\widetilde{g})}(T_{\widetilde{g}}) \times T_{\widetilde{g}}\widetilde{g}$ and $(T_{\widetilde{g}}\widetilde{g} \subset) \overline{\langle T_{\widetilde{g}}, \widetilde{g} \rangle^{zar}}$ is diagonalizable (see the proof of Proposition 3.4.1), $\widetilde{\phi}^u \in Z_{Z_{R_u(\widetilde{G})}(\widetilde{g})}(T_{\widetilde{g}}) \subset R_u(\widetilde{G})$ and $\widetilde{\phi}^{ss} \in T_{\widetilde{g}}\widetilde{g} \subset \widetilde{L}$. In particular $\widetilde{\phi}^R = \widetilde{\phi}^u$, $\widetilde{\phi}^L = \widetilde{\phi}^{ss}$. On the other hand, $p(\phi) = \widetilde{\phi}$ implies $p(\phi^R) = \widetilde{\phi}^R$, $p(\phi^L) = \widetilde{\phi}^L$ and (since the Jordan decomposition is preserved by morphisms of algebraic groups [B91, I, 4.4, Thm. (4)]) $p(\phi^u) = \widetilde{\phi}^u$, $p(\phi^{ss}) = \widetilde{\phi}^{ss}$. As a result, $\phi^L(\phi^{ss})^{-1} = (\phi^R)^{-1}\phi^u =: r \in R$. As $\phi^{ss} \in G$ is semisimple and normalizes R, the R-conjugacy class $C_R(\phi^{ss}) \subset G$ of ϕ^{ss} is closed in G and the canonical map $R/Z_R(\phi^{ss}) \to C_R(\phi^{ss})$, $\rho \to \rho\phi^{ss}\rho^{-1}$ is an isomorphism [B91, III, 9.1, 9.2]. But by assumption $Z_R(\phi^{ss}) = 1$, thus $C_R(\phi^{ss}) \subset p^{-1}(\widetilde{\phi}^{ss})$ is a closed subvariety of dimension $\dim(R)$. On the other hand, from the isomorphism $R \tilde{\to} p^{-1}(\widetilde{\phi}^{ss})$, $\rho \to \rho\phi^{ss}$, $p^{-1}(\widetilde{\phi}^{ss})$ is irreducible of dimension $\dim(R)$. This shows $C_R(\phi^{ss}) = p^{-1}(\widetilde{\phi}^{ss}) = R\phi^{ss}$. In particular, there exists $\rho := \rho_{\phi^{ss}} \in R$ such that $\rho\phi^{ss}\rho^{-1} = r\phi^{ss} = \phi^L \in L$. As a result, replacing ϕ with $\rho\phi\rho^{-1}$, one may assume $\phi^{ss} \in L$ as claimed.

- From now on, assume $\phi^{ss} \in L$. Write $\phi = \phi^R \phi^L$ with $\phi^R \in R_u(G)$, $\phi^L \in L$ and $\phi^u = \phi^{u,R}\phi^{u,L}$ with $\phi^{u,R} \in R_u(G)$, $\phi^{u,L} \in L$. Since $\phi^{ss} \in L$, we have $\phi^R = \phi^{u,R}$ and (using again that the Jordan decomposition is preserved by morphisms of algebraic groups) $\phi^{ss} = \phi^{L,ss}$, $\phi^{u,L} = \phi^{L,u}$. But as $\phi^{ss} = \phi^{L,ss}$ and $\phi^{u,L} = \phi^{L,u}$ commute and ϕ^{ss} and ϕ^u commute, this shows ϕ^{ss} and $(\phi^R =)\phi^{u,R} = \phi^u(\phi^{u,L})^{-1}$ commute. On the other hand, $\widetilde{\phi} \in Z_{\widetilde{G}}(\widetilde{g})$ imposes $\phi g \phi^{-1} g^{-1} =: r \in R$. This can be rewritten as:

  $$r^{-1}\phi^R(\phi^L g(\phi^L)^{-1})(\phi^R)^{-1}(\phi^L g^{-1}(\phi^L)^{-1}) = g\phi^L g^{-1}(\phi^L)^{-1} \in L \cap R_u(G) = 1$$

  hence $\phi^L \in Z_G(g)$ and $\phi^R g(\phi^R)^{-1}g^{-1} = r \in R$. But $\phi^L \in Z_G(g)$ implies $\phi^{ss}(= \phi^{L,ss}) \in Z_G(g)$. This shows ϕ^{ss} commutes with both g and ϕ^R hence with r. As $R^{\phi^{ss}} = 1$ by assumption, $r = 1$ and $\phi^R \in Z_G(g)$. This shows $\phi = \phi^R\phi^L \in Z_G(g)$ as desired.

□

3.6. Assume G acts faithfully on a finite-dimensional Q-vector space V and that V is endowed with a filtration $S_\bullet V$: $V = S_1 V \supsetneq S_2 V \supsetneq \cdots \supsetneq S_s V \supsetneq S_{s+1} V = 0$ defined by a cocharacter $\omega : \mathbb{G}_{m,Q} \to G \hookrightarrow GL_V$. By this, we mean the following. For every $n \in \mathbb{Z}$ let $V_\omega(n) := \cap_{x \in \mathbb{G}_{m,Q}} \ker(\omega(x) - x^n Id) \subset V$ denote the Q-vector subspace over which ω acts with weight n and let $\mathcal{S}_\omega(V) := \{n \in \mathbb{Z} \mid V_\omega(n) \neq 0\} \subset \mathbb{Z}$ denote the set of weights of ω appearing in V. Then, ordering the elements of $\mathcal{S}_\omega(V)$ as $\sigma_1 > \cdots > \sigma_s$, one has $S_i V = \oplus_{n \leq \sigma_i} V_\omega(n)$, $i = 1, \ldots, s$. Let $H \subset G$ be a subgroup of the stabilizer of $S_\bullet V$ in G containing the centralizer $Z_G(\omega)$ of the image of ω in G (*e.g.* the stabilizer itself). Let $\Psi \subset H$ be a union of H-conjugacy classes and let $\Phi \subset G$ for the union of G-conjugacy classes generated by Ψ in G.

Lemma. *Let $g \in H^{ss}$ and assume that $\overline{\Psi^{zar}} \supset H^\circ g$. Then $\overline{\Phi^{zar}} \supset G^\circ g$.*

Proof. Let $T^\circ \subset G$ be a maximal torus containing the image of ω; in particular $T^\circ \subset Z_G(\omega) \subset H$ and T° is a maximal torus of H. Let $L_H \subset H$ be a Levi subgroup containing T° and $B_H^\circ \subset L_H$ a Borel subgroup of L_H containing T°. Write $C_H := N_{L_H}(B_H^\circ) \cap N_{L_H}(T^\circ) \subset L_H$. From 3.2.2, one may assume $g \in C_H$. From Fact 3.2.6, $T_g := Z_{T^\circ}(g)^\circ \subset Z_{L_H^\circ}(g)$ is a maximal torus of $Z_H(g)$ with corresponding Cartan subgroup $C_{H,g} := Z_{Z_H(g)^\circ}(T_g)^\circ \subset Z_H(g)^\circ$. Let $L \subset G$ be a Levi subgroup containing L_H and $B^\circ \subset L$ a Borel subgroup of L containing B_H° (hence T°). Set $C := N_L(B^\circ) \cap N_L(T^\circ) \subset L$. Then $C_H \subset C$ with $C_H^\circ = C^\circ = T^\circ$ and, again, $T_g = Z_{T^\circ}(g)^\circ \subset Z_{L^\circ}(g)$ is a maximal torus of $Z_G(g)$ with corresponding Cartan subgroup $C_g := Z_{Z_G(g)^\circ}(T_g)^\circ \subset Z_G(g)^\circ$.

- We claim that $C_{H,g} = C_g$. Indeed, since L_H is reductive (hence linearly reductive) and contains the image of ω, L_H centralizes the image of ω. (Observe that the linear reductivity yields an L_H-equivariant isomorphism $V\tilde{\to}\oplus_{1\leq i\leq s} S_iV/S_{i+1}V$ and that by definition of $S_\bullet V$, ω is pure on each graded piece $S_iV/S_{i+1}V$.) In particular, $g\in L_H$ centralizes the image of ω. This shows the image of ω is contained in T_g. But then C_g also centralizes the image of ω; in particular $C_g\subset Z_G(\omega)\cap Z_G(g)\subset H\cap Z_G(g) = Z_H(g)$, $C_g\subset Z_H(g)^\circ$, and $C_g\subset C_{H,g}$. Conversely, the inclusion $Z_{H^\circ}(g)^\circ\subset Z_{G^\circ}(g)^\circ$ immediately implies that $C_{H,g}\subset C_g$.

- From (3.3.2.2), $\overline{\Psi}^{zar}\supset H^\circ g$ implies $\overline{(\Psi\cap C_{H,g}g)}^{zar} = C_{H,g}g$. Since $\Psi\cap C_{H,g}g = \Psi\cap C_g g\subset \Phi\cap C_g g$ this implies $\overline{(\Phi\cap C_g g)}^{zar} = C_g g$, which, applying again (3.3.2.2), implies $\overline{\Phi}^{zar}\supset G^\circ g$ as desired.

□

4. Preliminary reductions

4.1. To prove Theorem 1.4, one may assume X is smooth over k. This follows from the following applied to the smooth locus of X.

Fact. *Let $U\subset X$ be a dense open subset. Then for every $\mathcal{C}\in\mathcal{C}^\dagger(X,Q)$ the restriction functor $\mathcal{C}^\dagger(X,Q)\to\mathcal{C}^\dagger(U,Q)$ induces an isomorphism $G(\mathcal{C}|_U)\tilde{\to}G(\mathcal{C})$ of algebraic groups.*

Proof. (Sketch) Since the induced morphism $G(\mathcal{C}|_U)\to G(\mathcal{C})$ is compatible with the closed immersions $G(\mathcal{C}|_U)\hookrightarrow GL(\mathcal{C}_{\overline{x}})$ and $G(\mathcal{C})\hookrightarrow GL(\mathcal{C}_{\overline{x}})$, it is a closed immersion. Thus, it is enough to show the surjectivity. For $Q = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_{\mathrm{u}}$, this directly follows from the fact that the functorial morphism $\pi_1(U)\to\pi_1(X)$ is surjective and that $G(\mathcal{C})$ (resp. $G(\mathcal{C}|_U)$ is the Zariski-closure of $W(X)$ (resp. $W(U)$) acting on $\mathcal{C}_{\overline{x}}$. For $Q = \overline{\mathbb{Q}}_p$, this follows from the fact that the restriction functor $\mathcal{C}^\dagger(X,Q)\to\mathcal{C}^\dagger(U,Q)$ is fully faithful [Ts12] and a Tannakian trick - see [AE19, §4.6] for details. □

4.2. So from now on and unless otherwise mentioned, we will assume that X is smooth over k.

4.2.1. **Fact.** *Let $\mathcal{C}$ be a Q-coefficient on X. For every $x\in|X|$ there is a canonical diagram, commutative up to conjugacy and whose right vertical arrow is continuous*

$$\begin{array}{ccccc} W(x) & & \longrightarrow & & \pi_1(X) \\ \downarrow & & & & \downarrow{\scriptstyle cont.} \\ G(x^*\mathcal{C}) & \longrightarrow & G & \twoheadrightarrow & \pi_0(G) \end{array}$$

Proof. (Sketch) See the discussions in [D'A20a, §3.3], [DrKed17, App. B] and [HP18, §6]. The key points are the following. Let Q_X denote the trivial Q-coefficient on X. Let $\mathcal{C}^f(X,Q)\subset\mathcal{C}^{(\dagger)}(X,Q)$ denote the full subcategory of finite Q-coefficients, that is those $\mathcal{C}\in\mathcal{C}^{(\dagger)}(X,Q)$ such that $G(\mathcal{C})$ is finite and let $\mathcal{C}^{isotr}(X,Q)\subset\mathcal{C}^{(\dagger)}(X,Q)$ denote the full subcategory of isotrivial Q-coefficients, that is those $\mathcal{C}\in\mathcal{C}^{(\dagger)}(X,Q)$ such that $p^*\mathcal{C} = Q_{\tilde{X}}^{\mathrm{rank}(\mathcal{C})}$ for some Galois étale cover $p:\tilde{X}\to X$.

- (4.2.1.1) $\mathcal{C}^f(X,Q) = \mathcal{C}^{isotr}(X,Q)\subset\mathcal{C}^{(\dagger)}(X,Q)$.

- (4.2.1.2) The Tannakian group of $\mathcal{C}^{isotr}(X,Q)\subset\mathcal{C}^{(\dagger)}(X,Q)$ is the pro-algebraic group $\varprojlim\pi_1(X)/U$, where the limit is over all normal open subgroups of $\pi_1(X)$.

The inclusion $\mathcal{C}^{isotr}(X,Q)\subset\mathcal{C}^f(X,Q)$ in (4.2.1.1) and (4.2.1.2) follow from étale descent, namely that for every Galois étale cover $p:\tilde{X}\to X$, the functor $p^*:\mathcal{C}^{(\dagger)}(X,Q)\to\mathcal{C}^{(\dagger)}(\tilde{X},Q)$ factors as a composite of an equivalence of $\otimes$-categories

$$p^*:\mathcal{C}^{(\dagger)}(X,Q)\to\mathcal{C}^{(\dagger)}(\tilde{X},Q)^{Aut(\tilde{X}|X)}$$

onto the category $\mathcal{C}^{(\dagger)}(\tilde{X},Q)^{Aut(\tilde{X}|X)}$ of Q-coefficients in $\mathcal{C}^{(\dagger)}(\tilde{X},Q)$ equipped with descent data with respect to $p:\tilde{X}\to X$ and the forgetful functor $\mathcal{C}^{(\dagger)}(\tilde{X},Q)^{Aut(\tilde{X}|X)}\to\mathcal{C}^{(\dagger)}(\tilde{X},Q)$. This is tautological for $\overline{\mathbb{Q}}_\ell$- and $\overline{\mathbb{Q}}_{\mathrm{u}}$-coefficients. For $\overline{\mathbb{Q}}_p$-coefficients, see [O84, Thm. 4.5], [E02, Thm. 1]. The inclusion $\mathcal{C}^f(X,Q)\subset\mathcal{C}^{isotr}(X,Q)$ in (4.2.1.1) amounts to showing that for every $\mathcal{C}\in\mathcal{C}^f(X,Q)$ there exists a connected étale cover $p:\tilde{X}\to X$ such that $p^*\mathcal{C}$ is trivial. For $\overline{\mathbb{Q}}_\ell$-coefficients (resp. $\overline{\mathbb{Q}}_p$-coefficients), this follows from the fact that finite $\overline{\mathbb{Q}}_\ell$-coefficients (resp. $\overline{\mathbb{Q}}_p$-coefficients) are étale (resp. unit-root since the eigenvalues of φ_x acting on $\mathcal{C}_{\overline{x}}$

are roots of unity) hence correspond to finite-dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1(X)$ (resp.[3] $\overline{\mathbb{Q}}_p^\sigma$-representations of $\pi_1(X)$, where σ is a lifting of the Frobenius [Cr87, 2.2, Thm.]). For $\overline{\mathbb{Q}}_\mathfrak{u}$-coefficients, by definition, the finite-dimensional $\overline{\mathbb{Q}}_\mathfrak{u}$-representation of $\pi_1(X)$ corresponding to an almost $\mathfrak{u}$-tame sheaf factors through a topologically finitely generated (profinite) quotient; in particular, if the representation has finite image, its kernel is automatically open [NS07a], [NS07b]. □

4.2.2. **Corollary.** *The resp. part of Conjecture 1.3 follows from the non-resp. part of Conjecture 1.3.*

Proof. Indeed, by the (classical) Cebotarev density theorem [P97, Thm. B.9], for every conjugacy class $\overline{\Delta} \subset \pi_0(G)$ with inverse image $\Delta := p_{G^\circ}^{-1}(\overline{\Delta}) \subset G$, $\delta^u(S_\Delta) \leq \delta(|X|_\Delta) = \frac{|\overline{\Delta}|}{|\pi_0(G)|}$ while, using the decomposition $S = \sqcup_{\overline{\Delta}} S_\Delta$, one gets $\delta^u(S) \leq \sum_{\overline{\Delta}} \delta^u(S_\Delta)$. Therefore $\delta^u(S) = 1$ implies that for every conjugacy class $\overline{\Delta} \subset \pi_0(G)$, $\delta^u(S_\Delta) = \frac{|\overline{\Delta}|}{|\pi_0(G)|} > 0$. In particular, from the non-resp. part of Theorem 1.4 (applied to S_Δ), there exists $g \in \Delta$ such that $\overline{\Phi_{S_\Delta}}^{zar} \supset G^\circ g$. But every other connected component of G contained in Δ is of the form $\gamma G^\circ g \gamma^{-1}$ for some $\gamma \in G(Q)$ and $\overline{\Phi_{S_\Delta}}^{zar} (= \gamma \overline{\Phi_{S_\Delta}}^{zar} \gamma^{-1}) \supset \gamma G^\circ g \gamma^{-1}$. □

Remark. Using the inequalities $\delta^u(S) \leq \sum_{\overline{\Delta}} \delta^u(S_\Delta)$ and $\delta^u(S_\Delta) \leq \delta^u(|X|_\Delta) = \frac{|\overline{\Delta}|}{|\pi_0(G)|}$, one can relate more precisely $\delta^u(S)$ to the number of connected components in which Φ_S is Zariski-dense (possibly depending on the structure of $\pi_0(G)$). For instance $\overline{\Phi_S}^{zar}$ contains at least $\lceil \delta^u(S)|\pi_0(G)| \rceil$ connected components of G *etc.*

4.2.3. The proof of Corollary 4.2.2 also shows that to prove the non-resp. part of Conjecture 1.3 and after possibly shrinking S, one may assume (i) $\Phi_S \subset \Delta := p_{G^\circ}^{-1}(\overline{\Delta})$ for a conjugacy class $\overline{\Delta} \subset \pi_0(G)$ and (ii) every connected component in Δ has a representative in Φ_S^{ss}.

5. Étale $\overline{\mathbb{Q}}_\ell$-coefficients

5.1. **Theorem 1.4 for étale $\overline{\mathbb{Q}}_\ell$-coefficients.** When $Q = \overline{\mathbb{Q}}_\ell$ and $\mathcal{C}$ is an étale $\overline{\mathbb{Q}}_\ell$-coefficient on X, Theorem 1.4 is a consequence of the (classical) Cebotarev density theorem. Indeed, write $G := G(\mathcal{C})$, which we identify with the Zariski-closure of the image $G^{an} := G(\mathcal{C})^{an}$ of the continuous $\overline{\mathbb{Q}}_\ell$-representation V of $\pi_1(X)$ corresponding to $\mathcal{C}$. For every closed point $x \in |X|$, let $\Phi_x^{an} \subset G^{an}$ denote the G^{an}-conjugacy class of the image φ_x of a geometric Frobenius attached to x so that Φ_x^{an} is Zariski-dense in the G-conjugacy class Φ_x defined in Subsection 1.4. In particular, for every closed subset $C \subset G$ which is a union of conjugacy classes, the subset S_C defined in Subsection 1.4 can also be described as

$$(5.1.1)\quad S_C = \{x \in |X| \mid \Phi_x^{an} \subset G^{an} \cap C\}.$$

Without loss of generality one may assume V arises from a continuous Q_ℓ-representation V_ℓ for a finite extension Q_ℓ of $\mathbb{Q}_\ell$. Let Z_ℓ denote the ring of integers of Q_ℓ. Fix a G^{an}-stable Z_ℓ-lattice $\Lambda_\ell \subset V_\ell$, set $G(Z_\ell) := G(Q_\ell) \cap \mathrm{GL}(\Lambda_\ell)$ and let $\mu : \mathcal{B}(G(Z_\ell)) \to [0, |\pi_0(G)|]$ denote the Haar measure on $G(Z_\ell)$ normalized so that $\mu(G^\circ(Z_\ell)) = 1$, where $\mathcal{B}(G(Z_\ell))$ denotes the Borel algebra on $G(Z_\ell)$. Assume $C := \overline{\Phi_S}^{zar}$ does not contain any connected component of G. Since $G(Z_\ell)(\supset G^{an})$ is Zariski-dense in G, $\mu(C(Z_\ell)) = 0$ [Se12, Prop. 5.12]. On the other hand, since $C(Z_\ell) \subset G(Z_\ell)$ is analytically closed, $0 < \delta^u(S) \leq \delta^u(|X|_{C(Z_\ell)}) \leq \mu(C(Z_\ell)) = 0$, where the last inequality is [Se12, Thm. 6.8] (using the description (5.1.1) of $|X|_{C(Z_\ell)}$). Whence a contradiction.

To prove Theorem 1.4 for arbitrary motivic Q-coefficients we will use that every semisimple motivic Q-coefficient admits an étale $\overline{\mathbb{Q}}_\ell$-companion for some prime $\ell \neq p$.

5.2. **Existence of étale $\overline{\mathbb{Q}}_\ell$-companions.** Let now Q be any of $\overline{\mathbb{Q}}_\mathfrak{l}$ for some prime $\mathfrak{l} \neq p$, $\overline{\mathbb{Q}}_\mathfrak{u}$ for some $\mathfrak{u} \in \mathcal{U}$ or $\overline{\mathbb{Q}}_p$. The field of coefficients $Q_\mathcal{C}$ of a Q-coefficient $\mathcal{C}$ is the $\mathbb{Q}$-subextension of Q generated by the coefficients of the $\det(Id - T\varphi_x|\mathcal{C}_{\overline{x}})$, $x \in |X|$.

Given an isomorphism $\iota : Q \tilde{\to} \mathbb{C}$, a Q-coefficient $\mathcal{C}$ is said to be ι-pure of weight $w \in \mathbb{R}$ if for every $x \in |X|$ and eigenvalue α of φ_x acting on $\mathcal{C}_{\overline{x}}$ one has $|\iota\alpha| = |k(x)|^{w/2}$. A Q-coefficient $\mathcal{C}$ is said to be pure of weight

[3] More precsisely, Crew's theorem is for convergent F-isocrystals but since X is assumed to be smooth over k, the functor $\alpha : \mathcal{C}^\dagger(X, \overline{\mathbb{Q}}_p) \to \mathcal{C}(X, \overline{\mathbb{Q}}_p)$ is fully faithful (see Subsection 1.1) hence to show that a finite overconvergent F-isocrystal $\mathcal{C}^\dagger$ is isotrivial it is enough to show that the convergent F-isocrystal $\alpha(\mathcal{C}^\dagger)$ is. (Note that $\alpha(\mathcal{C}^\dagger)$ is also finite, as $G(\alpha(\mathcal{C}^\dagger)) \subset G(\mathcal{C}^\dagger)$ ($\subset GL(\mathcal{C}^\dagger_{\overline{x}})$).

w if it is ι-pure of weight $w \in \mathbb{R}$ for every $\iota : Q \tilde{\to} \mathbb{C}$; this forces $Q_{\mathcal{C}} \subset \overline{\mathbb{Q}}$.

Fix a prime $\ell \neq p$ and isomorphisms $\iota : Q \tilde{\to} \mathbb{C}$, $\iota_\ell : \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{C}$. A Q-coefficient $\mathcal{C}$ and a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_\ell$ are said to be compatible or companions (with respect to ι, ι_ℓ) if

$$\iota_\ell \det(Id - T\varphi_x | \mathcal{C}_{\ell,\overline{x}}) = \iota \det(Id - T\varphi_x | \mathcal{C}_{\overline{x}}),\ x \in |X|.$$

Recall X is assumed to be smooth over k.

5.2.1. **Fact.** (Companion conjecture, [D80, (2.2.10)], [Dr78], [L02], [A18b], [D12], [Dr12], [Ked18b], [AE19], [C19a] - see also [C19b] for a survey) *Let $\mathcal{I}$ be an irreducible motivic Q-coefficient with finite determinant (i.e. $\det(\mathcal{I}) \in \mathcal{C}^f(X, Q)$). The following hold.*

- *(5.2.1.1) $\mathcal{I}$ is pure of weight 0;*
- *(5.2.1.2) $Q_{\mathcal{I}}$ is a finite extension of $\mathbb{Q}$;*
- *(5.2.1.3) There exists an étale $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{I}_\ell$ which is compatible (with respect to ι, ι_ℓ) with $\mathcal{I}$.*

In (5.2.1.3) $\mathcal{I}_\ell$ is automatically irreducible (hence unique) with finite determinant.

5.2.2. Let $p_X : X \to \mathrm{spec}(k)$ denote the structural morphism and given $\alpha \in Q^\times$, let $Q^{(\alpha)}$ denote the Q-coefficient on $\mathrm{spec}(k)$ corresponding to φ acting on Q by multiplication by α. For a Q-coefficient $\mathcal{C}$ on X, write $\mathcal{C}^{(\alpha)} := \mathcal{C} \otimes p_X^* Q^{(\alpha)}$ for the 'twist of $\mathcal{C}$ by α'. The following is a consequence of class field theory.

5.2.2.1. **Fact.** ([D80, (1.3.4)], [A18a, Lem. 6.1], [C19a, Prop. 6.1.2]) *Every rank-1 motivic Q-coefficient on X is a twist of a finite Q-coefficient.*

In particular, applying Fact 5.2.2.1 to the determinant and using Fact 5.2.1, one gets

5.2.2.2. **Corollary.** *Every irreducible motivic Q-coefficient on X is a twist of an irreducible motivic Q-coefficient with finite determinant hence, in particular, is ι-pure.*

5.2.2.3. **Fact.** (Grothendieck's unipotent monodromy theorem, [D80, (1.3.8)], [Cr92, Thm. 4.9] (and [D'A20a, Thm. 3.4.4] for higher-dimensional varieties), [C19a, 6.1.3]) *Let $\mathcal{C}$ be a motivic Q-coefficient on X. Then the radical of $G(\overline{\mathcal{C}})$ is unipotent.*

5.2.3. **Corollary.** *Let $\mathcal{C}$ be a semisimple motivic Q-coefficient on X. Then for every prime $(p \neq)\ell \gg 0$, there exists an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and a (necessarily unique) semisimple étale $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_\ell$ which is compatible with $\mathcal{C}$ (with respect to ι, ι_ℓ).*

Proof. From Corollary 5.2.2.2, one can write $\mathcal{C} = \oplus_{i\in I} \mathcal{I}_i^{(\alpha_i)}$ with $\mathcal{I}_i$ an irreducible motivic Q-coefficient with finite determinant and $\alpha_i \in Q^\times$, $i \in I$. By (5.2.1.3) for every $\ell \neq p$ and isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and for every $i \in I$, there exists an étale $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{I}_{i,\ell}$ compatible with $\mathcal{I}_i$. As pointed out, $\mathcal{I}_{i,\ell}$ is automatically irreducible hence, by construction, $\mathcal{C}_\ell := \oplus_{i\in I} \mathcal{I}_{i,\ell}^{(\iota_\ell^{-1}\iota(\alpha_i))}$ is a semisimple motivic $\overline{\mathbb{Q}}_\ell$-coefficient on X compatible with $\mathcal{C}$. From Lemma 5.2.4 below, for $p \neq \ell \gg 0$ one can furthermore choose $\iota_\ell : \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{C}$ in such a way that the $\iota_\ell^{-1}\iota(\alpha_i)$ are ℓ-adic units that is $\mathcal{C}_\ell$ is an étale $\overline{\mathbb{Q}}_\ell$-coefficient. □

5.2.4. **Lemma.** *Let $0 \neq \alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Then for every prime $\ell \gg 0$ there exists a field isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{C}$ such that $\iota_\ell^{-1}(\alpha_1), \ldots, \iota_\ell^{-1}(\alpha_m)$ are ℓ-adic units.*

Proof. By the Noether normalization lemma there exists $t_1, \ldots, t_r \in \mathbb{Q}[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}]$, algebraically independent over $\mathbb{Q}$ and such that the extension $\mathbb{Q}[t_1, \ldots, t_r] \subset \mathbb{Q}[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}]$ is finite. For some integer $N \geq 1$, the extension $\mathbb{Q}[t_1, \ldots, t_r] \hookrightarrow \mathbb{Q}[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}]$ restricts to a finite extension $\mathbb{Z}[1/N][t_1, \ldots, t_r] \hookrightarrow \mathbb{Z}[1/N][\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}]$. Fix a prime $\ell \not| N$. Since $\mathbb{Z}_\ell$ is uncountable, one can find $t_{1,\ell}, \ldots, t_{r,\ell} \in \mathbb{Z}_\ell$ algebraically independent over $\mathbb{Q}$, whence an embedding $\mathbb{Z}[1/N][t_1, \ldots, t_r] \hookrightarrow \mathbb{Z}_\ell$. Localizing at the zero-ideals, one obtains a commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}_\ell & \hookleftarrow & \mathbb{Z}[1/N][t_1, \ldots, t_r] & \xrightarrow{\text{finite}} & \mathbb{Z}[1/N][\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}] \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}_\ell & \hookleftarrow & \mathbb{Q}(t_1, \ldots, t_r) & \xrightarrow{\text{finite}} & \mathbb{Q}(\alpha_1, \ldots, \alpha_m)
\end{array}$$

hence, taking a connected component Q_ℓ of $\mathbb{Q}(\alpha_1,\ldots,\alpha_m)\otimes_{\mathbb{Q}(t_1,\ldots,t_r)}\mathbb{Q}_\ell$, a commutative diagram of fields

$$\begin{array}{ccc} \mathbb{Q}(t_1,\ldots,t_r) & \xrightarrow{\text{finite}} & \mathbb{Q}(\alpha_1,\ldots,\alpha_m)\,. \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell & \xrightarrow{\text{finite}} & Q_\ell \end{array}$$

Let Z_ℓ denote the ring of integers of Q_ℓ. Since $\mathbb{Z}[1/N][t_1,\ldots,t_r]\subset\mathbb{Z}[1/N][\alpha_1^{\pm1},\ldots,\alpha_m^{\pm1}]$ is finite and Z_ℓ is normal, one obtains a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[1/N][t_1,\ldots,t_r] & \xrightarrow{\text{finite}} & \mathbb{Z}[1/N][\alpha_1^{\pm1},\ldots,\alpha_m^{\pm1}]) \\ \downarrow & \swarrow \exists! & \downarrow \\ Z_\ell & \longrightarrow & Q_\ell, \end{array}$$

where the diagonal dotted arrow is automatically injective. Eventually, using that $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ have the same transcendence degree over $\mathbb{Q}$, the above diagram extends as

$$\begin{array}{ccccccc} \mathbb{Z}[1/N][t_1,\ldots,t_r] & \xrightarrow{\text{finite}} & \mathbb{Z}[1/N][\alpha_1^{\pm1},\ldots,\alpha_m^{\pm1}]) & \longrightarrow & Q & \xrightarrow[\simeq]{\iota} & \mathbb{C} \\ \downarrow & \swarrow \exists! & \downarrow & & & \iota_\ell \nearrow \simeq & \\ Z_\ell & \longrightarrow & Q_\ell & \longrightarrow & \overline{\mathbb{Q}}_\ell, & & \end{array}$$

where the right up right dotted arrow is an isomorphism. □

6. Theorem 1.4 for semisimple motivic Q-coefficients

6.1. Let now Q be any of $\overline{\mathbb{Q}}_{\mathfrak{l}}$ for some prime $\mathfrak{l}\neq p$, $\overline{\mathbb{Q}}_{\mathfrak{u}}$ for some $\mathfrak{u}\in\mathcal{U}$ or $\overline{\mathbb{Q}}_p$. Let $\mathcal{C}$ be a semisimple motivic Q-coefficient on X. Fix an isomorphism $\iota: Q\tilde{\to}\mathbb{C}$. By Corollary 5.2.3, there exists a prime $\ell\neq p$ and an isomorphism $\iota_\ell:\overline{\mathbb{Q}}_\ell\tilde{\to}\mathbb{C}$ such that the semisimple $\overline{\mathbb{Q}}_\ell$-companion $\mathcal{C}_\ell$ of $\mathcal{C}$ is an étale $\overline{\mathbb{Q}}_\ell$-coefficient on X. Write $G:=G(\mathcal{C})$, $G_\ell:=G(\mathcal{C}_\ell)$, which are reductive.

6.2. **Fact.** (*E.g.* [C19b, §8])
- (6.2.1) *The group G is connected if and only if the group G_ℓ is connected. In particular, if $\widetilde{X}\to X$ is the étale cover corresponding to the kernel of $\pi_1(X)\twoheadrightarrow\pi_0(G_\ell)$, the groups $\pi_0(G)$ and $\pi_0(G_\ell)$ are canonically isomorphic to $Aut(\widetilde{X}/X)$.*
- (6.2.2) *The companion correspondence $(\mathcal{O}b(\langle\mathcal{C}\rangle)/\simeq)\tilde{\to}(\mathcal{O}b(\langle\mathcal{C}_\ell\rangle)/\simeq)$ induces a canonical semiring isomorphism $\mathbb{C}^+[\iota G]\tilde{\to}\mathbb{C}^+[\iota_\ell G_\ell]$, characterized by the fact that it preserves local L-functions and maps irreducible representations to irreducible representations.*

6.3. **Lemma.** *Theorem 1.4 holds for $\mathcal{C}$ if and only if for every conjugacy class $\overline{\Delta}\subset Aut(\widetilde{X}/X)$ (notation as in (6.2.1)),* $\dim(\chi(G)_{\overline{\Delta}})=\dim(\chi(G_\ell)_{\overline{\Delta}})$.

Proof. By definition of compatibility, for every subset $S\subset|X|$, $\mathfrak{Z}_S:=\iota\chi(\Phi_S^{\mathcal{C}})=\iota_\ell\chi(\Phi_S^{\mathcal{C}_\ell})$; set $Z_S:=\overline{\mathfrak{Z}_S}^{zar}$. For a conjugacy class $\overline{\Delta}\subset Aut(\widetilde{X}/X)$ with inverse images $\Delta:=p_{G^\circ}^{-1}(\overline{\Delta})\subset G$, $\Delta_\ell:=p_{G_\ell^\circ}^{-1}(\overline{\Delta})\subset G_\ell$, write

$$S_{\overline{\Delta}}:=S_{\Delta}^{\mathcal{C}}=S_{\Delta_\ell}^{\mathcal{C}_\ell}\subset|X|.$$

Then one always has $Z_{S_{\overline{\Delta}}}\subset\iota\chi(G)_{\overline{\Delta}}$ (resp. $Z_{S_{\overline{\Delta}}}\subset\iota_\ell\chi(G_\ell)_{\overline{\Delta}}$), and, from Proposition 3.4.1 (and Corollary 4.2.2), Theorem 1.4 for $\mathcal{C}$ (resp. $\mathcal{C}_\ell$) is equivalent to

$$\delta^u(S_{\overline{\Delta}})>0\Rightarrow Z_{S_{\overline{\Delta}}}=\iota\chi(G)_{\overline{\Delta}}\ (\text{resp. } Z_{S_{\overline{\Delta}}}=\iota_\ell\chi(G_\ell)_{\overline{\Delta}}).$$

From the (classical) Cebotarev density theorem, one always has $\delta^u(|X|_{\overline{\Delta}})>0$ hence from Theorem 1.4 for étale $\overline{\mathbb{Q}}_\ell$-coefficients (Subsection 5.1) applied to $S=|X|$, one gets the inclusion $\iota_\ell\chi(G_\ell)_{\overline{\Delta}}(=Z_{|X|_{\overline{\Delta}}})\subset\iota\chi(G)_{\overline{\Delta}}$. On the other hand, from (3.4.1.1) $\iota_\ell\chi(G_\ell)_{\overline{\Delta}},\iota\chi(G)_{\overline{\Delta}}\subset\mathbb{P}_{r,\mathbb{C}}$ are closed irreducible subsets hence

$$\dim(\chi(G)_{\overline{\Delta}})=\dim(\chi(G_\ell)_{\overline{\Delta}})\Leftrightarrow\iota_\ell\chi(G_\ell)_{\overline{\Delta}}=\iota\chi(G)_{\overline{\Delta}}.$$

□

6.4. Let us show that for every conjugacy class $\overline{\Delta} \subset Aut(\widetilde{X}/X)$ (notation as in (6.2.1)), $\dim(\chi(G)_{\overline{\Delta}}) = \dim(\chi(G_\ell)_{\overline{\Delta}})$ that is, from (3.4.1.1), for some (equivalently every) quasi-Cartan $T_\ell \subset G_\ell := G(\mathcal{C}_\ell)$, $T \subset G := G(\mathcal{C})$ and $g_\ell \in T_\ell$, $g \in T$ lifting an element $\overline{g}$ of $\overline{\Delta}$, $Z_{T_\ell^\circ}(g_\ell)^\circ$ and $Z_{T^\circ}(g)^\circ$ have the same dimension. For this, fix a Borel $B_\ell^\circ \subset G_\ell^\circ$, a maximal torus $T_\ell^\circ \subset B_\ell^\circ$ and $g_\ell \in T_\ell := N_{G_\ell}(B_\ell^\circ) \cap N_{G_\ell}(T_\ell^\circ)$, $g \in G$ lifting an element $\overline{g} \in \overline{\Delta}$.

Since $\mathcal{C}|_{\widetilde{X}}$ and $\mathcal{C}_\ell|_{\widetilde{X}}$ are again semisimple and compatible, (6.2.2) yields a canonical semiring isomorphism $\Sigma : \mathbb{C}^+[\iota G^\circ] \tilde{\to} \mathbb{C}^+[\iota_\ell G_\ell^\circ]$, characterized by the fact that it preserves local L-functions and maps irreducible representations to irreducible representations. It then follows from [KaLV14, Thm. 1.2] that $\Sigma : \mathbb{C}^+[\iota G^\circ] \tilde{\to} \mathbb{C}^+[\iota_\ell G_\ell^\circ]$ is induced by an isomorphism of algebraic groups $\sigma : \iota_\ell G_\ell^\circ \tilde{\to} \iota G^\circ$, which is unique up to conjugacy. On the other hand, for every $\overline{g} \in Aut(\widetilde{X}/X)$ the pullback functors $\overline{g}^* : \langle \mathcal{C}_\ell|_{\widetilde{X}} \rangle \to \langle \overline{g}^* \mathcal{C}_\ell|_{\widetilde{X}} \rangle$, $\overline{g}^* : \langle \mathcal{C}|_{\widetilde{X}} \rangle \to \langle \overline{g}^* \mathcal{C}|_{\widetilde{X}} \rangle$ induce a canonical commutative diagram of semiring isomorphisms - all mapping irreducible representations to irreducible representations,

$$\begin{array}{ccc} \mathbb{C}^+[\iota G^\circ] & \xrightarrow[\simeq]{\Sigma} & \mathbb{C}^+[\iota_\ell G_\ell^\circ] \\ {\scriptstyle \iota g - \iota g^{-1}} \uparrow \simeq & & \simeq \uparrow {\scriptstyle \iota_\ell g_\ell - \iota_\ell g_\ell^{-1}} \\ \mathbb{C}^+[\iota G^\circ] & \xrightarrow[\Sigma]{\simeq} & \mathbb{C}^+[\iota_\ell G_\ell^\circ], \end{array}$$

whence a diagram of isomorphisms of algebraic groups

$$\begin{array}{ccc} \iota G^\circ & \xleftarrow[\simeq]{\sigma} & \iota_\ell G_\ell^\circ \\ {\scriptstyle \iota g - \iota g^{-1}} \downarrow \simeq & & \simeq \downarrow {\scriptstyle \iota_\ell g_\ell - \iota_\ell g_\ell^{-1}} \\ \iota G^\circ & \xleftarrow[\sigma]{\simeq} & \iota_\ell G_\ell^\circ, \end{array}$$

which is unique and commutative up to conjugacy hence, up to replacing g by a G°-translate, can be assumed to be commutative. In particular, g normalizes $T^\circ := \iota^{-1}\sigma(\iota_\ell T_\ell^\circ) \subset B^\circ := \iota^{-1}\sigma(\iota_\ell B_\ell^\circ)$ (that is, $g \in T := N_G(B^\circ) \cap N_G(T^\circ)$) and σ maps isomorphically $\iota_\ell Z_{T_\ell^\circ}(g_\ell)$ onto $\iota Z_{T^\circ}(g)$.

This concludes the proof of Theorem 1.4 for semisimple motivic Q-coefficients.

7. Reduction of Theorem 1.4 to motivic Q-coefficients with semisimple geometric monodromy

Fix a field isomorphism $\iota : Q \tilde{\to} \mathbb{C}$.

7.1. **The ι-weight filtration.**

7.1.1. For a motivic Q-coefficient $\mathcal{C}$ on X let $H^1(\overline{X}, \mathcal{C})$ denote

- if $Q = \overline{\mathbb{Q}}_\ell$ for some $\ell \in \mathcal{L}$: the étale cohomology group $H^1(\overline{X}, \mathcal{C}) := \varprojlim_n (H^1(\overline{X}, \mathcal{H}/\lambda^n)) \otimes_{Z_\ell} \overline{\mathbb{Q}}_\ell$, where $\mathcal{H}$ is an étale sheaf representing $\overline{\mathcal{C}}$ over the ring of integers Z_ℓ of a finite extension of $\mathbb{Q}_\ell$ and λ a uniformizer. (*e.g.* [D80, (1.1)])
- if $Q = \overline{\mathbb{Q}}_{\mathfrak{u}}$ for some $\mathfrak{u} \in \mathcal{U}$: the étale cohomology group $H^1(\overline{X}, \mathcal{C}) := (\prod_{\ell \in \mathcal{L}} H^1(\overline{X}, \mathcal{M}_\ell)) \otimes \overline{\mathbb{Q}}_{\mathfrak{u}}$, where $\underline{\mathcal{M}} = (\mathcal{M}_\ell)_{\ell \in \mathcal{L}}$ is an almost $\mathfrak{u}$-tame sheaf on X representing $\mathcal{C}$ ([C19a, §3.6]).
- if $Q = \overline{\mathbb{Q}}_p$: the rigid cohomology group $H^1(\overline{X}, \mathcal{C}) := H^1_{rig}(X/K, \mathcal{E}) \otimes_K \overline{\mathbb{Q}}_p$, where $\mathcal{E} = (\mathcal{E}, \lambda)$ is an overconvergent F-isocrystal over a finite field extension K of $W(k) \otimes \mathbb{Q}$ representing $\mathcal{C}$ (*e.g.* [A18b, §4.1], [AE19, §1], [Ked06, §2.1]).

The Q-vector space $H^1(\overline{X}, \mathcal{C})$ is finite-dimensional and equipped with an action of the Frobenius φ which satisfies the fundamental property that if $\mathcal{C}$ is ι-pure of weight w then the action of φ on $H^1(\overline{X}, \mathcal{C})$ has ι-weights $w + 1 + n$ with $n \geq 0$ integers ([D80], [Ked06] and [ACa18], [C19a]).

7.1.2. Let $\mathcal{C}_1, \mathcal{C}_2$ be motivic Q-coefficients on X. One has an exact sequence

$$(7.1.2.1)\quad 0 \to H^0(\overline{X}, \mathcal{C}_1 \otimes \mathcal{C}_2^\vee)_\varphi \to Ext_{\mathcal{C}^\dagger(X,Q)}(\mathcal{C}_2, \mathcal{C}_1) \to H^1(\overline{X}, \mathcal{C}_1 \otimes \mathcal{C}_2^\vee)^\varphi,$$

We briefly recall the construction of (7.1.2.1). The group $Ext_{\mathcal{C}^\dagger(X,Q)}(\mathcal{C}_2,\mathcal{C}_1)$ is the group of classes of extensions in $\mathcal{C}^\dagger(X,Q)$ and the right arrow is

$$Ext_{\mathcal{C}^\dagger(X,Q)}(\mathcal{C}_2,\mathcal{C}_1)\overset{\overline{(-)}}{\to} Ext_{\overline{\mathcal{C}}^\dagger(X,Q)}(\overline{\mathcal{C}}_2,\overline{\mathcal{C}}_1)\hookrightarrow H^1(\overline{X},\mathcal{C}_1\otimes\mathcal{C}_2^\vee)$$

By construction, its image lies in $H^1(\overline{X},\mathcal{C}_1\otimes\mathcal{C}_2^\vee)^\varphi$. A class $[\mathcal{E}]\in Ext_{\mathcal{C}^\dagger(X,Q)}(\mathcal{C}_2,\mathcal{C}_1)$ becomes trivial on $\overline{X}$ if and only if one has a splitting $s:\overline{\mathcal{C}}_1\to\overline{\mathcal{E}}$ and the possible splittings are then the $s-f$ with $f\in \mathrm{Hom}_{\overline{\mathcal{C}}^\dagger(X,Q)}(\overline{\mathcal{C}}_2,\overline{\mathcal{C}}_1)$. The class $[\mathcal{E}]$ is trivial if and only if one can choose f in such a way that $s-f\in \mathrm{Hom}_{\mathcal{C}^\dagger(X,Q)}(\mathcal{C}_2,\mathcal{C}_1)$ that is $\varphi\cdot s-s=\varphi\cdot f-f$.

7.1.3. A Q-coefficient on X is said to be ι-mixed if it is a successive extension of ι-pure Q-coefficients. By considering Jordan-Hölder filtrations, it follows from Corollary 5.2.2.2 that every motivic Q-coefficient $\mathcal{C}$ on X is ι-mixed. This and 7.1.1, 7.1.2 formally imply the existence on every motivic Q-coefficient $\mathcal{C}$ of a ι-weight filtration in $\mathcal{C}^\dagger(X,Q)$ - automatically unique and functorial

$$\mathcal{C}:=W_1\mathcal{C}\supsetneq W_2\mathcal{C}\supsetneq\cdots\supset W_r\mathcal{C}\supsetneq W_{r+1}\mathcal{C}=0$$

such that $Gr_i^W(\mathcal{C})=:W_i\mathcal{C}/W_{i+1}\mathcal{C}$ is ι-pure of weight w_i, $i=1,\dots,r$ with $w_1>w_2>\cdots>w_r$ ([D80, (3.4.1), (3.4.6), (3.4.7)], [Ked18a, Cor. 10.5], [C19a, 15.1.2]).

7.2. Let $\mathcal{C}$ be a motivic Q-coefficient on X. Write $G:=G(\mathcal{C})$, $\overline{G}:=G(\overline{\mathcal{C}})$ and $\varphi\in\Gamma:=G/\overline{G}$ for 'the' geometric Frobenius. As $\Gamma=\overline{\langle\varphi\rangle}^{zar}$, $\Gamma=R_u(\Gamma)\times\Gamma^{red}$, with $R_u(\Gamma)\simeq\mathbb{G}_{a,Q}^\epsilon$ and $\epsilon=0,1$, and Γ^{red} of multiplicative type.

Lemma. *The following are equivalent.*
- *(7.2.1) $G=R_u(G)\times G^{red}$;*
- *(7.2.2) $\overline{G}$ is semisimple;*
- *(7.2.3) $\mathcal{C}$ is a direct sum of ι-pure motivic Q-coefficients.*

Proof. We show (7.2.1) ⇔ (7.2.2) ⇔ (7.2.3). Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two motivic Q-coefficients on X. Let W_i denote the set of ι-weights appearing in the ι-weight filtration of $\mathcal{C}_i$, $i=1,2$. We use the exact sequence (7.1.2.1).

(1) Assume $W_1=W_2=\{w\}$. Then $\mathcal{C}_1\otimes\mathcal{C}_2^\vee$ is ι-pure of weight 0. Hence $H^1(\overline{X},\mathcal{C}_1\otimes\mathcal{C}_2^\vee)$ has ι-weights ≥ 1 and $H^1(\overline{X},\mathcal{C}_1\otimes\mathcal{C}_2^\vee)^\varphi=0$. In particular, if $\mathcal{C}$ is ι-pure then $\overline{G}$ is reductive hence semisimple (Fact 5.2.2.3). This shows (7.2.3) ⇒ (7.2.2).

(2) Assume $W_1\cap W_2=\emptyset$. Then $H^0(X,\mathcal{C}_1\otimes\mathcal{C}_2^\vee)_\varphi=0$. In particular, an extension $0\to\mathcal{C}_1\to\mathcal{C}\to\mathcal{C}_2\to 0$ splits in $\mathcal{C}^\dagger(X,Q)$ if and only if the resulting extension $0\to\overline{\mathcal{C}}_1\to\overline{\mathcal{C}}\to\overline{\mathcal{C}}_2\to 0$ splits in $\overline{\mathcal{C}}^\dagger(X,Q)$. By induction on the length of the weight filtration of $\mathcal{C}$, this shows (7.2.2) ⇒ (7.2.3).

Assume $\overline{G}$ is semisimple. Then the unipotent radical $R_u(G)$ of G injects into Γ hence the morphism $G\twoheadrightarrow\Gamma$ restricts to an isomorphism $R_u(G)\tilde{\to}R_u(\Gamma)$. On the other hand,

$$L:=p^{-1}(\Gamma^{red})=\ker(G\twoheadrightarrow\Gamma=R_u(\Gamma)\times\Gamma^{red}\twoheadrightarrow R_u(\Gamma))\subset G.$$

is a normal subgroup splitting the short exact sequence

$$1\to R_u(G)\to G\to G^{red}\to 1.$$

Hence

$$G=R_u(G)\times L\simeq R_u(G)\times G^{red}.$$

This shows (7.2.2) ⇒ (7.2.1). For (7.2.1) ⇒ (7.2.2), since $\overline{G}$ is normal in G and the radical of $\overline{G}$ is unipotent by Fact 5.2.2.3, it is enough to show that $\overline{G}\subset\ker(G\to R_u(G))$. Fix a faithful representation V of $R_u(G)$ and let $\mathcal{V}$ be the corresponding motivic Q-coefficient on X. Since $G(\mathcal{V})=R_u(G)$ is unipotent, $\mathcal{V}$ is pure of weight 0 hence $G(\mathcal{V}|_{\overline{X}})$ is semisimple by (1) above. As $G(\mathcal{V}|_{\overline{X}})$ is also unipotent by construction, this forces $G(\mathcal{V}|_{\overline{X}})=1$. □

7.3. Consider the ι-weight filtration $\mathcal{C}:=W_1\mathcal{C}\supsetneq W_2\mathcal{C}\supsetneq\cdots\supset W_r\mathcal{C}\supsetneq W_{r+1}\mathcal{C}=0$ in $\mathcal{C}^\dagger(X,Q)$; recall that this means that $Gr_i^W(\mathcal{C}):=W_i\mathcal{C}/W_{i+1}\mathcal{C}$ is ι-pure of weight w_i, $i=1,\dots,r$ with $w_1>w_2>\cdots>w_{r+1}$. Set $\widetilde{\mathcal{C}}:=\oplus_{1\leq i\leq r}Gr_i^W(\mathcal{C})$ and $\widetilde{G}:=G(\widetilde{\mathcal{C}})$. Then $R:=\ker(G\twoheadrightarrow\widetilde{G})\subset R_u(G)$ while, from (7.2.3) ⇒ (7.2.1), $\widetilde{G}\simeq\mathbb{G}_{a,Q}^{\widetilde{\epsilon}}\times G^{red}$ with $\widetilde{\epsilon}=0,1$.

7.3.1. **Lemma.** *For every $x \in |X|$, $R^{\varphi_x^{ss}} = 1$.*

Proof. Let V denote the Q-representation of G corresponding to $\mathcal{C}$ and $V = W_1 \supsetneq W_2 \supsetneq \cdots \supsetneq W_r \supsetneq W_{r+1} = 0$ the filtration on V induced by the weight filtration on $\mathcal{C}$. This defines a descending filtration by closed normal subgroups on G:

$$F^jG = \ker(G \to \prod_{1\leq i\leq r-j+1} \mathrm{GL}(W_i/W_{i+j})),\ 0 \leq j \leq r.$$

(In particular, $F^0G = G$, $F^1G = R$ and $F^rG = 1$). By construction, the embeddings

$$F^jG/F^{j+1}G \hookrightarrow \prod_{1\leq i\leq r-j} \mathrm{Hom}(W_i/W_{i+1}, W_{i+j}/W_{i+j+1}),\ \ 0 \leq j \leq r-1$$

are G-equivariant hence φ_x-equivariant. But since $w_i > w_{i+j}$,

$$\mathrm{Hom}(W_i/W_{i+1}, W_{i+j}/W_{i+j+1})^{\varphi_x^{ss}} = 0,\ \ 1 \leq j \leq r-1,\ 1 \leq i \leq r-j.$$

□

7.3.2. It follows from Lemma 7.3.1 and Lemma 3.5.2 applied to the extension $1 \to R \to G \to \widetilde{G} \to 1$ and $\Phi = \Phi_S \subset G$ for any subset $S \subset |X|$ that Theorem 1.4 holds for $\mathcal{C}$ if and only if Theorem 1.4 holds for $\widetilde{\mathcal{C}}$.

8. Theorem 1.4 for motivic Q-coefficients with semisimple geometric monodromy

Fix an isomorphism $\iota : Q\tilde{\to}\mathbb{C}$. Let $\mathcal{C}$ be a motivic Q-coefficient on X. We retain the notation of Section 7. Assume $\overline{G}$ is semisimple hence $G = R_u(G) \times G^{red}$ and $R_u(G)\tilde{\to}R_u(\Gamma) \simeq \mathbb{G}_{a,Q}^{\epsilon}$ with $\epsilon = 0, 1$ (Lemma 7.2). From Theorem 1.4 for semisimple motivic Q-coefficients (Section 6), one may assume $\epsilon = 1$. Fix the isomorphism $R_u(\Gamma) \simeq \mathbb{G}_{a,Q}$ so that $\varphi \in \Gamma = R_u(\Gamma) \times \Gamma^{red}$ maps to $1 \in \mathbb{G}_{a,Q}$. Then for every $x \in |X|$, $\varphi_x \in G = R_u(G) \times G^{red}$ maps to $n_x := [k(x) : k] \in \mathbb{G}_{a,Q}$. Fix a prime $\ell \neq p$ and an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell\tilde{\to}\mathbb{C}$ such that $\mathcal{C}^{ss}$ admits a semisimple étale $\overline{\mathbb{Q}}_\ell$-companion $\mathcal{C}_\ell^{ss}$ (Corollary 5.2.3). Write $\mathcal{U}_\ell$ for the étale $\overline{\mathbb{Q}}_\ell$-coefficient on X defined by the 2-dimensional representation $\pi_1(X) \twoheadrightarrow \pi_1(k) \overset{\varphi\mapsto U}{\to} \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and set $\mathcal{C}_\ell^+ := \mathcal{U}_\ell \oplus \mathcal{C}_\ell^{ss}$. Write $G_\ell := G(\mathcal{C}_\ell^+)$. By construction $G_\ell = R_u(G_\ell) \times G_\ell^{red}$ with $R_u(G_\ell) = G(\mathcal{U}_\ell) = \mathbb{G}_{a,\overline{\mathbb{Q}}_\ell}$, $G_\ell^{red} = G(\mathcal{C}_\ell^{ss})$ and $\mathcal{C}_\ell^+$ is a $\overline{\mathbb{Q}}_\ell$-companion of $\mathcal{C}^+ := \mathcal{C} \oplus \mathbb{I}^2$ (recall $\mathbb{I}$ denotes the trivial motivic Q-coefficient on X) in the sense that

$$(8.1)\ \ \iota_\ell\psi(\varphi_x) = (n_x, \iota_\ell\det(Id - T\varphi_x|\mathcal{C}_{\ell,\overline{x}}^+)) = (n_x, \iota\det(Id - T\varphi_x|\mathcal{C}_{\overline{x}}^+)) = \iota\psi(\varphi_x),\ x \in |X|.$$

Let $S \subset |X|$ such that $\delta^u(S) > 0$. Without loss of generality, we may assume $\Phi_S^{\mathcal{C}} \subset \Delta := p_{G^\circ}^{-1}(\overline{\Delta})$ for some conjugacy class $\overline{\Delta} \subset \pi_0(G) = \pi_0(G^{red}) = \pi_0(G_\ell^{red}) = \pi_0(G_\ell)$. From the (classical) Cebotarev density theorem and (3.4.1.2) (applied to $\Phi_S^{\mathcal{C}_\ell} \subset G_\ell$),

$$\overline{\iota_\ell\psi(\Phi_S^{\mathcal{C}_\ell})^{zar}} = \iota_\ell\psi(G_\ell)_{\overline{\Delta}}$$

while from (8.1), $\iota_\ell\psi(\Phi_S^{\mathcal{C}_\ell}) = \iota\psi(\Phi_S^{\mathcal{C}}) \subset \psi(G)_{\overline{\Delta}}$. But since $\dim(\psi(G)_{\overline{\Delta}}) = 1+\dim(\chi(G^{red})_{\overline{\Delta}})$, $\dim(\psi(G_\ell)_{\overline{\Delta}}) = 1+\dim(\chi(G_\ell^{red})_{\overline{\Delta}})$, one has $\dim(\psi(G)_{\overline{\Delta}}) = \dim(\psi(G_\ell)_{\overline{\Delta}})$ from Theorem 1.4 for $(\mathcal{C}^+)^{ss}$ and Lemma 6.3. This shows that $\overline{\psi(\Phi_S^{\mathcal{C}})^{zar}} = \psi(G)_{\overline{\Delta}}$ since $\psi(G)_{\overline{\Delta}}$ is irreducible hence, from (3.4.1.2) (applied to $\Phi_S^{\mathcal{C}} \subset G$), that $\overline{\Phi_S^{\mathcal{C}}}^{zar} = \Delta$. This concludes the proof of Theorem 1.4 for motivic Q-coefficients with semisimple geometric monodromy hence, from 7.3.2, the proof of Theorem 1.4.

9. $\overline{\mathbb{Q}}_p$-coefficients

In this section, we assume X is smooth over k.

9.1. Write $v : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ for the p-adic valuation normalized by $v(|k|) = 1$. Let $\mathcal{C} \in \mathcal{C}(X, \overline{\mathbb{Q}}_p)$. For every $x \in |X|$, the slopes of $\mathcal{C}$ at x are the $\frac{v(\alpha)}{[k(x):k]} \in \mathbb{Q}$ for α running over the set of eigenvalues (counted with multiplicities) of φ_x acting on $\mathcal{C}_{\overline{x}}$. One says that $\mathcal{C}$ is isoclinic of slope σ if $\frac{v(\alpha)}{[k(x):k]} = \sigma$ for every $x \in |X|$ and eigenvalue α of φ_x acting on $\mathcal{C}_{\overline{x}}$.

9.1.1. **Fact.** ([K79, Thm. 2.3.1, Cor. 2.6.3]; see also [Ked18a, Thm. 3.12, Cor. 4.2]) *For every $\mathcal{C} \in \mathcal{C}(X, \overline{\mathbb{Q}}_p)$ there exists a dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration in $\mathcal{C}(U, \overline{\mathbb{Q}}_p)$- automatically unique and functorial*

$$S_\bullet(\mathcal{C}|_U) : \ \mathcal{C}|_U = S_1\mathcal{C}|_U \supsetneq S_2\mathcal{C}|_U \supsetneq \cdots \supsetneq S_s\mathcal{C}|_U \supsetneq S_{s+1}\mathcal{C}|_U = 0$$

such that $Gr_i^S(\mathcal{C}|_U) := S_i\mathcal{C}|_U / S_{i+1}\mathcal{C}|_U$ is isoclinic of slope σ_i, $i = 1, \dots, s$ with $\sigma_1 > \sigma_2 > \cdots > \sigma_s$.

Remark. ([Am19, Prop. 6.5.1.4.3]) If $\mathcal{C} \in \mathcal{C}(X, \overline{\mathbb{Q}}_p)$ has a slope filtration then for every dense open subscheme $U \subset X$ the canonical closed immersion $G(\mathcal{C}|_U) \subset G(\mathcal{C})$ is an isomorphism.

9.1.2. With the notation of Fact 9.1.1,

Lemma. *The slope filtration $S_\bullet(\mathcal{C}|_U)$ is defined by a cocharacter $\omega : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G(\mathcal{C}|_U) \subset G(\mathcal{C})$.*

Proof. Fix a closed point $x \in |U|$ and a $\overline{k}$-point $+$ on U. Fix a $\overline{\mathbb{Q}}_p$-basis $\underline{\epsilon} = \underline{\epsilon}_1, \dots, \underline{\epsilon}_s$ of $\mathcal{C}_+$ adapted to the slope filtration $S_\bullet(\mathcal{C}|_U)_+$ (namely, $\underline{\epsilon}_s$ is a $\overline{\mathbb{Q}}_p$-basis of $(S_s\mathcal{C}|_U)_+$, $\underline{\epsilon}_{s-1}, \underline{\epsilon}_s$ is a $\overline{\mathbb{Q}}_p$-basis of $(S_{s-1}\mathcal{C}|_U)_+$ *etc.*) and such that $T_x := \overline{\langle \varphi_x^{ss} \rangle^{zar}}$ appears as a subgroup of the corresponding diagonal torus $\mathbb{D} \subset GL(\mathcal{C}_+) \simeq GL_{r,\overline{\mathbb{Q}}_p}$. Let $\underline{\lambda} = \underline{\lambda}_1, \dots, \underline{\lambda}_s$ denote the basis of $X^*(\mathbb{D})$ corresponding to $\underline{\epsilon}$ (*i.e.* $\tau\epsilon_{i,j} = \lambda_{i,j}(\tau)\epsilon_{i,j}$ for every $\tau \in \mathbb{D}$) and $\underline{\chi} = \underline{\chi}_1, \dots, \underline{\chi}_s$ the dual basis of $\underline{\lambda}$ in $X_*(\mathbb{D})$ (*i.e.* $\lambda_{i,j} \circ \chi_{i',j'} = \delta_{(i,j),(i',j')}$). Let $\underline{\alpha}_1, \dots, \underline{\alpha}_s$ denote the eigenvalues of φ_x with $\underline{\alpha}_i = \alpha_{i,1}, \dots, \alpha_{i,r_i}$ the eigenvalues of φ_x acting on $S_i\mathcal{C}_+/S_{i+1}\mathcal{C}_+$, $i = 1, \dots, s$. Then, identifying $X^*(\mathbb{D}) \simeq \mathbb{Z}^{r_1} \oplus \cdots \oplus \mathbb{Z}^{r_s}$ by means of $\underline{\lambda}$ and setting

$$Q_x := \{\underline{a} = \underline{a}_1, \dots, \underline{a}_s \in X^*(\mathbb{D}) \mid \underline{\alpha}_1^{\underline{a}_1} \cdots \underline{\alpha}_s^{\underline{a}_s} = 1\},$$

$$T_x = \bigcap_{\underline{a} \in Q_x} \ker(\underline{a}) \subset \mathbb{D}.$$

By definition of the slope filtration, $v(\alpha_{i,j}) = [k(x) : k]\sigma_i$ with $\sigma_1 > \cdots > \sigma_s$. So taking the p-adic valuation, the relation $\underline{\alpha}_1^{\underline{a}_1} \cdots \underline{\alpha}_s^{\underline{a}_r} = 1$ imposes

$$\sum_{1 \leq i \leq s} \sigma_i \sum_{1 \leq j \leq r_i} a_{i,j} = 0.$$

In particular, T_x - hence *a fortiori* $G(\mathcal{C})$ - contains the image of the cocharacter

$$\omega = N \sum_{1 \leq i \leq s} \sigma_i \sum_{1 \leq j \leq r_i} \chi_{i,j}$$

defining the filtration $S_\bullet(\mathcal{C}|_U)_+$, where N stands for the minimal common denominator of $\sigma_1, \dots, \sigma_s \in \mathbb{Q}$. □

9.2. **Parabolicity conjectures.** The following is a generalization to arbitrary convergent $\overline{\mathbb{Q}}_p$-coefficients of a question stated by Crew [Cr92, §4, p. 460]. It was suggested to us by Ambrosi (in [HP18], a variant of it - [HP18, Conj. 10.1] - is attributed to Pink).

9.2.1. **Conjecture.** (Generalized parabolicity conjecture) *Let $\mathcal{C} \in \mathcal{C}(X, \overline{\mathbb{Q}}_p)$. For every dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration $S_\bullet(\mathcal{C}|_U)$ on U, $G(\mathcal{C}|_U) \subset G(\mathcal{C})$ is the stabilizer of $S_\bullet(\mathcal{C}|_U)$ in $G(\mathcal{C})$.*

It was recently solved by D'Addezio for †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficients. More precisely, D'Addezio proved the following which, in particular, answers positively Crew's original question. That D'Addezio's results implies Conjecture 9.2.1 is the content of Lemma 9.2.3 below.

9.2.2. **Fact.** (Crew's parabolicity conjecture) (D'Addezio [D'A20b, Thm. 1.1.1]; see also [Ts19] for preliminary results on the minimal slope conjecture of Kedlaya) *Let $\mathcal{C}^\dagger \in \mathcal{C}^\dagger(X, \overline{\mathbb{Q}}_p)$. Assume $\mathcal{C} := \alpha(\mathcal{C}^\dagger) \in \mathcal{C}(X, \overline{\mathbb{Q}}_p)$ admits a slope filtration $S_\bullet(\mathcal{C})$ on X. Then $G(\mathcal{C}) \subset G(\mathcal{C}^\dagger)$ is the stabilizer of $S_\bullet(\mathcal{C})$ in $G(\mathcal{C}^\dagger)$.*

9.2.3. **Lemma.** *Fact 9.2.2 implies Conjecture 9.2.1 for †-extendable convergent* $\overline{\mathbb{Q}}_p$*-coefficients.*

Proof. Let $\mathcal{C}^\dagger \in \mathcal{C}^\dagger(X,\overline{\mathbb{Q}}_p)$; write $\mathcal{C} := \alpha(\mathcal{C}^\dagger) \in \mathcal{C}(X,\overline{\mathbb{Q}}_p)$. Since X is smooth, it follows from [Ked07, Thm. 5.2.1 and Prop. 5.3.1] that for every dense open subscheme $U \subset X$, the canonical closed immersion $G(\mathcal{C}^\dagger|_U) \hookrightarrow G(\mathcal{C}^\dagger)$ is an isomorphism. Let $U \subset X$ be any dense open subscheme such that $\mathcal{C}|_U$ admits a slope filtration $S_\bullet(\mathcal{C}|_U)$ on U. Then one has a commutative diagram of closed immersions

$$\begin{array}{ccc} G(\mathcal{C}|_U) & \hookrightarrow & G(\mathcal{C}^\dagger|_U) \\ \downarrow & & \downarrow \simeq \\ G(\mathcal{C}) & \hookrightarrow & G(\mathcal{C}^\dagger) \end{array}$$

Fact 9.2.2 for $\mathcal{C}^\dagger|_U$ implies that $G(\mathcal{C}|_U) \subset G(\mathcal{C}^\dagger|_U) \simeq G(\mathcal{C}^\dagger)$ is the stabilizer of $S_\bullet(\mathcal{C}|_U)$. Hence, *a fortiori*, $G(\mathcal{C}|_U) \subset G(\mathcal{C})(\subset G(\mathcal{C}^\dagger))$ is the stabilizer of $S_\bullet(\mathcal{C}|_U)$ that is, Conjecture 9.2.1 holds for $\mathcal{C}$. □

For our purpose, we only need the following consequence of Conjecture 9.2.1.

9.2.4. **Conjecture.** (Weak generalized parabolicity conjecture) *Let* $\mathcal{C} \in \mathcal{C}(X,\overline{\mathbb{Q}}_p)$*. For every dense open subscheme* $U \subset X$ *such that* $\mathcal{C}|_U$ *admits a slope filtration* $S_\bullet(\mathcal{C}|_U)$ *on* U*,* $G(\mathcal{C}|_U) \subset G(\mathcal{C})$ *contains the centralizer in* $G(\mathcal{C})$ *of the cocharacter* $\omega : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G(\mathcal{C}|_U) \subset G(\mathcal{C})$ *defining the slope filtration* $S_\bullet(\mathcal{C}|_U)$ *(see Lemma 9.1.2).*

From Remark 9.1.1, if Conjecture 9.2.1 (resp. Conjecture 9.2.4) holds for one dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration then it holds for every dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration.

9.3. Fact 9.2.2 and Lemma 9.2.3 reduce Theorem 1.5 for †-extendable convergent $\overline{\mathbb{Q}}_p$-coefficients to the following.

9.3.1. **Proposition.** *Let* $\mathcal{C} \in \mathcal{C}(X,\overline{\mathbb{Q}}_p)$*. Then Conjecture 9.2.4 for* $\mathcal{C}$ *implies Conjecture 1.3 for* $\mathcal{C}$*.*

The arguments used to prove Proposition 9.3.1 also yield

9.3.2. **Proposition.** *Fact 9.2.2 implies Conjecture 1.3 for motivic* $\overline{\mathbb{Q}}_p$*-coefficients.*

Proposition 9.3.2 provides an alternative purely p-adic and "elementary" proof of Conjecture 1.3 for motivic $\overline{\mathbb{Q}}_p$-coefficients.

9.4. **Proof of Propositions 9.3.1, 9.3.2.** Our starting point is Fact 1.3.3. Let $\mathcal{C} \in \mathcal{C}(X,\overline{\mathbb{Q}}_p)$ (resp. $\mathcal{C}^\dagger \in \mathcal{C}^\dagger(X,\overline{\mathbb{Q}}_p)$ and set $\mathcal{C} := \alpha(\mathcal{C}^\dagger) \in \mathcal{C}(X,\overline{\mathbb{Q}}_p)$).

Lemma. *Assume* $\mathcal{C}$ *admits a slope filtration* $S_\bullet(\mathcal{C})$*:* $\mathcal{C} = S_1\mathcal{C} \supsetneq S_2\mathcal{C} \supsetneq \cdots \supsetneq S_s\mathcal{C} \supsetneq S_{s+1}\mathcal{C} = 0$ *on* X*. Then Conjecture 1.3 holds for* $\mathcal{C}$*.*

Proof. Write $\widetilde{\mathcal{C}} := \oplus_{1\leq i\leq s} S_i\mathcal{C}/S_{i+1}\mathcal{C}$, $G := G(\mathcal{C})$, $\widetilde{G} = G(\widetilde{\mathcal{C}})$ so that one has a short exact sequence

$$1 \to R \to G \xrightarrow{p} \widetilde{G} \to 1$$

with $R \subset R_u(G)$. (For the surjectivity of p, see [DM82, Prop. 2.21].) Write also $\Phi := \Phi_S^{\mathcal{C}}$, $\widetilde{\Phi} := \Phi_S^{\widetilde{\mathcal{C}}}(= p(\Phi))$. Since there is no non-trivial morphism between isoclinic convergent $\overline{\mathbb{Q}}_p$-coefficients with different slopes, $R^g = 1$ for every $g \in \Phi^{ss}$ (see the argument in the proof of Lemma 7.3.1 with 'weight' replaced by 'slope'). From Subsection 4.2.3, one may assume $\Phi \subset \Delta := p_{G^\circ}^{-1}(\overline{\Delta})$ for a conjugacy class $\overline{\Delta} \subset \pi_0(G)$. From Conjecture 1.3 for $\widetilde{\mathcal{C}}$ (Fact 1.3.3), for every $g \in \Phi^{ss}$ with image $\widetilde{g} := p(g) \in \widetilde{\Phi}^{ss}$, $\overline{\widetilde{\Phi}^{zar}} \supset \widetilde{G}^\circ\widetilde{g}$. From Lemma 3.5.2 this implies $\overline{\Phi^{zar}} \supset G^\circ g$. This concludes the proof of the lemma. □

Remark. D'Addezio pointed out that, using the functoriality of the slope filtration, one can show that $p : G \to \widetilde{G}$ admits a canonical splitting (corresponding to the retraction of the natural inclusion functor $\langle Gr^S(\mathcal{E})\rangle^\otimes \subset \langle\mathcal{E}\rangle^\otimes$ by the functor sending an object $\mathcal{E}' \in \langle\mathcal{E}\rangle^\otimes$ to its graded quotient $Gr^S(\mathcal{E}')$) whose image contains the centralizer of ω (see [D'A20b, (Proof of) Prop. 5.1.4] for details) so that, at least when G is connected, one could invoke directly Corollary 3.3.2. rather than Lemma 3.5.2.

We return to the case where $\mathcal{C}$ is arbitrary. Fix a dense open subscheme $U \subset X$ such that $\mathcal{C}|_U$ admits a slope filtration $S_\bullet(\mathcal{C}|_U)$ on U (Fact 9.1.1) and assume $G(\mathcal{C}|_U)$ contains the centralizer $Z_{G(\mathcal{C})}(\omega)$ in $G(\mathcal{C})$ of the image of the cocharacter $\omega : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G(\mathcal{C}|_U) \subset G(\mathcal{C})$ defining $S_\bullet(\mathcal{C}|_U)$ (resp. is the stabilizer of $S_\bullet(\mathcal{C}|_U)$ in $G(\mathcal{C}^\dagger)$). Since $\delta^u(S \cap U) = \delta^u(S) > 0$, one may assume $S = S \cap U$. Write $\Psi := \Phi_S^{\mathcal{C}|_U}$ and $\Phi := \Phi_S^{\mathcal{C}}$ (resp. $\Phi := \Phi_S^{\mathcal{C}^\dagger}$). From Lemma 9.4 there exists $g \in \Psi^{ss}$ such that $\overline{\Psi^{zar}} \supset G(\mathcal{C}|_U)^\circ g$ hence Proposition 9.3.1 (resp. Proposition 9.3.2) follows from Corollary 4.2.2, Lemma 9.1.2 and Lemma 3.6 applied with $H := G(\mathcal{C}|_U) \subset G := G(\mathcal{C})$ (resp. $H := G(\mathcal{C}|_U) \subset G := G(\mathcal{C}^\dagger|_U) = G(\mathcal{C}^\dagger)$, see the proof of Lemma 9.2.3).

9.5. **Proof of the assertion in Remark 1.5.1.** Let $\mathcal{C} = \mathcal{E}_1 \oplus \mathcal{E}_2$ with $\mathcal{E}_1$, $\mathcal{E}_2$ satisfying Conjecture 9.2.4 (*e.g.* †-extendable or admitting a slope filtration) and assume that for every dense open subscheme $U \subset X$ such that both $\mathcal{E}_1|_U$ and $\mathcal{E}_2|_U$ admit a slope filtration the canonical morphism $G(\mathcal{E}_1|_U, \mathcal{E}_2|_U) \to G(\mathcal{E}_1, \mathcal{E}_2)$ is a closed immersion (where we set $G(\mathcal{E}_1, \mathcal{E}_2) := G(\langle \mathcal{E}_1 \rangle \cap \langle \mathcal{E}_2 \rangle))$. From Proposition 9.3.1 it suffices to prove $\mathcal{C}$ satisfies Conjecture 9.2.4. For this, observe that $G(\mathcal{C}) = G(\mathcal{E}_1) \times_{G(\mathcal{E}_1,\mathcal{E}_2)} G(\mathcal{E}_2)$ (*cf.* [HP18, Prop. 3.6(c)]) and that, for every dense open subscheme $U \subset X$,

$$G(\mathcal{C}|_U) = G(\mathcal{E}_1|_U) \times_{G(\mathcal{E}_1|_U,\mathcal{E}_2|_U)} G(\mathcal{E}_2|_U) = G(\mathcal{E}_1|_U) \times_{G(\mathcal{E}_1,\mathcal{E}_2)} G(\mathcal{E}_2|_U) \hookrightarrow G(\mathcal{E}_1) \times_{G(\mathcal{E}_1,\mathcal{E}_2)} G(\mathcal{E}_2) = G(\mathcal{C}).$$

Here the second equality follows from the assumption that the canonical morphism $G(\mathcal{E}_1|_U, \mathcal{E}_2|_U) \to G(\mathcal{E}_1, \mathcal{E}_2)$ is a closed immersion. Fix a dense open subscheme $U \subset X$ such that both $\mathcal{E}_1|_U$ and $\mathcal{E}_2|_U$ admit a slope filtration. By construction / definition, for $i = 1, 2$ the cocharacter $\omega : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G(\mathcal{C}|_U) \subset G(\mathcal{C})$ defining the slope filtration on $\mathcal{C}|_U$ (*cf.* Lemma 9.1.2) composed with the projection $G(\mathcal{C}) \twoheadrightarrow G(\mathcal{E}_i)$ yields the cocharacter $\omega_{\mathcal{E}_i} : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G(\mathcal{E}_i|_U) \subset G(\mathcal{E}_i)$ defining the slope filtration on $\mathcal{E}_i|_U$. In particular,

$$Z_{G(\mathcal{C})}(\omega) \subset Z_{G(\mathcal{E})}(\omega_{\mathcal{E}_1}) \times_{G(\mathcal{E}_1,\mathcal{E}_2)} Z_{G(\mathcal{E}_2)}(\omega_{\mathcal{E}_2}) \subset G(\mathcal{E}_1|_U) \times_{G(\mathcal{E}_1,\mathcal{E}_2)} G(\mathcal{E}_2|_U) = G(\mathcal{C}|_U),$$

which shows $\mathcal{C}$ satisfies Conjecture 9.2.4 as well.

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