DIOPHANTINE PROBLEMS AND p-ADIC PERIOD MAPPINGS AFTER B. LAWRENCE AND A. VENKATESH

PROOF OF [LV18, PROP. 10.6]

The aim of this note is to review the proof of [LV18, Prop. 10.6]. The formulation of [LV18, Prop. 10.6] is derived from the one of [LV18, Thm. 10.1], which is used for the applications to hyper surfaces. But [LV18, Prop. 10.6] is actually a special case of a more general statement - Theorem 2.2 below, a by-product of [LV18, Prop. 11.3 and §11.6] - which can be formulated in terms of filtrations on arbitrary reductive groups. As pointed out in [LV18, §1.5], Theorem 2.2 is probably far from optimal. In Section 1 I introduce the material required to formulate and perform the proof of Theorem 2.2, in particular I briefly review the combinatorics of parabolic subgroups, the notion of filtrations as equivalence classes of cocharacters and two related constructions - the weight and induced filtrations. I state Theorem 2.2 in Section 2 and make a few remarks about it - see in particular 2.4. The proof of Theorem 2.2 is performed in Section 3. I followed the main guidelines of the proof in [LV18, §11], elaborating on some points which were not completely clear to me - see *e.g.* the lemma and remark in Subsection 3.1 and the final remark in Subsection 3.2.

1. Filtrations in reductive groups

Let k be an algebraically closed field of characteristic 0 and G a connected reductive group over k. A subgroup of G always means a closed algebraic subgroup of G.

References: [B91], [DM91], [M17].

1.1. **Recollection on parabolic subgroups.**

1.1.1. Recall that for a connected subgroup $P\subset G$ the following are equivalent[1]

(i) G/P is projective;
(ii) P contains a Borel subgroup of G;
(iii) $P=N_G(R_u(P))$.

and that such a subgroup is called parabolic. Let $\mathcal{P}_G$ denote the set of parabolic subgroups of G and for a subgroup $H\subset G$, let $\mathcal{P}_G(H)\subset\mathcal{P}_G$ denote the subset of parabolic subgroups of G containing H. The group G acts by conjugacy on $\mathcal{P}_G$ and for every Borel subgroup $B\subset G$, $\mathcal{P}_G(B)$ is finite with $\mathcal{P}_G(B)\tilde{\to}\mathcal{P}_G/G$ (in other words, $\mathcal{P}_G(B)$ is a system of representatives of $\mathcal{P}_G/G$).

1.1.2. One can describe explicitly parabolic subgroups of G in terms of roots as follows. Fix a maximal torus $T\subset G$ and let $\Phi:=\Phi(G,T)$ the corresponding root system; for $\alpha\in\Phi$ let $U_\alpha\subset G$ denote the corresponding root group. A subset $\Psi\subset\Phi$ is called closed if it satisfies the following equivalent properties
- (i) for every $\alpha,\beta\in\Psi$ $\mathbb{Z}\alpha+\mathbb{Z}\beta\cap\Phi\subset\Psi$.
- (ii) for every $\alpha\in\Phi$, $U_\alpha\subset G_\Psi:=\langle T,U_\gamma\mid\gamma\in\psi\rangle\Rightarrow\alpha\in\Psi$.

The map $\Psi\to G_\Psi$ induces a bijective correspondance between closed subsets $\Psi\subset\Phi$ and connected subgroups $T\subset H\subset G$. Furthermore, under this correspondance
(1) symmetric closed subsets $\Psi=-\Psi\subset\Phi$ correspond to connected reductive subgroups $T\subset H\subset G$;
(2) closed subsets $\Psi\subset\Phi$ such that for every $\alpha\in\Phi$, $\alpha\in\Psi$ or $-\alpha\in\Psi$ correspond to parabolic subgroups $P\in\mathcal{P}_G(T)$. In particular a parabolic subgroup $P=G_\Psi\in\mathcal{P}_G(T)$ contains a unique Levi subgroup $L_P:=G_{(\Psi\cap-\Psi)}$ containing T.

[1] Since for an arbitrary subgroup $H\subset G$ one always has $H\subset N_G(H)\subset N_G(R_u(H))$, Property (iii) implies in particular that for $P\in\mathcal{P}_G$, $P=N_G(P)=N_G(R_u(P))$.

1.1.3. For a subset $\Psi \subset \Phi$ the following are equivalent

- $\Psi \subset \Phi$ is closed and for every $\alpha \in \Phi$, $\alpha \in \Psi$ or $-\alpha \in \Psi$;
- There exists $\chi \in X_*(T)$ such that $\Psi = \Phi^{\chi \geq 0} := \{\alpha \in \Phi \mid \langle \alpha, \chi \rangle \geq 0\}$.

In other words, one has a parametrization $X_*(T) \twoheadrightarrow \mathcal{P}_G(T)$. The Levi factor L_P of $P := G_{\Phi^{\chi\geq 0}}$ containing T is then the centralizer of χ in G.

The parabolic subgroup $P := G_{\Phi^{\chi\geq 0}}$ depends only on the image of χ in $X_*(T) \hookrightarrow X_*(G)$ and not on T. More precisely, let $T' \subset$ be another maximal torus containing the image of χ. Since $T, T' \subset Z_G(T \cap T') \subset Z_G(\chi) = L_P$ are maximal tori of the connected group $Z_G(T \cap T')$ there exists $g \in Z_G(T \cap T') \subset Z_G(\chi)$ such that $T' = gTg^{-1}$. Then $\Phi' := \Phi(G, T') = \{g \cdot \alpha = \alpha(g^{-1} - g) \mid \alpha \in \Phi\}$. This shows that $g \cdot \Phi^{\chi \geq 0} = \Phi'^{\,\chi\geq 0}$ hence that $G_{\Phi'^{\chi\geq 0}} = gG_{\Phi^{\chi\geq 0}}g^{-1} = G_{\Phi^{\chi\geq 0}}$ since $g \in L_P \subset P = G_{\Phi^{\chi\geq 0}}$. As a result, the parametrization $X_*(T) \twoheadrightarrow \mathcal{P}_G(T)$ extends as

$$\begin{array}{ccc} X_*(T) & \twoheadrightarrow & \mathcal{P}_G(T) \\ \big\downarrow & & \big\downarrow \\ X_*(G) & \underset{P_-}{\twoheadrightarrow} & \mathcal{P}_G. \end{array}$$

For every $\chi, \chi' \in X_*(G)$, one says that $\chi \sim \chi'$ if $P_\chi = P_{\chi'}$ and $\chi' \in P_\chi \chi \subset X_*(G)$. Note that the resulting map $X_*(G)/\sim \twoheadrightarrow \mathcal{P}_G$ is far from being injective. This is because $\mathcal{P}_G$ 'does not see the numbering' of the filtration but only the dimension of the graded pieces while elements in $X_*(G)/\sim$ 'see the numbering'.

1.2. **Relation with the usual notion of filtration.** The set $X_*(G)/\sim$ has to be regarded as the generalization of the classical notion of filtration. More precisely, given a finite dimensional k-vector space V, a cocharacter $\chi \in X_*(GL_V)$ defines a filtration $F^p_\chi(V) := \oplus_{n\geq p} V_\chi(n)$, $p \in \mathbb{Z}$. Conversely, given a filtration $F^\bullet(V) = V = F^{-\infty}V \supset \cdots F^pV \supset F^{p+1}V \supset \cdots \supset F^{+\infty}V = 0$, any choice of a splitting $V = \oplus_{n\in\mathbb{Z}} V_F(n)$ of $F^\bullet$ defines a (necessarily unique) cocharacter $\chi \in X_*(GL_V)$ such that $V_\chi(n) = V_F(n)$, $n \in \mathbb{Z}$ and two cocharacters $\chi, \chi' \in X_*(GL_V)$ define the same filtration if and only if $P_\chi := \mathrm{Stab}_{GL_V}(F^\bullet_\chi) = \mathrm{Stab}_{GL_V}(F^\bullet_{\chi'}) =: P_{\chi'}$ and χ, χ' are P_χ-conjugate.

More generally, the Tannakian category $Rep_k(\mathbb{G}_{m,k})$ is $\otimes$-equivalent to the Tannakian category $Gr^{\mathbb{Z}}Vect_k$ of finite-dimensional $\mathbb{Z}$-graded k-vector spaces and one has a natural $\otimes$-functor $Gr^{\mathbb{Z}}Vect_k, \to Fil^{\mathbb{Z}}Vect_k$ to the $\otimes$-category $Fil^{\mathbb{Z}}Vect_k$ of finite-dimensional k-vector spaces endowed with a descending filtration indexed by $\mathbb{Z}$ so that for an arbitrary morphism $\varphi : G_1 \to G_2$ of algebraic groups over k, one has

$$\begin{array}{ccccc} X_*(G_1) & \xrightarrow{\ \tilde{\ }\ } & \mathrm{Hom}^\otimes(Rep_k(G_1), Gr^{\mathbb{Z}}Vect_k) & \longrightarrow & \mathrm{Hom}(Rep_k(G_1), Fil^{\mathbb{Z}}Vect_k) \\ {\scriptstyle \varphi\circ -}\big\downarrow & & \big\downarrow & & \big\downarrow \\ X_*(G_2) & \xrightarrow{\ \tilde{\ }\ } & \mathrm{Hom}^\otimes(Rep_k(G_2), Gr^{\mathbb{Z}}Vect_k) & \longrightarrow & \mathrm{Hom}(Rep_k(G_2), Fil^{\mathbb{Z}}Vect_k) \end{array}$$

Elements in $\mathrm{Hom}(Rep_k(G_2), Fil^{\mathbb{Z}}Vect_k)$ are called filtration on V. In more down-to-earth terms, every cocharacter $\chi \in X_*(G)$ defines a filtration $F^p_\chi(V) := \oplus_{n\geq p} V_\chi(n)$, $p \in \mathbb{Z}$ on each finite-dimensional k-representation V of G and these filtrations are functorial, compatible with the formation of duals and tensor products and exact in the sense that the functor $V \to Gr^\bullet_F(V)$ is exact. Conversely, every filtration $F^\bullet : Rep_k(G) \to Fil^{\mathbb{Z}}Vect_k$ whose essential image satisfies the above conditions arises from a (non-unique) cocharacter $\chi \in X_*(G)$ - sometimes called a splitting of $F^\bullet$.

For instance, if $G = Aut(V, \langle -, - \rangle)$ for a finite-dimensional k-vector space V equipped with a non-degenerated k-bilinear form $\langle -, - \rangle : V \otimes_k V \to k$, the G-equivariant isomorphism $V \tilde{\to} V^\vee$, $v \to \langle v, - \rangle$ imposes that the filtrations $F^\bullet V$ defined by elements in $X_*(G)$ satisfy

$$\langle F^n V, - \rangle = F^n(V^\vee) = \ker(V \twoheadrightarrow (F^{-n}V)^\vee)$$

that is $F^nV = (F^{-n}V)^\perp$ and, actually, this is the unique condition those filtrations have to satisfy. If $G = GAut(V, \langle -,-\rangle)$ and $\nu \in X^*(G)$ is the character defined by $\langle g-, g-\rangle = \nu(g)\langle -,-\rangle$, $g \in G$, for $\chi \in X_*(G)$ let $r_\chi : \mathbb{G}_{m,k} \xrightarrow{\chi} G \xrightarrow{\nu} \mathbb{G}_{m,k} \in \mathbb{Z}$. Then the G-equivariant isomorphism $V \tilde{\to} V^\vee \otimes k(\nu)$, $v \to \langle v, -\rangle$ imposes that the filtrations $F^\bullet V$ defined by elements in $X_*(G)$ satisfy

$$\langle F^nV, -\rangle = F^n(V^\vee \otimes k(\nu)) = \sum_{p\in\mathbb{Z}} F^{n+p}(V^\vee) \otimes F^{-p}(k(\nu)) = F^{n-r}V = \ker(V \twoheadrightarrow (F^{r-n}V)^\vee)$$

that is $F^nV = (F^{r-n}V)^\perp$ *etc.*

Every filtration $F^\bullet : Rep_k(G) \to Fil^{\mathbb{Z}}Vect_k$ on G induces a filtration by closed normal subgroups

$$F^pG = \cap_V \ker(G \to \oplus_{n\in\mathbb{Z}} F^nV/F^{n+p}V),\ p \in \mathbb{Z}$$

so that $F^{-1}G = G$, $F^pG \subset R_u(G)$, $p \geq 1$ and $F^0G = F^{-1}G \rtimes Z_G(\chi)$ for every $\chi \in X_*(G)$ splitting $F^\bullet : Rep_k(G) \to Fil^{\mathbb{Z}}Vect_k$. When G is reductive, one has the following explicit description

Proposition. *Assume G is reductive. Consider the filtration*

$$F^pG = \ker(G \to \oplus_{n\in\mathbb{Z}} F^n\mathfrak{g}/F^{n+p}\mathfrak{g}),\ p \in \mathbb{Z}$$

induced from the filtration $F^\bullet(\mathfrak{g})$ on the adjoint representation $\mathfrak{g} := Lie(G)$ of G.

1. *For every representation V of G, $F^0G = \mathrm{Stab}_G(F^\bullet V)$ and $F^0G \in \mathcal{P}_G$ with $Lie(F^0G) = F^0\mathfrak{g}$;*
2. *For every representation V of G, $F^1G = \ker(F^0G \to GL(gr^\bullet_F(V)))$ and $F^1G = R_u(F^0G)$ with $Lie(F^1G) = F^1\mathfrak{g}$;*
3. *For every $\chi \in X_*(G)$ splitting $F^\bullet : Rep_k(G) \to Fil^{\mathbb{Z}}Vect_k$, $Z_G(\chi) \subset F^0G$ is a Levi factor of F^0G; the composite $\chi^{red} : \mathbb{G}_{m,k} \xrightarrow{\chi} F^0G \to (F^0G)^{red}$ is central;*
4. *For every $\chi, \chi' \in X_*(G)$ $F^\bullet_\chi = F^\bullet_{\chi'}$ if and only if $F^0_\chi G = F^0_{\chi'}G =: P$ and $\chi^{red} = \chi'^{\,red}$ that is, χ, χ' are $R_u(P)$-conjugate.*

1.3. **Weight.** Since G is reductive, the canonical morphisms $Z(G)^\circ \to G \to G^{ab}$ is an isogeny hence it induces a commutative diagram

$$\begin{array}{ccc} & X_*(G) & \\ {}^{\omega}\swarrow & & \searrow \\ X_*(Z(G)^\circ) \otimes \mathbb{Q} & \xrightarrow{\ \simeq\ } & X_*(G^{ab}) \otimes \mathbb{Q} \end{array}$$

The map $\omega_G : X_*(G) \to X_*(G) \otimes \mathbb{Q} \to X_*(Z(G)^\circ) \otimes \mathbb{Q}$ factors through the weight map $\omega_G : X_*(G)/\sim \to X_*(Z(G)^\circ) \otimes \mathbb{Q}$.

Examples. (cont.)
(1) If $G = GL(V)$, $\omega_G(F^\bullet V) = \sum_{p\in\mathbb{Z}} p\dim(F^pV)/\dim(V)$;
(2) If $G = GAut(V, \langle -,-\rangle)$, $\omega_G(F^\bullet V)(= \sum_{p\in\mathbb{Z}} p\dim(F^pV)/\dim(V)) = \frac{r}{2}$.

1.4. **Intersections of parabolic subgroups and induced filtration.**

Proposition 1. *Let $P, Q \in \mathcal{P}_G$. The group $P\cap Q$ is connected and contains a maximal torus T of G. Fix Levi $T \subset L_P \subset P$, $T \subset L_Q \subset Q$. Then $P\cap Q = (L_P\cap L_Q)\cdot(L_P\cap R_u(Q))\cdot(L_Q\cap R_u(P))\cdot(R_u(P)\cap R_u(Q))$.*

Proposition 2. *Let $H \subset G$ be a connected subgroup of maximal rank. Then, the parabolic subgroups of H are the $P \cap H$ for P a parabolic subgroup of G containing a maximal torus in H and the Levi subgroups of such a $P\cap H$ are the $L_P\cap H$ for $L_P \subset P$ a Levi subgroup of P containing a maximal torus of H.*

Let $Q \in \mathcal{P}_G$ and $\chi \in X_*(G)$. Since $Q \cap P_\chi$ contains a maximal torus T of G, up to replacing χ by a P_χ-conjugate (which does not affect the $\sim$-class of χ in $X_*(G)$) one may assume $\chi \in X_*(Q \cap P)$. Let $p_{Q^{red}} : Q \twoheadrightarrow Q^{red} := Q/R_u(Q)$ denote the canonical projection. Then,

Lemma. *The $\sim$-class $[\chi]_{Q^{red}}$ of $p_{Q^{red}} \circ \chi$ in $X_*(Q^{red})$ only depends on the $\sim$-class of χ in $X_*(G)$.*

Proof. We have to show that if $\chi \sim \chi'$ in $X_*(G)$ and χ' has image in $Q \cap P_\chi$ then $[\chi]_{Q^{red}} = [\chi']_{Q^{red}}$.
We first show that χ, χ' are conjugate under $Q \cap P_\chi$. Since $\chi \sim \chi'$ there exists $g \in P_\chi$ such that $\chi' = g\chi(-)g^{-1}$ hence the image of χ' is contained in Q implies that $g^{-1}Q \in (P_\chi Q/P_\chi)^\chi$. But $Q \in (P_\chi Q/P_\chi)^\chi$ and $Z_G(\chi) \subset P_\chi$ acts transitively on $(P_\chi Q/Q^\chi$ so that there exists $p \in Z_G(\chi) \subset P_\chi$, $q \in Q$ such that $g^{-1} = pq$ hence $q \in P_\chi \cap Q$. To see why $Z_G(\chi) \subset P_\chi$ acts transitively on $(P_\chi Q/Q)^\chi$, recall that $Z_G(\chi) \subset P_\chi$ is a Levi subgroup so that it is enough to show that the unique χ-fixed point of $R_u(P_\chi)Q/Q$ is Q. But by definition of P_χ, $R_u(P_\chi)$ is generated by the root groups U_α, $\alpha \in \Phi = \Phi(G,T)$ such that $\langle \alpha, \chi \rangle > 0$. For every $u \in U_\alpha$, $\chi(t)uQ = uQ$ if and only if $\chi(t)u\chi(t)^{-1} \in uQ$. But as $\langle \alpha, \chi \rangle > 0$, the morphism $\phi : \mathbb{G}_{m,k} \to U_\alpha$, $t \to \chi(t)u\chi(t)^{-1}$ extends to a morphism $\phi : \mathbb{A}^1_{m,k} \to U_\alpha$ mapping 0 to Id so that $Id \in uQ$ *i.e.* $u \in Q$.
So, now, we can write $\chi' = g\chi(-)g^{-1}$ for some $g \in Q \cap P_\chi$. Using the decomposition in Proposition 1 and that L_P centralizes χ, one may furthermore assume $g = rs$ with $r \in R_u(Q)$ and $s \in R_u(P_\chi) \cap L_Q$. Then $p_{Q^{red}} \circ \chi' = p_{Q^{red}}(s) p_{Q^{red}} \circ \chi(-) p_{Q^{red}}(s)^{-1}$. But from Proposition 2 (applied with $H = L_Q$) and the definition of P_χ one easily checks that

$$P_{p_{Q^{red}} \circ \chi} = P_\chi \cap L_Q \cdot R_u(Q)/R_u(Q).$$

As a result $p_{Q^{red}} \circ \chi' \sim p_{Q^{red}} \circ \chi$ in Q^{red}. □

1.5. **Balanced filtrations.** Let $P \in \mathcal{P}_G$ and $L_P \subset P$ a Levi subgroup.

1.5.1. Since P hence L_P contains a maximal torus of G, L_P contains $Z(G)$ whence a canonical commutative diagram

$$\begin{array}{ccccccc}
Z(P^{red})^\circ & \xleftarrow{\simeq} & Z(L_P)^\circ & \hookrightarrow & L_P & \twoheadrightarrow & L_P^{ab} \\
 & & \uparrow & & \downarrow & & \downarrow \\
 & & Z(G)^\circ & \longrightarrow & G & \twoheadrightarrow & G^{ab}
\end{array}$$

which induces

$$\begin{array}{ccccc}
X_*(Z(P^{red})^\circ) \otimes \mathbb{Q} & \xleftarrow{\simeq} & X_*(Z(L_P)^\circ) \otimes \mathbb{Q} & \xrightarrow{\simeq} & X_*(L_P^{ab}) \otimes \mathbb{Q} \\
 & {\scriptstyle c_P} \nwarrow & \uparrow & & \downarrow \\
 & & X_*(Z(G)^\circ) \otimes \mathbb{Q} & \xrightarrow[\simeq]{} & X_*(G^{ab}) \otimes \mathbb{Q}
\end{array}$$

The set of filtrations of G balanced with respect to P is the equalizer of

$$\begin{array}{ccccc}
 & & X_*(P^{red})/\sim & & \\
 & {\scriptstyle (-)_{P^{red}}} \nearrow & & \searrow {\scriptstyle \omega_{P^{red}}} & \\
X_*(G)/\sim & & & & X_*(Z(P^{red})^\circ) \otimes \mathbb{Q} \\
 & {\scriptstyle \omega_G} \searrow & & \nearrow {\scriptstyle c_P} & \\
 & & X_*(Z(G)^\circ) \otimes \mathbb{Q} & &
\end{array}$$

that is the subset $X_*(G,P) \subset X_*(G)/\sim$ of all $[\chi] \in X_*(G)/\sim$ such that $c_P(\omega_G([\chi])) = \omega_{P^{red}}([\chi]_{P^{red}})$ in $X_*(Z(P^{red})^\circ) \otimes \mathbb{Q}$.

1.5.2. For $\chi : \mathbb{G}_{m,k} \to T \subset L_P \subset P$, $\chi \in X_*(G,P)$ implies that for every

$$\alpha \in \ker(X^*(P) = X^*(P^{ab}) = X^*(L_P^{ab}) \to X^*(Z(L_P)) \to X^*(Z(G)^\circ)),$$

$\langle \alpha, \chi \rangle = 0$. We will apply this observation with $\alpha : P \to \mathbb{G}_{m,k}$ the determinant of the adjoint action of P on $Lie(R_u(P))$.

2. Reformulation of what we want to prove

2.1. Consider the set $\mathcal{T}$ of triples $\Theta := (P_0, \varphi_0, \mu_0)$ where $P_0 \in \mathcal{P}_G$, $\varphi_0 \in P_0^{red}$, $\mu_0 \in X_*(P_0^{red})/\sim$; set $R_0 := P_{\mu_0} \in \mathcal{P}_{P_0^{red}}$. G acts on $\mathcal{T}$ by

$$g \cdot \Theta = (gP_0g^{-1}, g\varphi_0 g^{-1}, g[\mu_0]g^{-1}),$$

where we write again $g - g^{-1} : P_0^{red} \to P_0^{red}$ for the right vertical arrow in the diagram below.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_u(P_0) & \longrightarrow & P_0 & \longrightarrow & P_0^{red} & \longrightarrow & 1 \\ & & \downarrow & & \simeq \downarrow g-g^{-1} & & \simeq \downarrow g-g^{-1} & & \\ 1 & \longrightarrow & R_u(gP_0g^{-1}) = gR_u(P_0)g^{-1} & \longrightarrow & gP_0g^{-1} & \longrightarrow & (gP_0g^{-1})^{red} & \longrightarrow & 1 \end{array}$$

Let $\varphi \in G$ and $\chi \in X_*(G)$; write $Q := P_\chi \in \mathcal{P}_G$. We want to compute the dimension of the subvariety $X(\varphi, \Theta, \chi) \subset G/Q$ of all gQg^{-1} such that there exists $P \in \mathcal{P}_G$ satisfying the following properties:

(1) $\varphi \in P$;
(2) $g[\chi]g^{-1} \in X_*(G, P)$;
(3) $(P, p_{P^{red}}(\varphi), (g[\chi]g^{-1})_{P^{red}}) \in G \cdot \Theta$.

Fix a a maximal torus $T \subset Q$ containing the image of χ; set $\Phi := \Phi(G, T)$ and define the Hodge numbers of Q as the multiset $\mathcal{H}(\chi) := \{\langle \alpha, \chi \rangle \mid \alpha \in \Phi\}$ to which we add 0 with the multiplicity $\mathrm{dim}T$. Order the elements in $\mathcal{H}(Q)$ as

$$h_\chi^1 \leq h_\chi^2 \leq \cdots \leq h_\chi^m$$

(so that $m = |\Phi| + \dim(T) = \dim(G)$) and define the function 'sum of topmost $-$ Hodge numbers':

$$\begin{array}{cccc} T_\chi : & \mathbb{Z} \cap [1, m] & \to & \mathbb{Z} \\ & n & \to & \displaystyle\sum_{m-n+1 \leq i \leq m} h_\chi^i \end{array}$$

2.2. **Theorem.** *Assume $\varphi \in G$ is semisimple. Let $c \in \mathbb{Z}_{\geq 1}$ such that*

(i) $c + \mathrm{dim}Z_G(\varphi) \leq \dim(G/Q)$;
(ii) $T_\chi(c + \mathrm{dim}Z_G(\varphi)) + T_\chi(c + \mathrm{dim}Z_G(\varphi) + \frac{\mathrm{dim}L_Q}{2}) < \mathrm{dim}G/Q$.

Then $\mathrm{dim}X(\chi, \varphi, \Theta) < \mathrm{dim}G/Q - c$.

2.3. **Remark.** Writing $\varphi = \varphi^u \varphi^{ss}$ for the Jordan decomposition of φ in G and using that the Jordan decomposition is unique and preserved by morphism of algebraic groups one has: $\varphi \in P \Rightarrow \varphi^{ss} \in P$, $\varphi_0 \in P_0 \Rightarrow \varphi_0^{ss} \in P_0$ and $p_{P_0^{red}}(g\varphi g^{-1}) = \varphi_0 \Rightarrow p_{P_0^{red}}(g\varphi^{ss}g^{-1}) = \varphi_0^{ss}$. Hence $X(\varphi, \Theta) \subset X(\varphi^{ss}, (P_0, \varphi_0^{ss}, R_0))$ so that, in particular, if the assumptions of Theorem 2.2 are satisfied for φ^{ss}, one has $\mathrm{dim}X(\chi, \varphi, \Theta) \leq \mathrm{dim}X(\chi, \varphi^{ss}, \Theta) \leq \mathrm{dim}G/Q - c$.

2.4. We check that the Hodge numbers introduced in [LV18, Thm. 10.1, Prop. 10.6] correspond with those defined in 2.1. This follows from the Proposition in Subsection 1.2. With the notation of [LV18, §10], write $V := V_0 \otimes \mathbb{C}$ and $G := GAut(V, \langle -, - \rangle)$. Then, as graded vector spaces, $\mathfrak{g} := Lie(G) \simeq \mathbb{C} \oplus S^2(V)$ if $\langle -, - \rangle$ is antisymmetric and $\simeq \mathbb{C} \oplus \bigwedge^2(V)$ if $\langle -, - \rangle$ is symmetric (identifying $V \otimes V^\vee$ with $V^{\otimes 2}$ *via* the non-degenerate pairing $\langle -, - \rangle$). Write h^p for the $\mathbb{C}$-dimension of the $(p, -p)$ component of the Hodge filtration on $\mathfrak{g}$. In [LV18, Thm. 10.1, Prop. 10.6], the set of Hodge numbers for G is defined as the multiset of all $p \in \mathbb{Z}$ endowed with the multiplicity h^p. Write

$$V = F^{-\infty}V \supset F^p \supset F^{p+1} \supset \cdots \supset \cdots F^{+\infty}V = 0$$

for the Hodge filtration on V. Recall it induces a filtration by closed normal subgroups on G by

$$F^pG = G,\ p \leq -1\ F^pG := \{g \in G \mid gF^nV \subset F^{n+p}V,\ n \in \mathbb{Z}\},\ p \geq 0$$

and that the filtration on $\mathfrak{g}$ induced by $F^\bullet V$ *via* the isomorphism $\mathfrak{g} \simeq \mathbb{C} \oplus S^2(V)$ or $\simeq \mathbb{C} \oplus \bigwedge^2(V)$ coincides with the filtration

$$F^p\mathfrak{g} = \mathfrak{g},\ p \leq -1\ F^p\mathfrak{g} = Lie(F^pG) = \{g \in \mathfrak{g} \mid gF^nV \subset F^{n+p}V,\ n \in \mathbb{Z}\},\ p \geq 0.$$

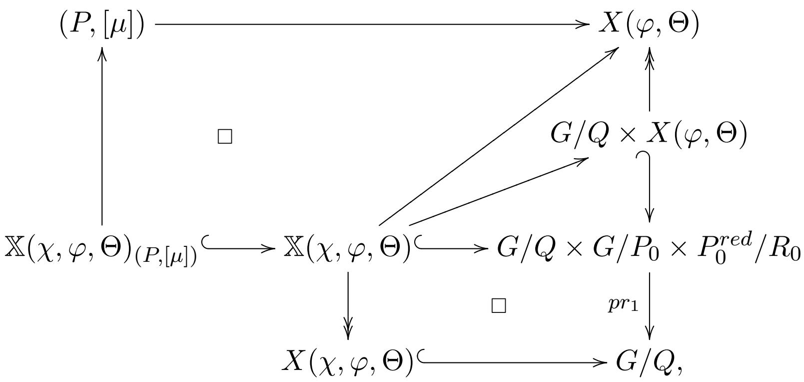
Let $\chi \in X_*(G)$ be a representative of the Hodge filtration. Fix a Borel subgroup $B \subset P_\chi$ and a maximal torus $T \subset B$ such that $\chi \in X_*(T)$; write $\Phi := \Phi(G,T)$. In terms of χ, $F^pV = \oplus_{n \geq p} V_\chi(n)$. Let $\alpha \in \Phi$, $u \in \mathfrak{u}_\alpha := Lie(U_\alpha)$ and $v \in V_\chi(n)$. Then $\chi(t)u(v) = Ad(\chi(t))(u)(\chi(t)v) = t^{\langle \alpha, \chi \rangle + n} u(v)$. This shows that $\mathfrak{u}_\alpha \subset F^{\langle \alpha,\chi \rangle}\mathfrak{g} \setminus F^{\langle \alpha,\chi \rangle + 1}\mathfrak{g}$. In particular, the dimension of $gr_F^p(\mathfrak{g})$ (te h^p of [LV18, Thm. 10.1, Prop. 10.6]) is

$$|\{\alpha \in \Phi \mid \langle \alpha, \chi \rangle = p\}|$$

(and for $p = 0$ one has to add the dimension of T, since it is not counted among the roots). So both definitions of Hodge numbers are consistant.

3. Proof of Theorem 2.2

To prove Theorem 2.2, one consider the following diagram



where $X(\varphi,\Theta) \subset G/P_0 \times P_0^{red}/R_0$ classifies all possible pairs $(gP_0g^{-1}, g[\mu_0]g^{-1})$ such that $\varphi \in gP_0g^{-1}$ and $(gP_0g^{-1}, p_{Pred}(\varphi), g[\mu_0]g^{-1}) \in G \cdot \Theta$. The proof decomposes in two steps:
- Step 1: Show that $\dim(X(\varphi,\Theta)) \leq \dim Z_G(\varphi)$;
- Step 2: For every $(P,[\mu]) \in X(\varphi,\Theta)$, show that $\dim(\mathbb{X}(\chi,\varphi,\Theta)_{(P,[\mu])}) \leq \dim(G/Q) - (c + \dim Z_G(\varphi))$.

3.1. **Step 1.** Let $X(\varphi) \subset G/P_0$ denote the set of all $P = gP_0g^{-1}$ such that $\varphi \in P$. The group $Z_G(\varphi)$ acts by conjugacy on $X(\varphi)$. Since φ is assumed to be semisimple, we have

Lemma. *$X(\varphi)/Z_G(\varphi)$ is finite.*

Proof. Write $S := \overline{\varphi^{\mathbb{Z}}}^{zar} \subset G$; it is a multiplicative subgroup. Fix a G-conjugate P of P_0 such that $S \subset P$. Using that $N_G(P) = P$, for $g \in G$ we have $S \subset gPg^{-1}$ if and only if $gPg^{-1} \in (G/P)^S$. So what we want to prove is that $Z_G(\varphi) = Z_G(S)$ has only finitely many orbits when acting on $(G/P)^S$. For this, it is enough to show that the $Z_G(S)$-orbits of $(G/P)^S$ are open (then they will be automatically closed since they form a partition of $(G/P)^S$). By homogeneity, it is enough to show the $Z_G(S)$-orbit of P is open that is that the map $Z_G(S) \to (G/P)^S$, $z \to zPz^{-1}$ has open image. To show this, it is enough to show that $Z_G(S) \to (G/P)^S$ is smooth. Again, by homogeneity, it is enough to show it is smooth at $Id \in Z_G(S)$. Since both $Z_G(S)$ and $(G/P)^S$ are smooth, it is enough to check that the differential map $\mathfrak{g}^{Ad(\varphi)-1} \to (\mathfrak{g}/\mathfrak{p})^{Ad(\varphi)-1}$, $z \to [z,\mathfrak{p}]$ is surjective. This follows from the surjectivity of $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{p}$ and the fact that $Ad(\varphi)$ is *semisimple* since $\varphi \in G$ is. $\square$

Fix $P \in X(\varphi)$. Then P is in the image of $X(\varphi,\Theta) \to G/P_0$ forces (*) $P = gP_0g^{-1}$ for some $g \in G$ such that $gp_{Pred}(\varphi)g^{-1} = \varphi_0$. If this holds for P it holds for every elements in $Z_G(\varphi) \cdot P \subset X(\varphi)$. Note that

$$Z_G(\varphi) \cap N_G(P) = Z_G(\varphi) \cap P = Z_P(\varphi)$$

so that

$$\dim(Z_G(\varphi) \cdot P) = \dim(Z_G(\varphi)) - \dim(Z_P(\varphi)).$$

So let $P_1, \dots, P_r \in X(\varphi)$ denote a system of representatives of the $Z_G(\varphi)$-orbits $O_1, \dots, O_r$ in $X(\varphi)$ which furthermore satisfy (*). Consider the diagram

$$\begin{array}{ccccc} G/P_0 \times P_0^{red}/R_0 & \hookleftarrow & X(\varphi,\Theta) & \hookleftarrow & X(\varphi,\Theta)_P \\ \downarrow & & \downarrow & \square & \downarrow \\ G/P_0 & \hookleftarrow X(\varphi) \hookleftarrow & \sqcup_{1\leq i\leq r} O_i & \hookleftarrow & P \end{array}$$

where $P = gP_0g^{-1}$ for some $g \in G$ such that $gp_{P^{red}}(\varphi)g^{-1} = \varphi_0$. For $\gamma \in P$, write $\gamma^{red} := p_{P^{red}}(\gamma) \in P^{red}$. Then, for every $\gamma \in G$ one has $\gamma \cdot (P, \varphi^{red}, [\mu]) = (P_0, \varphi_0, [\mu_0]) = (g^{-1}Pg, g^{-1}\varphi^{red}g, [\mu_0])$ if and only if $g\gamma \in N_G(P) = P$, $(g\gamma)^{red} \in Z_{P^{red}}(\varphi^{red})$ and $[\mu] = \gamma^{-1}[\mu_0]\gamma = (g\gamma)^{-1}g[\mu_0]g^{-1}g\gamma$ so that

$$X(\varphi,\Theta)_P \subset Z_{P^{red}}(\varphi^{red}) \cdot g[\mu_0]g^{-1} \subset P^{red}/gR_0g^{-1}.$$

In particular,

$$\mathrm{dim}X(\varphi,\Theta)_P \leq \dim(Z_{P^{red}}(\varphi^{red})) - \dim(Z_{gR_0g^{-1}}(\varphi^{red})) \leq \dim(Z_{P^{red}}(\varphi^{red})) \leq \dim(Z_P(\varphi)).$$

(For the last inequality, observe that since φ is assumed to be semisimple, it is contained in a Levi factor L_P of $P = R_u(P) \rtimes L_P$ so that $\dim(Z_{P^{red}}(\varphi^{red})) = \dim(Z_{L_P}(\varphi)) \leq \dim(Z_P(\varphi))$). To conclude,

$$\dim(X(\varphi,\Theta)) \leq \max\{\mathrm{dim}O_i + \mathrm{dim}X(\varphi,\Theta)_P \mid P \in O_i,\ 1 \leq i \leq r\}.$$

But for $P \in O_i$,

$$\mathrm{dim}O_i + \mathrm{dim}X(\varphi,\Theta)_P = \dim(Z_G(\varphi)\cdot P) + \mathrm{dim}X(\varphi,\Theta)_P \leq \dim(Z_G(\varphi)) - \dim(Z_P(\varphi)) + \dim(Z_P(\varphi)) = \dim(Z_G(\varphi)).$$

Remark. Write $P_i = g_iP_0g_i^{-1} \in O_i$ for a representative of an element in O_i and consider the Cartesian diagram

$$\begin{array}{ccc} Z_G(\varphi) \times G & \xrightarrow{(z,\gamma)\to\gamma z g_i} & G \\ \uparrow & \square & \uparrow \\ \Delta_i & \longrightarrow & p_{P_0^{red}}^{-1}(Z_{P_0^{red}}(\varphi_0)) \end{array}$$

Then $X(\varphi,\Theta) \times_{X(\varphi)} O_i$ identifies with the image of $\Delta_i \to Z_G(\varphi) \cdot P_i \times Z_{P_0^{red}}(\varphi_0) \cdot \varphi_0 \subset G/P \times P_0^{red}/R_0$, $(z,\gamma) \to (z \cdot P_i, (\gamma z g_i) \cdot R_0)$. In particular, it has a structure of algebraic variety and it indeed makes sense to talk about its dimension.

3.2. **Step 2.** We now fix $(P,[\mu]) \in X(\Theta,\varphi)$ and consider

$$\mathbb{X}(\chi,\varphi,\Theta)_{(P,[\mu])} := \{gQg^{-1} \in G/Q \mid g[\chi]g^{-1} \in X_*(G,P) \text{ and } (g[\chi]g^{-1})_{P^{red}} = [\mu]\}.$$

Since one may replace Q by any G-conjugate, one may assume there exists a Borel subgroup $B \subset P \cap Q$. Fix a maximal torus $T \subset B$. Let $\Phi := \Phi(G,T)$ and write $\Phi^+ \subset \Phi_P, \Phi_Q \subset \Phi$ for the sets of roots corresponding to B, P and Q respectively. Let also $\Delta \subset \Phi$ denote the set of simple roots defined by Φ^+ and $\Delta_P, \delta_Q \subset \Delta$ the subset of simple roots α such that $-\alpha \in \Phi_P$ and $-\alpha \in \Phi_Q$ respectively. Since $Q = P_\chi$ one has in particular

$$\Phi_Q = \{\alpha \in \Phi \mid \langle \alpha, \chi \rangle \geq 0\} \text{ and } \langle \Delta_Q, \chi \rangle = 0.$$

The proof of Step 2 decomposes as follows. Let $W := W(G,T) := N_G(T)/T$ denote the Weyl group of (G,T).

- Step 2.1: show that $G = PW_{P,Q}Q$, where

$$W_{P,Q} := \{w \in W \mid w^{-1}\Delta_P \subset \Phi^+,\ w\Delta_Q \subset \Phi^+\}.$$

Indeed, if $W_P = \langle s_\alpha \mid \alpha \in \Delta_P \rangle$, $W_Q = \langle s_\alpha \mid \alpha \in \Delta_Q \rangle$ so that $P = BW_PB$, $Q = BW_QB$, it is enough to show that $W = W_PW_{P,Q}W_Q$ since, then, $G = BWB = BW_PW_{P,Q}W_QB \subset PW_{P,Q}Q \subset G$. Recall that for $w \in W$, its length can be described as $\ell(w) = |\{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}|$ and that for $\alpha \in \Delta$ $\ell(ws_\alpha) = \ell(w) + 1$ if and only if $w\alpha \in \Phi^+$ and $\ell(ws_\alpha) = \ell(w) - 1$ if and only if $w\alpha \in \Phi^-$. So, if w is an element of minimal length in W_PwW_Q for every $\alpha \in \Delta_P$, $\ell(w^{-1}s_\alpha) = \ell(s_\alpha w) = \ell(w) + 1$ hence

$w^{-1}\alpha \in \Phi^+$ while for every $\alpha \in \Delta_Q$, $\ell(s_\alpha w) = \ell(w)+1$ hence $w\alpha \in \Phi^+$.

- Step 2.2: Write $X_w := \mathbb{X}(\chi,\varphi,\Theta)_P \cap (PwQ/Q) \subset G/Q$ and observe that either $X_w = \emptyset$ or $X_w = PwQ/Q$. Since
$$\mathbb{X}(\chi,\varphi,\Theta)_P = \bigsqcup_{w\in W_{P,Q}} X_w,$$
it is enough to show that for every $w \in W_{P,Q}$ such that $X_w \neq \emptyset$ and every $[\mu] \in X_*(P^{red})/\sim$, $\dim((X_w)_{[\mu]}) < \dim(G/Q) - (c + \dim Z_G(\varphi))$.

- Step 2.3: $\dim((X_w)_{[\mu]}) = \dim(R_u(P)) - \dim(wQw^{-1} \cap R_u(P))$.

Indeed, let $T \subset P \cap wQw^{-1}$ be a maximal torus such that $w\cdot\chi \in X_*(T)$ and let $L_P \subset P$, $L_{wQw^{-1}} \subset Q$ the Levi subgroups containing T of P and wQw^{-1} respectively. Then the parabolic subgroup attached to $p_{P^{red}} \circ w\cdot\chi$ is $(L_{wQw^{-1}} \cap L_P)\cdot R_u(P)/R_u(P)$ (see proof of the lemma in Subsection 1.4). Let $p_\mu \in P$ such that $p_\mu wQ(p_\mu w)^{-1} \in (X_w)_{[\mu]}$. Then for every $p \in P$, $pwQ(pw)^{-1} \in (X_w)_{[\mu]}$ if and only if $[pw\cdot\chi]_{P^{red}} = [\mu] = [p_\mu w\cdot\chi]_{P^{red}}$ that is if and only if $p^{-1}p_\mu \in (L_{wQw^{-1}} \cap L_P)\cdot R_u(P)$. As a result, using again that $N_G(wQw^{-1}) = wQw^{-1}$
$$\dim((X_w)_{[\mu]}) = \dim((L_{wQw^{-1}} \cap L_P)\cdot R_u(P)) - \dim(wQw^{-1} \cap ((L_{wQw^{-1}} \cap L_P)\cdot R_u(P))).$$
But from Proposition 1, $wQw^{-1} \cap ((L_{wQw^{-1}} \cap L_P)\cdot R_u(P))) = (L_{wQw^{-1}} \cap L_P)\cdot(wQw^{-1} \cap R_u(P))$. Note that, here as in Proposition 1, the '$\cdot$' really means direct product decomposition as algebraic varieties.

- Step 2.4: for simplicity, write $e := c + \dim Z_G(\varphi)$. Assume by contradiction that there exists $w \in W_{P,Q}$ such that $X_w \neq \emptyset$ and $[\mu] \in X_*(P^{red})/\sim$ such that $\dim((X_w)_{[\mu]}) \geq \dim(G/Q) - e$. By Step 2.3,
$$\begin{aligned}\dim(G/Q) - e \leq \dim((X_w)_{[\mu]}) &= \dim(R_u(P)) - \dim(wQw^{-1} \cap R_u(P)) \\ &= |\Phi \setminus \Phi_P| - |\{\beta \in \Phi_Q \mid -w\beta \in \Phi \setminus \Phi_P\}| \\ &= |\Phi \setminus \Phi_P| - |\{\alpha \in \Phi \setminus \Phi_P \mid -w^{-1}\alpha \in \Phi_Q\}| \\ &= |\{\alpha \in \Phi \setminus \Phi_P \mid -w^{-1}\alpha \in \Phi \setminus \Phi_Q\}|.\end{aligned}$$
Set
$$S := \{\alpha \in \Phi \setminus \Phi_P \mid -w^{-1}\alpha \in \Phi \setminus \Phi_Q\},\quad S' := (\Phi \setminus \Phi_P)\setminus S = \{\alpha \in \Phi \setminus \Phi_P \mid -w^{-1}\alpha \in \Phi_Q\}.$$
Then, we have
$$\dim(G/Q) - e \leq |S|$$
and
$$|S'| = \dim(G/P) - |S| \leq \dim(G/P) - \dim(G/Q) + e = \dim Q - \dim P + e \leq \dim(Q/B) + e \leq \frac{\dim(L_Q)}{2} + e$$
On the other hand, the condition $X_w \neq \emptyset$ implies in particular that $w\cdot\chi \in X_*(G,P)$ so that (see 1.5.2),
$$\sum_{\alpha\in\Phi\setminus\Phi_P} \langle \alpha, w\chi\rangle = 0$$
or, equivalently
$$\sum_{\alpha\in S} \langle w^{-1}\alpha, \chi\rangle = -\sum_{\alpha\in S'} \langle w^{-1}\alpha, \chi\rangle$$
By definition of S, S', the terms $\langle w^{-1}\alpha, \chi\rangle$, $\alpha \in S$ are > 0 while the terms $-\langle w^{-1}\alpha, \chi\rangle$, $\alpha \in S'$ are ≥ 0. As a result, by the definition of Hodge numbers,
$$(2.4.1)\quad -\sum_{\alpha\in S'} \langle w^{-1}\alpha, \chi\rangle \leq T(|S'|) \leq T(\frac{\dim(L_Q)}{2} + e)$$
and
$$(2.4.2)\quad \sum_{\alpha\in S} \langle w^{-1}\alpha, \chi\rangle = \sum_{-\beta\in\Phi\setminus\Phi_Q \mid w\beta\in\Phi\setminus\Phi_P} \langle \beta, \chi\rangle \geq \sum_{-\beta\in\Phi\setminus\Phi_Q} \langle \beta, \chi\rangle - T(e) \geq \dim(G/Q) - T(e).$$

(2.4.1), (2.4.2) contradict the numerical assumption 2.2 (ii).

Remark. (??) While I could see quite easily why $X(\varphi,\Theta)$ and the $\mathbb{X}(\chi,\varphi,\Theta)_{(P,[\mu])}$ are algebraic subvarieties of $G/P_0\times P_0^{red}/R_0$ and G/Q respectively, I could not see why $X(\chi,\varphi,\Theta)$ is an algebraic subvariety of G/Q (or[2] why $\cup_{(P,[\mu])\in X(\varphi,\Theta)}\mathbb{X}(\chi,\varphi,\Theta)_{(P,[\mu])}=\mathbb{X}(\chi,\varphi,\Theta)$ is an algebraic subvariety of $G/Q\times G/P_0\times P_0^{red}/R_0$). This seems to be implicit in [LV18] and crucial to conclude the proof. So I'd be happy to have an argument for this fact. Maybe one can say something like this though it looks too complicated. Up to replacing χ and P_0 by G-conjugates one may assume $\mathbb{X}(\chi,\varphi,\Theta)$ contains an element of the form $(Q,P_0,*)$ with $S:=\mathrm{Im}(\chi)\subset Q\cap P_0$. Fix as above $T\subset B\subset P_0\cap Q$; write $\Phi:=\Phi(G,T)$. Consider the Cartesian diagram

$$\begin{array}{ccccc} Z & \longrightarrow & & & (G/P)^S \\ \big\downarrow & & \square & & \big\downarrow \\ G\times G & \xrightarrow[(g_Q,g_P)\to g_Qg_P^{-1}]{} & G & \xrightarrow[g\to g\cdot P]{} & G/P \end{array}$$

By construction, for every $(g_Q,g_P)\in Z$, the image of $g_Q\chi(-)g_Q^{-1}$ is contained in $g_PP_0g_P^{-1}$ so that it makes sense to consider $\langle\alpha,g_Q\chi(-)g_Q^{-1}\rangle$ for $\alpha\in X^*(g_PP_0g_P^{-1})$. Furthermore, we now that $X(\chi,\varphi,\Theta)$ is contained in the image of the first projection $pr_1:Z\hookrightarrow G\times G\to G\to G/Q$. Now, since $Z(G)^\circ$ is normal in G, for every $\alpha\in\ker(X^*(P_0)\to X^*(Z(G)^\circ))$ and $g\in G$, $\alpha(g^{-1}-g)\in\ker(X^*(gP_0g^{-1})\to X^*(Z(G)^\circ))$. So fix a $\mathbb{Z}$-basis $\alpha_1,\dots,\alpha_r$ of $\ker(X^*(P_0)\to X^*(Z(G)^\circ))$ and consider the closed subvariety[3]

$$\mathbb{Z}(\chi,\varphi,\Theta):=\{(g_P,g_Q,\gamma)\in Z\cap(pr_2\times pr_3)^{-1}(X(\varphi,\Theta))\mid\langle\alpha_i(g_P^{-1}-g_P),g_Q\chi(-)g_Q^{-1}\rangle=0,\ i=1,\dots,r\}$$

By definition of being balanced with respect to P, $\mathbb{X}(\chi,\varphi,\Theta)$ is in the image of

$$\mathbb{Z}(\chi,\varphi,\Theta)\hookrightarrow G\times G\times P_0^{red}\to G/Q\times G/P_0\times P_0^{red}/R_0$$

...

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[2] Note that $pr_1:G/Q\times G/P_0\times P_0^{red}/R_0\to G/Q$ is closed.

[3] Actually, in [LV18], they consider the slightly larger subvariety where the α_i, $i=1,\dots,r$ are replaced by $\sum_{\alpha\in\Phi\setminus\Phi_{P_0}}\alpha$.