ON $\mathbb{Q}$-COMPATIBILITY OF CERTAIN FAMILIES OF $\mathbb{Q}_\ell$-LOCAL SYSTEMS

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In honor of Minhyong Kim's 60th birthday

ABSTRACT. Let k be a number field, X a smooth, separated, geometrically connected variety over k and $f: Y \to X$ a smooth projective morphism. Fix an infinite place $\infty: k \hookrightarrow \mathbb{C}$ and let $f_\infty^{\mathrm{an}}: Y_\infty^{\mathrm{an}} \to X_\infty^{\mathrm{an}}$ denote the resulting morphism of complex analytic spaces. The Hodge conjecture predicts that, for every integers $i \geq 0$, j, generic motivated $\mathbb{Q}$-local subsystems (in the sense of André) in the Tannakian category generated by the polarizable $\mathbb{Q}$-VHS $\mathcal{V}_\infty := R^i f_{\infty *}^{\mathrm{an}} \mathbb{Q}(j)$ should give rise to $\mathbb{Q}$-compatible families of $\mathbb{Q}_\ell$-local systems on X. We prove this conjecture under mild assumptions for a specific generic motivated $\mathbb{Q}$-local subsystem $\overline{\mathcal{H}}_\infty$ of $\mathcal{V}_\infty \otimes \mathcal{V}_\infty^\vee$ closely related to the "tangent space" $\overline{\mathcal{L}}_\infty$ of $\mathcal{V}_\infty$.

Given a scheme S, we write $|S|$ for the set of closed points and $\pi_1(S)$ for its étale fundamental group[1]; when $S = \mathrm{spec}(R)$ for a ring R, we sometimes write $\pi_1(R) = \pi_1(S)$.

For a number field k, let $\mathcal{O}_k \subset k$ denote the ring of integers. For a finite place v of k, let k_v denote the completion of k at v and set $\mathfrak{m}_v \subset \mathcal{O}_{k_v} \twoheadrightarrow \kappa_v$ for the maximal ideal, ring of integers and residue field of k_v respectively.

Let k be a field and X a smooth and geometrically connected variety (throughout this paper, a variety means a scheme separated, reduced and of finite type) over k. For a field extension $k \hookrightarrow K$, we usually write $X_K := X \times_k K$. If k is a number field, $x \in |X|$ with residue field $k(x)$, and v is a finite place of $k(x)$, let $x_v : \mathrm{spec}(k(x)_v) \to \mathrm{spec}(k(x)) \xrightarrow{x} X$ denote the resulting $k(x)_v$-point.

1. Introduction

1.1. $\mathbb{Q}$-compatibility and motivic local systems. We first define the notion of $\mathbb{Q}$-compatiblity used in this paper. Let k be a number field; write $S_k := \mathrm{spec}(\mathcal{O}_k) \to S_{\mathbb{Q}} := \mathrm{spec}(\mathbb{Z})$.

Let $\ell \in |S_{\mathbb{Q}}|$, $v \in |S_k|$ with residue characteristic p, and $\mathcal{V}_\ell$ a $\mathbb{Q}_\ell$-local system on $\mathrm{spec}(k_v)$.

- If $\ell \neq p$ and $\mathcal{V}_\ell$ is unramified, let $\chi_{\mathcal{V}_\ell} \in \mathbb{Q}_\ell[T]$ denote the characteristic polynomial of the geometric Frobenius $\varphi_{\mathcal{V}_\ell} : \mathcal{V}_{\ell,\bar{x}} \to \mathcal{V}_{\ell,\bar{x}}$;
- If $\ell = p$ and $\mathcal{V}_p$ is crystalline , let $\chi_{\mathcal{V}_p} \in k_{v,0}[T]$ denote the characteristic polynomial of the m_vth power of the crystalline Frobenius $\varphi_{\mathcal{V}_p} := \phi_{\mathcal{V}_p}^{m_v} : D_{\mathrm{cris}}(\mathcal{V}_p) \to D_{\mathrm{cris}}(\mathcal{V}_p)$, where $m_v := [k_{v,0} : \mathbb{Q}_p]$ is the degree of the maximal unramified extension $k_{v,0}$ of $\mathbb{Q}_p$ contained in k_v and $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\pi_1(k_v)) \to \mathrm{FM}_{k_v}(\phi)$ is Fontaine's (fully faithful) period functor from the category of crystalline $\mathbb{Q}_p$-representations of $\pi_1(k_v)$ to the category of filtered ϕ-modules over k_v.

One says that a family of $\mathbb{Q}_\ell$-local systems $\mathcal{V} := (\mathcal{V}_\ell)_{\ell \in |S_{\mathbb{Q}}|}$ on $\mathrm{spec}(k_v)$ is $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$) if $\mathcal{V}_\ell$ is unramified for $\ell \neq p$ and crystalline for $\ell = p$, and if the polynomial $\chi_{\mathcal{V}} := \chi_{\mathcal{V}_\ell}$ is in $\mathbb{Q}[T]$ and independent of ℓ (resp. and for every root α of χ_{x_v} and infinite place $\infty : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$, $|\alpha|_\infty = |\kappa_v|^{\frac{w}{2}}$).

Let $\ell \in |S_{\mathbb{Q}}|$ and $\mathcal{V}_\ell$ a $\mathbb{Q}_\ell$-local system on $\mathrm{spec}(k)$. Let $U_{k,\mathcal{V}_\ell} \subset |S_k|$ denote the set of all $v \in |S_k|$ such that, $x_v^* \mathcal{V}_\ell$ is unramified if $\ell \neq p$ and $x_v^* \mathcal{V}_p$ is crystalline, where p denotes the residue characteristic of v. One says that a family of $\mathbb{Q}_\ell$-local systems $\mathcal{V} := (\mathcal{V}_\ell)_{\ell \in |S_{\mathbb{Q}}|}$ on $\mathrm{spec}(k)$ is $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$) if there exists a non-empty open subset $U_{k,\mathcal{V}} \subset |S_k|$ such that $U_{k,\mathcal{V}} \subset \cap_\ell U_{k,\mathcal{V}_\ell}$ and for every $v \in U_{k,\mathcal{V}}$

[1] As fiber functors will play no part in the following, we will in general omit them from our notation.

the resulting family $x_v^*\mathcal{V}$ is $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$).

Let X be a smooth and geometrically connected variety over k. One says that a family of $\mathbb{Q}_\ell$-local systems $\mathcal{V} := (\mathcal{V}_\ell)_{\ell\in|S_\mathbb{Q}|}$ on X is (pointwise) $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$) if $x^*\mathcal{V}$ is $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$), $x \in |X|$ and that it is almost (pointwise) $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$) if there exists a connected étale cover $f : X' \to X$ such that $f^*\mathcal{V}$ is (pointwise) $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{Z}$).

The fundamental example of $\mathbb{Q}$-compatible families of $\mathbb{Q}_\ell$-local systems on X are those of the form $\mathcal{V}_\ell = R^if_*\mathbb{Q}_\ell(j)$ for a smooth projective morphism $f : Y \to X$ and integers $i \geq 0$, j (these are pure of weight $w = i - 2j$) [D74], [F89], [KM74]. More generally, if $f : Y \to X$ is of relative dimension d every codimension d algebraic cycle ϵ on the generic fiber of $f^{(2)} : Y \times_X Y \to X$ inducing an idempotent correspondence gives rise to a $\mathbb{Q}$-compatible subfamily $\epsilon\mathcal{V} := (\epsilon\mathcal{V}_\ell)_{\ell\in|S_\mathbb{Q}|} \subset \mathcal{V}$ (Lemma 10); we will say that such (families of) $\mathbb{Q}_\ell$-local systems are motivic. Conversely, for a given prime ℓ, a central problem in arithmetic geometry is to determine which $\mathbb{Q}_\ell$-local systems on X are motivic. A weaker but still almost completely open variant of this problem is to determine which sub $\mathbb{Q}_\ell$-local systems $\mathcal{W}_\ell$ of $T(\mathcal{V}_\ell) := \oplus_{m,n\geq 0}(\mathcal{V}_\ell^{\otimes m}) \otimes (\mathcal{V}_\ell^\vee)^{\otimes m}$ are motivic. A necessary condition for a $\mathbb{Q}_\ell$-local system $\mathcal{W}_\ell$ to be motivic is that it lies in a $\mathbb{Q}$-compatible family while a conjectural sufficient condition is that it be motivated in the sense of André [An96]. As motivic cycles are motivated, an intermediate problem is whether every motivated $\mathbb{Q}_\ell$-local system $\mathcal{W}_\ell$ lies in an almost $\mathbb{Q}$-compatible family. Actually, by definition / construction, a motivated $\mathbb{Q}_\ell$-local system $\mathcal{W}_\ell$ automatically comes as part of a canonical family $\mathcal{W} = (\mathcal{W}_\ell)_{\ell\in|S_\mathbb{Q}|}$ of motivated $\mathbb{Q}_\ell$-local systems on X which is expected to be motivic, so the problem boils down to proving that $\mathcal{W}$ is almost $\mathbb{Q}$-compatible. This is the question we want to address, for a specific motivated family $\mathcal{W}$, which we will define in Subsection 1.4 as a special case of the more general construction explained in Subsection 1.2.

1.2. **Constructing generic motivated cycles.** Let $f : Y \to X$ be a smooth projective morphism of relative dimension d. Fix an infinite place $\infty : k \hookrightarrow \mathbb{C}$, let $(-)_\infty$ denote the base-change functor along $\mathrm{spec}(\mathbb{C}) \overset{\infty}{\to} \mathrm{spec}(k)$ and $(-)^{\mathrm{an}}$ the analytification functor from varieties over $\mathbb{C}$ to complex analytic spaces. Consider the pure polarizable $\mathbb{Q}$-variation of Hodge structure ($\mathbb{Q}$-VHS for short) $\mathcal{V}_\infty := R^{2d}f_{\infty *}^{\mathrm{an}(2)}\mathbb{Q}(d)$. For $x \in X_\infty^{\mathrm{an}}$, let $\overline{G}_\infty \subset \mathrm{GL}_{V_\infty}$ denote the Zariski-closure of the image $\overline{\Pi}_\infty$ of the topological fundamental group $\pi_1(X_\infty^{\mathrm{an}})$ acting on $V_\infty := \mathcal{V}_{\infty,x}$; it is a semisimple subgroup of GL_{V_∞}; in particular, the normalizer $N_\infty := \mathrm{Nor}_{\mathrm{GL}_{V_\infty}}(\overline{G}_\infty^\circ) \subset \mathrm{GL}_{V_\infty}$ of its neutral component $\overline{G}_\infty^\circ$ in GL_{V_∞} is reductive. As $\overline{G}_\infty^\circ$ is a normal subgroup of the generic Mumford-Tate group $G_\infty \subset \mathrm{GL}_{V_\infty}$ of $\mathcal{V}_\infty$ [An92, Thm. 1], one has $G_\infty \subset N_\infty$ hence the classes in $V_\infty^{N_\infty}$ are G_∞-fixed *viz* generic Hodge classes.

Fix $\ell \in |S_\mathbb{Q}|$ and consider the $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell := R^{2d}f_*^{(2)}\mathbb{Q}_\ell(d)$. For $x \in X$, let $\overline{G}_\ell \subset \mathrm{GL}_{V_\ell}$ denote the Zariski-closure of the image $\overline{\Pi}_\ell$ of $\pi_1(X_{\bar{k}})$ acting on $V_\ell := \mathcal{V}_{\ell,\bar{x}}$. Modulo the singular-étale comparison isomorphism $V_{\infty\mathbb{Q}_\ell} \tilde{\to} V_\ell$, the group $\overline{G}_\ell$ identifies with $\overline{G}_{\infty\mathbb{Q}_\ell}$ so that $N_\ell := \mathrm{Nor}_{\mathrm{GL}_{V_\ell}}(\overline{G}_\ell^\circ) = N_{\infty\mathbb{Q}_\ell} \subset \mathrm{GL}_{V_\ell}$. Let $G_\ell \subset \mathrm{GL}_{V_\ell}$ denote the Zariski-closure of the image Π_ℓ of $\pi_1(X)$ acting on V_ℓ. As $\pi_1(X_{\bar{k}})$ is a closed normal subgroup of $\pi_1(X)$, the group $\overline{G}_\ell$ is a normal subgroup of $G_\ell \subset \mathrm{GL}_{V_\ell}$ hence $G_\ell \subset N_\ell$ and the classes in $(V_\infty^{N_\infty})_{\mathbb{Q}_\ell} \simeq V_\ell^{N_\ell} \subset V_\ell$ are G_ℓ°-fixed *viz* generic Tate classes.

Eventually, let $G_{\mathrm{mot}} \subset \mathrm{GL}_{V_\infty}$ denote André's generic motivated motivic Galois group [An96, §5.2]; this is a reductive group such that $\overline{G}_\infty^\circ \subset G_\infty \subset G_{\mathrm{mot}}^\circ$ with $\overline{G}_\infty^\circ$ normal in G_{mot} [An96, §5.3] and, modulo the singular-étale comparison isomorphism $V_{\infty\mathbb{Q}_\ell} \tilde{\to} V_\ell$, $\overline{G}_\ell^\circ \subset G_\ell^\circ \subset G_{\mathrm{mot}\mathbb{Q}_\ell}^\circ$. To sum it up, one has $G_\infty \subset G_{\mathrm{mot}} \subset N_\infty$ hence, for a class in V_∞,

$$N_\infty\text{-fixed} \Rightarrow \text{generic motivated} \Rightarrow \text{generic Hodge};$$

and, modulo the singular-étale comparison isomorphism $V_{\infty\mathbb{Q}_\ell} \tilde{\to} V_\ell$, one has $G_\ell^\circ \subset G_{\mathrm{mot}\mathbb{Q}_\ell} \subset N_{\infty\mathbb{Q}_\ell}$ hence, for a class in V_ℓ,

$$N_{\infty\mathbb{Q}_\ell}\text{-fixed} \Rightarrow \mathbb{Q}_\ell\text{-linear combination of generic motivated} \Rightarrow \text{generic Tate}.$$

1.3. **The main conjecture.** Returning to the case where $\mathcal{V}_\infty := R^i f^{\mathrm{an}}_{\infty *}\mathbb{Q}(j)$, $\mathcal{V}_\ell := R^i f_*\mathbb{Q}_\ell(j)$, $\ell \in |S_\mathbb{Q}|$ for some smooth projective morphism $f : Y \to X$ and integers $i \geq 0, j$, every finite-dimensional N_∞-subrepresentation $W_\infty \subset T(V_\infty)$ is generically motivated (as $G_{\mathrm{mot}} \subset N_\infty$), and gives rise to a polarizable $\mathbb{Q}$-VHS $\mathcal{W}_\infty$ on X^{an}_∞ (as $G_\infty \subset N_\infty$) and to a family of $\mathbb{Q}_\ell$-local systems $\mathcal{W} := (\mathcal{W}_\ell)_{\ell\in|S_\mathbb{Q}|}$ on X (as $G_\ell \subset N_{\infty\mathbb{Q}_\ell}$).

The following is implicitly formulated in [An96, §0.7] in the more general setting of motivated local systems (not only those arising from N_∞-subrepresentations); it is a consequence of the Hodge conjecture (Corollary 12).

Conjecture 1. *The family of $\mathbb{Q}_\ell$-local systems $\mathcal{W}$ is almost $\mathbb{Q}$-compatible.*

Example 2. The polarizable $\mathbb{Q}$-VHS $\overline{\mathcal{L}}_\infty \subset \mathcal{E}_\infty := \mathcal{V}_\infty \otimes \mathcal{V}_\infty^\vee$ corresponding to the N_∞-subrepresentation $\overline{L}_\infty := \mathrm{Lie}(\overline{G}_\infty) \subset E_\infty := \mathcal{E}_{\infty,x}$ is motivated with $\mathbb{Q}_\ell$-incarnation the $\mathbb{Q}_\ell$-local system $\overline{\mathcal{L}}_\ell \subset \mathcal{E}_\ell := \mathcal{V}_\ell \otimes \mathcal{V}_\ell^\vee$ corresponding to the N_ℓ-subrepresentation $\overline{L}_\ell := \mathrm{Lie}(\overline{G}_\ell) \subset E_\ell := \mathcal{E}_{\ell,x}$. This is true, more generally, replacing $\overline{L}_\infty$ by any of its isotypical Lie ideal.

1.4. **A specific generic motivated cycle.** We retain the notations of Subsection 1.2.

1.4.1. *Construction.* Define $\overline{H}_\infty := \cap_{U\lhd_{f.i.}\overline{\Pi}_\infty}\mathbb{Q}[U] \subset E_\infty$, where the intersection is over all normal finite index subgroups U of $\overline{\Pi}_\infty$ and $\overline{H}_\ell := \cap_{U\lhd_{op}\overline{\Pi}_\ell}\mathbb{Q}_\ell[U] \subset E_\ell$, where the intersection is over all normal open subgroups U of $\overline{\Pi}_\ell$ respectively. By construction

$$Z_{\overline{H}_\infty}(E_\infty) = E_\infty^{\overline{G}^\circ_\infty},\ \ Z_{\overline{H}_\ell}(E_\ell) = E_\ell^{\overline{G}^\circ_\ell}.$$

Lemma 3. *The canonical map $\overline{H}_\infty \hookrightarrow E_\ell$ induced by $\overline{\Pi}_\infty \to \overline{\Pi}_\ell \hookrightarrow E_\ell$ yields an isomorphism of $\mathbb{Q}_\ell$-algebras*

$$\overline{H}_\infty \otimes_\mathbb{Q} \mathbb{Q}_\ell \tilde{\to} \overline{H}_\ell.$$

Proof. By left exactness of $-\otimes_\mathbb{Q}\mathbb{Q}_\ell$, the morphism of $\mathbb{Q}_\ell$-algebras $\overline{H}_\infty\otimes_\mathbb{Q}\mathbb{Q}_\ell \to E_\ell$ is injective; we are to prove that its image is $\overline{H}_\ell$. For a finite index subgroup $U \subset \overline{\Pi}_\infty$, let $U_\ell \subset \overline{\Pi}_\ell$ denote the closure of its image in $\overline{\Pi}_\ell$. As $\mathbb{Q}$ is dense in $\mathbb{Q}_\ell$ and E_ℓ is finite-dimensional, the isomorphism $E_\infty\otimes_\mathbb{Q}\mathbb{Q}_\ell\tilde{\to}E_\ell$ restricts to an isomorphism of $\mathbb{Q}_\ell$-algebras $\mathbb{Q}[U]\otimes_\mathbb{Q}\mathbb{Q}_\ell\tilde{\to}\mathbb{Q}_\ell[U_\ell]$. On the other hand, for every normal, open subgroup $U \subset \overline{\Pi}_\ell$, the inverse image $U_\infty \subset \overline{\Pi}_\infty$ of U *via* $\overline{\Pi}_\infty \to \overline{\Pi}_\ell$ is a normal, finite index subgroup and, as $\overline{\Pi}_\infty$ is dense in $\overline{\Pi}_\ell$, one gets $U_{\infty,\ell} = U$. Thus, using that the (filtered) intersections of subspaces of a finite-dimensional vector space commute with base change[2], one eventually gets:

$$\overline{H}_\infty \otimes_\mathbb{Q} \mathbb{Q}_\ell = (\cap_{U\lhd_{f.i.}\overline{\Pi}_\infty}\mathbb{Q}[U]) \otimes_\mathbb{Q} \mathbb{Q}_\ell = \cap_{U\lhd_{f.i.}\overline{\Pi}_\infty}(\mathbb{Q}[U] \otimes_\mathbb{Q} \mathbb{Q}_\ell) = \cap_{U\lhd_{f.i.}\overline{\Pi}_\infty}\mathbb{Q}_\ell[U_\ell] = \cap_{U\lhd_{op}\overline{\Pi}_\ell}\mathbb{Q}_\ell[U] = \overline{H}_\ell$$

as desired. □

Let $\overline{\mathcal{H}}_\infty \subset \mathcal{E}_\infty$ denote the $\mathbb{Q}$-local subsystem corresponding to the $\pi_1(X^{\mathrm{an}}_\infty)$-representation $\overline{H}_\infty \subset E_\infty$.

Lemma 4. *Then $\overline{\mathcal{H}}_\infty \subset \mathcal{E}_\infty$ is motivated.*

Proof. It is enough to show that $\overline{H}_\infty$ is N_∞-stable. As already observed in the proof of Lemma 3, $\overline{H}_\infty = \mathbb{Q}[U]$ for $U \subset \overline{\Pi}_\infty$ a sufficiently small normal subgroup of finite index, which we may assume to be contained in $\overline{G}^\circ_\infty(\mathbb{Q})$. Let $n := \dim_\mathbb{Q}(V_\infty)$, $N := \dim_\mathbb{Q}(\overline{H}_\infty)$. Let $\mathbb{A}^N_\mathbb{Q} \simeq \underline{\overline{H}}_\infty \subset \underline{E}_\infty \simeq \mathbb{A}^{n^2}_\mathbb{Q}$ denote the varieties over $\mathbb{Q}$ underlying $\overline{H}_\infty \subset E_\infty$ and let GL_{V_∞} acts by conjugacy on $\underline{E}_\infty$. It is enough to show that the subgroup $N_\infty \subset \mathrm{GL}_{V_\infty}$ stabilizes $\underline{\overline{H}}_\infty \subset \underline{E}_\infty$. Letting GL_{V_∞} acts trivially on $\mathbb{A}^N_\mathbb{Q}$ and diagonally on $(\underline{E}_\infty)^N$, the morphism of $\mathbb{Q}$-varieties

$$\begin{array}{cccc} p: & \mathbb{A}^N_\mathbb{Q} \times_\mathbb{Q} (\underline{E}_\infty)^N & \to & \underline{E}_\infty \\ & (\underline{a}, \underline{f}) & \mapsto & \sum_{1\leq i\leq N} a_i f_i. \end{array}$$

is GL_{V_∞}-equivariant. As, by definition, $N_\infty \subset \mathrm{GL}_{V_\infty}$ stabilizes the subvariety $\mathbb{A}^N_\mathbb{Q}\times_\mathbb{Q}(\overline{G}^\circ_\infty)^N \subset \mathbb{A}^N_\mathbb{Q}\times_\mathbb{Q}(\underline{E}_\infty)^N$, it is enough to show that $p : \mathbb{A}^N_\mathbb{Q} \times_\mathbb{Q} (\underline{E}_\infty)^N \to \underline{E}_\infty$ has image contained in $\underline{\overline{H}}_\infty$ and that the resulting morphism $p : \mathbb{A}^N_\mathbb{Q} \times_\mathbb{Q} (\overline{G}^\circ_\infty)^N \to \underline{\overline{H}}_\infty$ is dominant. That $p : \mathbb{A}^N_\mathbb{Q} \times_\mathbb{Q} (\underline{E}_\infty)^N \to \underline{E}_\infty$ has image contained in $\underline{\overline{H}}_\infty$ follows from the facts that $\mathbb{Q}^N \times U^N \subset p^{-1}(\overline{H}_\infty)$ and $\mathbb{Q}^N \times U^N \subset \mathbb{A}^N_\mathbb{Q} \times_\mathbb{Q} (\overline{G}^\circ_\infty)^N$ is Zariski-dense. That the resulting morphism $p : \mathbb{A}^N_\mathbb{Q} \times_\mathbb{Q} (\overline{G}^\circ_\infty)^N \to \underline{\overline{H}}_\infty$ is dominant follows from the fact its image contains $\underline{\overline{H}}_\infty(\mathbb{Q}) = \overline{H}_\infty = p(\mathbb{Q}^N \times U^N)$ (as the N-dimensional $\mathbb{Q}$-vector space $\overline{H}_\infty = \mathbb{Q}[U]$ admits a basis contained in U). □

[2] Indeed, by duality of finite-dimensional vector spaces, this is reduced to the commutativity of (filtered) unions (or, more generally, colimits) and base change, which is well known.

Lemma 5. *One has $Z(\overline{H}_\infty)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \simeq Z(\overline{H}_\ell)$; in particular $d(\mathcal{V}) := \dim_{\mathbb{Q}_\ell}(Z(\overline{H}_\ell)) = \dim_{\mathbb{Q}}(Z(\overline{H}_\infty))$ is independent of ℓ, $\infty : k \hookrightarrow \mathbb{C}$.*

Proof. This follows from $Z(\overline{H}_\infty) = \overline{H}_\infty \cap Z_{\overline{H}_\infty}(E_\infty) = \overline{H}_\infty \cap (E_\infty)^{\overline{G}^\circ_\infty}$, $Z(\overline{H}_\ell) = \overline{H}_\ell \cap Z_{\overline{H}_\ell}(E_\ell) = \overline{H}_\ell \cap (E_\ell)^{\overline{G}^\circ_\ell}$, Lemma 3, $((E_\infty)^{\overline{G}^\circ_\infty})_{\mathbb{Q}_\ell} = (E_{\infty\mathbb{Q}_\ell})^{\overline{G}^\circ_{\infty\mathbb{Q}_\ell}} = (E_\ell)^{\overline{G}^\circ_\ell}$ and the exactness of $-\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. □

1.4.2. *Main result.* For every $\ell \in |S_{\mathbb{Q}}|$, let $\overline{\mathcal{H}}_\ell \subset \mathcal{E}_\ell := \mathcal{V}_\ell \otimes \mathcal{V}_\ell^\vee$ denote the $\mathbb{Q}_\ell$-local system corresponding to the $\pi_1(X)$-representation $\overline{H}_\ell \subset E_\ell$. From Lemma 4 and Conjecture 1, the family $\overline{\mathcal{H}} := (\overline{\mathcal{H}}_\ell)_{\ell \in |S_{\mathbb{Q}}|}$ should be almost $\mathbb{Q}$-compatible. The main result of this paper is the proof of this fact modulo a mild condition on the center of $\overline{H}_\ell$.

Theorem 6. *Assume $d(\mathcal{V}) = 1$. Then Conjecture 1 holds for $\overline{\mathcal{H}}$ namely, $\overline{\mathcal{H}}$ is almost $\mathbb{Q}$-compatible.*

Remark 7. We are not able to prove that $\overline{\mathcal{L}} = (\overline{\mathcal{L}}_\ell)_{\ell \in |S_{\mathbb{Q}}|}$ (Example 2) is almost $\mathbb{Q}$-compatible though $\overline{\mathcal{L}}$ and $\overline{\mathcal{H}}$ are related. More precisely, using the ℓ-adic exponential / logarithm, it is easy to prove that $\overline{H}_\ell$ is the largest subalgebra of E_ℓ containing $\overline{L}_\ell$.

Our initial motivation for Theorem 6 arises from the following consequence of the unramified Fontaine-Mazur conjecture and which, in the case of motivic $\mathbb{Q}_\ell$-local systems, is also predicted by the Mumford-Tate conjecture (see [CT25] for details). It roughly predicts that, for a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X, the arithmetic and geometric parts of the corresponding representation of $\pi_1(X)$ are far from commuting, unless $\mathcal{V}_\ell$ is geometrically isotrivial. More precisely, for a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X and for every $x \in |X|$, let $\Pi_{\ell,x} \subset \Pi_\ell$ denote the image of $\pi_1(x)$ acting on $V_\ell := \mathcal{V}_{\ell,\bar{x}}$ through $\pi_1(x) \to \pi_1(X)$ and $G_{\ell,x} \subset G_\ell$ its Zariski closure.

Conjecture 8. *Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X. Assume there exists $x \in |X|$ such that $G^\circ_{\ell,x} \subset Z_{G_\ell}(\overline{G}^\circ_\ell)$. Then $\overline{G}^\circ_\ell = 1$.*

By considering the $\mathbb{Q}_\ell$-local system $\overline{\mathcal{H}}_\ell$ attached to $\mathcal{V}_\ell$, Conjecture 8 for $\mathcal{V}_\ell$ is reduced to Conjecture 9 below for $\overline{\mathcal{H}}_\ell$.

Conjecture 9. *Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X. Assume there exists $x \in |X|$ such that $G^\circ_{\ell,x} = 1$. Then $G^\circ_\ell = 1$.*

In [CT25], we prove Conjecture 9 when $\mathcal{V}_\ell$ is part of a $\mathbb{Q}$-compatible family $\mathcal{V}$ of $\mathbb{Q}_\ell$-local systems on X. In particular, Theorem 6 implies Conjecture 8 for motivic $\mathbb{Q}_\ell$-local system of the form $\mathcal{V}_\ell = R^i f_* \mathbb{Q}_\ell(j)$ under the assumption $d(\mathcal{V}) = 1$.

The paper is organized as follows. In Section 2, we review some basic facts from the specialization theory of étale and crystalline cohomologies in the good reduction setting. In particular we give there the detailed argument showing that Conjecture 1 follows from the Hodge conjecture - Corollary 12 (which is certainly well-known to experts but for which we could not find a suitable reference) and prove that global sections are compatible with specialization (Lemma 14 and Lemma 15). In Section 3, we gather the ingredients from linear algebra - in particular the key lemma 17 - involved in the proof of Theorem 6. The proof of Theorem 6 is carried out in Section 4.

2. Review of specialization in the good reduction setting

Let $v \in |S_k|$ with residue characteristic p. For a morphism $\mathcal{X} \to S := \mathrm{spec}(\mathcal{O}_{k_v})$, we use the notation in the following Cartesian diagram for the generic and special fibers respectively.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_v \arrow[l] \arrow[d] \\ \mathrm{spec}(k_v) \arrow[r] & S & \mathrm{spec}(\kappa_v) \arrow[l] \end{tikzcd}$$

2.1. **Specialization and cycle class map.**

2.1.1. Let $\mathcal{Y} \to S$ be a smooth, proper, geometrically connected morphism of relative dimension d. Fix an integer $r \geq 0$ and Let $sp_v : \mathrm{CH}^r(Y) \to \mathrm{CH}^r(\mathcal{Y}_v)$ denote the specialization map for codimension r $\mathbb{Q}$-algebraic cycles modulo rational equivalence ([Fu75, 4.4], [Fu84, 20.3]). For every prime ℓ and integer $r \geq 0$, let also $c_\ell : \mathrm{CH}^r(Y) \to \mathrm{H}^{2r}(Y_{\bar{k}_v}, \mathbb{Q}_\ell)(r)$ and $c_{v,\ell} : \mathrm{CH}^r(\mathcal{Y}_v) \to \mathrm{H}^{2r}(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell)(r)$, $\ell \neq p$, $c_{v,\mathrm{cris}} : \mathrm{CH}^r(\mathcal{Y}_v) \to \mathrm{H}^{2r}_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0})(r)$ denote the cycle class maps in étale $\mathbb{Q}_\ell$-cohomology and crystalline cohomology respectively. One can compare the $\mathbb{Q}_\ell$-cohomology groups of Y with:

(2.1.1) if $\ell \neq p$: the $\mathbb{Q}_\ell$-cohomology groups of $\mathcal{Y}_v$. More precisely, for every integers $i \geq 0$, j, smooth proper base-change yields a functorial[3] equivariant specialization diagram

$$\begin{tikzcd} \pi_1(k_v) \arrow[r, two heads] \arrow[dr] & \pi_1(S) \arrow[r, "\simeq"] \arrow[d] & \pi_1(\kappa_v) \arrow[d] \\ & \mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_\ell)(j) \arrow[r, "\simeq"] & \mathrm{H}^i(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell)(j), \end{tikzcd}$$

In particular, for $i = 2r$, $j = r$, the following diagram commutes

$$\begin{tikzcd} \mathrm{H}^{2r}(Y_{\bar{k}}, \mathbb{Q}_\ell)(r) \arrow[r, "\simeq"] & \mathrm{H}^{2r}(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell)(r) \\ \mathrm{CH}^r(Y) \arrow[u, "c_\ell"] \arrow[r, "sp_v"] & \mathrm{CH}^r(\mathcal{Y}_v) \arrow[u, "c_{v,\ell}"'] \end{tikzcd}$$

and is compatible with the intersection product on Chow groups and the cup-product on cohomology groups.

(2.1.2) if $\ell = p$, the rational crystalline cohomology group of $\mathcal{Y}_v$. More precisely, for every integers $i \geq 0$, j the p-adic étale - crystalline comparison theorem yields a functorial isomorphism of filtered ϕ-module $D_{\mathrm{cris}}(\mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_p(j))) \tilde{\to} \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0})(j)$. In particular, for $i = 2r$, $j = r$, the following diagram commutes

$$\begin{tikzcd} D_{\mathrm{cris}}(\mathrm{H}^{2r}(Y_{\bar{x}}, \mathbb{Q}_p)(r)) \arrow[r, "\simeq"] & \mathrm{H}^{2r}_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0})(r) \\ \mathrm{CH}^r(Y) \arrow[u, "c_p"] \arrow[r, "sp_v"] & \mathrm{CH}^r(\mathcal{Y}_v), \arrow[u, "c_{v,\mathrm{cris}}"'] \end{tikzcd}$$

and is compatible with the intersection product on Chow groups and the cup-product on cohomology groups.

Write $p_i : Y^2 \to Y$ for the ith projection, $i = 1, 2$. Fix an algebraic cycle $\epsilon \in CH^d(Y^2)$ such that the induced morphism

$$e_\ell : \mathrm{H}(Y_{\bar{k}_v}, \mathbb{Q}_\ell) \to \mathrm{H}(Y_{\bar{k}_v}, \mathbb{Q}_\ell) \ \ \alpha \mapsto p_{2*}(p_1^*\alpha \cup c_\ell(\epsilon)),$$

is idempotent (note that this property is independent of $\ell \in |S_{\mathbb{Q}}|$).

Lemma 10. *The family of $\pi_1(k_v)$-representations $W = (W_\ell := \mathrm{im}(e_\ell))_{\ell \in |S_{\mathbb{Q}}|}$ is $\mathbb{Q}$-compatible.*

Proof. To simplify notation, set $V_\ell := \mathrm{H}(Y_{\bar{k}_v}, \mathbb{Q}_\ell)$. If $\ell \neq p$, the specialization properties recalled in (2.1.1) show that $e_\ell : V_\ell \to V_\ell$ identifies with the idempotent

$$\text{(2.1.1')} \ \ e_{v,\ell} : \mathrm{H}(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell) \to \mathrm{H}(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell), \ \ \alpha \mapsto p_{2*}(p_1^*\alpha \cup c_{v,\ell}(sp_v(\epsilon))).$$

If $\ell = p$, note that $\mathrm{im}(e_p)$ is crystalline (as a subrepresentation of the crystalline representation V_p) hence, by exactness of $D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k_v)) \to \mathrm{FM}_{k_v}(\phi)$, the image of $D_{\mathrm{cris}}(e_p) : D_{\mathrm{cris}}(V_p) \to D_{\mathrm{cris}}(V_p)$ coincides with $D_{\mathrm{cris}}(\mathrm{im}(e_p))$ in $\mathrm{FM}_{k_v}(\phi)$ while, on the other hand, the specialization properties recalled in (2.1.2) show that $D_{\mathrm{cris}}(e_p) : D_{\mathrm{cris}}(V_p) \to D_{\mathrm{cris}}(V_p)$ identifies with the idempotent

$$\text{(2.1.2')} \ \ e_{v,p} : \mathrm{H}_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0}) \to \mathrm{H}_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0}), \ \ \alpha \mapsto p_{2*}(p_1^*\alpha \cup c_{v,\mathrm{cris}}(\epsilon_v)).$$

Let $\Phi^n_{x_v} \in CH^d(\mathcal{Y}^2_{\bar{v}})$ denote the graph of the nth power of the geometric Frobenius $Fr^n_v : \mathcal{Y}_{\bar{v}} \to \mathcal{Y}_{\bar{v}}$. Using (2.1.1'), (2.1.2'), one can compute the traces of the powers $\varphi^n_{W_\ell}$, $n \geq 1$ of the geometric Frobenius φ_{W_ℓ} acting on $\mathrm{im}(e_\ell)$ using the Lefschetz trace formula for $\mathbb{Q}_\ell$-cohomology (if $\ell \neq p$) and crystalline cohomology (if $\ell = p$) to get,

$$\sum_{0 \leq i \leq 2d} (-1)^i tr(\varphi^n_{W_\ell} \circ e_{v,\ell} | \mathrm{H}^i(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell)) = \Phi^n_{x_v} \cup {}^t\epsilon_v = \sum_{0 \leq i \leq 2d} (-1)^i tr(\varphi^n_{W_p} \circ e_{v,p} | \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0}), \ \ \ell \neq p$$

[3] *viz* compatible with pullbacks, pushforwards, Poincaré duality, Künneth decomposition, cup products and cycle class maps.

whence

$$\prod_{0\leq i\leq 2d}\det(1-\varphi_{W_\ell}\circ e_{v,\ell}T|\mathrm{H}^i(\mathcal{Y}_{\overline{v}},\mathbb{Q}_\ell))^{(-1)^{i+1}}=\prod_{0\leq i\leq 2d}\det(1-\varphi_{W_p}\circ e_{v,p}T|\mathrm{H}^i(\mathcal{Y}_v/k_{v,0}))^{(-1)^{i+1}},\ \ \ell\neq p$$

and, by purity, [D74], [KM74]

$$\det(1-\varphi_{W_\ell}\circ e_{v,\ell}T|\mathrm{H}^i(\mathcal{Y}_{\overline{v}},\mathbb{Q}_\ell)=\det(1-\varphi_{W_p}\circ e_{v,p}T|\mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v/k_{v,0})),\ \ i\geq 0,\ \ell\neq p.$$

□

2.1.2. Recall the statement of the Hodge conjecture.

Conjecture 11. (Hodge [Ho52]) *Let Y be a smooth, projective variety over $\mathbb{C}$. For every integer $i\geq 0$, the image of the cycle class map $c_\infty:\mathrm{CH}^i(Y)\to\mathrm{H}^{2i}(Y^{\mathrm{an}},\mathbb{Q}(i))$ coincides with the sub $\mathbb{Q}$-vector space $\mathrm{Hodge}^i(Y)\subset\mathrm{H}^{2i}(Y^{\mathrm{an}},\mathbb{Q}(i))$ of Hodge classes.*

Corollary 12. *Conjecture 11 implies Conjecture 1.*

Remark 13. Actually, one only needs Conjecture 11 in middle degree for a smooth compactification of a sufficiently large fibered power of $f:Y\to X$.

Proof. We retain the notation of Subsection 1.2. By Künneth formula and Poincaré duality, and up to replacing $f:Y\to X$ by a suitable fibered power and increasing i, one may assume $\mathcal{W}_\infty\subset\mathcal{V}_\infty:=R^if^{\mathrm{an}}_\infty\mathbb{Q}(j)$ for some smooth projective morphism $f:Y\to X$ and, as $\mathbb{Q}$-compatibility is invariant under Tate twist, that $j=0$. As almost $\mathbb{Q}$-compatibility is also insensitive under base change by finite covers, one may assume $\overline{G}_\infty$ is connected; this ensures $\overline{G}_\infty\subset G_\infty$. As $G_\infty\subset N_\infty$, $W_\infty:=\mathcal{W}_{\infty,x}$ is a G_∞-subrepresentation of $V_\infty:=\mathcal{V}_{\infty,x}$ or, equivalently, $\mathcal{W}_\infty$ is a sub $\mathbb{Q}$-VHS of the polarizable $\mathbb{Q}$-VHS $\mathcal{V}_\infty$. Hence, one has a (canonical if one chooses a polarization) decomposition $\mathcal{V}_\infty=\mathcal{W}_\infty\oplus(\mathcal{W}_\infty)^\perp$ of $\mathbb{Q}$-VHS whence an idempotent morphism $e_\infty:\mathcal{V}_\infty\twoheadrightarrow\mathcal{W}_\infty\hookrightarrow\mathcal{V}_\infty$ of $\mathbb{Q}$-VHS such that $\mathcal{W}_\infty=\mathrm{im}(e_\infty)$. For every $x_\infty\in X^{\mathrm{an}}_\infty$ above $x\in|X|$, let $G_{x,\infty}\subset G_\infty$ denote the Mumford-Tate group of the polarizable $\mathbb{Q}$-Hodge structure $x^*\mathcal{V}_\infty$. The idempotent $e_\infty:\mathcal{V}_\infty\to\mathcal{V}_\infty$ corresponds to a G_∞-equivariant idempotent $e_\infty:V_\infty\to V_\infty$ hence to a Hodge class $e_{x,\infty}\in\mathrm{H}^{2d}((Y^{\mathrm{an}}_{x,\infty})^2,\mathbb{Q}(d))$ (as $e_\infty:V_\infty\to V_\infty$ is $G_{x,\infty}$-equivariant) lying in $\mathrm{H}^0(X^{\mathrm{an}}_\infty,R^{2d}f^{\mathrm{an}}_{\infty*}\mathbb{Q}(d))$ (as $e_\infty:V_\infty\to V_\infty$ is $\overline{G}_\infty$-equivariant). Fix a smooth compactification $Y\hookrightarrow Y^{\mathrm{cpt}}$ of Y and consider the canonical commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{2d}((Y^{\mathrm{cpt,an}}_\infty)^2,\mathbb{Q})(d) \arrow[r, "|_{(Y^{\mathrm{an}}_\infty)^2}"] & \mathrm{H}^{2d}((Y^{\mathrm{an}}_\infty)^2,\mathbb{Q})(d) \arrow[r, "\epsilon"] & \mathrm{H}^0(X^{\mathrm{an}}_\infty,R^{2d}f^{\mathrm{an}}_{\infty*}\mathbb{Q}(d)) \arrow[r, hook, "(-)_x"] & \mathrm{H}^{2d}((Y^{\mathrm{an}}_{x,\infty})^2,\mathbb{Q})(d) \\
\mathrm{CH}^d((Y^{\mathrm{cpt}}_\infty)^2) \arrow[u, "c"] \arrow[rrr, "|_{(Y_{x,\infty})^2}"'] & & & \mathrm{CH}^d((Y_{x,\infty})^2) \arrow[u, "c_{x_\infty}"']
\end{tikzcd},$$

where $\epsilon:\mathrm{H}^{2d}(Y^{\mathrm{an}}_\infty,\mathbb{Q}(d))\to\mathrm{H}^0(X^{\mathrm{an}}_\infty,R^{2d}f^{\mathrm{an}}_{\infty*}\mathbb{Q}(d))$ is the edge morphism

$$\mathrm{H}^{2d}(Y^{\mathrm{an}}_\infty,\mathbb{Q}(d))\twoheadrightarrow E^{0,2d}_\infty\hookrightarrow E^{0,2d}_2=\mathrm{H}^0(X^{\mathrm{an}}_\infty,R^{2d}f^{\mathrm{an}}_{\infty*}\mathbb{Q}(d))$$

from the Leray spectral sequence for $f^{\mathrm{an}}_\infty:Y^{\mathrm{an}}_\infty\to X^{\mathrm{an}}_\infty$. From the theorem of the fixed part [D71, Thm. (4.1.1)], the morphism $\epsilon\circ-|_{(Y^{\mathrm{an}}_\infty)^2}:\mathrm{H}^{2d}((Y^{\mathrm{cpt,an}}_\infty)^2,\mathbb{Q})(d)\to\mathrm{H}^0(X^{\mathrm{an}}_\infty,R^{2d}f^{\mathrm{an}}_{\infty*}\mathbb{Q}(d))$ is surjective so that $e_\infty\in\mathrm{H}^0(X^{\mathrm{an}}_\infty,R^{2d}f^{\mathrm{an}}_{\infty*}\mathbb{Q}(d))$ lifts to a Hodge class $e^{\mathrm{cpt}}_\infty\in\mathrm{H}^{2d}((Y^{\mathrm{cpt,an}}_\infty)^2,\mathbb{Q})(d)$ hence, by Conjecture 11 to an algebraic class $\widetilde{e}^{\mathrm{cpt}}_\infty\in\mathrm{CH}^d((Y^{\mathrm{cpt}}_\infty)^2)$. As every algebraic cycle on $(Y^{\mathrm{cpt}}_\infty)^2$ is actually defined over a finitely generated extension K of k and as cycle class maps are compatible with specialization ([Fu75, 4.4], [Fu84, 20.3]), the images of $c_\infty:CH^d((Y^{\mathrm{cpt}}_\infty)^2)\to\mathrm{H}^{2d}((Y^{\mathrm{cpt,an}}_\infty)^2,\mathbb{Q})(d)$ and of $CH^d((Y^{\mathrm{cpt}}_{\bar{k}})^2)\to CH^d((Y^{\mathrm{cpt}}_\infty)^2)\to\mathrm{H}^{2d}((Y^{\mathrm{cpt,an}}_\infty)^2,\mathbb{Q})(d)$ coincide so that one may assume $\widetilde{e}^{\mathrm{cpt}}_\infty=\widetilde{e}^{\mathrm{cpt}}\in CH^d((Y^{\mathrm{cpt}}_{\bar{k}})^2)$. Up to replacing k by a finite field extension, one may even assume $\widetilde{e}^{\mathrm{cpt}}\in CH^d((Y^{\mathrm{cpt}})^2)$. On the other hand, writing $\widetilde{e}_x:=\widetilde{e}^{\mathrm{cpt}}|_{(Y_x)^2}$, it follows from the definition of $\mathcal{W}_\ell$ that W_ℓ identifies with the image of the idempotent morphism

$$e_{x,\ell}:\mathrm{H}^i(Y_{\overline{x}},\mathbb{Q}_\ell)\to\mathrm{H}^i(Y_{\overline{x}},\mathbb{Q}_\ell),\ \alpha\mapsto p_{2*}(p_1^*(\alpha)\cup c_{x,\ell}(\widetilde{e}_x)).$$

The assertion thus follows from Lemma 10. □

2.2. **Specialization and global sections.**

2.2.1. Let $\mathcal{X}\to S$ be a smooth, geometrically connected morphism. Fix $\ell\in|S_\mathbb{Q}|$ and a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X.

2.2.1.1. Assume $p \neq \ell$ and $\mathcal{V}_\ell$ extends to a $\mathbb{Q}_\ell$-local system on $\mathcal{X}$; set $\mathcal{V}_{\ell,v} := \mathcal{V}_\ell|_{\mathcal{X}_v}$. Assume also that there exists a smooth, proper geometrically connected morphism $\mathcal{X}^{\mathrm{cpt}} \to S$ and an open S-immersion $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}}$ such that $\mathcal{D} := \mathcal{X}^{\mathrm{cpt}} \setminus \mathcal{X} \to S$ is a relative normal crossing divisor.

Lemma 14. *One has a canonical $\pi_1(S)$-equivariant isomorphism $\mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_\ell) \tilde{\to} \mathrm{H}^0(\mathcal{X}_{\bar{v}}, \mathcal{V}_{\ell,v})$.*

Proof. As $\mathcal{X}$ is regular and the generic points of $\mathcal{D}$ have characteristic 0, the action of $\pi_1(\mathcal{X}_v)$ on V_ℓ factors through the tame fundamental group $\pi_1(\mathcal{X}) \tilde{\to} \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}; \mathcal{D}) \tilde{\leftarrow} \pi_1^{\mathrm{t}}(\mathcal{X}_v^{\mathrm{cpt}}; \mathcal{D}_v)$. The assertion then follows from the surjectivity of the tame specialization morphism $sp : \pi_1^{\mathrm{t}}(X_{\bar{k}_v}^{\mathrm{cpt}}, D_{\bar{k}_v}) \twoheadrightarrow \pi_1^{\mathrm{t}}(\mathcal{X}_{\bar{v}}^{\mathrm{cpt}}, \mathcal{D}_{\bar{v}})$ in the tame specialization diagram [G71, XIII]

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\mathrm{t}}(X_{\bar{k}_v}^{\mathrm{cpt}}, D_{\bar{k}_v}) \arrow[r] \arrow[d, "\mathrm{sp}"] & \pi_1^{\mathrm{t}}(X^{\mathrm{cpt}}, D) \arrow[r] \arrow[d] & \pi_1(k_v) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1^{\mathrm{t}}(\mathcal{X}_{\bar{v}}^{\mathrm{cpt}}, \mathcal{D}_{\bar{v}}) \arrow[r] \arrow[d, equal] & \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}, \mathcal{D}) \arrow[r] & \pi_1(S) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1^{\mathrm{t}}(\mathcal{X}_{\bar{v}}^{\mathrm{cpt}}, \mathcal{D}_{\bar{v}}) \arrow[r] & \pi_1^{\mathrm{t}}(\mathcal{X}_v^{\mathrm{cpt}}, \mathcal{D}_v) \arrow[r] \arrow[u, "\simeq"] & \pi_1(v) \arrow[r] \arrow[u, "\simeq"] & 1
\end{tikzcd}$$

□

2.2.1.2. Assume $p = \ell$ and $\mathcal{V}_p = \otimes_{1 \leq a \leq r} R^{i_a} \mathfrak{f}_{a*}\mathbb{Q}_p$, where $\mathfrak{f}_a : Y_a \to X$ is the generic fiber of some smooth projective morphism $f_a : \mathcal{Y}_a \to \mathcal{X}$, $a = 1, \dots, r$ and integers $i_1, \dots, i_r \geq 0$. Assume furthermore that $\mathcal{X} \to S$ admits a section $x : S \to \mathcal{X}$ and that for every $a = 1, \dots, r$ there exist a smooth, projective geometrically connected morphism $\mathcal{Y}_a^{\mathrm{cpt}} \to S$ and an open S-immersion $\mathcal{Y}_a \hookrightarrow \mathcal{Y}_a^{\mathrm{cpt}}$. Set $\mathcal{V}_{p,v} := \otimes_{1 \leq a \leq r} R^{i_a} f_{a,v,\mathrm{cris}*}\mathcal{O}_{\mathcal{Y}_v/k_{v,0}}$.

Lemma 15. *One has a canonical isomorphism $D_{\mathrm{cris}}(\mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p)) \tilde{\to} \mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v})$ of filtered ϕ-modules over k_v.*

Proof. Assume first $r = 1$, $i_1 =: i$ and $f_1 =: f$. The restriction morphism $\mathrm{H}^i(Y_{\bar{k}}^{\mathrm{cpt}}, \mathbb{Q}_p) \overset{(0)}{\to} \mathrm{H}^i(Y_{\bar{x}}, \mathbb{Q}_p)$ factors as

$$(*) \quad \begin{tikzcd}
\mathrm{H}^i(Y_{\bar{k}_v}^{\mathrm{cpt}}, \mathbb{Q}_p) \arrow[r, "(0)"] \arrow[d, "(1)"'] & \mathrm{H}^i(Y_{\bar{x}}, \mathbb{Q}_p) = \mathcal{V}_{p,\bar{x}}, \\
\mathrm{H}^i(Y_{\bar{k}_v}, \mathbb{Q}_p) \arrow[r, "(2)"] & \mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p) \arrow[u, hook, "(3)"']
\end{tikzcd}$$

where (2) is the edge morphism $\mathrm{H}^i(Y_{\bar{k}_v}, \mathbb{Q}_p) \twoheadrightarrow E_\infty^{0,i} \hookrightarrow E_2^{0,i} = \mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p)$ from the Leray spectral sequence for $\mathfrak{f} : Y \to X$, (1) is the restriction morphism and (3) is taking the stalk at $\bar{x}$, which is injective since the functor "stalk at $\bar{x}$" is faithful (as $X_{\bar{k}_v}$ is connected). Furthermore, (3) induces an isomorphism $\mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p) \tilde{\to} \mathcal{V}_{p,\bar{x}}^{\pi_1(X_{\bar{k}_v})} \hookrightarrow \mathcal{V}_{p,\bar{x}}$. From the theorem of the fixed part [D71, Thm. (4.1.1)] and singular-étale comparison, $(2) \circ (1)$ is surjective hence, in particular, $E_\infty^{0,i} = E_2^{0,i}$ and the image of $\mathrm{H}^i(Y_{\bar{k}_v}^{\mathrm{cpt}}, \mathbb{Q}_p) \overset{(0)}{\to} \mathcal{V}_{p,\bar{x}}$ identifies $\pi_1(k_v)$-equivariantly with $\mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p)$. As $\mathcal{Y}^{\mathrm{cpt}} \to S$, $\mathcal{Y}_x \to S$ are both smooth projective morphisms, the $\pi_1(k_v)$-representations $\mathrm{H}^i(Y_{\bar{k}_v}^{\mathrm{cpt}}, \mathbb{Q}_p)$, $\mathcal{V}_{p,\bar{x}}$ are crystalline hence, by exactness of $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\pi_1(k_v)) \to \mathrm{FM}_{k_v}(\phi)$, the image of the morphism $D_{\mathrm{cris}}(\mathrm{H}^i(Y_{\bar{k}_v}^{\mathrm{cpt}}, \mathbb{Q}_p)) \overset{D_{\mathrm{cris}}((0))}{\to} D_{\mathrm{cris}}(\mathcal{V}_{p,\bar{x}})$ identifies with $D_{\mathrm{cris}}(\mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p))$ as filtered ϕ-modules over k_v. On the other hand, by compatibility of D_{cris} with pullbacks, the morphism $D_{\mathrm{cris}}(\mathrm{H}^i(Y_{\bar{k}_v}^{\mathrm{cpt}}, \mathbb{Q}_p)) \overset{D_{\mathrm{cris}}((0))}{\to} D_{\mathrm{cris}}(\mathcal{V}_{p,\bar{x}})$ identifies with the restriction morphism $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v^{\mathrm{cpt}}/k_{v,0}) \overset{(0)}{\to} \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_{v,x_v}/k_{v,0})$ in crystalline cohomology, which factors again as

$$(**) \quad \begin{tikzcd}
\mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v^{\mathrm{cpt}}/k_{v,0}) \arrow[rr, "(0)"] \arrow[dr, "(2)\circ(1)"'] & & \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_{v,x_v}/k_{v,0}), \\
& \mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v}) \arrow[ur, hook, "(3)"'] &
\end{tikzcd}$$

with $(1), (2), (3)$ defined as in $(*)$. In particular (3) is injective (again because the functor "stalk at x_v" is faithful) and $(2) \circ (1)$ is surjective by the theorem of the fixed part in crystalline cohomology [M19, Sec. 2, esp. Thm. 2.5]. This proves $D_{\mathrm{cris}}(\mathrm{H}^0(X_{\bar{k}_v}, \mathcal{V}_p))$ identifies with the image

$$\mathrm{im}(\mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v^{\mathrm{cpt}}/k_{v,0}) \stackrel{(0)}{\to} \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_{v,x_v}/k_{v,0})) \simeq \mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v})$$

as filtered ϕ-modules over k_v. For the general case, applying (**) to

$$\mathcal{Y}^{\mathrm{cpt}} := \mathcal{Y}_1^{\mathrm{cpt}} \times \cdots \times \mathcal{Y}_r^{\mathrm{cpt}} \hookleftarrow \mathcal{Y} := \mathcal{Y}_1 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{Y}_r \stackrel{f}{\to} \mathcal{X}$$

with $i = i_1 + \cdots + i_r$, and using Künneth decomposition, one gets a commutative diagram:

$$\begin{tikzcd}
\bigoplus\limits_{j_1+\cdots+j_r=i} \bigotimes\limits_{a=1}^{r} \mathrm{H}^{j_a}_{\mathrm{cris}}(\mathcal{Y}^{\mathrm{cpt}}_{a,v}/k_{v,0}) \arrow[rr, "(0)"] \arrow[dr, "(2)\circ(1)"'] & & \bigoplus\limits_{j_1+\cdots+j_r=i} \bigotimes\limits_{a=1}^{r} \mathrm{H}^{j_a}_{\mathrm{cris}}(\mathcal{Y}_{a,v,x_v}/k_{v,0}). \\
& \bigoplus\limits_{j_1+\cdots+j_r=i} \mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v}) \arrow[ur, hook, "(3)"'] &
\end{tikzcd}$$

As (0) is the restriction and (3) is taking the stalk at x_v, both (0) and (3) are compatible with the Künneth direct sum decompositions. As (2) is the edge morphism arising from the Leray spectral sequence for $f_v : \mathcal{Y}_v \to \mathcal{X}_v$ in crystalline cohomology, it is *a priori* unclear that $(2) \circ (1)$ is compatible with the Künneth direct sum decompositions but as (3) is injective, this formally follows from the compatibility of (0) and (3). The claim follows from the surjectivity of $(2) \circ (1)$ and the injectivity of (3) by taking the component $(j_1, \ldots, j_r) = (i_1, \ldots, i_r)$ of the above diagram. $\square$

Remark 16. The proof of Lemma 15 uses the existence of a section of $\mathcal{X} \to S$ but the statement of Lemma 15 remains valid without this assumption (using that, by formal smoothness, $\mathcal{X} \to S$ always admits a section after replacing S by a finite étale cover $S' \to S$ and that the morphisms $(2) \circ (1)$ in diagrams (*), (**) as well as the isomorphism $D_{\mathrm{cris}}(\mathrm{H}^i(Y^{\mathrm{cpt}}_{\bar{k}_v}, \mathbb{Q}_p)) \tilde{\to} \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}^{\mathrm{cpt}}_v/k_{v,0})$, are canonical - *i.e.* independent of the choice of a section - and defined over the base field k_v). As for our purpose we can reduce to the case where $\mathcal{X} \to S$ has a section, we do not elaborate.

3. Linear algebra

Let Q be a field of characteristic 0.

3.1. **Extracting eigenvalues.** For a finite-dimensional Q-representation V of $\mathbb{Z}$, let $\chi_V \in Q[T]$ denote the characteristic polynomial of 1 acting on V; let E_V denote the multiset of the roots of χ_V and $E_V^{\mathrm{red}} \subset \overline{Q}^{\times}$ the underlying set. For $i = 1, 2$, let $Q \hookrightarrow Q_i$ be a field extension and A_i, B_i, C_i three finite-dimensional Q_i-representations of $\mathbb{Z}$. Assume the following:

(1) One has $A_i \otimes_{Q_i} B_i \simeq C_i$ $\mathbb{Z}$-equivariantly, $i = 1, 2$;

(2) χ_{A_i}, χ_{C_i} lie in $Q[T]$, $i = 1, 2$ and $\chi_{A_1} = \chi_{A_2}$, $\chi_{C_1} = \chi_{C_2}$;

(3) By (2) one has $\Gamma := \langle E^{\mathrm{red}}_{A_1}, E^{\mathrm{red}}_{C_1} \rangle = \langle E^{\mathrm{red}}_{A_2}, E^{\mathrm{red}}_{C_2} \rangle \subset \overline{Q}^{\times}$ as subgroups of $\overline{Q}^{\times}$. Assume Γ is torsion-free.

Lemma 17. *Then χ_{B_i} also lies in $Q[T]$, $i = 1, 2$ and $\chi_{B_1} = \chi_{B_2}$.*

Proof. By assumption (2) $E_A := E_{A_1} = E_{A_2}$ and $E_C := E_{C_1} = E_{C_2}$ while, by assumption (1), one has $\Gamma = \langle E^{\mathrm{red}}_A, E^{\mathrm{red}}_{B_i} \rangle \subset \overline{Q}^{\times}$, $i = 1, 2$. By definition Γ is a finitely generated abelian group hence, by assumption (3), one also has $\Gamma \simeq \mathbb{Z}^r$ for some integer $r \geq 1$ so that the group algebra $\mathbb{Z}[\Gamma] \simeq \mathbb{Z}[T_1, T_1^{-1}, \ldots, T_r, T_r^{-1}]$ is an integral domain. Let $[E_A]$, $[E_C]$, $[E_{B_1}]$, $[E_{B_2}] \in \mathbb{Z}[\Gamma]$ denote the element representing the multisets E_A, E_C, E_{B_1}, E_{B_2} respectively. Then by assumption (1), $[E_A][E_{B_1}] = [E_C] = [E_A][E_{B_2}]$ in $\mathbb{Z}[\Gamma]$. As $\mathbb{Z}[\Gamma]$ is integral, this implies $[E_{B_1}] = [E_{B_2}]$ hence $\chi_{B_1} = \chi_{B_2}$ as polynomials in $\overline{Q}[T]$. It remains to prove that $\chi_{B_1} = \chi_{B_2}$ lies in $Q[T]$. But, for $i = 1, 2$, by assumption (2), E_A, E_C (or, equivalently, $[E_A]$, $[E_C]$) are fixed by the action of $\pi_1(Q)$ while by assumption (1) and the integrality of $\mathbb{Z}[\Gamma]$, $[E_{B_i}]$ (or, equivalently, E_{B_i}) is fixed by the action of $\pi_1(Q)$, which means χ_{B_i} lies in $Q[T]$, as Q is perfect. $\square$

Remark.

(1) The proof only exploits the integrality of $\mathbb{Z}[\Gamma] \simeq \mathbb{Z}[T_1, T_1^{-1}, \ldots, T_r, T_r^{-1}]$. It would be interesting to try and exploit the fact that $\mathbb{Z}[\Gamma]$ is the localization of $\mathbb{Z}[T_1, \ldots, T_r]$ at the multiplicative monoid generated by $T_1, \ldots, T_r$ hence, in particular, is a unique factorization domain.

(2) Set $\Gamma_A := \langle E_{A_1}^{\mathrm{red}}\rangle = \langle E_{A_2}^{\mathrm{red}}\rangle \subset \Gamma$. Then assumption (3) can be replaced by the following weaker assumption (3') Γ_A is torsion-free. Indeed, Lemma 17 amounts to proving that $[E_A]$ is a non-zero divisor in $\mathbb{Z}[\Gamma]$. Assumption (3') ensures $\mathbb{Z}[\Gamma_A]$ is integral. As $\mathbb{Z}[\Gamma_A]\to\mathbb{Z}[\Gamma]$ endows $\mathbb{Z}[\Gamma]$ with the structure of a free $\mathbb{Z}[\Gamma_A]$-module (with $\mathbb{Z}[\Gamma_A]$-basis any system of representatives of Γ/Γ_A), $\mathbb{Z}[\Gamma]$ is a flat $\mathbb{Z}[\Gamma_A]$-algebra; in particular, every non-zero element in $\mathbb{Z}[\Gamma_A]$ is a non-zero divisor in $\mathbb{Z}[\Gamma]$. This is in particular the case for $[E_A]\in\mathbb{Z}[\Gamma_A]$.

3.2. **Semisimple algebras.** Let V be a finite-dimensional Q-vector space and set $E := \mathrm{End}_Q(V)$. Let $H\subset E$ be a semisimple Q-subalgebra. Set

$$Z_E(H) := \{f\in E \mid fh = hf,\ h\in H\},\ \ Z(H) := Z_E(H)\cap H.$$

Lemma 18. *The canonical morphism of $Z(H)$-modules*

$$H\otimes_{Z(H)} Z_E(H)\to Z_E(Z(H))$$

is an isomorphism.

Proof. Let $\widehat{H}$ denote a system of representatives of the isomorphism classes (as left H-modules) of simple left ideals in H. For $I\in\widehat{H}$, set $D_I := \mathrm{End}_H(I)$, $Z_I := Z(D_I)$ and let $H_I\simeq I^{\oplus n_I}\subset H$ denote the I-isotypical component of H viewed as a left H-module. Then $H = \oplus_{I\in\widehat{H}} H_I$ and write $1 = \sum_{I\in\widehat{H}} e_I$ with $e_I\in H_I$, $I\in\widehat{H}$. With this notation, $H_I\subset H$, endowed with the product of H, carries a natural structure of central simple algebra over $Z_I = Z(H_I)$ with unit e_I, and the isotypical decomposition of $H=\oplus_{I\in\widehat{H}}H_I = \oplus_{I\in\widehat{H}} e_I H$ as left H-modules gives an isomorphism $H\simeq\prod_{I\in\widehat{H}} H_I$ of rings. Write $E_I := e_I E e_I$, $I\in\widehat{H}$. One immediately checks that the natural morphisms

$$Z_E(H)\to\prod_{I\in\widehat{H}} Z_{E_I}(H_I),\ \ f\mapsto (e_I f)_{I\in\widehat{H}}$$

and

$$Z_E(Z(H))\to\prod_{I\in\widehat{H}} Z_{E_I}(Z(H_I)),\ \ f\mapsto (e_I f)_{I\in\widehat{H}}$$

are isomorphisms of rings so that one gets a canonical commutative diagram of $Z(H)$-modules

$$\begin{tikzcd} H\otimes_{Z(H)} Z_E(H) \arrow[r] \arrow[d, "\simeq"] & Z_E(Z(H)) \arrow[d, "\simeq"] \\ \prod_{I\in\widehat{H}} H_I\otimes_{Z(H_I)} Z_{E_I}(H_I) \arrow[r] & \prod_{I\in\widehat{H}} Z_{E_I}(Z(H_I)). \end{tikzcd}$$

This reduces the proof of Lemma 18 to the case where H is a simple ring, which follows from [B58, §10, 2., Cor. of Thm. 2]. □

In particular, if $Z(H) = QId$ then one has a canonical isomorphism of Q-modules $H\otimes_Q Z_E(H)\tilde{\to}E$.

4. Proof of Theorem 6

4.1. **Adding level.** Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X and $\Pi_\ell\subset\mathrm{GL}(V_\ell)$ the image of $\pi_1(X)$ acting on $V_\ell := \mathcal{V}_{\ell,\bar{x}}$. Consider the following level condition

Lev($\mathcal{V}_\ell$) There exists a Π_ℓ-stable $\mathbb{Z}_\ell$-lattice $V_\ell^\circ\subset V_\ell$ such that $\Pi_\ell\subset Id+\tilde{\ell}\mathrm{End}_{\mathbb{Z}_\ell}(V_\ell^\circ)$, where $\tilde{\ell}=4$ if $\ell=2$ and $\tilde{\ell}=\ell$ otherwise.

Note that Condition Lev($\mathcal{V}_\ell$) can always be achieved after replacing $\mathcal{V}_\ell$ on X with $\alpha^*\mathcal{V}_\ell$ on X' for the connected étale cover $\alpha : X'\to X$ trivializing the local system $\mathcal{V}_\ell^\circ/\tilde{\ell}$, where $\mathcal{V}_\ell^\circ$ denotes the $\mathbb{Z}_\ell$-local system corresponding to the $\pi_1(X)$-stable $\mathbb{Z}_\ell$-lattice $V_\ell^\circ\subset V_\ell$.

Condition Lev($\mathcal{V}_\mathfrak{l}$) for a single $\mathfrak{l}\in|S_\mathbb{Q}|$ implies the following. Recall that $G_\ell\subset\mathrm{GL}_{V_\ell}$ denotes the Zariski-clsoure of Π_ℓ and, for every $x\in|X|$, $G_{\ell,x}\subset G_\ell$ denotes the Zariski closure of the image $\Pi_{\ell,x}\subset\Pi_\ell$ of $\pi_1(x)$ acting on V_ℓ through $\pi_1(x)\to\pi_1(X)$. For every $x\in|X|$ and $v\in U_{k(x),x^*\mathcal{V}}$ with residue characteristic $p\neq\mathfrak{l}$, the subgroup $\Xi_{x_v}\subset\overline{\mathbb{Q}}^\times$ generated by the roots of $\chi_{x_v} := \chi_{\mathcal{V}_\mathfrak{l},x_v}$ is contained in $1+\tilde{\mathfrak{l}}\mathbb{Z}_\mathfrak{l}$ hence torsion-free.

4.2. **$\mathbb{Q}$-compatibility of isotrivial tensors.** Let $\mathfrak{f}_a : Y_a \to X$ be a smooth projective morphism, $a = 1, \dots, r$. For every $\ell \in |S_{\mathbb{Q}}|$, integer j and r-tuples $\underline{i} = (i_1, \dots, i_r)$, $\underline{n} = (n_1, \dots, n_r)$ of integers with $i_1, \dots, i_r \geq 0$, let $I := \{1 \leq a \leq r \mid n_a \geq 0\}$ and set

$$\mathcal{V}_\ell := \otimes_{a \in I} (R^{i_a} \mathfrak{f}_{a*} \mathbb{Q}_\ell)^{\otimes n_a} \otimes_{a \notin I} (R^{i_a} \mathfrak{f}_{a*} \mathbb{Q}_\ell)^{\vee \otimes -n_a}(j)$$

and let $\mathcal{C}_\ell \subset \mathcal{V}_\ell$ denote the (geometrically constant) sub-$\mathbb{Q}_\ell$-local system corresponding to the $\pi_1(X)$-submodule $\mathrm{H}^0(X_{\bar{k}}, \mathcal{V}_\ell) = V_\ell^{\pi_1(X_{\bar{k}})} \subset V_\ell := \mathcal{V}_{\ell, \bar{x}}$. By construction, $\mathcal{C}_\ell$ is the pullback along the structural morphism $X \to \mathrm{spec}(k)$ of a $\mathbb{Q}_\ell$-local system on $\mathrm{spec}(k)$, which we again denote by $\mathcal{C}_\ell$.

Proposition 19. *The family $\mathcal{C} := (\mathcal{C}_\ell)_{\ell \in |S_{\mathbb{Q}}|}$ is a $\mathbb{Q}$-compatible family of $\mathbb{Q}_\ell$-local systems on X.*

Proof. By Poincaré duality, one may assume that $I = \{1, \dots, r\}$, up to increasing r, that $n_1 = \cdots = n_r = 1$ and, as $\mathbb{Q}$-compatibility is invariant under Tate twist, that $j = 0$. Fix smooth, normal crossing compactifications $X \hookrightarrow X^{\mathrm{cpt}}$, $Y_a \hookrightarrow Y_a^{\mathrm{cpt}}$, $a = 1, \dots, r$ [N62], [N63], [H64] and a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$ such that for every $a = 1, \dots, r$ one has Cartesian diagrams

$$\begin{tikzcd} Y_a \arrow[r, "\mathfrak{f}_a"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X^{\mathrm{cpt}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{spec}(k) \arrow[d] \\ \mathcal{Y}_a \arrow[r, "f_a"'] & \mathcal{X} \arrow[r, hook] & \mathcal{X}^{\mathrm{cpt}} \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} Y_a \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y_a^{\mathrm{cpt}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{spec}(k) \arrow[d] \\ \mathcal{Y}_a \arrow[r, hook] & \mathcal{Y}_a^{\mathrm{cpt}} \arrow[r] & U \end{tikzcd}$$

with $\mathcal{X} \to U$ smooth, $f_a : \mathcal{Y}_a \to \mathcal{X}$ smooth projective, $a = 1, \dots, r$, and $\mathcal{X} \to \mathcal{X}^{\mathrm{cpt}} \to U$, $\mathcal{Y}_a \to \mathcal{Y}_a^{\mathrm{cpt}} \to U$, $a = 1, \dots, r$ relative smooth normal crossing compactifications; set $\mathcal{D} := \mathcal{X}^{\mathrm{cpt}} \setminus \mathcal{X}$ and $D := X^{\mathrm{cpt}} \setminus X$. By smooth proper base change $\mathcal{V}_\ell = \otimes_{1 \leq a \leq r} R^{i_a} \mathfrak{f}_{a*} \mathbb{Q}_\ell$ extends to the $\mathbb{Q}_\ell$-local system $\otimes_{1 \leq a \leq r} R^{i_a} f_{a*} \mathbb{Q}_\ell$, which we again denote $\mathcal{V}_\ell$, on $\mathcal{X}[\frac{1}{\ell}]$. Fix $x \in |X|$; without loss of generality one may assume $k(x) = k$ and, up to shrinking U, that $x : \mathrm{spec}(k) \to X$ extends to a U-point $x : U \to \mathcal{X}$. Let $v \in U_p$ and for every $a = 1, \dots, r$ consider the base-change diagram

$$\begin{tikzcd} \mathcal{Y}_{a,v} \arrow[r, "f_{a,v}"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{X}_v \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{X}_v^{\mathrm{cpt}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathrm{spec}(\kappa_v) \arrow[d, hook] \arrow[ll, bend right, "x_v"'] \\ \mathcal{Y}_{a,\mathcal{O}_{k_v}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{\mathcal{O}_{k_v}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{\mathcal{O}_{k_v}}^{\mathrm{cpt}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{spec}(\mathcal{O}_{k_v}) \arrow[d] \arrow[ll, bend right, "x_v"'] \\ \mathcal{Y}_a \arrow[r, "f_a"'] & \mathcal{X} \arrow[r, hook] & \mathcal{X}^{\mathrm{cpt}} \arrow[r] & U. \arrow[ll, bend left, "x"] \end{tikzcd}$$

Write $\mathcal{V}_{\ell,v} := \mathcal{V}_\ell|_{\mathcal{X}_v}$, $\ell \neq p$ and $\mathcal{V}_{p,v} := \bigotimes_{1 \leq i \leq r} R^{i_a} f_{a,v,\mathrm{cris},*} \mathcal{O}_{\mathcal{Y}_v/k_{v,0}}$. From [A23, Thm. 2.1.1.2], $\mathcal{V}_{p,v}$ lifts to a unique (up to isomorphism) overconvergent F-isocrystal $\mathcal{V}_{p,v}^\dagger$ on $\mathcal{X}_v$.

From Lemma 14 (resp. Lemma 15), the characteristic polynomial $\chi_{\mathcal{C}_\ell, x_v} \in \mathbb{Q}_\ell[T]$ of the geometric Frobenius $\varphi_{\mathcal{C}_\ell, x_v} : \mathcal{C}_{\ell, \bar{x}} \to \mathcal{C}_{\ell, \bar{x}}$ identifies with the one of the geometric Frobenius $\varphi_{x_v} : \mathrm{H}^0(\mathcal{X}_{\overline{v}}, \mathcal{V}_\ell) \to \mathrm{H}^0(\mathcal{X}_{\overline{v}}, \mathcal{V}_\ell)$, $\ell \neq p$ (resp. the characteristic polynomial $\chi_{\mathcal{C}_p, x_v, p} \in k_{v,0}[T]$ of the m_vth power of the crystalline Frobenius $\varphi_{\mathcal{C}_p, x_v} := \phi_{x_v}^{m_v} : D_{\mathrm{cris}}(x_v^* \mathcal{C}_p) \to D_{\mathrm{cris}}(x_v^* \mathcal{C}_p)$ identifies with the one of the m_vth power of the crystalline Frobenius $\varphi_{x_v} := \phi_{x_v}^{m_v} : \mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v}) \to \mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v})$). Let d denote the dimension of X. The fact that $\chi_{\mathcal{C}_\ell, x_v}$ lies in $\mathbb{Q}[T]$ and is independent of $\ell \in |S_{\mathbb{Q}}|$ now classically follows from

(i) Lefschetz trace formula for cohomology with compact support:

$$L(\mathcal{V}_{\ell,v}^\vee, T) = \prod_{0 \leq w \leq 2d} \det(TId - \varphi_{x_v} | \mathrm{H}_c^w(\mathcal{X}_{\overline{v}}, \mathcal{V}_{\ell,v}^\vee))^{(-1)^{w+1}}, \ \ell \neq p$$

$$L(\mathcal{V}_{p,v}^{\dagger\vee}, T) = \prod_{0 \leq w \leq 2d} \det(TId - \varphi_{x_v} | \mathrm{H}_{\mathrm{rig},c}^w(\mathcal{X}_v / k_{v,0}, \mathcal{V}_{p,v}^{\dagger\vee}))^{(-1)^{w+1}} \quad \text{[EL93, Thm. 6.3]}$$

plus the fact that the L-functions $L(\mathcal{V}_{p,v}^{\dagger\vee}, T)$ and $L(\mathcal{V}_{p,v}^\vee, T)$ coincide. (Note that $\mathrm{H}^0_{\mathrm{cris}}(\mathcal{X}_v, \mathcal{V}_{p,v}) \simeq \mathrm{H}^0_{\mathrm{rig}}(\mathcal{X}_v / k_{v,0}, \mathcal{V}_{p,v}^\dagger)$).

(ii) The $\mathbb{Q}$-compatibility of $(\mathcal{V}_{\ell,v})_{\ell\in|S_{\mathbb{Q}}|}$ [D74], [KM74], which ensures that the left-hand sides of (i) is independent of $\ell\in|S_{\mathbb{Q}}|$;

(iii) The fact that $\mathrm{H}^w_c(\mathcal{X}_{\overline{v}},\mathcal{V}_\ell^\vee)$ is mixed of weights $\leq w-i$, for $w<2d$ while $\mathrm{H}^{2d}_c(\mathcal{X}_{\overline{v}},\mathcal{V}_\ell^\vee)$ is pure of weight $2d-i$ [D80, Thm. (3.3.1)] and the similar statement for rigid cohomology [Ke06b, Thm. 6.6.2]. Here $i=i_1+\cdots+i_r$ is the weight of $\mathcal{V}_\ell$.

(iv) Poincaré duality: $\mathrm{H}^{2d}_c(\mathcal{X}_{\overline{v}},\mathcal{V}_{\ell,v}^\vee)^\vee(-d)\tilde{\to}\mathrm{H}^0(\mathcal{X}_{\overline{v}},\mathcal{V}_{\ell,v}),\ \ell\neq p$

$$\mathrm{H}^{2d}_{\mathrm{rig},c}(\mathcal{X}_v/k_{v,0},\mathcal{V}_{p,v}^{\dagger\vee})^\vee(-d)\tilde{\to}\mathrm{H}^0_{\mathrm{rig}}(\mathcal{X}_v/k_{v,0},\mathcal{V}_{p,v}^{\dagger\vee})\quad \text{[Ke06a, Thm. 1.2.3]}$$

□

Remark 20. Actually, the proof of Proposition 19 also shows the (a priori stronger) fact that $\mathcal{C}$ is $\mathbb{Q}$-compatible as a family of $\mathbb{Q}_\ell$-local systems over $\mathrm{spec}(k)$. If X has a k-rational point, this immediately follows from Proposition 19. Otherwise, fix models $\mathcal{Y}_a\to\mathcal{X}\hookrightarrow\mathcal{X}^{\mathrm{cpt}}\to U$ over some non-empty open subscheme U of $\mathrm{spec}(\mathcal{O}_k)$ as in the proof of Proposition 19. By formal smoothness of $\mathcal{X}\to U$ and the Weil bounds, for $p\gg 0$ and every $v|p$ in U, $\mathcal{X}$ has an $\mathcal{O}_{k_v}$-point. One can then conclude as in the proof of Proposition 19.

4.3. **Proof of Theorem 6.** Recall that $\mathcal{V}=(\mathcal{V}_\ell:=R^if_*\mathbb{Q}_\ell(j))_{\ell\in S_{\mathbb{Q}}}$ for some smooth projective morphism $f:Y\to X$ of relative dimension d and that $\mathcal{E}:=(\mathcal{E}_\ell:=\mathcal{V}_\ell\otimes\mathcal{V}_\ell^\vee)_{\ell\in S_{\mathbb{Q}}}$. Both $\mathcal{V}$ and $\mathcal{E}$ are $\mathbb{Q}$-compatible and pure of weight $i-2j$ and 0, respectively. We are to prove that the corresponding family $\overline{\mathcal{H}}\subset\mathcal{E}$ is almost $\mathbb{Q}$-compatible.

As the assumptions and conclusions of Theorem 6 are invariant under base-change by a connected étale cover, one may assume that

- $\overline{G}^\circ_\infty=\overline{G}_\infty$ so that $\overline{G}^\circ_\ell=\overline{G}_\ell$, $\ell\in|S_{\mathbb{Q}}|$.
- Condition $\mathrm{Lev}(\mathcal{V}_\ell)$ holds for at least one $\ell\in|S_{\mathbb{Q}}|$.

For every $\ell\in|S_{\mathbb{Q}}|$, let $\mathcal{C}_\ell$ denote the isotrivial $\mathbb{Q}_\ell$-local system on X corresponding to the $\pi_1(X)$-submodule

$$E_\ell^{\pi_1(X_{\bar{k}})}=E_\ell^{\overline{G}_\ell}=Z_{E_\ell}(\overline{H}_\ell)\subset E_\ell.$$

From Proposition 19, $\mathcal{C}=(\mathcal{C}_\ell)_{\ell\in|S_{\mathbb{Q}}|}$ is $\mathbb{Q}$-compatible. As $\pi_1(X_{\bar{k}})$ acts semisimply on V_ℓ, $\overline{H}_\ell$ is a semisimple $\mathbb{Q}_\ell$-algebra and, by assumption $Z(\overline{H}_\ell)=\mathbb{Q}_\ell$ so that by Lemma 18 one has a canonical isomorphism $\overline{\mathcal{H}}_\ell\otimes\mathcal{C}_\ell\tilde{\to}\mathcal{E}_\ell$ of $\mathbb{Q}_\ell$-local systems on X. As $\mathcal{E}$ and $\mathcal{C}$ are both $\mathbb{Q}$-compatible, and as $\mathrm{Lev}(\mathcal{V}_\ell)$ ensures the torsion-freeness in Condition (3) of Subsection 3.1, the assertion follows from Lemma 17.

References

[A23] E. Ambrosi, *Specialization of Neron-Severi groups in positive characteristic*, Annales Sc. École Normale Sup., to appear. arXiv:1810.06481

[An92] Y. André, *Mumford-Tate groups of mixed Hodge-structures and the theorem of the fixed part*, Compositio Math. **82**, p. 1–24, 1992.

[An96] Y. André, *Pour une théorie inconditionnelle des motifs*, Publ. Math. I.H.E.S. **83**, 1996. Sém. Bourbaki 1995/1996 **815**, Astérisque **241**, S.M.F., 1997, p. 273–311.

[B58] N. Bourbaki, *Algèbre Chap. VIII (Modules et Anneaux semisimples)*, Hermann, 1958.

[CT25] A. Cadoret and A. Tamagawa, *Rigidity properties of ramification and applications*, In preparation.

[D71] P. Deligne, *Théorie de Hodge: II*, Inst. Hautes Etudes Sci. Publ. Math. **40**, p. 5–57, 1971.

[D74] P. Deligne, *La conjecture de Weil: I*, Inst. Hautes Etudes Sci. Publ. Math. **43**, p. 273–307, 1974.

[D80] P. Deligne, *La conjecture de Weil: II*, Inst. Hautes Etudes Sci. Publ. Math. **52**, p. 137–252, 1980.

[EL93] J.Y. Etesse and B. Le Stum, *Fonctions L associées aux F-isocristaux surconvergent I. Interprétation cohomologique,* Math. Ann. **296**, p. 557–576, 1993.

[F89] G. Faltings, *Crystalline cohomology and p-adic Galois representations*, in Algebraic analysis, geometry, and number theory, Johns Hopkins Univ. Press, p. 25–80, 1989.

[Fu75] W. Fulton, *Rational equivalence on singular varieties.* Publ. Math. I.H.E.S. **45**, p.147–167, 1975.

[Fu84] W. Fulton, *Intersection Theory.* E.G.M. **2**, Springer, 1984.

[G71] A. Grothendieck, *Revêtements étales et groupe fondamental - S.G.A.1*, L.N.M. **224**, Springer-Verlag, 1971.

[H64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. **79**, p. 109–203, 205–326, 1964.

[Ho52] W. Hodge, *The topological invariants of algebraic varieties*, in Proceedings of the International Congress of Mathematicians (Cambridge, MA, 1950), 1, p. 182–192, 1952.

[KM74] N. M. Katz and W. Messing, *Some consequences of the Riemann Hypothesis for varieties over finite fields*, Inventiones Math. **23**, p. 73–77, 1974.

[Ke06a] K.S. Kedlaya, *Finiteness of rigid cohomology with coefficients*, Duke Math. Journal **134**, p. 15–97, 2006.

[Ke06b] K.S. Kedlaya, *Fourier transforms and p-adic 'Weil II'*, Compos. Math. **142**, p.1426–1450, 2006.

[M19] M. Morrow, *A Variational Tate Conjecture in crystalline cohomology*, J. Eur. Math. Soc. **21**, p. 3467–3511, 2019.

[N62] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto **2**, p.1–10, 1962.

[N63] M. Nagata, *A generalization of the imbedding problem*, J. Math. Kyoto **3**, p.89–102, 1963.

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