Proofs of Lemmas 10.4 and 10.5

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1 Small Frobenius centraliser

Let $X \to Y$ be a smooth and proper morphism of $\mathbb{Z}[S^{-1}]$-schemes whose fibres are geometrically connected of dimension d. For $y \in Y(\mathbb{Z}[S^{-1}])$ write $V_y = \mathrm{H}^d(X_{y,\mathbb{C}}, \mathbb{Q})^{\mathrm{prim}}$. Write $V_0 = V_{y_0}$.

Let $\mathbf{G}'$ be the reductive $\mathbb{Q}$-group consisting of the automorphisms of V_0 that multiply the intersection form by a constant. Let $\varphi_0 : S^1 \to \mathbf{G}'(\mathbb{C})$ be the homomorphism describing the Hodge structure on V_0.

If $y \in Y(\mathbb{Z}[S^{-1}])$, then X_y has good reduction modulo any $\ell \notin S$ so we can consider the primitive crystalline cohomology group $\mathrm{H}^d_{\mathrm{cris}}(X_{y,\mathbb{F}_\ell}/\mathbb{Q}_\ell)^{\mathrm{prim}}$ of the reduction mod ℓ. It carries a Frobenius, which is linear and not just semilinear because $\mathbb{F}_\ell$ is a prime field. Hence

$$F_y^{\mathrm{cris},\ell} \in \mathrm{GAut} := \mathrm{GAut}\ \mathrm{H}^d_{\mathrm{cris}}(X_{y,\mathbb{F}_\ell}/\mathbb{Q}_\ell)^{\mathrm{prim}},$$

where GAut is the group of the automorphisms that multiply the intersection form by a constant.

Recall that the semisimplifications are taken with respect to the reductive groups $\mathbf{G}'$ or GAut (as introduced by Serre in "Complète réductibilité"). This means that a representation is called *irreducible* if its image is not contained in a proper parabolic subgroup (equivalently, if the group is orthogonal or symplectic, an invariant subspace must be isotropic). A representation is called *completely reducible* if any parabolic containing the image has a Levi factor also containing the image. A *semisimplification* of a representation is defined by taking the minimal parabolic containing the image and projecting to a Levi factor. The result is well defined up to conjugation. The Zariski closure of the semisimplification is a reductive group (in char 0), see Serre, *op. cit.*

Lemma 10.4 *There exists an integer L with the following property:*
for any $y \in Y(\mathbb{Z}[S^{-1}])$ there is a prime $\ell < L$ not in S such that

$$\dim Z_{\mathrm{GAut}}((F_y^{\mathrm{cris},\ell})^{\mathrm{ss}}) \leq \dim Z_{\mathbf{G}'(\mathbb{C})}(\varphi_0).$$

Proof The superscript Zar will denote the Zariski closure.

Step 1.

Fix $p \notin S$ and let $\rho_{y,p} : G_{\mathbb{Q}} \to \mathbf{G}'(\mathbb{Q}_p)$ be the continuous representation in $\mathrm{H}^d_{\text{ét}}(X_{y,\overline{\mathbb{Q}}}, \mathbb{Q}_p)^{\text{prim}} \simeq V_y \otimes_{\mathbb{Q}} \mathbb{Q}_p$. It is continuous, unramified outside S and pure of weight $d/2$. By Faltings' lemma there are only finitely many possibilities for the semisimplification $\rho^{\text{ss}}_{y,p}$.

Let $\mathbf{H} = \rho^{\text{ss}}_{y,p}(G_{\mathbb{Q}})^{\text{Zar}} \subset \mathbf{G}'$, and let $\mathbf{H}^\circ \subset \mathbf{H}$ be the connected component of the identity. Since $\rho^{\text{ss}}_{y,p}$ is semisimple, $\mathbf{H}^\circ$ is a reductive group.

Recall that an element in $\mathbf{H}^\circ(\overline{\mathbb{Q}}_p)$ is called *regular* if its centraliser has the least possible dimension (equal to the rank of $\mathbf{H}^\circ$). A *semisimple* element is called *very regular* if its centraliser taken in $\mathrm{Aut}(V_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p)$ has the least possible dimension.

Let $\mathbf{T}_0 \subset \mathbf{H}^\circ$ be a maximal torus defined over $\overline{\mathbb{Q}}_p$. Consider the adjoint representation of the Lie group $\mathrm{Aut}(V_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p)$ on its Lie algebra, and restrict it to $\mathbf{T}_0$. Let Φ be the characters of $\mathbf{T}_0$ that show up in this representation of $\mathbf{T}_0$. It is clear that $t \in \mathbf{T}_0(\overline{\mathbb{Q}}_p)$ is very regular if and only if $\alpha(t) \neq 1$ for all roots $\alpha \in \Phi$. Then the centraliser of t is the centraliser of $\mathbf{T}_0$, which means that t is regular in $\mathbf{H}^\circ$. Since any semisimple element is contained in a maximal torus, this implies that a very regular element is regular in $\mathbf{H}^\circ$. This also implies that a very regular element of $\mathbf{H}^\circ(\overline{\mathbb{Q}}_p)$ is also very regular as an element of $\mathbf{G}'(\overline{\mathbb{Q}}_p) \subset \mathrm{Aut}(V_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p)$.

The set of very regular elements is a non-empty Zariski open subset of $\mathbf{H}^\circ$. Indeed, the function $f = \prod_{\alpha \in \Phi}(\alpha(t) - 1)$ is a regular function on $\mathbf{T}_0$ which is invariant under the Weyl group W. By a result of Steinberg (Cor. 6.4 in Steinberg's "Regular elements in semisimple groups", Appendix 1 to Ch. 3 of Serre's "Galois cohomology"), the algebra $\overline{\mathbb{Q}}_p[\mathbf{T}_0/W]$ is naturally isomorphic to the algebra of class functions on $\mathbf{H}^\circ$ (i.e., regular functions invariant under conjugation). Thus f comes from a class function on $\mathbf{H}^\circ$, and the locus of very regular elements is the set of non-zeros of this function.

As $\rho^{\text{ss}}_{y,p}$ is continuous and $\rho^{\text{ss}}_{y,p}(G_{\mathbb{Q}})$ is Zariski dense in $\mathbf{H}$, the preimage of the set of very regular elements under $\rho^{\text{ss}}_{y,p}$ is non-empty and open in $G_{\mathbb{Q}}$. Since $\rho^{\text{ss}}_{y,p}$ is unramified outside S, by Chebotarev density theorem the images of Frobenii at primes outside S are dense in $\rho^{\text{ss}}_{y,p}(G_{\mathbb{Q}})$, so there is a prime ℓ such that $\rho^{\text{ss}}_{y,p}(\mathrm{Frob}_\ell)$ is a very regular element of $\mathbf{H}^\circ$. By Faltings' lemma, for given S, p, and dimension of the representation, there are only finitely many possibilities for $\rho^{\text{ss}}_{y,p}$. So such an ℓ is bounded by a constant L depending only on these.

Step 2.

Now consider the restriction of $\rho_{y,p}$ to $G_{\mathbb{Q}_p}$. Then $\rho_{y,p}(G_{\mathbb{Q}_p})^{\text{Zar}}$ is an algebraic subgroup of $\mathbf{G}'$. The claim of Step 2 is that this algebraic group contains a subgroup $\mathbf{S}$ which is conjugate, over $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$, to the Hodge torus $\varphi_0(S^1)^{\text{Zar}} \subset \mathbf{G}'(\mathbb{C})$.

The group $\rho_{y,p}(G_{\mathbb{Q}_p})^{\mathrm{Zar}}$ is interpreted as the group associated to the neutral Tannakian category generated by the $G_{\mathbb{Q}_p}$-module $V_y \otimes_{\mathbb{Q}} \mathbb{Q}_p$ (i.e., the automorphism group of the natural fibre functor). The attached filtered ϕ-module gives another fibre functor on this category; it gives rise to another Tannakian group. By Deligne–Milne, the fibre functors on the category of representations of a group bijectively correspond to the torsors of this group. Thus over $\overline{\mathbb{Q}}_p$ the two Tannakian groups become isomorphic.

But Wintenberger [W] shows that the Hodge filtration is canonically split, and this allows one to define an associated cocharacter φ_W acting as a scalar on each quotient. Using an isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ both φ_W and φ_0 can be considered as maps to $\mathbf{G}'(\mathbb{C})$. They both preserve the Hodge filtration and act as multiplication by the same number on the quotients. This implies their conjugacy by Lemma 2.5.

Step 3.

The claim of Step 2 formally implies

$$\dim Z_{\mathbf{G}'(\mathbb{Q}_p)}(\mathbf{S}) = \dim Z_{\mathbf{G}'(\mathbb{C})}(\varphi_0).$$

Moreover, $\mathbf{S}$ is conjugate to a subgroup of $\rho^{\mathrm{ss}}_{y,p}(G_{\mathbb{Q}})^{\mathrm{Zar}}$. For this take a minimal parabolic subgroup $P = MU$ containing $\rho_{y,p}(G_{\mathbb{Q}})$. Then $\rho^{\mathrm{ss}}_{y,p}$ is obtained by projecting $\rho_{y,p}$ to the Levi factor M. Hence $\rho^{\mathrm{ss}}_{y,p}(G_{\mathbb{Q}})^{\mathrm{Zar}}$ contains the projection of $\rho_{y,p}(G_{\mathbb{Q}})^{\mathrm{Zar}}$. By Lemma 2.5 any torus in P is conjugate under U to its projection to M. (This uses the fact that all Levi subgroups are conjugate under U.)

By the same argument we get that $\rho_{y,p}(\mathrm{Frob}_\ell)^{\mathrm{ss}}$ is conjugate to $\rho^{\mathrm{ss}}_{y,p}(\mathrm{Frob}_\ell)$ in $\mathbf{H}^\circ(\overline{\mathbb{Q}}_p)$. By Step 1 this last element is very regular (and $\mathbf{S}$ is topologically generated by one element), so

$$\dim Z_{\mathbf{G}'(\overline{\mathbb{Q}}_p)}(\rho_{y,p}(\mathrm{Frob}_\ell)^{\mathrm{ss}}) \leq \dim Z_{\mathbf{G}'(\overline{\mathbb{Q}}_p)}(\mathbf{S}) = \dim Z_{\mathbf{G}'(\mathbb{C})}(\varphi_0).$$

Step 4.

By Katz–Messing [KM], the eigenvalues of $F_y^{\mathrm{cris},\ell}$ are the same as of the Frobenius Frob_ℓ acting on $\mathrm{H}^d_{\text{ét}}(X_{y,\overline{\mathbb{F}}_\ell}, \mathbb{Q}_p)^{\mathrm{prim}}$ for any prime $p \neq \ell$. Thus $\rho_{y,p}(\mathrm{Frob}_\ell)^{\mathrm{ss}}$ and $(F_y^{\mathrm{cris},\ell})^{\mathrm{ss}}$ have the same characteristic polynomial and both scale the bilinear form by the same number ℓ.

Let us decompose $V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p = \oplus V_\lambda$, where the V_λ are eigenspaces for $\rho_{y,p}(\mathrm{Frob}_\ell)^{\mathrm{ss}}$. The bilinear form gives a perfect pairing $V_\lambda \times V_{\ell/\lambda} \to \mathbb{Q}_p$. The centraliser of $\rho_{y,p}(\mathrm{Frob}_\ell)^{\mathrm{ss}}$ in $\mathbf{G}'$ is the subgroup of $\prod \mathrm{Aut}(V_\lambda)$ preserving these pairings, so its dimension is uniquely determined by the eigenvalues taken with their multiplicities. The same applies to $(F_y^{\mathrm{cris},\ell})^{\mathrm{ss}}$. This gives the desired inequality.

Remark By the Jordan–Chevalley decomposition, the semisimple part α^{ss} of a linear transformation α can be written as $p(\alpha)$ for some polynomial $p(x)$. Hence

$$Z_{\mathrm{GAut}}(F_y^{\mathrm{cris},\ell}) \subset Z_{\mathrm{GAut}}((F_y^{\mathrm{cris},\ell})^{\mathrm{ss}}).$$

Thus we can drop "ss" in Lemma 10.4.

2 Not Zariski dense

Lemma 10.5 *Let ℓ be a prime not in S and let $y_0 \in Y(\mathbb{Z}[S^{-1}])$ be such that*

$$\dim Z_{\mathrm{GAut}}(F_{y_0}^{\mathrm{cris},\ell}) \leq \dim Z_{\mathbf{G}'(\mathbb{C})}(\varphi_0).$$

Then the set of $y \in Y(\mathbb{Z}[S^{-1}])$ in the residue disk of y_0 modulo ℓ such that the representations $\rho_{y,\ell}^{\mathrm{ss}}$ and $\rho_{y_0,\ell}^{\mathrm{ss}}$ are conjugate in $\mathbf{G}'$ is not Zariski dense in Y.

Proof In the proof we write p instead of ℓ. The proof proceeds by reducing this, via the p-adic version of Bakker–Tsimerman theorem, to essentially a linear algebra computation (Proposition 10.6).

Recall that $\rho_{y_0,p}^{\mathrm{ss}}$ is obtained by taking a maximal self-dual flag of $\rho_{y_0,p}$-stable subspaces of $\mathrm{H}^d_{\text{ét}}(X_{y_0}, \mathbb{Q}_p)^{\mathrm{prim}}$ such that each quotient is irreducible (and the middle quotient has no isotropic invariant subspace). We refer to this as the "semisimplification flag". Since $\rho_{y,p}^{\mathrm{ss}}$ and $\rho_{y_0,p}^{\mathrm{ss}}$ are conjugate in $\mathbf{G}'$, there is a similar filtration for $\rho_{y,p}^{\mathrm{ss}}$ with isomorphic respective quotients.

Since the morphism $X_y \to \mathrm{Spec}(\mathbb{Z}[S^{-1}])$ is smooth and proper, our representation is cristalline. The functor $D_{\mathrm{cris}} : ? \mapsto (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} ?)^{G_{\mathbb{Q}_p}}$ sends the cristalline $G_{\mathbb{Q}_p}$-representation $\mathrm{H}^d_{\text{ét}}((X_y)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ to the filtered ϕ-module

$$\mathrm{H}^d_{\mathrm{dR}}((X_y)_{\mathbb{Q}_p}) \cong \mathrm{H}^d_{\mathrm{cris}}((X_y)_{\mathbb{F}_p}/\mathbb{Q}_p),$$

where ϕ is the semilinear Frobenius which in fact is linear here (as we work over $K = \mathbb{Q}$ so that the residue field is a prime field $\mathbb{F}_p$), and the filtration is the Hodge filtration on de Rham cohomology. Since y is in the residue disk of y_0, the $\mathbb{Q}_p$-vector spaces $\mathrm{H}^d_{\mathrm{dR}}((X_y)_{\mathbb{Q}_p})$ are canonically identified with $\mathrm{H}^d_{\mathrm{dR}}((X_{y_0})_{\mathbb{Q}_p})$ since $\mathrm{H}^d_{\mathrm{cris}}((X_y)_{\mathbb{F}_p}/\mathbb{Q}_p)$ depends only on the fibre $(X_y)_{\mathbb{F}_p} = (X_{y_0})_{\mathbb{F}_p}$ over the residue field $\mathbb{F}_p$. This identification preserves ϕ, the intersection forms, and the primitive subspaces.

Let us denote by F_0 the descreasing Hodge filtration on $V = \mathrm{H}^d_{\mathrm{dR}}((X_{y_0})_{\mathbb{Q}_p})^{\mathrm{prim}}$. Let $\mathfrak{H}_p$ be the set of self-dual flags in V with subspaces of the same dimensions as the subspaces in F_0.

Fontaine's functor D_{cris} transforms the $G_{\mathbb{Q}_p}$-invariant semisimplification flags for y_0 and y into corresponding flags in V. Let us denote them by $\mathfrak{f}_0$ and $\mathfrak{f}$, respectively. Because of $G_{\mathbb{Q}_p}$-invariance the graded quotients of $\mathfrak{f}_0$ and $\mathfrak{f}$ are filtered ϕ-modules; by assumption they are isomorphic in each degree (in the middle quotient the isomorphism preserves the bilinear form).

Sending y to the Hodge filtration on $\mathrm{H}^d_{\mathrm{dR}}((X_y)_{\mathbb{Q}_p})^{\mathrm{prim}}$, which is canonically identified with V, we obtain a period map

$$\Phi_p \ : \ \text{the residue disk at } y_0 \text{ modulo } p \longrightarrow \mathfrak{H}_p$$

Let $\mathfrak{S} \subset \mathfrak{H}_p$ be the space of filtrations F on V such that there is another self-dual filtration $\mathfrak{f}$ with the following properties:

(1) $\mathfrak{f}$ is ϕ-stable;

(2) the filtration induced by F on each graded quotient $\mathrm{gr}_j^{\mathfrak{f}}$ has weight $d/2$;

(3) for each j there is an isomorphism of filtered ϕ-modules (i.e. of vector spaces respecting the Hodge filtration and Frobenius)

$$(\mathrm{gr}_j^{\mathfrak{f}},\ \text{filtration induced by } F) \simeq (\mathrm{gr}_j^{\mathfrak{f}_0},\ \text{filtration induced by } F_0),$$

where the isomorphism of the middle graded quotient preserves the bilinear form.

Remark Recall that the *weight* of a decreasing filtration F on V is

$$(\dim V)^{-1} \sum_{i \geq 0} i \dim \mathrm{gr}^i(V).$$

That (2) holds for the Hodge filtration of y follows from Lemma 2.9 since $G_{\mathbb{Q}}$-representation $\rho_{y,p}^{\mathrm{ss}}$ is cristalline at p and pure of weight $d/2$. (This is the calculation that the Hodge–Tate weight of a continuous character of $G_{\mathbb{Q}} \to \mathbb{Q}_p^*$ unramified outside S, pure of weight d and locally algebraic at p equals $d/2$. This is applied to $\det(V)$).

Proposition 10.6 says that if the Hodge numbers of the adjoint filtration of F on the Lie algebra of $\mathrm{GAut}(V)$ satisfy inequalities (10.23) and (10.24) with $e = \dim(Y)$, then $\mathrm{codim}(\mathfrak{S}) \geq \dim(Y)$. Then Lemma 9.3 ($p$-adic transcendence of the period map) implies that $\Phi_p^{-1}(\mathfrak{S})$ is not Zariski dense in Y. This proves Lemma 10.5.

References

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