Representations of étale fundamental groups and specialization of algebraic cycles

In honor of Gerhard Frey's 75th birthday

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We describe an elementary strategy to study the locus where a finite family of linearly independent 1-cohomology classes for the étale fundamental group remains linearly independent under specialization. We apply this strategy to the injectivity of the specialization morphism on the second graded piece of the ℓ-adic Abel-Jacobi filtration on Chow groups varying in the fibers of a smooth projective morphism. When the base scheme is a curve this provides a generalization (to Chow groups and arbitrary finitely generated fields of characteristic 0) of a theorem of Silverman about the sparsity of the jumping locus of the rank in the fibers of an abelian scheme.

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1. Introduction

1.1. Let k be a field of characteristic $p \geq 0$. A variety over k means a separated scheme of finite type over k. Let S be a smooth, geometrically connected variety over k. Set $\overline{S} := S \times_k \overline{k}$. Let η denote the generic point of S, $|S|$ the set of closed points of S. For a subset $\Sigma \subset |S|$ and an integer $d \geq 1$, let $\Sigma^{\leq d}$ denote the set of all $s \in \Sigma$ with $[k(s) : k] \leq d$ (in particular, $|S|^{\leq 1} = S(k)$). For every $s \in S$, fix a geometric point $\overline{s}$ over s and an étale path $\overline{\eta} \tilde{\to} \overline{s}$, which induces an isomorphism $\pi_1(s, \overline{s}) \simeq \pi_1(s, \overline{\eta})$. By functoriality of the étale fundamental group, $s \in S$ regarded as a morphism $s : \mathrm{spec}(k(s)) \to S$ induces a continuous group morphism $\sigma_s : \pi_1(s, \overline{s}) \simeq \pi_1(s, \overline{\eta}) \to \pi_1(S, \overline{\eta})$, which is injective if $s \in |S|$. As the choice of base points will basically play no part in the following, we will omit them from the notation unless necessary.

For a field K and an (any) algebraic closure $\overline{K}$ of K, write $\pi_1(K) := \pi_1(\mathrm{spec}(K), \mathrm{spec}(\overline{K}))$ for the absolute Galois group of K.

1.2. Fix a prime $\ell \neq p$ and V a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous action of $\pi_1(S)$. Write Π_V, $\overline{\Pi}_V$ and $\Pi_{s,V}$ for the image of $\pi_1(S)$, $\pi_1(\overline{S})$ and $\pi_1(s)$ acting on V respectively. Let $S(V) \subset |S|$ denote the set of all $s \in |S|$ such that $\Pi_{s,V} \subset \Pi_V$ is open. For a finite-dimensional $\mathbb{Q}_\ell$-subvector space $E \hookrightarrow \mathrm{H}^1(\pi_1(S), V)$, let $S(E, V) \subset |S|$ denote the set of all $s \in |S|$ such that

$$res_s|_E : E \hookrightarrow \mathrm{H}^1(\pi_1(S), V) \overset{res_s}{\to} \mathrm{H}^1(\pi_1(s), V)$$

is injective, where res_s denotes the restriction morphism with respect to $\sigma_s : \pi_1(s) \to \pi_1(S)$.

The problem addressed in this note is to describe the arithmetico-geometric structure of $S(E, V)$ when k is 'arithmetically rich' that is, essentially, with a "large absolute Galois group $\pi_1(k)$".

This is motivated by studying the variation of certain type of arithmetico-geometric invariants in families of smooth, proper varieties. Here is an example. Let $A \to S$ be an abelian scheme. The Kummer map and the Néron extension property for abelian schemes give rise to a canonical commutative diagram

$$\begin{tikzcd}
A_\eta(k(\eta))\otimes\mathbb{Q} \arrow[r, hook, "\kappa_\ell"] & \mathrm{H}^1(\pi_1(\eta), V_\ell(A_{\overline{\eta}})) \\
A(S)\otimes\mathbb{Q} \arrow[r, hook] \arrow[u, "\simeq"] \arrow[d, "sp_s"] & \mathrm{H}^1(\pi_1(S), V_\ell(A_{\overline{\eta}})) \arrow[u, "\simeq", "infl"'] \arrow[d, "res_s"] \\
A_s(k(s)))\otimes\mathbb{Q} \arrow[r, hook, "\kappa_\ell"] & \mathrm{H}^1(\pi_1(s), V_\ell(A_{\overline{s}})),
\end{tikzcd}$$

where $V_\ell(-) := (\varprojlim - [\ell^n]) \otimes \mathbb{Q}_\ell$ denotes the $\mathbb{Q}_\ell$-Tate module of $-$. Taking $V = V_\ell(A_{\overline{\eta}})$ and $E = A_\eta(k(\eta)) \otimes \mathbb{Q}$, the locus $S(E, V) \subset |S|$ is the set of all $s \in |S|$ where the specialization map $sp_s : A_\eta(k(\eta)) \otimes \mathbb{Q} \to A(S) \otimes \mathbb{Q}$ is injective and, in particular, it is contained in the set of all $s \in |S|$ where $\mathrm{rank}(A_\eta(k(\eta))) \leq \mathrm{rank}(A_s(k(s)))$.

1.3. In general, one expects $S(E, V)$ to be 'huge'. This relies on the observation that one can attach to (E, V) a "universal extension" $\tilde{E}$, which is a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous action of $\pi_1(S)$ fitting into a $\pi_1(S)$-equivariant short exact sequence $0 \to V \to \tilde{E} \to E \to 0$ (with E endowed with the trivial $\pi_1(S)$-action) and with the property that $S(\tilde{E}) \subset S(E, V)$; see Subsection 2.1 and Lemma 2.1.3 for details. The point is that, under mild assumptions on V, one expects $S(\tilde{E})$ to be 'huge' (see 3). More precisely, if k is Hilbertian, there always exists an integer $d \geq 1$ such that $S(\tilde{E})^{\leq d}$ is infinite (Fact 3.1.1). This is already enough for applications which only require the existence of a single closed point in $S(E, V)$ (see *e.g* Subsection 6.1). Under mild assumptions on V (satisfied by representations V arising from geometry as in the example of 1.2) and when k is finitely generated over $\mathbb{Q}$, one expects stronger abundance results for $S(E, V)$. For higher dimensional varieties S such results are still highly conjectural but when S is a curve, one can ensure that if $\overline{\Pi}_V$ has perfect Lie algebra then for every integer $d \geq 1$, $(|S| \setminus S(E, V))^{\leq d}$ is finite (Fact 3.1.2.1). To apply our strategy, we have to find conditions on V which can be checked in practice and which ensure that $\overline{\Pi}_{\tilde{E}}$ has perfect Lie algebra. This is basically the content of Lemma 2.2.

We also provide a variant of our results for families $\underline{V} = V_\ell$, $\ell \in \mathcal{L}$ of finite dimensional $\mathbb{F}_\ell$-vector spaces of bounded rank and endowed with a continuous action of $\pi_1(S)$ for ℓ varying in an infinite set $\mathcal{L}$ of primes.

1.4. The group-theoretical results described in Subsection 1.3 are stated and proved in Section 2. The remaining sections of the paper (Sections 4 - 6) are devoted to applications of these result to motivic representations (that is those arising from the étale cohomology of smooth proper morphisms $X \to S$). We discuss in particular the injectivity of the specialization map for the second graded piece of the ℓ-adic Abel-Jacobi filtration on Chow groups in the fibers of a smooth projective morphism $X \to S$. This provides a generalization (to Chow groups and arbitrary finitely generated fields) of a theorem of Silverman about the sparsity of the dropping locus of the rank in the fibers of an abelian scheme; the theorem of Silverman corresponds to the example of Subsection 1.2 (see Corollary 5.2.3 and, for the case of Abelian schemes, 6.2). We also show that the locus where the unipotent part of the Mumford-Tate group of a 1-motive degenerates is sparse (See **??**). The reader who is mostly interested in these applications can jump directly to Sections 4 - 6 and browse the formal Section 2 only when needed.

1.5. This note does not intend to exhaustivity but rather to present a method that could hopefully be applied to other situations. In particular, we only deal with $\mathbb{Q}_\ell$ or $\mathbb{F}_\ell$-coefficients but the method extend to more general coefficients. It would also be interesting to investigate what could be said for higher cohomology groups (for i smaller than the cohomological dimension of the fields $k(s)$, $s \in |S|$). The method itself (see Section 2) relies on elementary group-theoretic observations. What is *not* elementary and involves subtle arithmetico-geometric inputs are the specialization results (see Subsection 3) we inject in the method and the fact that the method can be applied to motivic representations (see Section 4). It seems unlikely that our sparsity results could be enhanced for arbitrary $E \subset \mathrm{H}^1(\pi_1(S), V)$. But when $E \subset \mathrm{H}^1(\pi_1(S), V)$ parametrizes cocycles of geometric origin (as it is the case in our application to Chow groups), one expects stronger sparsity results (algebraicity, finiteness, bounded height - see Subsection 6.2.4 for discussion). This illustrates both the generality of the method (which applies to arbitrary cohomology classes) and its limits (since it cannot see the difference between transcendental and geometric cohomology classes).

1.6. In a first version of this note, we also included additional applications to the case where the base field k has characteristic $p > 0$ (and for primes $\ell \neq p$). These applications were relying on specialization results in [CT19], [A18], which in turn were relying on [EEHK09, Thm. 2] - a tentative positive characteristic analogue of the Mordell conjecture. But just before resubmitting the revised version of this note, Akio Tamagawa exhibited a counter-example to [EEHK09, Thm. 2], showing that the statement of [EEHK09, Thm. 2] has to be refined (to include some isotriviality phenomena). The results of [CT19], [A18] should be adjusted consequently. These could then be (re)injected in the strategy described in this note to obtain similar applications when the base field k has characteristic $p > 0$.

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2. Representation-theoretic results

Let Λ denote $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$ for some prime ℓ and let Π be a profinite group. Write $\mathcal{M}\mathrm{od}_\Lambda(\Pi)$ for the (abelian) category of finitely generated Λ-modules equipped with a continuous Λ-linear action of Π. In particular, $\mathcal{M}\mathrm{od}_\Lambda := \mathcal{M}\mathrm{od}_\Lambda(1)$ is just the category of finitely generated Λ-modules. For $V \in \mathcal{M}\mathrm{od}_\Lambda(\Pi)$ let $\Pi_V \subset \mathrm{Aut}_\Lambda(V)$, $\Pi^V \subset \Pi'$ denote respectively the image and kernel of the corresponding morphism $\Pi \to \mathrm{Aut}_\Lambda(V)$ so that one has a short exact sequence of profinite groups

$$1 \to \Pi^V \to \Pi \to \Pi_V \to 1.$$

For a continuous morphism of profinite groups $\Gamma \to \Pi$, write $-|_\Gamma : \mathcal{M}\mathrm{od}_\Lambda(\Pi) \to \mathcal{M}\mathrm{od}_\Lambda(\Gamma)$ for the obvious restriction functor. Let $\mathrm{H}^i(\Pi, -) : \mathcal{M}\mathrm{od}_\Lambda(\Pi) \to \mathrm{Mod}_\Lambda$ denote the ith continuous cohomology group functor of Π (defined by means of the continuous i-cochains $Z^i(\Pi, V)$- see [T76]).

2.1. The universal extension $\tilde{E}$. Let Λ denote $\mathbb{Q}_\ell$ or $\mathbb{F}_\ell$ for some prime ℓ and let Π be a profinite group. Fix $V \in \mathcal{M}\mathrm{od}_\Lambda(\Pi)$. Recall that if $\Lambda = \mathbb{Q}_\ell$, Π_V being a closed subgroup of $\mathrm{Aut}_\Lambda(V)$ is a (compact) ℓ-adic Lie group; we write $\mathrm{Lie}(\Pi_V)$ for its Lie algebra (as a ℓ-adic Lie group).

2.1.1. *Construction.* Let $\mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(-, V) : \mathcal{M}\mathrm{od}_\Lambda \to Ab$ denote the functor sending $E \in \mathcal{M}\mathrm{od}_\Lambda$ to the abelian group $\mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(E, V)$ of equivalence classes of extensions

$$0 \to V \to \tilde{E} \to E \to 0$$

in $\mathcal{M}\mathrm{od}_\Lambda(\Pi)$. There is a canonical isomorphism of functors $\mathcal{M}\mathrm{od}_\Lambda \to \mathrm{Mod}_\mathbb{Z}$

$$\phi := \phi(\Pi, V) : \mathrm{Hom}_{\mathcal{M}\mathrm{od}_\Lambda}(-, \mathrm{H}^1(\Pi, V)) \tilde{\to} \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(-, V)$$

defined as follows. Let $\widetilde{-} : \mathrm{H}^1(\Pi, V) \hookrightarrow Z^1(\Pi, V)$ be a section in Mod_Λ of the canonical projection $Z^1(\Pi, V) \twoheadrightarrow \mathrm{H}^1(\Pi, V)$. For $E \in \mathcal{M}\mathrm{od}_\Lambda$ and a morphism $f : E \to \mathrm{H}^1(\Pi, V)$ in Mod_Λ, a representative of the isomorphism class $\phi(E)(f) \in \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(E, V)$ is given by $\tilde{E} = V \oplus E$ endowed with the Λ-linear action of Π given by $\pi\cdot(m \oplus e) = (\pi m + \widetilde{f(e)}(\pi)) \oplus e$, $\pi \in \Pi$. One easily checks that $\phi(E)(f)$ does not depend on the choice of the section $\widetilde{-} : \mathrm{H}^1(\Pi, V) \hookrightarrow Z^1(\Pi, V)$ and that the construction defines a morphism of functors $\mathcal{M}\mathrm{od}_\Lambda \to Ab$. To show that ϕ is an isomorphism, one can exhibit an explicit inverse $\psi := \psi(\Pi, V) : \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(-, V) \to \mathrm{Hom}_{\mathcal{M}\mathrm{od}_\Lambda}(-, \mathrm{H}^1(\Pi, V))$ as follows. For the isomorphism class $[\tilde{E}] \in \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(E, V)$ of an extension $\tilde{E}$, the morphism $\psi(E)([\tilde{E}]) : E \to \mathrm{H}^1(\Pi, V)$ sends $e \in E$ to the cohomology class of the 1-cocycle

$$\begin{array}{ccc} \Pi & \to & V \\ \pi & \to & \pi\tilde{e} - \tilde{e}, \end{array}$$

where $\widetilde{-} : E \to \tilde{E}$ is a section in $\mathcal{M}\mathrm{od}_\Lambda$ of the projection $\tilde{E} \twoheadrightarrow E$. Again, one easily checks that $\psi(E)([\tilde{E}])$ does not depend on the choice of $\widetilde{-} : E \to \tilde{E}$ and that $\psi(E) \circ \phi(E) = Id$, $\phi(E) \circ \psi(E) = Id$.

2.1.2. *Functoriality.* The construction of $\phi := \phi(\Pi, V)$ is functorial in Π, V. In particular, for short exact sequence of profinite groups $1 \to N \to \Pi \to \Pi/N \to 1$, one has a canonical inflation-restriction commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}_\Lambda}(-, \mathrm{H}^1(\Pi/N, V^N)) \arrow[r, "infl"] \arrow[d, "\phi(\Pi/N, V^N)"] & \mathrm{Hom}_{\mathrm{Mod}_\Lambda}(-, \mathrm{H}^1(\Pi, V)) \arrow[r, "res"] \arrow[d, "\phi(\Pi, V)"] & \mathrm{Hom}_{\mathrm{Mod}_\Lambda}(-, \mathrm{H}^1(N, V)) \arrow[d, "\phi(N, V)"] \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi/N)}(-, V^N) \arrow[r, "(V^N \hookrightarrow V)_*"'] & \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(-, V) \arrow[r, "|_N"'] & \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(N)}(-, V)
\end{tikzcd}$$

2.1.3. *Injectivity criterion for the restriction morphism on H^1.* Let $\Gamma \to \Pi$ be a morphism of profinite groups, let $V \in \mathcal{M}\mathrm{od}_\Lambda(\Pi)$ and $\iota : E \hookrightarrow \mathrm{H}^1(\Pi, V)$ a finitely generated Λ-submodule. Let $\tilde{E}$ be a representative of $\phi(E)(\iota) \in \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(E, V)$. Recall that $\Pi_{\tilde{E}} \subset \mathrm{Aut}_\Lambda(\tilde{E})$ denote the image of Π acting on $\Pi_{\tilde{E}}$ and that $\Pi^{\tilde{E}} = \ker(\Pi \to \Pi_{\tilde{E}})$.

Lemma. *Fix a closed normal subgroup $N \subset \Pi_{\tilde{E}}$. Assume $\Lambda = \mathbb{Q}_\ell$ (resp. $\Lambda = \mathbb{F}_\ell$) and*
(SS) *$\Pi_{\tilde{E}}/N$ acts semisimply on V^N (resp. and $\ell \gg 0$ compared with the Λ-rank of V^N).*
(O) *$\Gamma_{\tilde{E}} \cap N \subset N$ is open of index $[N : \Gamma_{\tilde{E}} \cap N]$ invertible in Λ.*
Then $E \overset{\iota}{\hookrightarrow} \mathrm{H}^1(\Pi, V) \overset{res}{\to} \mathrm{H}^1(\Gamma, V)$ is injective.

Proof. By construction, $\phi(E)(\iota)|_{\Pi^{\tilde{E}}} = 0$ hence by 2.1.2, $\iota : E \hookrightarrow \mathrm{H}^1(\Pi, V)$ factors through

$$(*) \quad \begin{tikzcd}
 & & E \arrow[dl, dotted, "\iota"'] \arrow[d, hook, "\iota"] \arrow[dr, "0"] & \\
0 \arrow[r] & \mathrm{H}^1(\Pi_{\tilde{E}}, V) \arrow[r, "infl"] \arrow[d, "res"] & \mathrm{H}^1(\Pi, V) \arrow[r, "res"] \arrow[d, "res"] & \mathrm{H}^1(\Pi^{\tilde{E}}, V) \arrow[d, "res"] \\
0 \arrow[r] & \mathrm{H}^1(\Gamma_{\tilde{E}}, V) \arrow[r, "infl"] & \mathrm{H}^1(\Gamma, V) \arrow[r, "res"] & \mathrm{H}^1(\Gamma \cap \Pi^{\tilde{E}}, V)
\end{tikzcd}$$

So it is enough to prove that the left vertical arrow of $(*)$ is injective. From the inflation-restriction diagram

$$(**)\quad \begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(\Pi_{\tilde{E}}/N, V^N) \arrow[r, "infl"] \arrow[d, "res"'] & \mathrm{H}^1(\Pi_{\tilde{E}}, V) \arrow[r, "res"] \arrow[d, "res"'] & \mathrm{H}^1(N, V) \arrow[d, "res"'] \\ 0 \arrow[r] & \mathrm{H}^1(\Gamma_{\tilde{E}} \cap N/N, V^N) \arrow[r, "infl"] & \mathrm{H}^1(\Gamma_{\tilde{E}}, V) \arrow[r, "res"] & \mathrm{H}^1(\Gamma_{\tilde{E}} \cap N, V) \end{tikzcd}$$

to show that $res : \mathrm{H}^1(\Pi_{\tilde{E}}, V) \to \mathrm{H}^1(\Gamma_{\tilde{E}}, V)$ is injective it is enough to show that $\mathrm{H}^1(\Pi_{\tilde{E}}/N, V^N) = 0$ and $res : \mathrm{H}^1(N, V) \to \mathrm{H}^1(\Gamma_{\tilde{E}} \cap N, V)$ is injective. The injectivity of $res : \mathrm{H}^1(N, V) \to \mathrm{H}^1(\Gamma_{\tilde{E}} \cap N, V)$ follows from (O) by corestriction while $\mathrm{H}^1(\Pi_{\tilde{E}}/N, V^N) = 0$ follows from (SS). More precisely, if $\Lambda = \mathbb{Q}_\ell$ then $\Pi_{\tilde{E}}/N$ is a compact ℓ-adic Lie group hence its ℓ-adic Lie algebra $\mathrm{Lie}(\Pi_{\tilde{E}}/N)$ is reductive. This implies $\mathrm{H}^1(\mathrm{Lie}(\Pi_{\tilde{E}}/N), V) = 0$ and the assertion follows from the injectivity of the canonical morphism $\mathrm{H}^1(\Pi_{\tilde{E}}/N, V^N) \to \mathrm{H}^1(\mathrm{Lie}(\Pi_{\tilde{E}}/N), V^N)$ [Se64, Prop. 12]. If $\Lambda = \mathbb{F}_\ell$, this is [N87, Thm. E]. $\square$

In applications, we will consider $N = \Pi_{\tilde{E}}$ (in which case Condition (SS) is empty) or $N = \ker(\Pi_{\tilde{E}} \twoheadrightarrow \Pi_V)$.

2.2. Transfer of perfectness. We retain the notation and conventions of Subsection 2.1. If $\Lambda = \mathbb{Q}_\ell$ (resp. $\Lambda = \mathbb{F}_\ell$) we say that $V \in \mathcal{M}\mathrm{od}_\Lambda(\Pi)$ has no Π-Lie-coinvariants (resp. has no Π-coinvariants) if $W_{\mathrm{Lie}(\Pi_V)} = 0$ for every subobject $W \subset V$ in $\mathcal{M}\mathrm{od}_\Lambda(\Pi)$ (resp. if $W_\Pi = 0$ for every subobject $W \subset V$ in $\mathcal{M}\mathrm{od}_\Lambda(\Pi)$).

In practice, to ensure Condition (O) of Lemma 2.1.3 holds for the groups $\Gamma(:= \pi_1(s)) \subset \Pi(:= \pi_1(S))$ we are interested in, we will use the following observation.

Lemma. *Let $E \in \mathcal{M}\mathrm{od}_\Lambda$ and $\tilde{E} \in \mathrm{Ext}^1_{\mathcal{M}\mathrm{od}_\Lambda(\Pi)}(E, V)$. Assume $\Lambda = \mathbb{Q}_\ell$ (resp. $\Lambda = \mathbb{F}_\ell$) and V has no Π-Lie-coinvariants (resp. no Π-coinvariants). Then $\mathrm{Lie}(\Pi_{\tilde{E}})^{ab} = \mathrm{Lie}(\Pi_V)^{ab}$ (resp. $\Pi_{\tilde{E}}^{ab} = \Pi_V^{ab}$), where $(-)^{ab}$ denote the abelianization functor for Lie algebras (resp. finite groups).*

Proof. We only give the proof for $\Lambda = \mathbb{Q}_\ell$; the proof for $\Lambda = \mathbb{F}_\ell$ is exactly similar, working in the category of groups instead of Lie algebras. Applying the Lie functor to the short exact sequence of ℓ-adic Lie groups $1 \to N \to \Pi_{\tilde{E}} \to \Pi_V \to 1$ we get the short exact sequence of $\mathbb{Q}_\ell$-Lie algebras

$$0 \to \mathrm{Lie}(N) \to \mathrm{Lie}(\Pi_{\tilde{E}}) \to \mathrm{Lie}(\Pi_V) \to 0$$

and, applying the abelianization functor, the short exact sequence

$$0 \to \mathrm{Lie}(N) \cap [\mathrm{Lie}(\Pi_{\tilde{E}}), \mathrm{Lie}(\Pi_{\tilde{E}})] \to \mathrm{Lie}(N) \to \mathrm{Lie}(\Pi_{\tilde{E}})^{ab} \to \mathrm{Lie}(\Pi_V)^{ab} \to 0.$$

To show $\mathrm{Lie}(N) \subset [\mathrm{Lie}(\Pi_{\tilde{E}}), \mathrm{Lie}(\Pi_{\tilde{E}})]$, it is enough to show $\mathrm{Lie}(N_E) = [\mathrm{Lie}(\Pi_{\tilde{E}}), \mathrm{Lie}(N)]$, which follows from

$$\mathrm{Lie}(N)/[\mathrm{Lie}(\Pi_{\tilde{E}}), \mathrm{Lie}(N)] = \mathrm{Lie}(N)_{\mathrm{Lie}(\Pi_{\tilde{E}})}$$

and $\mathrm{Lie}(N)_{\mathrm{Lie}(\Pi_{\tilde{E}})} = 0$ since $\mathrm{Lie}(N) \subset E^\vee \otimes V \simeq V^{\dim(E^\vee)}$ is a Π-invariant submodule. $\square$

3. Specialization of representations of the étale fundamental group and injectivity of the restriction morphism on H^1

Let S be a smooth, geometrically connected variety over k. We are now going to apply the results of Section 2 to the case where $\Pi := \pi_1(S)$ and $\Gamma = \Pi_s := \pi_1(s)$ for $s \in |S|$.

3.1. $\Lambda = \mathbb{Q}_\ell$**.** Let $H \in \mathcal{M}\mathrm{od}_{\mathbb{Q}_\ell}(\Pi)$. Recall that $S(H) \subset |S|$ denote the set of all $s \in |S|$ such that $\Pi_{s,H} \subset \Pi_H$ is open. By definition, when $H = \tilde{E}$ is the universal extension attached to a pair (E, V) (Subsection 2.1) Condition (O) of Lemma 2.1.3 is satisfied for $N = \Pi_{\tilde{E}}$, $\Gamma_{\tilde{E}} := \Pi_{s,\tilde{E}}$, $s \in S(\tilde{E})$. In particular, $S(\tilde{E}) \subset S(E, V)$.

When k is 'arithmetically rich' in the sense that it has a huge non-abelian absolute Galois group, one expect $S(H)^{\leq d}$ to be 'huge'.

3.1.1. The first result in this direction is the following elementary group-theoretical and very general observation.

3.1.1.1. Fact.[1] ([Se89, 10.6])*Assume k is Hilbertian of characteristic $p \geq 0$ and let $\mathcal{L}$ be a finite set of primes $\neq p$. For each $\ell \in \mathcal{L}$ fix $H_\ell \in \mathcal{M}\mathrm{od}_{\mathbb{Q}_\ell}(\Pi)$. Then there exists an integer $d \geq 1$ such that for infinitely many $s \in |S|^{\leq d}$, the images of Π_s and Π acting on $\oplus_{\ell \in \mathcal{L}} H_\ell$ coincide. In particular, $\cap_{\ell \in \mathcal{L}} S(H_\ell)^{\leq d}$ is infinite.*

Fact 3.1.1 applies in particular to fields k which are finitely generated over their prime field [Se89, 9.5 - Rem. 4), 5), 9.6].

3.1.1.2. Combining Fact 3.1.1 and Lemma 2.1.3, one gets:

Corollary. *Assume k is Hilbertian of characteristic $p \geq 0$ and let $\mathcal{L}$ be a finite set of primes $\neq p$. For each $\ell \in \mathcal{L}$, let $V_\ell \in \mathcal{M}\mathrm{od}_{\mathbb{Q}_\ell}(\Pi)$ and $E_\ell \subset H^1(\Pi, V_\ell)$ a finite-dimensional $\mathbb{Q}_\ell$-vector subspace. Then there exists an integer $d \geq 1$ such that $\cap_{\ell \in \mathcal{L}} S(E_\ell, V_\ell)^{\leq d}$ is infinite.*

3.1.2. When k is finitely generated and provided Π_H satisfies some mild assumption, one expects $S(H)^{\leq d}$ satisfies much stronger abundance results. For instance, if H is motivic, $d = 1$ and k is a number field, it should follow from the Bombieri-Lang conjecture and some (more tractable) conjectures on the geometric properties of some projective systems of étale covers of S attached to V that $(|S| \setminus S(H))(k)$ is not Zariski-dense in S (See *e.g.* the brief discussion in [CCh20, §2.2]). In this direction, one has the following unconditional result, which ultimately relies on Mordell and Mordell-Lang conjectures.

3.1.2.1. Fact. (([CT12], [CT13]) *Assume k is finitely generated over $\mathbb{Q}$ and S is a curve. Suppose (P)* $\mathrm{Lie}(\overline{\Pi}_H)$ *is perfect. Then for every integer $d \geq 1$, $(|S| \setminus S(H))^{\leq d}$ is finite.*

Remark.
(1) Furthermore, $\sup\{[\Pi_V : \Pi_{s,V}] \mid s \in (|S| \setminus S(V))^{\leq d}\} < +\infty$ but we will not need this uniformity result in our applications.
(2) For an analogue of Fact 3.1.2.1 to the char $p > 0$ case when $d = 1$ see [A18].

3.1.2.2. From Lemma 2.2, the condition that V has no $\overline{\Pi}$-Lie-coinvariants ensures Condition (P) of Fact 3.1.2.1 is satisfied by the universal extension $\tilde{E}$ attached to any pair (E, V). Combining this with Lemma 2.1.3 and Fact 3.1.2.1, one gets:

Corollary. *Assume k is finitely generated of characteristic* 0 *and S is a curve. Let $V \in \mathcal{M}\mathrm{od}_{\mathbb{Q}_\ell}(\Pi)$. Assume*

[1] More generally, if $\Pi \twoheadrightarrow \Pi''$ is any continuous quotient such that the Frattini subgroup of Π'' is open in Π then there exists an integer $d \geq 1$ such that for infinitely many $s \in |S|^{\leq d}$, the images of Π_s and Π in Π'' coincide. The condition that the Frattini subgroup of Π'' is open in Π is equivalent to the fact that the set of primes dividing the order of Π'' is finite and that the p-Sylow of Π'' are topologically finitely generated. In particular, it is satisfied if Π'' is a closed subgroup of a finite product of compact ℓ-adic Lie groups. See [Se89, 10.6, Prop. p. 148].

> *(P)* $\mathrm{Lie}(\overline{\Pi}_V)$ *is perfect;*
> *(cI)* V *has no* $\overline{\Pi}$*-Lie-coinvariants.*

Then, for every finite-dimensional $\mathbb{Q}_\ell$*-vector subspace* $E \subset H^1(\Pi, V)$ *and integer* $d \geq 1$, $(|S| \setminus S(E,V))^{\leq d}$ *is finite.*

3.2. $\Lambda = \mathbb{F}_\ell$**.** Let $H \in \mathcal{M}\mathrm{od}_{\mathbb{F}_\ell}(\Pi)$. Let $S(H) \subset |S|$ denote the set of all $s \in |S|$ such that $\Pi_{s,H} \subset \Pi_H$ is of index prime-to-ℓ. By definition, when $H = \tilde{E}$ is the universal extension attached to a pair (E, V) (Subsection 2.1) Condition (O) of Lemma 2.1.3 is satisfied for $N = \Pi_{\tilde{E}}$, $\Gamma_{\tilde{E}} := \Pi_{s,\tilde{E}}$, $s \in S(\tilde{E})$. In particular, $S(\tilde{E}) \subset S(E, V)$.

Let $\mathcal{L}$ be an infinite set of primes and let $\underline{H} = H_\ell$, $\ell \in \mathcal{L}$ be a family of elements in $\mathcal{M}\mathrm{od}_{\mathbb{F}_\ell}$ of uniformly bounded Λ_ℓ-rank $r_\ell \leq r$. Since $\mathcal{L}$ is infinite, Fact 3.1.1 (under its general form stated in footnote 1) fails. Still, one has the following analogue of Fact 3.1.2. For a finite subgroup $G \subset \mathrm{GL}(V_\ell)$, let $G^+ \subset G$ denote the (characteristic) subgroup of G generated by its order ℓ elements.

3.2.1. Fact. ([EHK12], [CT19, §7]) *Assume* k *is finitely generated over* $\mathbb{Q}$ *and* S *is a curve. Suppose* $\underline{H}$ *satisfies*

> *(U)* *There exists an open subgroup* $U \subset \overline{\Pi}$ *such that* $U_{H_\ell} = U^+_{H_\ell}$, $\ell \in \mathcal{L}$*;*
> *(P)* $\overline{\Pi}^+_{H_\ell}$ *is perfect for* $\ell \gg 0$.

Then for every integer $d \geq 1$ *there exists an integer* $B_d \geq 1$ *such that for* $\ell \gg 0$ *the set of all* $s \in |S|^{\leq d}$ *with* $[\Pi_{H_\ell} : \Pi_{s,H_\ell}] > B_d$ *is finite. In particular,* $(|S| \setminus S(H_\ell))^{\leq d}$ *is finite.*

Remark. For an analogue of the above Fact to the char $p > 0$ case see [CT19], [CT20].

3.2.2. Again, from Lemma 2.2, the condition that V_ℓ has no $\overline{\Pi}$-coinvariants ensures Condition (P) of Fact 3.2 is satisfied by the universal extension $\tilde{E}_\ell$ attached to any pair (E_ℓ, V_ℓ). Combining this with Lemma 2.1.3 and Fact 3.2, one gets:

Corollary. *Assume* k *is finitely generated of characteristic* 0 *and* S *is a curve. Let* $\mathcal{L}$ *be an infinite set of primes and let* $\underline{V} = V_\ell$, $\ell \in \mathcal{L}$ *be a family of elements in* $\mathcal{M}\mathrm{od}_{\mathbb{F}_\ell}(\Pi)$*. Assume*

> *(U)* *There exists an open subgroup* $U \subset \overline{\Pi}$ *such that* $U_{V_\ell} = U^+_{V_\ell}$, $\ell \in \mathcal{L}$*;*
> *(P)* $\overline{\Pi}^+_{V_\ell}$ *is perfect for* $\ell \gg 0$*;*
> *(cI)* $\underline{V}$ *has no* $\overline{\Pi}$*-quasi-coinvariants,*

Then, for every family of finitely generated $\mathbb{F}_\ell$*-submodules* $E_\ell \subset \mathrm{H}^1(\Pi, V_\ell)$, $\ell \in \mathcal{L}$ *of uniformly bounded* $\mathbb{F}_\ell$*-dimension* $r_\ell \leq r$ *and integer* $d \geq 1$, $(|S| \setminus S(V_\ell, E_\ell))^{\leq d}$*) is finite,* $\ell \gg 0$.

3.3. Base change. Let $\Lambda = \mathbb{Q}_\ell$ or $\mathbb{F}_\ell$ and $V \in \mathcal{M}\mathrm{od}_\Lambda(\Pi)$. Let $f : S' \to S$ be a Galois cover and write $\Pi' := \pi_1(S') \subset \Pi$. Let $s \in |S|$ and $s' \in |S'|$ lying over s. The restriction morphisms induce a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\Pi, V) \arrow[r, "res"] \arrow[d, "res_s"] & \mathrm{H}^1(\Pi', V) \arrow[d, "res_{s'}"] \\
\mathrm{H}^1(\Pi_s, V) \arrow[r, "res"] & \mathrm{H}^1(\Pi_{s'}, V)
\end{tikzcd}$$

and the inflation-restriction exact sequence shows that the horizontal arrows are injective provided $S' \to S$ has degree invertible in Λ. In particular, $f^{-1}(S(E,V)) = S(res(E), V|_{\Pi'})$.

This shows that the following are equivalent:
- Corollary 3.1.1.2 for V_ℓ, $\ell \in \mathcal{L}$ (resp. Corollary 3.1.2.2 for V);

- Corollary 3.1.1.2 for $V_\ell|_{\Pi'}$, $\ell \in \mathcal{L}$ (resp. Corollary 3.1.2.2 for $V|_{\Pi'}$) for some Galois cover $S' \to S$;
- Corollary 3.1.1.2 for $V_\ell|_{\Pi'}$, $\ell \in \mathcal{L}$ (resp. Corollary 3.1.2.2 for $V|_{\Pi'}$) for every Galois cover $S' \to S$.

and, similarly, that the following are equivalent:
- Corollary 3.2.2 for $\underline{V}$;
- Corollary 3.2.2 for $\underline{V}$ for some Galois cover $S' \to S$;
- Corollary 3.2.2 for $\underline{V}$ for every Galois cover $S' \to S$.

In particular, to apply Corollaries 3.1.1.2, 3.1.2.2, 3.2.2 we may freely replace S by a connected étale cover hence, in the setting[2] of Corollary 3.2.2 , replace (U) with the seemingly stronger assumption

$$(\mathrm{U}+)\ \overline{\Pi}_{V_\ell} = \overline{\Pi}^+_{V_\ell} \text{ for } \ell \gg 0.$$

3.4. Remarks about assumptions (P), (SS), (cI), (I).

3.4.1. (P) in Corollary 3.1.2.2 (resp. in Corollary 3.2.2) can be ensured by the stronger (resp. - see [CT17, Cor. 3.3]) assumption (SS) $Lie(\overline{\Pi}_V)$ is semisimple. (resp. (SS) $\overline{\Pi}$ acts semisimply on V_ℓ, $\ell \gg 0$). But (SS), contrary to (P), does not transfer in general from V to $\tilde{E}$ (Lemma 2.2).

3.4.2. In the setting of Corollary 3.1.2.2 (resp. of Corollary 3.2.2), consider the condition (I) $V^{\mathrm{Lie}(\overline{\Pi}_V)} = 0$ (resp. (I) $V_\ell^{\overline{\Pi}} = 0$, $\ell \gg 0$). Then, under (SS), (I) and (cI) are equivalent.

3.4.3. The restriction that $E \subset \mathrm{H}^1(\Pi, V)$ be a finite dimensional $\mathbb{Q}_\ell$-vector subspace in Corollaries 3.1.1.2, 3.1.2.2 (resp. that $E_\ell \subset \mathrm{H}^1(\Pi, V_\ell)$, $\ell \in \mathcal{L}$ be of uniformly bounded $\mathbb{F}_\ell$-dimension $r_\ell \leq r$ in Corollary 3.2.2) becomes tautological under (I) (resp. provided the V_ℓ, $\ell \in \mathcal{L}$ are of uniformly bounded $\mathbb{F}_\ell$-dimension, $\ell \in \mathcal{L}$). Indeed, if (I) holds, the inflation-restriction exact sequence implies that the restriction morphism $\mathrm{H}^1(\Pi, V) \to \mathrm{H}^1(\overline{\Pi}, V)$ is injective (resp. that the restriction morphism $\mathrm{H}^1(\Pi, V_\ell) \to \mathrm{H}^1(\overline{\Pi}, V_\ell)$ is injective, $\ell \gg 0$). Thus it is enough to prove that $\dim_{\mathbb{Q}_\ell}(\mathrm{H}^1(\overline{\Pi}, V)) < +\infty$ (resp. that $\dim_{\mathbb{F}_\ell}(\mathrm{H}^1(\overline{\Pi}, V_\ell))$ is uniformly bounded, $\ell \in \mathcal{L}$). In the non resp. situation, this follows from the fact that $\mathrm{H}^1(\overline{\Pi}, V) \tilde{\to} \mathrm{H}^1_{et}(S_{\overline{k}}, V)$. In the resp. situation, the fact that $\mathrm{H}^1(\overline{\Pi}, V_\ell) \tilde{\to} \mathrm{H}^1_{et}(S_{\overline{k}}, V)$ already shows that $E_\ell := \mathrm{H}^1(\overline{\Pi}, V_\ell)$ has finite $\mathbb{F}_\ell$-dimension. Let $\tilde{E}_\ell$ denote the universal extension attached to (E_ℓ, V_ℓ).
- (i) By construction, $N_\ell := \ker(\overline{\Pi}_{\tilde{E}_\ell} \twoheadrightarrow \overline{\Pi}_{V_\ell}) \subset \mathrm{Hom}_{\mathbb{F}_\ell}(E_\ell, V_\ell)$ is an elementary ℓ-group.
- (ii) By construction and the inflation-restriction exact sequence, we have a canonical isomorphism $\mathrm{H}^1(\overline{\Pi}_{\tilde{E}_\ell}, V_\ell) \tilde{\to} \mathrm{H}^1(\overline{\Pi}, V_\ell)$.

Now, recall $\overline{\Pi}$ is topologically finitely generated; let s denote the minimal number of topological generators of $\overline{\Pi}$. Then $\overline{\Pi}_{\tilde{E}_\ell}$ is generated by $\leq s$ elements. As $\mathrm{H}^1(\overline{\Pi}_{\tilde{E}_\ell}, V_\ell)$ is a quotient of the $\mathbb{F}_\ell$-module of 1-cocycles $Z^1(\overline{\Pi}_{\tilde{E}_\ell}, V_\ell)$ and as 1-cocyles are entirely determined by their image on generators of $\overline{\Pi}_{\tilde{E}_\ell}$, one has

$$\dim_{\mathbb{F}_\ell}(\mathrm{H}^1(\overline{\Pi}_{\tilde{E}_\ell}, V_\ell)) \leq \dim_{\mathbb{F}_\ell}(Z^1(\overline{\Pi}_{\tilde{E}_\ell}, V_\ell)) \leq s\dim_{\mathbb{F}_\ell}(V_\ell).$$

3.4.4. In the setting of Corollary 3.1.2.2 (resp. Corollary 3.2.2), (P) implies (I) for $V/V^{\mathrm{Lie}(\overline{\Pi}_V)}$ (resp. (U+) and (P) imply (I) for $V_\ell/V_\ell^{\overline{\Pi}}$, $\ell \in \mathcal{L}$). Indeed, if for every $v \in V$ the 1-cocycle $\mathrm{Lie}(\overline{\Pi}_V) \to V$, $g \to gv$ takes its value in $V^{\mathrm{Lie}(\overline{\Pi}_V)}$, it factors through $\mathrm{Lie}(\overline{\Pi}_V)^{ab}$, which is 0 by (P) (resp. if for every $v \in V_\ell$, $\ell \in \mathcal{L}$ the 1-cocycle $\overline{\Pi}_{V_\ell} \to V_\ell$, $\pi \to \pi v - v$ takes its value in $V_\ell^{\overline{\Pi}}$, it factors through ${\overline{\Pi}_{V_\ell}}^{ab}$, which is 0 by (U+) and (P)).

3.4.5. (cI) does not hold in general. However by (3.4.4), in the setting of Corollary 3.1.2.2, (SS) implies (cI) (=(I)) for $V/V^{\mathrm{Lie}(\overline{\Pi}_V)}$ and, in the setting of Corollary 3.2.2, (U+), (SS) imply (cI) (=(I)) for $V_\ell/V_\ell^{\overline{\Pi}}$, $\ell \in \mathcal{L}$.

4. Motivic representations

The formulation of the group-theoretical Corollaries 3.1.2.2, 3.2.2 is motivated by its possible applications to motivic representations that is (subquotients cut out by algebraic correspondance of) those of the form

- $V = \mathrm{H}^u(X_{\overline{\eta}}, \mathbb{Q}_\ell(v)) \in \mathcal{M}\mathrm{od}_{\mathbb{Q}_\ell}(\Pi)$;
- $V_\ell = \mathrm{H}^u(X_{\overline{\eta}}, \mu_\ell^{\otimes v}) \in \mathcal{M}\mathrm{od}_{\mathbb{F}_\ell}(\Pi)$, $\ell \neq p$.

for some smooth proper morphism $X \to S$ (smooth-proper base change - see [SGA4-III, Exp. XVI]).

We recall briefly the arguments ensuring assumptions (SS) (hence (P)), (U) and (I) (hence (cI)) and give references for more details.

- In Corollary 3.1.2.2, (P) is a consequence of (SS), which follows from comparison between Betti and étale cohomology and Deligne's semisimplicity theorem for variation of polarizable Hodge structures ([D71]).

- In Corollary 3.2.2,

  – The fact that the $\mathbb{F}_\ell$-rank of V_ℓ is uniformly bounded follows from the comparison between Betti and étale cohomology and the fact that Betti-cohomology with $\mathbb{Z}$-coefficients in finitely generated;
  – for $p \geq 0$: the torsion-freeness of $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)$ for $\ell \gg 0$ ([**?**]) and the Weil conjectures ([**?**]).

  – For (U), see [CT17].

  – (P) is a consequence of (SS) which follows from comparison between Betti and étale cohomology, the fact that Betti-cohomology with $\mathbb{Z}$-coefficients in finitely generated and the constructibility of semisimplicity - see [CT11, 2.2].

- (I) does not hold in general; this is closely related to isotriviality. For instance, if $u = 1$, by the geometric Lang-Néron theorem ([LN59]), (I) holds if and only if $\mathrm{Pic}^\circ(X_{\overline{\eta}})$ contains no non-trivial $\overline{k}$-isotrivial abelian subvariety.

5. Specialization of the ℓ-adic Abel-Jacobi filtration on Chow groups

For a noetherian scheme $\mathcal{X}$ and an integer $i \geq 0$, let $\mathrm{Z}^i(\mathcal{X})$ denote the group of codimension i algebraic cycles and $\mathrm{CH}^i(\mathcal{X})$ denote the Chow group of codimension i algebraic cycles modulo rational equivalence. Let X be a smooth projective geometrically connected scheme of dimension g over a field K of characteristic $p \geq 0$. Determining the structure of the groups $\mathrm{CH}^i(X)$, $i \geq 0$ is a difficult still widely open problem. In particular, these groups are not finitely generated in general and they are very sensitive to extensions of the base field K. If K is finitely generated over its prime field, one technics is to endow $\mathrm{CH}^i(X)$ with the so-called ℓ-adic Abel-Jacobi filtrations, $\ell \neq p$ and study the graded pieces of these, which embed in continuous Galois cohomology groups. To simplify the exposition, we work with $\mathbb{Q}_\ell$-coefficients. Our arguments can be basically transposed as they are to $\mathbb{F}_\ell$-coefficients provided $\ell \gg 0$. We refer to [J88] for the technical setting of continuous cohomology and to [R95] for a nice introduction to the formalism of ℓ-adic Abel-Jacobi filtrations and related motivic conjectures (see also [J90]).

5.1. Absolute setting. Let X be a smooth projective geometrically connected scheme of dimension g over a finitely generated field K of characteristic $p \geq 0$.

5.1.1. *Construction.* Consider the Hochschild-Serre spectral sequence for continuous étale cohomology

$$E_2^{j,2i-j} = \mathrm{H}^j(K, \mathrm{H}^{2i-j}(X_{\overline{K}}, \mathbb{Q}_\ell)(i)) \Rightarrow \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i)),$$

and pull-back the induced filtration

$$F^{i,0}_{H,\ell}(X) = \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i)) \supset F^{i,1}_{H,\ell}(X) \supset \cdots \supset F^{i,2i}_{H,\ell}(X) \supset 0$$

to $\mathrm{CH}^i(X) \otimes \mathbb{Q}$ *via* the cycle class map $cl^i_\ell : \mathrm{CH}^i(X) \otimes \mathbb{Q} \to \mathrm{H}^{2i}(X, \mathbb{Q}_\ell(i))$. The resulting filtration

$$F^{i,0}_{\ell}(X) = \mathrm{CH}^i(X) \otimes \mathbb{Q} \supset F^{i,1}_{\ell}(X) \supset \cdots \supset F^{i,2i}_{\ell}(X) \supset 0$$

is called the ℓ-adic Abel-Jacobi filtration and the canonical morphisms

$$a^{i,j}_\ell : F^{i,j}_\ell(X) \to F^{i,j}_{H,\ell}(X)/F^{i,j+1}_{H,\ell}(X) =: Gr^{i,j}_{H,\ell}(X),\ j \geq 0$$

the ℓ-adic Abel-Jacobi maps. Write $Gr^{i,j}_\ell(X) := F^{i,j}_\ell(X)/F^{i,j+1}_\ell(X)$. The ℓ-adic Abel-Jacobi maps induce an injective graded morphism $Gr^{i,\bullet}_\ell(X) \hookrightarrow Gr^{i,*}_{H,\ell}(X)$. It is known that the Hochschild-Serre spectral sequence degenerates at E_2 ([R95, Th. 1]) so, what we get, actually, is a morphism of graded groups

$$\bigoplus_{0 \leq j \leq 2i} Gr^{i,j}_\ell(X) \to \bigoplus_{0 \leq j \leq 2i} \mathrm{H}^j(K, \mathrm{H}^{2i-j}(X_{\overline{K}}, \mathbb{Q}_\ell)(i))$$

which is injective in degree $\leq 2i-1$.

Warning: *A priori*, the canonical morphism $Gr^{i,\bullet}_\ell(X) \otimes \mathbb{Q}_\ell \to Gr^{i,\bullet}_{H,\ell}(X)$ is neither injective nor surjective in general. In particular, the $\mathbb{Q}$-rank of $Gr^{i,\bullet}_\ell(X)$ might be infinite even though the $\mathbb{Q}_\ell$-rank of $Gr^{i,\bullet}_{H,\ell}(X)$ is finite.

5.1.2. *Example.* For codimension 1 and dimension 0-cycles, the filtration is rather explicit.

For an abelian variety A over a field K and an integer $n \geq 1$, let $A[n]$ denote the kernel of the multiplication-by-n morphism on $A(\overline{K})$. For a prime ℓ, write $T_\ell(A) := \varprojlim A[\ell^n]$, $V_\ell(A) := T_\ell(A) \otimes \mathbb{Q}_\ell$.

Let $\alpha : X \to \mathrm{Alb}_X$ denote the Albanese variety and Pic_X the Picard variety of X; let $\mathrm{Pic}^\circ_X \subset \mathrm{Pic}_X$ denote the connected component of Pic_X. One has canonical isomorphisms $\mathrm{H}^1(X_{\overline{K}}, \mathbb{Q}_\ell(1)) \tilde{\to} V_\ell(\mathrm{Pic}^\circ_X)$ and $\mathrm{H}^{2g-1}(X_{\overline{K}}, \mathbb{Q}_\ell(g)) \tilde{\to} V_\ell(\mathrm{Alb}_X)$. These isomorphisms induce commutative diagrams (see *e.g.* [R95, Appendix]), where κ_ℓ is the ℓ-adic Kummer morphism (induced by taking cohomology, projective limit and $- \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ from the projective system of

Kummer short exact sequences $0 \to \mathrm{Alb}_X[\ell^n] \to \mathrm{Alb}_X \xrightarrow{\ell^n} \mathrm{Alb}_X \to 0$, $n \geq 1$)

$$\begin{tikzcd}
 & Gr^{1,1}_{H,\ell}(X) = \mathrm{H}^1(K, V_\ell(\mathrm{Pic}^\circ_X)) \\
F^{1,1}_\ell(X) = \mathrm{Pic}^\circ(X) \otimes \mathbb{Q} \arrow[r] \arrow[ur, hook, "a^{1,1}_\ell = \kappa_\ell"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & F^{1,1}_{H,\ell}(X) \arrow[u, two heads] \arrow[d, hook] \\
F^{1,0}_\ell(X) = \mathrm{Pic}(X) \otimes \mathbb{Q} \arrow[r] \arrow[dr, "cl^1_\ell"'] & F^{1,0}_{H,\ell}(X) \arrow[d, two heads] \\
 & Gr^{1,0}_{H,\ell}(X) = \mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{\pi_1(K)}
\end{tikzcd}$$

$$\begin{tikzcd}
F^{g,1}_\ell(X) \arrow[r, "a^{g,1}_\ell"] \arrow[d, "\alpha"'] & Gr^{g,1}_{H,\ell}(X) = \mathrm{H}^1(K, V_\ell(\mathrm{Alb}_X)) \\
\mathrm{Alb}_X(K) \otimes \mathbb{Q} \arrow[ur, hook, "\kappa_\ell"']
\end{tikzcd}$$

5.2. Relative setting. Let S be a smooth, geometrically connected curve (to simplify) over a field k of characteristic $p \geq 0$. Let $f : X \to S$ be a smooth, projective morphism with geometrically connected fibers. Fix an integer $i \geq 1$ and a prime $\ell \neq p$. We first compare the ℓ-adic Abel-Jacobi filtrations on $\mathrm{CH}^i(X_\eta)$ and $\mathrm{CH}^i(X_s)$. This can be done using the arguments in [R95, 2], which we recall briefly.

5.2.1. For $s \in |S|$, let $\mathcal{O}_s$ denote the local ring of S at s and $\mathcal{O}^h_s$ its henselisation. Write $X(s) \to S(s)$ and $X(s)^h \to S(s)^h$ for the base-change of $X \to S$ *via* $S(s) := \mathrm{spec}(\mathcal{O}_s) \to S$ and $S(s)^h := \mathrm{spec}(\mathcal{O}^h_s) \to S$ respectively. Let η_s denote the generic point of $S(s)^h$. Then one has a canonical commutative specialization diagram ([Fu75, 4.4], [Fu84, 20.2, 20.3])

$$\begin{tikzcd}
 & \mathrm{CH}^i(X(s)) \arrow[dl, "\eta^*"'] \arrow[dr, "s^*"] & \\
\mathrm{CH}^i(X_\eta) \arrow[rr, "sp_s"'] & & \mathrm{CH}^i(X_s),
\end{tikzcd}$$

which is compatible with the cycle maps, that is such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{CH}^i(X_\eta) \arrow[rr, bend left, "sp_s"] \arrow[d] & \mathrm{CH}^i(X(s)) \arrow[l, "\eta^*"'] \arrow[r, "i^*"'] \arrow[d] & \mathrm{CH}^i(X_s) \arrow[d] \\
\mathrm{H}^{2i}(X_\eta, \mathbb{Q}_\ell(i)) \arrow[d] & \mathrm{H}^{2i}(X(s), \mathbb{Q}_\ell(i)) \arrow[l, "\eta^*"'] \arrow[r, "i^*"'] \arrow[d] & \mathrm{H}^{2i}(X_s, \mathbb{Q}_\ell(i)) \\
\mathrm{H}^{2i}(X_{\eta_s}, \mathbb{Q}_\ell(i)) & \mathrm{H}^{2i}(X(s)^h, \mathbb{Q}_\ell(i)) \arrow[l, "\eta_s^*"'] \arrow[ur, "\simeq", "i^*"'] &
\end{tikzcd}$$

One also has the Leray spectral sequence $E_2^{j,2i-j} = \mathrm{H}^j(S(s)^h, \mathrm{H}^{2i-j}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell(i)) \Rightarrow \mathrm{H}^{2i}(X(s)^h, \mathbb{Q}_\ell(i))$, which is compatible with the Hoschshild-Serre spectral sequences of 5.1.1 for X_s over $k(s)$ and X_{η_s} over $k(\eta_s)$. The key point is that the morphisms

$$\mathrm{H}^j(S(s)^h, \mathrm{H}^{2i-j}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell)(i)) \to \mathrm{H}^j(\eta_s, \mathrm{H}^{2i-j}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell)(i))$$

are injective ([R95, Lemma 2.13]; see also [CTHK97, Thm. B.2.1]). This shows that the Leray spectral sequence degenerates at E_2 (since the Hoschshild-Serre spectral sequence does) and then, by a straightforward induction on j, that

$$sp_s(F_\ell^{i,j}(X_\eta)) \subset F_\ell^{i,j}(X_s),\ \ j \geq 0$$

As a result, we obtain a commutative specialization diagram of graded groups

(5.2.1.1)
$$\begin{tikzcd}
\bigoplus_{j\geq 0} Gr_\ell^{i,j}(X_\eta) \arrow[r] \arrow[dd, "sp_s"] & \bigoplus\limits_{j\geq 0} \mathrm{H}^j(\Pi_\eta, \mathrm{H}^{2i-j}(X_{\overline{\eta}}, \mathbb{Q}_\ell(i))) \arrow[r] & \bigoplus\limits_{j\geq 0} \mathrm{H}^j(\Pi_{\eta_s}, \mathrm{H}^{2i-j}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell(i))) \\
 & \bigoplus\limits_{j\geq 0} \mathrm{H}^j(\Pi, \mathrm{H}^{2i-j}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell(i))) \arrow[r] \arrow[u, "infl"] \arrow[d, "res_s"] & \bigoplus\limits_{j\geq 0} \mathrm{H}^j(S(s)^h, R^{2i-j}f_*\mathbb{Q}_\ell(i)) \arrow[u, hook] \arrow[dl, "\simeq"] \\
\bigoplus_{j\geq 0} Gr_\ell^{i,j}(X_s) \arrow[r] & \bigoplus\limits_{j\geq 0} \mathrm{H}^j(\Pi_s, \mathrm{H}^{2i-j}(X_{\overline{s}}, \mathbb{Q}_\ell(i))) &
\end{tikzcd}$$

In (5.2.1.1), the central upper vertical arrow is the inflation map attached to the epimorphism $\Pi_\eta \twoheadrightarrow \Pi$ (recall that S is smooth hence normal) and the right vertical arrow is the one induced by the morphisms of sites $S(s)^h_{et} \to S_{et} \to \mathcal{B}\Pi$. Write $V := \mathrm{H}^{2i-1}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell(i))$.

5.2.2. Lemma. *The inflation morphism* $infl : \mathrm{H}^1(\Pi, V) \hookrightarrow \mathrm{H}^1(\Pi_\eta, V)$ *is an isomorphism.*

Proof. Let $S \hookrightarrow S^{cpt}$ denote the smooth compactification of S and set $\partial S := S^{cpt} \setminus S$. The kernel N of $\Pi_\eta \twoheadrightarrow \Pi$ is generated by the inertia groups at $s \in \partial S$. By the Hochschild-Serre spectral sequence, it is enough to show that $\mathrm{H}^1(N, V)^\Pi = 0$. By the smooth-proper base change theorem N acts trivially on V so that $\mathrm{H}^1(N, V)^\Pi = \mathrm{Hom}_\Pi(N, V)$. Hence it is enough to show that for every inertia group $I_s \subset N$ at $s \in \partial S$ one has $\mathrm{Hom}_{\Pi_s}(I_s^{(\ell)}, V) = 0$, where $(-)^{(\ell)}$ denotes pro-ℓ completion. But, as Π_s-modules, $I_s^{(\ell)} \simeq \mathbb{Z}_\ell(1)$ hence has Weil weight -2 whereas V has Weil weights -1. $\square$

5.2.3. Write $\widetilde{V} := V/V^{\mathrm{Lie}(\overline{\Pi}_V)}$. Since V is a semisimple $\overline{\Pi}$-module (see 4) that is V satisfies (SS), $\widetilde{V}^{\mathrm{Lie}(\overline{\Pi}_V)} = 0$. Equivalently, $\widetilde{V}$ satisfies (I). Since $\widetilde{V}$ also satisfies (SS), it satisfies (cI). Eventually, from 3.4.3, $\dim_{\mathbb{Q}_\ell}(\mathrm{H}^1(\Pi_\eta, \widetilde{V})) < +\infty$.

For $s \in S$, let $\widetilde{Gr}_\ell^{i,1}(X_s)$ denote the image of

$$F_\ell^{i,1}(X_s) \stackrel{a_\ell^{i,1}}{\to} \mathrm{H}^1(\Pi_s, V) \to \mathrm{H}^1(\Pi_s, \widetilde{V}).$$

For $s \in |S|$ the specialization morphism $sp_s : Gr_\ell^{i,1}(X_\eta) \to Gr_\ell^{i,1}(X_s)$ induces a commutative square

$$\begin{tikzcd}
\widetilde{Gr}_\ell^{i,1}(X_\eta) \arrow[r, hook] \arrow[d, "sp_s"] & \mathrm{H}^1(\Pi_\eta, \widetilde{V}) \arrow[d, "res_s"] \\
\widetilde{Gr}_\ell^{i,1}(X_s) \arrow[r, hook] & \mathrm{H}^1(\Pi_s, \widetilde{V})
\end{tikzcd}$$

If k is finitely generated over its prime field, the image of $Gr_\ell^{i,1}(X_\eta)$ in $\mathrm{H}^1(\Pi, V)$ always generates a $\mathbb{Q}_\ell$-submodule of finite $\mathbb{Q}_\ell$-rank ([R95, Prop. 2.5]). Combining these observations and Lemma 5.2.2, Corollary 3.1.1 and Corollary 3.1.2.2 yield the following results about specialization of $Gr_\ell^{i,1}$.

5.2.3.1. Corollary. *Assume k is Hilbertian. Let $\mathcal{L}$ be a finite set of primes not containing p. Set $V_\ell := \mathrm{H}^{2i-1}(X_{\overline{\eta}_s}, \mathbb{Q}_\ell(i))$, $\ell \in \mathcal{L}$ and let $E_\ell \subset Gr_\ell^{i,1}(X_\eta)$ whose image in $\mathrm{H}^1(\Pi, V_\ell)$ generates a finite-dimensional $\mathbb{Q}_\ell$-subvector space, $\ell \in \mathcal{L}$. Then there exists an integer $d \geq 1$ such that the specialization morphisms*

$$sp_s^1 : E_\ell \hookrightarrow Gr_\ell^{i,1}(X_\eta) \to Gr_\ell^{i,1}(X_s),\ \ell \in \mathcal{L}$$

are injective for infinitely many $s \in |S|^{\leq d}$.

In particular, if k is finitely generated over its prime field, there always exists $s \in |S|$ such that $\mathrm{rank}(Gr_\ell^{i,1}(X_\eta)) \leq \mathrm{rank}(Gr_\ell^{i,1}(X_s))$, $\ell \in \Sigma$ (where we allow infinite $\mathbb{Q}$-rank).

5.2.3.2. Corollary. *Assume k is finitely generated over its prime field and $p = 0$. Then for every integer $d \geq 1$ and all but finitely many $s \in |S|^{\leq d}$, the specialization morphism $\widetilde{sp}_s^1 : \widetilde{Gr}_\ell^{i,1}(X_\eta) \to \widetilde{Gr}_\ell^{i,1}(X_s)$ is injective.*

Remark. Since the specialization morphism $Gr_\ell^{i,0}(X_\eta) \to Gr_\ell^{i,0}(X_s)$ is injective for every $s \in |S|$, we may replace $Gr_\ell^{i,1}$ (resp. $\widetilde{Gr}_\ell^{i,1}$) with $Gr_\ell^{i,0} \oplus Gr_\ell^{i,1}$ (resp. $Gr_\ell^{i,0} \oplus \widetilde{Gr}_\ell^{i,1}$) in Corollaries 5.2.3.1, 5.2.3.2.

5.3. Note that our method only allows to tackle specialization issues for families over a 'geometric' basis that is schemes of finite type over a field k. Very little seems to be known for specialization over 'arithmetic' bases that is scheme of finite type over $\mathbb{Z}$. See however [Sc06] for a discussion about ℓ-primary torsion classes in Chow groups.

6. Applications, examples

6.1. About the ℓ-independence of the second step of the ℓ-adic Abel-Jacobi filtration in characteristic 0. Jannsen's injectivity conjecture ([J90, Conj. 9.15]; see also [BlK90, Conj. 5.3]) predicts that for a smooth, projective scheme X over a global field K the morphisms induced by the the ℓ-adic Abel-Jacobi maps

$$a_\ell^{i,1} : F_\ell^{i,1}(X) \otimes \mathbb{Q}_\ell \to \mathrm{H}^1(K, \mathrm{H}^{2i-1}(X_{\overline{K}}, \mathbb{Q}_\ell)(i))$$

are injective, $i \geq 0$. On the other hand, Beilinson's conjecture on the existence of filtrations on Chow groups predicts that, at least in characteristic 0, the ℓ-adic Abel-Jacobi filtration should be independent of ℓ ([J94, Lemma 2.7]). The following is a typical application of a specialization result as Corollary 5.2.3.1.

Corollary. *Assume Jannsen's injectivity conjecture for number fields. Then for every smooth, projective scheme X over a field K finitely generated over $\mathbb{Q}$, the second piece $F_\ell^{i,2}(X) \subset \mathrm{CH}^j(X) \otimes \mathbb{Q}$ of the ℓ-adic Abel-Jacobi filtration is independent of ℓ.*

Proof. By comparison between Betti and ℓ-adic cohomologies[2], $F_\ell^{i,1}(X) \subset \mathrm{CH}^i(X) \otimes \mathbb{Q}$ is independent of ℓ. We proceed by induction on the transcendence degree d of K. If $d = 0$, this follows from Jannsen's conjecture [J90, Conj. 9.15]. If $d \geq 1$, $K = k(\eta)$ for k a field finitely generated over $\mathbb{Q}$ of transcendence degree $d-1$ and η the generic point of a smooth, geometrically connected curve S over k. Up to replacing S by a non-empty open subscheme, one may assume X extends to a smooth projective scheme $\mathcal{X} \to S$. From Corollary 5.2.3.1, there exists $s \in |S|$ such that the restriction morphisms $res_{s,\ell} : \mathrm{H}^1(\Pi, V_\ell) \to \mathrm{H}^1(\Pi_s, V_\ell)$ and

[2]This is the only place where we use the assumption that K is of characteristic 0; it does not seem that for $p > 0$ one knows that $F_\ell^{i,1}(X) \subset \mathrm{CH}^i(X) \otimes \mathbb{Q}$ is independent of ℓ.

$res_{s,\ell'} : \mathrm{H}^1(\Pi, V_{\ell'}) \to \mathrm{H}^1(\Pi_s, V_{\ell'})$ are injective on the image of the corresponding first higher ℓ-adic Abel-Jacobi maps (here we write $V_\ell = \mathrm{H}^{2i-1}(X_{\overline{K}}, \mathbb{Q}_\ell(i))$). But, then

$$F_\ell^{i,2}(X) = sp_s^{-1}(F_\ell^{i,2}(\mathcal{X}_s) \cap F_\ell^{i,1}(X)) \overset{(*)}{=} sp_s^{-1}(F_{\ell'}^{i,2}(\mathcal{X}_s) \cap F_{\ell'}^{i,1}(X)) = F_{\ell'}^{i,2}(X),$$

where (*) follows from the ℓ-independence of $F_\ell^{i,1}(X) \subset \mathrm{CH}^i(X) \otimes \mathbb{Q}$ and the induction hypothesis applied to $\mathcal{X}_s$. □

6.2. Abelian schemes (5.1.2 cont.) Assume $X = A \to S$ is an abelian scheme. The specialization diagram of 5.2.3 reads as

$$\begin{tikzcd}
A_\eta(k(\eta)) \otimes \mathbb{Q} \arrow[rr, hook, "\kappa_\ell"] & & \mathrm{H}^1(\Pi_\eta, V_\ell(A_\eta)) \\
A(S) \otimes \mathbb{Q} \arrow[u, "\simeq"] \arrow[r, hook] \arrow[d, "sp_s"'] & \mathrm{H}^1(S, V_\ell(1)) \simeq & \mathrm{H}^1(\Pi, V_\ell(A_\eta)) \arrow[u, "\simeq", "infl"'] \arrow[d, "res_s"] \\
A_s(k(s))) \otimes \mathbb{Q} \arrow[rr, hook, "\kappa_\ell"] & & \mathrm{H}^1(\Pi_s, V_\ell(A_s)),
\end{tikzcd}$$

where the upper left isomorphism is the Néron extension property for abelian schemes. In particular, Corollary 5.2.3.1 and Corollary 5.2.3.2 together with the Lang-Néron theorem [LN59] imply respectively the following.

6.2.1. Corollary. *Assume k is finitely generated over its prime field. Then there exists an integer $d \geq 1$ such that the specialization morphism $sp_s : A_\eta(k(\eta)) \otimes \mathbb{Q} \to A_s(k(s)) \otimes \mathbb{Q}$ is injective for infinitely many $s \in |S|^{\leq d}$. In particular,* $\mathrm{rank}(A_\eta(k(\eta))) \leq \mathrm{rank}(A_s(k(s)))$ *for infinitely many $s \in |S|^{\leq d}$.*

6.2.2. Corollary. *Assume k is finitely generated over $\mathbb{Q}$ and X_η contains no non-zero $\overline{k}$-isotrivial abelian subvariety. Then for every integer $d \geq 1$ and all but finitely many $s \in |S|^{\leq d}$, the specialization map $sp_s : A_\eta(k(\eta)) \otimes \mathbb{Q} \to A_s(k(s)) \otimes \mathbb{Q}$ is injective. In particular,* $\mathrm{rank}(A_\eta(k(\eta))) \leq \mathrm{rank}(A_s(k(s)))$.

6.2.3. *Comparison with Silverman's theorem.* When k is a number field, one recovers Silverman's specialization theorem [Si83, Thm. C]. More precisely, in this setting, our method shows, more generally, that for all but finitely many $s \in |S|^{\leq d}$, the restriction morphism $res_s : \mathrm{H}^1(\Pi, V_\ell(X_\eta)) \to \mathrm{H}^1(\Pi_s, V_\ell(X_s))$ is injective and Corollary 6.2.2 appears as a special case of a more general specialization theorem about cohomology classes. On the other hand, Silverman's specialization theorem says more about algebraic classes, namely that the set of all $s \in |S|$ where the specialization map $sp_s : X_\eta(k(\eta)) \otimes \mathbb{Q} \to X_s(k(s)) \otimes \mathbb{Q}$ is non-injective is of bounded height.

Silverman's theorem follows from the non-degeneracy of the Néron-Tate height pairing $\langle -, - \rangle_{X_\eta} : X_\eta(k(\overline{\eta}))_{\mathbb{Q}} \times X_\eta(k(\overline{\eta}))_{\mathbb{Q}} \to \mathbb{R}$ at the generic fiber, the limit formula: for every $P, Q \in X(S)$

$$\lim_{s \in |S|,\, h_S(s) \to +\infty} \frac{\langle P_s, Q_s \rangle_{X_s}}{h_S(s)} = \langle P_\eta, Q_\eta \rangle_{X_\eta}.$$

and the Northcott property for heights over number fields.

So provided a good theory of heights is available for Chow groups, Silverman's argument should extend to $Gr_\ell^{1,i}(A_\eta)$. Several constructions of height pairings on Chow groups generalizing the Néron-Tate height pairing have been proposed ([Bei87], [Bl84], [GiS84]) but the constructions and properties (such as the non-degeneracy or limit formula) of these height pairings are still partly conjectural. However, for cycles geometrically algebraically equivalent to zero modulo incidence equivalence, which are parametrized by abelian varieties - the so-called higher Picard varieties (see [S79] for details) - one can essentially reduces to Silverman's theorem. More precisely, let

$$F^{i,1}_{inc}(A_\eta) \subset F^{i,1}_{alg}(A_\eta) \subset F^{i,1}_{\ell}(A_\eta)$$

denote the $\mathbb{Q}$-subvector spaces of cycles geometrically incidence equivalent to zero and geometrically algebraically equivalent to zero respectively. Set

$$F^{i,2}_{alg,\ell}(A_\eta) := F^{i,1}_{alg}(A_\eta) \cap F^{i,2}_{\ell}(A_\eta) \subset F^{i,1}_{alg}(A_\eta).$$

Then $F^{i,2}_{alg,\ell}(A_\eta) \subset F^{i,1}_{inc}(A_\eta)$ that is,

$$F^{i,1}_{alg}(A_\eta)/F^{i,1}_{inc}(A_\eta) \twoheadleftarrow F^{i,1}_{alg}(A_\eta)/F^{i,1}_{alg,\ell}(A_\eta) \hookrightarrow Gr^{i,1}_{\ell}(A_\eta).$$

Let $I \subset F^{i,1}_{alg}(A_\eta)/F^{i,1}_{alg,\ell}(A_\eta)$ be any $\mathbb{Q}$-vector subspace mapping injectively into $F^{i,1}_{alg}(A_\eta)/F^{i,1}_{inc}(A_\eta)$. Then, if the higher Picard variety $\mathrm{Pic}^i_{X_{\overline{\eta}}}$ contains no non-zero $\overline{k}$-isotrivial abelian subvariety, Silverman's theorem shows that the set of all $s \in |S|$ such that the induced morphism

$$I \subset Gr^{i,1}_{\ell}(A_\eta) \stackrel{sp_s}{\to} Gr^{i,1}_{\ell}(A_s)$$

is non-injective is of bounded height.

Remark: The geometric meaning of the condition that $\mathrm{Pic}^i_{A_{\overline{\eta}}}$ contains no non-zero $\overline{k}$-isotrivial abelian subvariety is unclear to us. We do not know if there is a simple criterion that ensures it.

6.2.4. *The set* $S^1(E,V)$. When k is finitely generated, $V := V_\ell(A_\eta)$ is a semisimple $\pi_1(S)$-module hence for every finite-dimensional $\mathbb{Q}_\ell$-vector subspace $E \subset \mathrm{H}^1(\Pi, V)$ one can apply Lemma 2.1.3 with $N = \ker(\Pi_{\tilde{E}} \to \Pi_V)$. Define $\Pi := \Pi_E : |S| \to \mathbb{Z}_{\geq 0}$ by $\Pi(s) = \dim(N) - \dim(N \cap \Pi_{s,\tilde{E}})$. Let $S^1(E,V) \subset |S|$ denote the set of all $s \in |S|$ such that $\Pi(s) = 0$. Then one has

$$S^0(E,V) := S(E,V) \supset S^1(E,V) \supset S^2(E,V) := S(\tilde{E}) = S^1(E,V) \cap S(V) \subset S(V)$$

6.2.4.1. Interpretation of $S^1(E,V)$ (abelian schemes). Let A be an abelian variety over a field K of characteristic $p \geq 0$. Fix a prime $\ell \neq p$ and consider $V := V_\ell(A_{\overline{K}}) \in \mathcal{M}\mathrm{od}_{\mathbb{Q}_\ell}(\Pi)$. Fix a free $\mathbb{Z}$-module $E_0 \subset A(K)$ of finite rank and set $E =: E_0 \otimes \mathbb{Q}_\ell$, which we identify with its image in $\mathrm{H}^1(\pi_1(K), V)$ *via* the ℓ-adic Kummer morphism. Then the 'universal' extension $\tilde{E}$ of 2.1.1 is the Tate module $V_\ell([\underline{E} \to A])$ of the 1-motive $[\underline{E} \to A]$ defined by E (see [Jo14]). Pick a $\mathbb{Z}$-basis $\underline{P} = (P_1, \ldots, P_r) \in E_0$. Let $A(\underline{P}) \subset X^r$ denote the connected component of the Zariski closure of $\mathbb{Z}\underline{P} \subset A^r$ and set $V(\underline{P}) := V_\ell(A(\underline{P})_{\overline{K}})$. Up to non-canonical isomorphism, $A(\underline{P})$ and $V(\underline{P})$ only depend on E and not on $\underline{P}$. By construction, $N \hookrightarrow \mathrm{Hom}_{\mathbb{Q}_\ell}(E,V) \simeq V^r$ and we have the following description of N ([Jo14, (Comment after) Thm. 2]),

Fact. (Unipotent part of the Mumford-Tate conjecture for 1-motives) *The inclusion $N \hookrightarrow V^r$ factors through $V(\underline{P})$ and induces an (a non-canonical) isomorphism of ℓ-adic Lie groups $N \otimes \mathbb{Q}_\ell \simeq V(\underline{P})$.*

We come back to our abelian scheme $A \to S$. If $E_0 \subset A(S)$ is a free $\mathbb{Z}$-module of finite rank and $\underline{P} = (P_1, \ldots, P_r) \in E_0$ a $\mathbb{Z}$-basis, for $s \in |S|$, one has

$$\Pi(s) = \dim(V(\underline{P}_\eta)) - \dim(V(\underline{P}_s)) = \dim(A(\underline{P}_\eta)) - \dim(A(\underline{P}_s)).$$

On the one hand, this shows that $S^1(E,V)$ is independent of ℓ and, on the other hand that

6.2.4.2. Corollary. *Assume k is finitely generated over $\mathbb{Q}$. Then the set of all $s \in S(k)$ (resp. for every integer $d \geq 1$ the set of all $s \in |S|^{\leq d}$) where the unipotent part $A(\underline{P}_s)$ of the motivic Galois group of $[\underline{E}_s \to A_s]$ degenerates is finite.*

6.2.5. *An example with $\mathbb{F}_\ell$-coefficients.* We conclude by a refinement of Corollary 6.2.2, using Corollary 3.2.2. For an abelian group M and a prime ℓ, let M_{tors} and $M[\ell^\infty]$ denote the torsion subgroup of M and the ℓ-Sylow of M_{tors} respectively. With the notation of 6.2, one has

6.2.5.1. Corollary. *Assume k is finitely generated over $\mathbb{Q}$ and A_η contains no non-zero $\overline{k}$-isotrivial abelian subvariety. Then for every integer $d \geq 1$ and all but finitely many $s \in |S|^{\leq d}$, the specialization morphism $sp_s : A_\eta(k(\eta)) \to A_s(k(s))$ is injective and satisfies* $\mathrm{coker}(sp_s)[\ell] = 0$.

Proof. For $s \in |S|$, the specialization map induces a morphism

$$sp_s : A_\eta(k(\eta))_{tors} \to A_s(k(s))_{tors}$$

which is injective. Since $A_\eta(k(\eta))$ is a finitely generated abelian group, $A_\eta(k(\eta))[\ell] = 0$ for $\ell \gg 0$. By the geometric Lang-Néron theorem ([LN59]), the assumption that $A_{\overline{\eta}}$ contains no non-zero $\overline{k}$-isotrivial abelian subvariety ensures (I). This, in turn, implies that for every integer $B \geq 1$, for $\ell \gg 0$ and every $0 \neq v_\ell \in A_\eta[\ell]$ one has $[\overline{\Pi}_{V_\ell} : \mathrm{Stab}_{\overline{\Pi}_{V_\ell}}(v_\ell)] > B^2$ ([CT16a, Lemma 2.8]). Then for every integer $d \geq 1$, there exists an integer $B \geq 1$ such that for $\ell \gg 0$ and all but finitely many $s \in |S|^{\leq d}$, $[\Pi_{V_\ell} : \Pi_{s,V_\ell}] \leq B$ 3.2 . Hence

$$B[\Pi_{s,V_\ell} : \mathrm{Stab}_{\Pi_{s,V_\ell}}(v_\ell)] \geq [\Pi_{V_\ell} : \mathrm{Stab}_{\Pi_{s,V_\ell}}(v_\ell)] \geq [\Pi_{V_\ell} : \mathrm{Stab}_{\Pi_{V_\ell}}(v_\ell)] \geq [\overline{\Pi}_{V_\ell} : \mathrm{Stab}_{\overline{\Pi}_{V_\ell}}(v_\ell)] > B^2,$$

which shows that $A_s(k(s))[\ell] = 0$. On the other hand, by Theorem **??**, for every integer $d \geq 1$, for $\ell \gg 0$ and for all but finitely many $s \in |S|^{\leq d}$, the specialization map $sp_s : A_\eta(k(\eta))/\ell \to A_s(k(s))/\ell$ is injective. The conclusion thus follows from the formal Lemma 6.2.5.2. □

6.2.5.2. Lemma. *Let R be a principal ideal domain, $\varphi : N \to M$ a morphism in* Mod_R *and let $0 \neq r \in R \setminus R^\times$ such that the induced morphism $\overline{\varphi} : N/r \to M/r$ is injective. If* $\ker(\varphi)_{tors} = M[r] = 0$ *then $\varphi : N \to M$ is injective and* $\mathrm{coker}(\varphi)[r] = 0$.

Proof. Apply the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "r"] \arrow[d, "\varphi"'] & M \arrow[r] \arrow[d, "\varphi"'] & M/r \arrow[r] \arrow[d, "\overline{\varphi}"'] & 0 \\
0 \arrow[r] & N \arrow[r, "r"] & N \arrow[r] & N/r \arrow[r] & 0
\end{tikzcd}$$ □

6.2.5.3. Remark. For ℓ fixed, increasing k produces lots of $s \in S(k)$ such that $\mathrm{coker}(sp_s)[\ell] \neq 0$. More precisely, if A_η contains no non-zero $\overline{k}$-isotrivial abelian subvariety, $A(S_{\overline{k}})$ is a finitely generated $\mathbb{Z}$-module. Up to replacing S by a finite cover, we may assume $A(S_{\overline{k}})$ has rank ≥ 1 and up to replacing k with a finite field extension, we may assume $X(S) = X(S_{\overline{k}})$. Then, $X(S)$ remains unchanged when k is replaced with a finite field extension. Pick $x \in A(S)$ a non-torsion, non-divisible section and $s \in S(k)$ such that $sp_s : A(S) \hookrightarrow A_s(k(s))$ is injective (after replacing k by a finite field extension, such an s always exists by Corollary 6.2.5.1). But then, up to replacing k by a finite field extension, we may assume $sp_s(x) =: x_s \in k(s)$ is ℓ-divisible that is there exists $a \in A_s(k(s))$ such that $x_s = \ell a$. Then $0 \neq a \in \mathrm{coker}(sp_s)[\ell]$ since, otherwise, there would exist $x' \in A(S)$ such that $a = sp_s(x')$ hence, by injectivity of $sp_s : A(S) \otimes \mathbb{Q} \hookrightarrow A_s(k(s)) \otimes \mathbb{Q}$, $x = \ell x'$. This would contradict our assumption on x.

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