Lawrence-Venkatesh Prop 10.2

The goal of this talk is to show that the conditions of ~~Prop~~ Thm 10.1 are satisfied for the universal family ~~of the~~ $X \to Y$ of hypersurfaces of large dimension and degree. Consequently ~~the~~ Thm 10.1 tells us that $Y(\mathbb{Z}[S^{-1}])$ is not Zariski dense in Y (for any finite set S of primes.)

More precisely:

~~Prop:~~ Let $X \to Y$ be the universal family of hypersurfaces of deg d in $\mathbb{P}^n$.

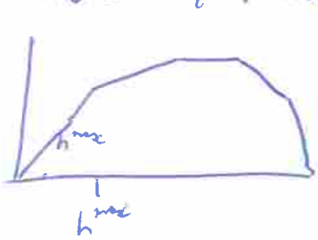
~~Let h^p denote the~~ Let $y_0 \in Y(\mathbb{Q})$, X_0 = fibre above y_0,
$V_0 = H^{n-1}(X_{0,\mathbb{C}}, \mathbb{Q})^{prim}$.

Prop: The image of the monodromy rep $\pi_1(Y_{\mathbb{C}}, y_0) \longrightarrow \mathrm{Aut}(V_0, \cup)$ is large i.e. its Zariski closure contains $\mathrm{Aut}(V_0, \cup)^\circ$.

Attached to X_0, we get an "adjoint HS" on $\mathrm{Lie}\, G\mathrm{Aut}(V_0, \cup)$
$\cong \mathbb{Q} \oplus \mathrm{Sym}^2 V_0 \quad (n-1)$
or $\mathbb{Q} \oplus \Lambda^2 V_0. \quad (n-1)$

Let $h^p = \dim((p,-p)$ cpt in adjoint HS$)$. for $1-n \le p \le n-1$.

Let $T(x)$ = sum of topmost x Hodge numbers of adj HS (interpolated linearly)
~~also~~ = y coordinate of Hodge polygon of adj HS at x.



Prop 10.2: ~~Thm 10.4 for~~ $\exists n_0, D_0(n)$ s.t. $\forall n \geq n_0$, $d \geq D_0(n)$, we have $p <$

(10.4) $\displaystyle\sum_{p>0} h^p \geq h^0 + \dim(Y)$. (Note: $\dim(Y) \ll h^0$.)

(10.5) $\displaystyle\sum_{p>0} p h^p > T(h^0 + \dim(Y)) + T(\tfrac{3}{2} h^0 + \dim(Y))$.

Hodge numbers

In order to calculate h^p;

First we calculate the Hodge numbers of $X_{0,\mathbb{C}}$, a smooth hypersurface of deg d in $\mathbb{P}^n$. (in $H^{n-1}(X_{0,\mathbb{C}}, \mathbb{Q})^{prim}$)

We may assume our hypersurface is the Fermat hypersurface

$$X_0^d + \dots + X_n^d = 0.$$

Some explicit calculations of differential forms show that

$$h^{pq} = \#\{(b_0, \dots, b_n) \in \mathbb{Z}^{n+1} : 0 < b_i < d, \ b_0 + \dots + b_n = (q+1)d\}$$

where $0 \le p, q \le n-1$, $p+q = n-1$.

$$= \#\{(b_1, \dots, b_n) \in \mathbb{Z}^n : 0 < b_i < d, \ qd < b_1 + \dots + b_n < (q+1)d\}$$

Thus $\dim H^{n-1}(X_{0,\mathbb{C}}, \mathbb{Q})^{prim} = \sum_{p=0}^{n-1} h^{p,n-p} = \#\{(b_1, \dots, b_n) \in \mathbb{Z}^n :$
$0 < b_i < d, \ d \nmid b_1 + \dots + b_n\}$

$\cancel{= \tfrac{1}{2} d^2 (1 + o(d))}$

$$\begin{cases} = (d-1)^n (1 + o_d(1)) \\ \sim d^n. \end{cases}$$

Fix n and let $d \to \infty$.

Look at $\dfrac{h^{pq}}{b_{n-1}(X_{0,\mathbb{C}})}$ $\sim$ Proportion of lattice pts in the hypercube $\{1, \dots, d-1\}^n$ which lie between the hyperplanes $\sum b_i = qd$, $\sum b_i = (q+1)d$

$=$ Proportion of pts of $\frac{1}{d}\mathbb{Z}^n$ which lie strictly inside the unit hypercube and between hyperplanes $\sum b_i = q$, $\sum b_i = q+1$

$p+$

$$\therefore \text{ As } d \to \infty, \quad \frac{h^{pq}}{b_{n-1}} \longrightarrow \mathrm{Vol}\left(\{(x_1,\dots,x_n) \in [0,1]^n : q < x_1 + \dots + x_n < q+1\}\right)$$

$$= \frac{1}{n!} A(n,p) \quad \text{where } A(n,p) \text{ is Eulerian number.}$$

$$=: \alpha_p(n)$$

(Laplace-Polya-Stanley)

The $(p,-p)$ part in adjoint HS is

$$\bigoplus_{\substack{p_1,p_2: \\ p_1+p_2 = p+n-1}} H^{p_1,q_1} \otimes H^{p_2,q_2} \Big/ \left(H^{p_1,q_1} \otimes H^{p_2,q_2} \simeq H^{p_2,q_2} \otimes H^{p_1,q_1}, \text{ either Sym or anti Sym in } \left(H^{(p+n-1)/2,\cdots}\right)^{\otimes 2} \right)$$

$$\text{So } h^p = \frac{1}{2} \sum_{p_1+p_2=p+n-1} h^{p_1,q_1} h^{p_2,q_2} \pm \frac{1}{2} h^{\cancel{p/2,\,n-1-p/2}\ (p+n-1)/2,\ (-p+n-1)/2}$$

$$\sim \frac{1}{2} d^{2n} \beta_p(n) \quad \text{where } \beta_p^{(n)} = \sum_{\substack{p_1+p_2 \\ = p+n-1}} \alpha_{p_1}^{(n)} \alpha_{p_2}(n)$$

Note that $\dim(Y) = \binom{n+d}{d-1} - 1 \sim \frac{d^{n+1}}{(n+1)!} = o(h^{p})$ as $d \to \infty$ for fixed n.

Let $X(n)$ = random variable which, for a uniformly distributed perm in S_n, ~~ess~~ takes the value $\#\{i : \sigma(i+1) > \sigma(i)\} - \frac{n-1}{2}$.

~~This is similar~~ Then $\alpha_0^{(n)}, \dots, \alpha_n^{(n)}$ give the probability distribution of $X(n)$ (shifted).

This is close to a binomial distribution:
it is sum of $n-1$ values $Y_i = \begin{cases} +\frac{1}{2} & \text{if } \sigma(i+1) > \sigma(i) \\ -\frac{1}{2} & \text{if } \sigma(i+1) < \sigma(i) \end{cases}$

Of course, Y_i takes values $\pm\frac{1}{2}$ with prob $\frac{1}{2}$ but the Y_i are not indep.

In fact: if $|i-j| > 1$, then Y_i, Y_j are indep.
If $|i-j| = 1$, then $\mathbb{P}(Y_i = Y_j) = \frac{1}{3}$ instead of $\frac{1}{2}$.

Hence $\mathrm{Var}(X(n)) = \sum_{i,j=1}^{n} \mathbb{E}(Y_i Y_j) = \underbrace{\frac{n-1}{4}}_{i=j} + \underbrace{2(n-2)\left(-\frac{1}{12}\right)}_{|i-j|=1} = \frac{n+1}{12}$

(vs $\frac{n-1}{4}$ for binomial — much more central!)

$\mathbb{E}(X(n)) = 0$.

~~Let $X'(n)$ = convolution~~

It is known that $X(n)/\sqrt{n}$ converges in distribution to $N\left(0, \frac{1}{12}\right)$

Let $X'(n)$ = convolution of $X(n)$ with itself = sum of two indep copies of $X(n)$.

Then $X'(n)$ has prob dist $\beta_p(n)$

and by previous facts $\mathbb{E}(X'(n)) = 0$, $\mathrm{Var}(X'(n)) = \frac{n+1}{6}$,

$$X'(n)/\sqrt{n} \xrightarrow[\text{in dist}]{} N\left(0, \frac{1}{6}\right).$$

(All we will use: $\exists$ limit in dist, β_p log concave)

Pf of (10.4): Thanks to convergence in dist,

$$\beta_0(n) = \mathbb{P}(X'(n) = 0) \to 0 \text{ as } n \to \infty,$$

$$\sum_{p>0} \beta_p(n) = \mathbb{P}(X'(n) > 0) \to \frac{1}{2}$$

~~So eventually~~

~~So for $d \geq D_0(n)$, $h^0 + \dim(Y) < 2h^0$~~

~~for $n \geq N_0$ $\leq 3\beta_0(n)$~~

~~$\leq \frac{1}{2}\sum \beta_p(n)$~~

So eventually $2d^{-2n}(h^0 + \dim(Y)) < \varepsilon$,

$$2d^{-2n} \sum_{p>0} h^p > \frac{1}{4}.$$

$\square$

To get (10.5), we observe a couple of facts:

(A) $\exists$ absolute $A>0$ s.t. $\sum_{p>0} p\beta_p(n) > A\sqrt{n}$

because $\sum_{p>0} \frac{p\beta_p(n)}{\sqrt{n}} \geq \sum_{p\geq\sqrt{n}} \beta_p(n) = \mathbb{P}(X(n)\geq\sqrt{n})$

$\longrightarrow \mathbb{P}(Z\geq 1)$ where $Z\sim N(0,\frac{1}{8})$.

(B) $\exists\, c>0$ s.t. $\forall n>>0$, $\exists\, p>\sqrt{n}$ s.t. $\beta_p > \frac{c}{\sqrt{n}}$.

because $\sum_{\sqrt{n}\leq p\leq 2\sqrt{n}} \beta_p(n) = \mathbb{P}(\sqrt{n}\leq X(n)\leq 2\sqrt{n})$

$\longrightarrow \mathbb{P}(1\leq Z\leq 2)$

(Explicit value $c=\frac{1}{40}$ can be obtained from $N(0,\frac{1}{8})$.)

(C) For $n>>0$, $\beta_0(n) < \frac{1}{c\sqrt{n}}$

Follows from (B) + $\beta_p(n)$ are log-concave.

Pf of (70.5)

We have $\frac{3}{2}h^0 + \dim(Y) \leq 2h^0$ for large enough d

$\leq \frac{2H}{c\sqrt{n}}$ by (C) where $H = \dim(\mathrm{adj}\, H_S)$

$\sim \frac{1}{2}d^{2n}$

while $\sum_{p>0} p h^p > AH\sqrt{n}$ by (A)

$\therefore$ s.t.p. $2T\left(\frac{2H}{c\sqrt{n}}\right) < AH\sqrt{n}$ for large n, d.

Let $\varepsilon = \frac{Ac}{8}$.

Split up contributions of Hodge numbers above and below εn to $T\left(\frac{2H}{c\sqrt{n}}\right)$:

$$T\left(\frac{2H}{c\sqrt{n}}\right) \leq \underbrace{(\varepsilon n)\frac{2H}{c\sqrt{n}}}_{\text{At most } \frac{2H}{c\sqrt{n}} \text{ Hodge numbers} \leq \varepsilon n} + \sum_{p>\varepsilon n} p h^p$$

$$= \frac{AH}{4}\sqrt{n} + \sum_{p>\varepsilon n} p h^p$$

$$H^{-1}\sum_{p>\varepsilon n} p h^p \longrightarrow \sum_{p>\varepsilon n} p\beta_p$$

$$\leq \sum_{p>\varepsilon n} p^2 \beta_p \frac{1}{\varepsilon n}$$

$$\leq \mathrm{Var}(X'(n))\frac{1}{\varepsilon n} \sim \frac{1}{6\varepsilon}.$$

$\therefore$ For large n,

$$T\left(\frac{2H}{c\sqrt{n}}\right) \leq \frac{AH}{4}\sqrt{n} + H \left(1 + \frac{1}{3\epsilon}\right) \quad \text{for large } d, n$$

$$\leq \frac{AH}{2}\sqrt{n} \quad \text{for large } n, d. \qquad \square$$