GHOSTS AND FAMILIES OF ABELIAN VARIETIES WITH A COMMON ISOGENY FACTOR

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Abstract. Let S be an integral variety over a finitely generated field k, with generic point η, and $A \to S$ an abelian scheme. The Hilbert irreducibility theorem and the Tate conjectures imply that the following local-global principle always holds if k is infinite. Given an abelian variety $\mathfrak{A}$ over k, for every closed point $s \in S$, $\mathfrak{A}$ is a geometric isogeny factor of A_s if and only if $\mathfrak{A} \times_k k(\eta)$ is a geometric isogeny factor of A_η. If k is finite, the problem is more subtle. We construct an obstruction - the ghost of $A \to S$ - which controls completely the failure of the above local-global principle and is a motive built out from the weight zero part of the representation of the geometric monodromy on the ℓ-adic Tate module of A_η ($\ell \neq p$). In particular, this enables us to show that the above local-global principle fails for certain abelian schemes built by Katz and Bültel.

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1. introduction

1.1. **Notation.** Given a field K, write $\overline{K}$ for its algebraic closure (which we assume to be fixed) and $\pi_1(K)$ for its absolute Galois group. Let k be a field of characteristic $p \geq 0$ and let S be a smooth, separated and geometrically connected scheme of finite type over k. Set $\overline{S} := S \times_k \overline{k}$. Let η denote the generic point of S, and $|S|$ the set of closed points of S. For $t \in S$ with residue field $k(t)$, regarded as a morphism $t = \mathrm{Spec}(k(t)) \to S$, let $\overline{t}$ denote the geometric point $\overline{t} = \mathrm{Spec}(\overline{k(t)}) \to S$; by functoriality of étale fundamental group, these induce a morphism of profinite groups $\pi_1(t, \overline{t}) \to \pi_1(S, \overline{t}) \simeq \pi_1(S, \overline{\eta})$ (well defined up to conjugacy[1]), which is injective if $t \in |S|$. Let ℓ be a prime $\neq p$.

For an algebraic group G over a field, let $G^\circ \subset G$ denote its neutral component and $G \twoheadrightarrow \pi_0(G) := G/G^\circ$ its group of connected components. For $g \in G$, we write $g = g^{ss}g^u = g^u g^{ss}$ for its multiplicative Jordan decomposition with $g^{ss} \in G$ its semisimple component, $g^u \in G$ its unipotent component.

1.2. **Abelian schemes.** Let $A \to S$ be an abelian scheme. The starting point of this note is the following question, originally addressed by Rössler and Szamuely in [RSza19, Rem. 4.2.2] for k a finite field: Assume the $A_{\overline{s}}$, $s \in |S|$ have a non-trivial common $\overline{k}$-isogeny factor. Does this imply that $A_{\overline{\eta}}$ has a non-trivial $\overline{k}$-isotrivial $\overline{k(\eta)}$-isogeny factor? Slightly more precisely, one can ask whether the following local-global principle holds.

Local-global principle G-I. *Let $\overline{\mathfrak{A}}$ be an abelian variety over $\overline{k}$. Assume that for every $s \in |S|$, $\overline{\mathfrak{A}}$ is a $\overline{k}$-isogeny factor of $A_{\overline{s}}$. Then $\overline{\mathfrak{A}} \times_{\overline{k}} \overline{k(\eta)}$ is a $\overline{k(\eta)}$-isogeny factor of $A_{\overline{\eta}}$.*

One can also upgrade[2] **G-I** as follows.

Local-global principle G-II. *Let $B_1, \dots, B_r \to S$ be abelian schemes. Assume that for every $s \in |S|$ there exists $1 \leq i_s \leq r$ such that $B_{i_s,\overline{s}}$ is a $\overline{k}$-isogeny factor of $A_{\overline{s}}$. Then there exists $1 \leq i \leq r$ such that $B_{i,\overline{\eta}}$ is a $\overline{k(\eta)}$-isogeny factor of $A_{\overline{\eta}}$.*

1.3. **Representation-theoretic formulation.** For a profinite group Δ, let $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$ denote the category of finitely generated $\mathbb{Q}_\ell$-modules equipped with a continuous $\mathbb{Q}_\ell$-linear action of Δ. For $V \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$, let $\Delta_V \subset \mathrm{Aut}_{\mathbb{Q}_\ell}(V)$ denote the image of the corresponding morphism $\Delta \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V)$. For a continuous homomorphism of profinite groups $\phi : \Gamma \to \Delta$, write ϕ^* or $-|_\Gamma : \mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta) \to \mathrm{Mod}_{\mathbb{Q}_\ell}(\Gamma)$ for

[1] The choice of geometric points does not play any part in this paper; we implicitly assume that étale paths are fixed between them and we will usually omit them from the notation for étale fundamental group.

[2] More precisely, up to replacing k by a finite field extension, $\overline{\mathfrak{A}}$ can be assumed to descend to $\mathfrak{A}$ over k, and **G-II** with $i = 1$ and $B_1 = \mathfrak{A} \times_k S \to S$ implies **G-I**.

the obvious restriction functor. This applies especially to the morphism of profinite groups $f:\pi_1(Y)\to\pi_1(S)$ induced by a morphism of k-schemes $f:Y\to S$, where Y is a smooth, separated, connected scheme of finite type over k. In particular, the notation s^* or $-|_{\pi_1(s)}:\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))\to\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(s))$ will refer to the morphism $s:\pi_1(s)\to\pi_1(S)$ induced by $s:\mathrm{Spec}(k(s))\to S$ while the notation a_S^* or $-|_{\pi_1(S)}:\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))\to\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$ (resp. a_s^* or $-|_{\pi_1(s)}:\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))\to\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(s))$) will refer to the morphism $a_S:\pi_1(S)\to\pi_1(k)$ (resp. $a_s:\pi_1(s)\to\pi_1(k)$) induced by the structural morphism $S\to\mathrm{Spec}(k)$ (resp. $\mathrm{Spec}(k(s))\to\mathrm{Spec}(k)$). For $V\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$, write V^{ss} for its semisimplification in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$.

Let $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ denote the category of *almost* Δ*-modules*, defined as follows. The objects of $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ are the elements in

$$\mathrm{colim}_U\mathrm{Mod}_{\mathbb{Q}_\ell}(U),$$

where the colimit is over all open subgroups of $U\subset\Delta$. Given $V_1,V_2\in\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$,

$$\mathrm{Hom}_{\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)}(V_1,V_2)=\mathrm{colim}_U\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{Q}_\ell}(U)}(V_1|_U,V_2|_U),$$

where the colimit is over all open subgroups of $U\subset\Delta$ such that V_1,V_2 come from and are regarded as objects of $\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$. Note that $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ is abelian and that one can regard $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$ as a subcategory of $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ *via* the tautological faithful functor $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)\to\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$. We will say that a property of an object M (resp. of a morphism $M\to N$) in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$ is persistent if, for every open subgroup $U\subset\Delta$, it is preserved by the restriction functor $-|_U:\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)\to\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$. For a persistent property P, we will say that an object M (resp. a morphism $M\to N$) in $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ has almost P if there exists an open subgroup $U\subset\Delta$ such that $M\in\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$ (resp. $M,N\in\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$ and $M\to N$ is a morphism in $\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$) and M (resp. $M\to N$) has P in $\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$. We will say that an object M (resp. a morphism $M\to N$) in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)$ has almost P if its image *via* $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Delta)\to\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ has almost P. For instance, the property of being semisimple is persistent[3] so that one can define the semisimplification M^{ss} of $M\in\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ as the image in $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)$ of the U-semisimplification of M, where $U\subset\Delta$ is any open subgroup such that $M\in\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$. For a continuous homomorphism of profinite groups $\phi:\Gamma\to\Delta$, write ϕ^* or $-|_\Gamma:\mathrm{aMod}_{\mathbb{Q}_\ell}(\Delta)\to\mathrm{aMod}_{\mathbb{Q}_\ell}(\Gamma)$ for the obvious restriction functor.

Let $V\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$. With the above terminology, consider the following representation-theoretic local-global principles:

Local-global principle R-I. *Assume there exists* $W\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))$ *such that for every* $s\in|S|$, $a_s^*W(=s^*a_S^*W)$ *is almost a submodule of* s^*V*. Then* a_S^*W *is almost a submodule of* V*.*

Local-global principle R-II. *Assume there exists* $W_1,\dots,W_r\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$ *and, for every* $s\in|S|$, $1\le i_s\le r$ *such that* $s^*W_{i_s}$ *is almost a submodule of* s^*V*. Then there exists* $1\le i\le r$ *such that* W_i *is almost a submodule of* V*.*

and the following "up to semisimplification" variants:

Local-global principle R-I'. *Assume there exists a semisimple* $W\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))$ *such that for every* $s\in|S|$, $a_s^*W(=s^*a_S^*W)$ *is almost a submodule of* $(s^*V)^{ss}$*. Then* a_S^*W *is almost a submodule of* V^{ss}*.*

Local-global principle R-II'. *Assume there exists* $W_1,\dots,W_r\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$ *and, for every* $s\in|S|$, $1\le i_s\le r$ *such that* $(s^*W_{i_s})^{ss}$ *is almost a submodule of* $(s^*V)^{ss}$*. Then there exists* $1\le i\le r$ *such that* W_i^{ss} *is almost a submodule of* V^{ss}*.*

1.3.1. **Remark.** Let Y be a smooth, separated, geometrically connected scheme of finite type over k and $f:Y\to S$ a k-morphism such that the induced morphism of profinite groups $\pi_1(Y)\to\pi_1(S)$ has open image (*e.g.* $f:Y\to S$ is dominant, or is a generic hyperplane section, *etc.*). Then **R-I** (resp. **R-II**) holds if it holds for $f^*V\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(Y))$ (resp. $f^*V,f^*W_1,\dots,f^*W_r\in\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(Y))$). When f is surjective, the converse is also true. Similar assertions hold for **R-I'**, **R-II'**.

[3] Since it is equivalent to the connected component of the Zariski-closure of the image of Δ to be reductive - here, we use that $\mathbb{Q}_\ell$ is a field of characteristic 0. For a more elementary argument, see *e.g.* (the proof of) [BuH06, 2.7 (1)].

1.3.2. For an abelian variety A over a field K of characteristic $p \geq 0$ and a prime $\ell \neq p$, let

$$T_\ell(A_{\overline{K}}) := \lim_n A[\ell^n]_{\overline{K}} \simeq \mathrm{H}^1(A_{\overline{K}}, \mathbb{Z}_\ell)^\vee$$

denote the ℓ-adic Tate module of $A_{\overline{K}}$ and set $V_\ell(A_{\overline{K}}) := T_\ell(A_{\overline{K}}) \otimes \mathbb{Q}_\ell$. When k is finitely generated, the Tate conjectures for abelian varieties, which we recall in Fact 1.3.2 below, ensure[4] the following equivalences

$$\textbf{G-I} \text{ for } A \to S \Leftrightarrow \textbf{R-I} \text{ for } V := V_\ell(A_{\overline{\eta}}) \Leftrightarrow \textbf{R-I'} \text{ for } V := V_\ell(A_{\overline{\eta}})$$

and

$$\textbf{G-II} \text{ for } A \to S \Leftrightarrow \textbf{R-II} \text{ for } V := V_\ell(A_{\overline{\eta}}) \Leftrightarrow \textbf{R-II'} \text{ for } V := V_\ell(A_{\overline{\eta}})$$

Fact. (Tate conjectures for abelian varieties) *Let K be a finitely generated field of characteristic $p \geq 0$ and $\ell \neq p$ a prime. Then,*
- (Semisimplicity) *For every abelian variety A over K, $V_\ell(A_{\overline{K}})$ is a semisimple $\pi_1(K)$-module.*
- (Fullness) *For every pair of abelian varieties A, B over K, the natural morphism of $\mathbb{Z}_\ell$-modules*

$$\mathrm{Hom}_K(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(A_{\overline{K}}), T_\ell(B_{\overline{K}}))^{\pi_1(K)}$$

is an isomorphism.

For the proof see [Ta66] (for K finite), [Z75], [Z76], [Mo77] (for $p > 0$), [F83], [FW92, IV] (for K a number field) and [FW92, VI] (for $p = 0$).

The difficulty of proving the above local-global principles depends on the arithmetic complexity of k.

1.4. **Fields containing an infinite finitely generated subfield.** If k is Hilbertian - *e.g.* an infinite finitely generated field, then **R-II** always holds.

Proposition 1. *Assume k is Hilbertian. Then **R-II** holds.*

If k contains an infinite finitely generated field (equivalently, k is not an algebraic extension of a finite field) then **G-II** always holds though **R-I** (hence *a fortiori* **R-II**) may fail (as for instance when $k = \overline{k}$).

Corollary 2. *Assume k contains an infinite finitely generated field. Then **G-II** holds.*

Proposition 1 is a consequence of the Hilbertian property and a Frattini argument of Serre. Corollary 2 follows from Proposition 1 by standard descent arguments and the Tate conjectures for abelian varieties. The proofs of Proposition 1 and Corollary 2 are given in Section 2.

1.5. **Finite fields.** If k is finite, there is a dichotomy depending on whether the dimension of W (resp. of the W_i, $i = 1, \dots, r$) in **R-I'** (resp. **R-II'**) is equal to or strictly smaller than the dimension of V.

1.5.1. If the dimension of W (resp. of the W_i, $i = 1, \dots, r$) in **R-I'** (resp. **R-II'**) is equal to the dimension of V, the following enhanced form of **R-II'** holds.

Proposition 3. *Let k be a finite field. Fix a prime $\ell \neq p$. Let $V, W_1, ..., W_r \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$; write G for the Zariski-closure of the image of $\pi_1(S)$ acting on $V \oplus W_1 \oplus \cdots \oplus W_r$. Assume that the set of all $s \in |S|$ for which there exists $1 \leq i_s \leq r$ such that $(s^* W_{i_s})^{ss}$ and $(s^* V)^{ss}$ are almost isomorphic has upper density[5] $> 1 - \frac{1}{|\pi_0(G)|}$.*
(1) *Then there exists $1 \leq i \leq r$ such that W_i^{ss} and V^{ss} are almost isomorphic.*
(2) *Assume $W_i|_{\pi_1(\overline{S})} \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(\overline{S}))$ is trivial for $i = 1, \dots, r$ and $V|_{\pi_1(\overline{S})}$ is semisimple in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(\overline{S}))$. Then $V|_{\pi_1(\overline{S})}$ is almost trivial.*

Proposition 3 is a consequence of (an ℓ-adic version of) the Cebotarev density theorem and the fact that semisimple modules are determined by their characteristic polynomials; its proof is given in Section 5. For an enhancement of Proposition 3, see Remark 5.4.

From Proposition 3 applied to $V = V_\ell(A_{\overline{\eta}})$, $W_i = V_\ell(B_{i,\overline{\eta}})$, $i = 1, \dots, r$ and Fact 1.3.2 (for $p > 0$), one deduces the following enhanced version of **G-II**.

[4] taking $V := V_\ell(A_{\overline{\eta}})$ and $W = V_\ell(\overline{\mathfrak{A}})$ (resp. $W_i := V_\ell(B_{i,\overline{\eta}})$, $i = 1, \dots, r$) in the **-I** case (resp. in the **-II** case.

[5] In this paper, the density $\delta_S(\Sigma)$ (resp. upper density $\delta_S^u(\Sigma)$) of a subset $\Sigma \subset |S|$ always refers to Dirichlet density (resp. Dirichlet upper density) - see Section 4.

Corollary 4. *Let k be a finite field. Let $A, B_1, \ldots, B_r \to S$ be abelian schemes; write G for the Zariski-closure of the image of $\pi_1(S)$ acting on $V_\ell(A_{\overline{\eta}}) \oplus V_\ell(B_{1,\overline{\eta}}) \oplus \cdots \oplus V_\ell(B_{r,\overline{\eta}})$. Assume that the set of all $s \in |S|$ for which there exists $1 \leq i_s \leq r$ such that $B_{i_s,\overline{s}}$ is $\overline{k}$-isogenous to $A_{\overline{s}}$ has upper density $> 1 - \frac{1}{|\pi_0(G)|}$. Then there exists $1 \leq i \leq r$ such that $B_{i,\overline{\eta}}$ is $\overline{k(\eta)}$-isogenous to $A_{\overline{\eta}}$.*

Since elliptic curves are automatically simple, Corollary 4 shows **G-II** always holds for elliptic curves.

1.5.2. If $\dim(W) < \dim(V)$, **R-I** (hence *a fortiori* **R-II**) fails in general (Subsection 3.1) as well as **R-I'** (hence *a fortiori* **R-II"**), even for motivic representation (Subsection 3.2). However, when V arises as the ℓth component $\mathcal{V}_{\ell,\overline{\eta}}$ of a $\mathbb{Q}$-rational compatible family $\mathcal{V}_\ell$, $\ell \neq p$ of (pointwise pure) $\mathbb{Q}_\ell$-local systems on S - typically $V_\ell = R^u f_* \mathbb{Q}_\ell(v)$, $\ell \neq p$ for a smooth proper morphism $f : X \to S$ and integers $u \geq 0$, v (but see Subsection 6.6), we show that the failure of **R-I'** is measured by a 'hidden motive', which we call the *ghost* of V. This is the content of Proposition 5 and Theorem 6.

Recall k is a finite field. Let $F \in \pi_1(k)$ denote the geometric Frobenius. For $s \in |S|$, let $k(s)$ denote the residue field at s, $n_s := [k(s) : k]$, and $F_s \in \pi_1(s)$ the geometric Frobenius (that is $F_s = F^{n_s}$); we also denote by F_s its image in $\pi_1(S)$ *via* $\pi_1(s) \hookrightarrow \pi_1(S)$ (See footnote 1).

Fix a $\mathbb{Q}$-rational compatible family V_ℓ, $\ell \neq p$ in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$, pointwise pure of weight $w \in \mathbb{Z}$; let $\rho_\ell : \pi_1(S) \to \mathrm{GL}(V_\ell)$ denote the continuous representation corresponding to V_ℓ and $\overline{G}_\ell \subset \mathrm{GL}_{V_\ell}$ the Zariski-closure of $\rho_\ell(\pi_1(\overline{S})) \subset \mathrm{GL}(V_\ell)$, $\ell \neq p$.

Proposition 5. *Fix a maximal torus $\overline{T}_\ell \subset \overline{G}_\ell$. After possibly replacing k with a finite field extension which is independent of ℓ, $V_\ell^{\overline{T}_\ell}$ is canonically equipped with a structure of $\pi_1(k)$-module - denoted by $\Psi(V_\ell)$, which is semisimple, independent of $\overline{T}_\ell$ up to isomorphism and such that for every maximal torus $T_\ell \subset G_\ell$ containing $\overline{T}_\ell$, $s \in |S|$ and $g \in G(\overline{\mathbb{Q}}_\ell)$ such that $g\rho_\ell(F_s)^{ss}g^{-1} \in T_\ell(\overline{\mathbb{Q}}_\ell)$, the action of F^{n_s} on $V_\ell^{\overline{T}_\ell} \otimes \overline{\mathbb{Q}}_\ell$ via $\Psi(V_\ell)$ coincides with the action of $g\rho_\ell(F_s)^{ss}g^{-1} \in T_\ell(\overline{\mathbb{Q}}_\ell)$ via the natural action of $T_\ell/\overline{T}_\ell$ on $V_\ell^{\overline{T}_\ell}$. In particular, for every $s \in |S|$, $a_s^*\Psi(V_\ell)$ is a submodule of $(s^*V_\ell)^{ss}$.*

The proof of Proposition 5 is carried out in Subsection 6.3. The following encapsulates the main properties of the family $\Psi(V_\ell)$, $\ell \neq p$.

Theorem 6. *With the notation of Proposition 5, and after possibly replacing k by a finite field extension such that the $\Psi(V_\ell)$, $\ell \neq p$ are well-defined, the family of semisimple $\pi_1(k)$-modules $\Psi(V_\ell)$, $\ell \neq p$ satisfies the following properties.*

(1) *Let W be a semisimple $\pi_1(k)$-module in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))$. Assume the set of all $s \in |S|$ such that a_s^*W is almost a submodule of $(s^*V_\ell)^{ss}$ has upper density 1. Then W is almost a submodule of $\Psi(V_\ell)$;*
(2) *The family $\Psi(V_\ell) \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))$, $\ell \neq p$ is $\mathbb{Q}$-rational, compatible, and pointwise pure of weight w.*

We will call the almost $\pi_1(k)$-module $\Psi(V_\ell)$ the *ghost attached to V_ℓ* (Definition 11). Proposition 5 and Theorem 6 (1) say that $\Psi(V_\ell)$ is the largest common almost factor of the $(s^*V_\ell)^{ss}$, $s \in |S|$. Theorem 6 (2) says that it is of motivic nature.

The proof of Theorem 6, which is the most technical part of the paper, is carried out in Subsection 6.5.

In terms of ghosts, **R-I'** for V_ℓ can be reformulated as follows.

Local-global principle R-I". *The equality $\Psi(V_\ell)(= V_\ell^{\overline{T}_\ell}) = V_\ell^{\overline{G}_\ell^\circ}$ holds. Equivalently, $\overline{G}_\ell^\circ$ act on V_ℓ with no non-trivial zero weights.*

We feel the introduction of ghosts and their use in the reformulation **R-I"** of **R-I'** is the most striking contribution of this paper.

When $V_\ell = V_\ell(A_{\overline{\eta}})$ for an abelian scheme $A \to S$, the ghost corresponds to the largest possible common $\overline{k}$-isogeny factor of the fibers $A_{\overline{s}}$, $s \in S$ (see Section 7). When A_η satisfies Zarhin's microweights conjecture ([Z85, Conj. 0.4] - see Conjecture 17), $\Psi(V_\ell)$ corresponds to the largest (weakly) $\overline{k}$-isotrivial isogeny factor of A_η, so that when Zarhin's microweights conjecture holds (see Corollary 19) **G-I** holds.

Let G_ℓ denote the Zariski-closure of the image of $\pi_1(S)$ acting on V_ℓ. We can also show that, if G_ℓ acts on the Tate module of its simple isogeny factors with a 1-dimensional center (this occurs for types I, II, III in Albert's classification) then the only possible common $\overline{k}$-isogeny factors of the $A_{\overline{s}}$ are supersingular (Proposition 20).

In the other direction, Bültel showed that, up to isogeny, *every* faithful representation of any given semisimple algebraic group appears as a direct factor of the geometric monodromy of the generic fiber of an abelian scheme [B05, Thm.1.2], [B13, Lem. B.3], [B22] (hence, in particular, Zarhin's microweights conjecture fails). Combined with our theory of ghosts, this provides a sample of counter-examples to **G-I**. Earlier, more sporadic, counter-examples, due to Katz [K90, Thm. 9.1.1], [K04, Thm. (3.2), (4.12)], arise from the naive Fourier transform of Artin motives. See Subsection 7.3 and Appendix A for more details and references.

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2. Base fields containing an infinite finitely generated field

2.1. **Proof of Proposition 1.** Write $M := \oplus_{1\leq i\leq r} W_i^\vee \otimes V$ and for $i = 1, \dots, r$, let $\Sigma_i \subset W_i^\vee \otimes V$ denote the open subset corresponding to injective morphisms $W_i \hookrightarrow V$. We are to show that there exists an open subgroup $\Pi \subset \pi_1(S)$ such that $(\sqcup_{1\leq i\leq r}\Sigma_i)^\Pi \neq \emptyset$. By assumption, it is enough to show that there exists $s \in |S|$ such that $\pi_1(s)_M$ is open in $\pi_1(S)_M$ (Recall the notation of Subsection 1.3). The existence of such an s (such that, even, $\pi_1(s)_M = \pi_1(S)_M$) is explained in [S81, §1]; it follows from the Hilbertian property of k (in [S81, §1], k is a number field but the argument only requires k to be Hilbertian) and the fact that $\pi_1(S)_M$ is a compact ℓ-adic Lie group, which ensures that its Frattini subgroup is open (see [S89, §10.6]).

Remark. The proof of Proposition 1 already suggests that the assumption '*for every* $s \in |S|$, *there exists* (...)' can be weakened. For instance, when $p = 0$, k is finitely generated, S is a curve and $\mathrm{Lie}(\pi_1(\overline{S})_M)$ is perfect, it is enough to assume [CT13, Thm. 1.1] that there exist an integer $d \geq 1$, and infinitely many $s \in S$ with $[k(s) : k] \leq d$ such that there exists $1 \leq i_s \leq r$ such that $W_{i_s}|_{\pi_1(s)}$ is almost a submodule of $V|_{\pi_1(s)}$. The assumption that $\mathrm{Lie}(\pi_1(\overline{S})_M)$ is perfect is satisfied when the W_i, $i = 1, ..., r$ and V are motivic [D71, Cor. 4.2.9 (a)], [CT12b, Thm. 5.7]. See also [MP12] (especially Prop. 1.15 and Rem. 1.17) for p-adic variants.

2.2. **Proof of Corollary 2.** If k is itself infinite finitely generated, Corollary 2 directly follows from Proposition 1 and Fact 1.3.2. Otherwise, as S is of finite type over k and $A, B_1, \dots, B_r$ are of finite type over S, there exists an infinite finitely generated field $k^\# \subset k$ such that $A, B_1, \dots, B_r \to S$ are defined over $k^\#$. So, in view of Proposition 1, to prove Corollary 2 it is enough to prove the following. Let k be an infinite finitely generated field and let Ω be an algebraically closed field containing k. Assume that for every $s \in |S|$, there exists $1 \leq i_s \leq r$ such that B_{i_s} is an isogeny factor of A_s over Ω. Then B_{i_s} is an isogeny factor of A_s over $\overline{k(s)}$. This in turn follows from the case of infinite finitely generated base fields observing that for every pair of abelian varieties $\mathfrak{A}$, $\mathfrak{B}$ over a field F

- The scheme of homomorphisms $Sch/_F \to Ab$, $T \to \mathrm{Hom}_{AbSch/_T}(\mathfrak{B}_T, \mathfrak{A}_T)$ is representable by a commutative étale group scheme over F. In particular, for every field extension K of F, every K-morphism from $\mathfrak{B}$ to $\mathfrak{A}$ is automatically defined over a finite (separable) field extension of F.
- By faithfully flat descent, for every F-morphism $\phi : \mathfrak{B} \to \mathfrak{A}$ and every field extension K of F, the following are equivalent:
  (1) $\phi : \mathfrak{B} \to \mathfrak{A}$ induces an isogeny onto its image;

(2) $\phi_K : \mathfrak{B}_K \to \mathfrak{A}_K$ induces an isogeny onto its image.

3. Finite fields - Counterexamples to R-I, R-II, R-I', R-II'

3.1. If k is finite, **R-I** (hence *a fortiori* **R-II**) cannot hold without additional assumptions.

For instance, one has to work with semisimplifications[6], as shown by the following counterexample to **R-I**. Let $S = \mathbb{G}_{m,k}$, $\chi : \pi_1(S) \twoheadrightarrow \pi_1(k) \to \mathbb{Z}_\ell^\times$ the ℓ-adic cyclotomic character and $\psi : \pi_1(S) \to \mathbb{Z}_\ell(1)$ a 1-cocyle lifting a class in

$$\mathrm{H}^1(\pi_1(S), \mathbb{Z}_\ell(1)) \simeq \lim_n \mathbb{G}_m(S)/\mathbb{G}_m(S)^{\ell^n}$$

whose restriction in $\mathrm{H}^1(\pi_1(\overline{S}), \mathbb{Z}_\ell(1))$ is non-zero (for instance, take ψ lifting the generic Kummer class that is the image of $T \in \mathbb{G}_m(S) \simeq k[T, T^{-1}]^\times$ in $\mathrm{H}^1(\pi_1(S), \mathbb{Z}_\ell(1))$). Let V denote the $\pi_1(S)$-module defined by

$$\begin{pmatrix} \chi & \psi \\ 0 & 1 \end{pmatrix}$$

and W the $\pi_1(S)$-module defined by

$$\begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}.$$

By our choice of ψ, $V|_{\pi_1(\overline{S})}$ is not almost trivial in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(\overline{S}))$ hence W and V are not almost isomorphic in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$, while, as k is finite, s^*W and s^*V are isomorphic in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(s))$ for every $s \in |S|$ (indeed, $\pi_1(s)$ is a procyclic group generated by the Frobenius $\varphi_s : x \mapsto x^{|k(s)|}$ and $\chi(\varphi_s) \neq 1$).

Hence the right formulations of **R-I**, **R-II** should rather be **R-I'**, **R-II'**.

3.2. But **R-I'** also fails, even for motivic representations (that is subquotients of $\mathrm{H}^u(X_{\overline{\eta}}, \mathbb{Q}_\ell(v))$ for $X \to S$ a smooth proper morphism and $u \geq 0, v$ integers). For instance, consider an abelian scheme $A \to S$ of relative dimension $g \geq 1$ with large geometric monodromy that is such that the Zariski-closure of the image of $\pi_1(\overline{S})$ acting on $V_\ell(A_{\overline{\eta}})$ is $\mathrm{Sp}_{2g,\mathbb{Q}_\ell}$. Set

$$E := \mathrm{End}(V_\ell(A_{\overline{\eta}})) \simeq V_\ell(A_{\overline{\eta}})^\vee \otimes V_\ell(A_{\overline{\eta}}) \subset \mathrm{H}^{2g}(A_{\overline{\eta}} \times A_{\overline{\eta}}, \mathbb{Q}_\ell(g))$$

and $V := E/\mathbb{Q}_\ell Id$; then, for each connected étale cover $S' \to S$, $E^{\pi_1(\overline{S'})} = \mathbb{Q}_\ell Id$ and $V^{\pi_1(\overline{S'})} = 0$ (semisimplicity), while $V^{\pi_1(s)} \neq 0$ for any $s \in |S|$ (as either the Frobenius endomorphism of A_s gives a non-trivial element of $V^{\pi_1(s)}$, or $E^{\pi_1(s)} = E$ and $V^{\pi_1(s)} = V$).

* * *

Unless otherwise stated, the following notation will be used in the remaining part of the paper.

Let k be a finite field and let $F \in \pi_1(k)$ denote the geometric Frobenius. For every integer $m \geq 1$, let $k \subset k_m$ be the unique field extension of degree $m = [k_m : k]$ (in a given algebraic closure $\overline{k}$ of k).

Let S be a smooth, separated and geometrically connected scheme of finite type over k. For $s \in |S|$, let $k(s)$ denote the residue field at s, $n_s := [k(s) : k]$, and $F_s \in \pi_1(s)$ the geometric Frobenius (that is $F_s = F^{n_s}$); we also denote by F_s its image in $\pi_1(S)$ *via* $\pi_1(s) \hookrightarrow \pi_1(S)$ (See footnote 1).

4. Density and upper density

In this Section we discuss some properties of upper Dirichlet density that will be used in the proof of Proposition 3 and Theorem 6, and might be of independent interest.

[6] By Fact 1.3.2, if A is an abelian variety over a finitely generated field k the action of $\pi_1(k)$ on $V_\ell(A_{\overline{k}})$ is semisimple. More generally, the Grothendieck-Serre conjecture predicts that the action of $\pi_1(k)$ on $\mathrm{H}^u(X_{\overline{k}}, \mathbb{Q}_\ell(v))$ should be semisimple for every smooth, proper scheme X over k but this conjecture is still widely open.

We refer to [P97, Appendix B] for the notion and basic properties of Dirichlet densities. For a subset $\Sigma \subset |S|$, the series

$$F_\Sigma(t) = \sum_{s\in\Sigma} |k(s)|^{-t}$$

converges absolutely and locally uniformly for $\mathrm{Re}(t) > \dim(S)$. Write

$$x_\Sigma(t) := \frac{F_\Sigma(t)}{F_{|S|}(t)},\ t \in \mathbb{R}_{>\dim(S)}$$

and $s_\Sigma(t) := \sup\{x_\Sigma(t') \mid \dim(S) < t' < t\}$, $t \in \mathbb{R}_{>\dim(S)}$. Let

$$\delta^u_S(\Sigma) := \lim_{t\to\dim(S),\, t\in\mathbb{R}_{>\dim(S)}} s_\Sigma(t)$$

denote the (Dirichlet) upper density of Σ. By definition $0 \le \delta^u_S(\Sigma) \le 1$ and, if the limit $\delta_S(\Sigma)$ of $x_\Sigma(t)$, for $t \to \dim(S)$, $t \in \mathbb{R}_{>\dim(S)}$ exists, then $\delta_S(\Sigma) = \delta^u_S(\Sigma)$ and one says that Σ has (Dirichlet) density $\delta_S(\Sigma)$.

Lemma 7.
(1) *If* $\Sigma = \bigcup_{1\le i\le r} \Sigma_i$ *then* $\delta^u_S(\Sigma) > 0 \Leftrightarrow$ *there exists* $1 \le i \le r$ *such that* $\delta^u_S(\Sigma_i) > 0$.
(2) *If* Σ_2 *has density then* $\delta^u_S(\Sigma_1\cap\Sigma_2) \ge \delta^u_S(\Sigma_1)+\delta_S(\Sigma_2)-1$. *If, moreover,* $\delta_S(\Sigma_2) = 1$, *then* $\delta^u_S(\Sigma_1\cap\Sigma_2) = \delta^u_S(\Sigma_1)$.
(3) *(If* $\delta^u(\Sigma) > \frac{1}{2}$, *then the gcd of* n_s, $s \in \Sigma$ *is* 1.
(4) *If* $f : S' \to S$ *is a connected finite étale cover and* $\delta^u_S(\Sigma) = 1$ *(resp.* $\delta^u_S(\Sigma) > 1 - \frac{1}{\deg(f)}$*), then* $\delta^u_{S'}(f^{-1}(\Sigma)) = 1$ *(resp.* $\delta^u_{S'}(f^{-1}(\Sigma)) > 0$*).*
(5) *If* $f : S' \to S$ *is a connected finite étale cover, then* $\delta^u_S(\Sigma) \ge \frac{1}{\deg(f)}\delta^u_{S'}(f^{-1}(\Sigma))$.

Proof. (1): The $\Rightarrow$ implication follows from the inequality

$$\delta^u_S(\cup_{1\le i\le r}\Sigma_i) \le \sum_{1\le i\le r} \delta^u_S(\Sigma_i)$$

while the $\Leftarrow$ implication follows from the fact that $\Sigma' \subset \Sigma$ implies $\delta^u_S(\Sigma') \le \delta^u_S(\Sigma)$.

(2): By definition of upper density, there exists a strictly decreasing sequence $\phi : \mathbb{Z}_{\ge1} \to \mathbb{R}_{>\dim(S)}$ such that $\phi(n) \to \dim(S)$ and $x_{\Sigma_1}(\phi(n)) \to \delta^u_S(\Sigma_1)$ $(n \to \infty)$. Then

$$x_{\Sigma_1\cap\Sigma_2}(\phi(n)) = x_{\Sigma_1}(\phi(n))+x_{\Sigma_2}(\phi(n))-x_{\Sigma_1\cup\Sigma_2}(\phi(n)) \ge x_{\Sigma_1}(\phi(n))+x_{\Sigma_2}(\phi(n))-1 \to \delta^u_S(\Sigma_1)+\delta_S(\Sigma_2)-1$$

Hence $\delta^u_S(\Sigma_1\cap\Sigma_2) \ge \delta^u_S(\Sigma_1) + \delta_S(\Sigma_2) - 1$. This concludes the proof of the first assertion, and the second assertion follows immediately from the first.

(3): As $\delta^u(\Sigma) > \frac{1}{2} > 0$, Σ is non-empty. Take $s_0 \in \Sigma$ and let $p_1,\dots,p_r$ be the prime factors of n_{s_0}. For each $i = 1,\dots,r$, consider the connected finite étale Galois cover $S_{k_{p_i}} \to S$ corresponding to $\pi_1(S) \twoheadrightarrow \pi_1(k) = \hat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/p_i\mathbb{Z}$. As $\delta^u(\Sigma) > \frac{1}{2} \ge \frac{1}{p_i}$, there exists $s_i \in \Sigma$ such that the image of $F_{s_i} \in \pi_1(S)$ in $\mathbb{Z}/p_i\mathbb{Z}$ (which is $n_{s_i} \in \mathbb{Z}/p_i\mathbb{Z}$) is distinct from $0 \in \mathbb{Z}/p_i\mathbb{Z}$, by the Cebotarev density theorem [P97, Thm. B.9]. Then the gcd of $n_{s_0}, n_{s_1},\dots,n_{s_r}$ is 1, hence so is that of n_s, $s \in \Sigma$.

(4): Let $|S'|^{split} \subset |S'|$ denote the set of $s' \in |S'|$ such that $[k(s') : k(f(s'))] = 1$ and set $|S|^{split} := f(|S'|^{split})$ and $|S|^{nonsplit} := |S| \setminus |S|^{split}$. Set $\Sigma' := f^{-1}(\Sigma)$, $(\Sigma')^{split} = \Sigma' \cap |S'|^{split}$, $\Sigma^{split} = \Sigma \cap |S|^{split}$ and $\Sigma^{nonsplit} = \Sigma \cap |S|^{nonsplit}$. Let $\mathcal{G}$ (resp. $\mathcal{H}$) be the Galois group of the Galois closure $\tilde{S} \to S$ of $f : S' \to S$ (resp. the cover $\tilde{S} \to S'$ induced naturally). Let Γ denote the set of conjugacy class of $\mathcal{G}$. For each $J \in \Gamma$, write $|S|_J$ for the set of closed points $s \in |S|$ such that the image of $F_s \in \pi_1(S)$ in $\mathcal{G}$ lies in J and set $\Sigma_J := \Sigma \cap |S|_J$. For each $J \in \Gamma$, take $j \in J$ and set

$$n_J := |(\mathcal{G}/\mathcal{H})^{\langle j\rangle}| = \frac{|\{g \in \mathcal{G} \mid g^{-1}jg \in \mathcal{H}\}|}{|\mathcal{H}|} = \frac{|J\cap\mathcal{H}||\mathcal{G}|}{|J||\mathcal{H}|}.$$

Then one has $F_{(\Sigma')^{split}}(t) = \sum_{J\in\Gamma} n_J F_{\Sigma_J}(t)$, while $F_{\Sigma^{split}}(t) = \sum_{J\in\Gamma, n_J>0} F_{\Sigma_J}(t)$.

We first prove:

(6) $\lim_{t\to \dim(S),\, t\in \mathbb{R}_{>\dim(S)}} \frac{F_{|S'|}(t)}{F_{|S|}(t)} = 1.$
(7) $|S|^{split} \subset |S|$ has density $\delta_S(|S|^{split}) \geq \frac{1}{\deg(f)}$.

(6): Since $F_{|S'|^{split}}(t) = \sum_{J\in\Gamma} n_J F_{|S|_J}(t)$, one has

$$\frac{F_{|S'|}(t)}{F_{|S|}(t)} = \frac{F_{|S'|}(t)}{F_{|S'|^{split}}(t)} \cdot \frac{F_{|S'|^{split}}(t)}{F_{|S|}(t)} = \frac{F_{|S'|}(t)}{F_{|S'|^{split}}(t)} \cdot \sum_{J\in\Gamma} n_J \frac{F_{|S|_J}(t)}{F_{|S|}(t)},$$

hence, from [P97, Prop. B.8 and Thm. B.9],

$$\lim_{t\to \dim(S),\, t\in \mathbb{R}_{>\dim(S)}} \frac{F_{|S'|}(t)}{F_{|S|}(t)} = \sum_{J\in\Gamma} n_J \frac{|J|}{|\mathcal{G}|} = \sum_{J\in\Gamma} \frac{|J\cap \mathcal{H}|}{|\mathcal{H}|} = 1.$$

(7): One has

$$\frac{F_{|S|^{split}}(t)}{F_{|S|}(t)} = \sum_{J\in\Gamma, n_J>0} \frac{F_{|S|_J}(t)}{F_{|S|}(t)},$$

hence, from [P97, Thm. B.9],

$$\delta_S(|S|^{split}) = \sum_{J\in\Gamma, n_J>0} \frac{|J|}{|\mathcal{G}|} = \frac{|\cup_{g\in\mathcal{G}}\, g\mathcal{H}g^{-1}|}{|\mathcal{G}|} \geq \frac{|\mathcal{H}|}{|\mathcal{G}|} = \frac{1}{\deg(f)}.$$

To prove (4), first, suppose $\delta^u_S(\Sigma) = 1$. Then there exists a strictly decreasing sequence $\phi : \mathbb{Z}_{\geq 1} \to \mathbb{R}_{>\dim(S)}$ such that $\phi(n) \to \dim(S)$ and $x_\Sigma(\phi(n)) \to \delta^u_S(\Sigma) = 1$ $(n\to\infty)$. Then

$$0 \leq \sum_{J\in\Gamma} x_{|S|_J\setminus\Sigma_J}(\phi(n)) = x_{|S|\setminus\Sigma}(\phi(n)) = x_{|S|}(\phi(n)) - x_\Sigma(\phi(n)) \to 1-1 = 0.$$

Thus, for each $J\in\Gamma$, one must have

$$\lim_{n\to\infty} (x_{|S|_J}(\phi(n)) - x_{\Sigma_J}(\phi(n))) = \lim_{n\to\infty} x_{|S|_J\setminus\Sigma_J}(\phi(n)) = 0,$$
$$\delta_S(|S|_J) = \lim_{n\to\infty} x_{|S|_J}(\phi(n)) = \lim_{n\to\infty} x_{\Sigma_J}(\phi(n)) \leq \delta^u_S(\Sigma_J) \leq \delta_S(|S|_J),$$

hence

$$\lim_{n\to\infty} x_{\Sigma_J}(\phi(n)) = \delta^u_S(\Sigma_J) = \delta_S(|S|_J).$$

Observe

$$x_{|S'|^{split}}(t) = \frac{\sum_{J\in\Gamma} n_J F_{|S|_J}(t)}{F_{|S'|}(t)} = \frac{F_{|S|}(t)}{F_{|S'|}(t)} \sum_{J\in\Gamma} n_J x_{|S|_J}(t),$$
$$x_{(\Sigma')^{split}}(t) = \frac{\sum_{J\in\Gamma} n_J F_{\Sigma_J}(t)}{F_{|S'|}(t)} = \frac{F_{|S|}(t)}{F_{|S'|}(t)} \sum_{J\in\Gamma} n_J x_{\Sigma_J}(t).$$

From these, (6) and [P97, Prop. B.8 and Thm. B.9], one has

$$1 = \delta_{S'}(|S'|^{split}) = \sum_{J\in\Gamma} n_J \delta_S(|S|_J) = \lim_{n\to\infty} \sum_{J\in\Gamma} n_J x_{\Sigma_J}(\phi(n)) = \lim_{n\to\infty} x_{(\Sigma')^{split}}(\phi(n)) \leq \delta^u_{S'}((\Sigma')^{split}) \leq \delta^u_{S'}(\Sigma') \leq 1,$$

hence $\delta^u_{S'}(\Sigma') = 1$.

Next, suppose $\delta^u_S(\Sigma) > 1 - \frac{1}{\deg(f)}$. Then, as $\Sigma = \Sigma^{nonsplit} \sqcup \Sigma^{split} \subset |S|^{nonsplit} \sqcup \Sigma^{split}$, one has

$$1 - \frac{1}{\deg(f)} < \delta^u_S(\Sigma) \leq \delta_S(|S|^{nonsplit}) + \delta^u_S(\Sigma^{split}) \leq 1 - \frac{1}{\deg(f)} + \delta^u_S(\Sigma^{split}),$$

where the last inequality follows from (7), hence $\delta^u_S(\Sigma^{split}) > 0$. Thus, one has

$$\delta^u_{S'}(\Sigma') \geq \delta^u_{S'}((\Sigma')^{split}) \geq \delta^u_S(\Sigma^{split}) > 0,$$

where the second inequality follows from (6) and

$$x_{(\Sigma')^{split}}(t) = \frac{F_{|S|}(t)}{F_{|S'|}(t)} \sum_{J\in\Gamma} n_J x_{\Sigma_J}(t) \geq \frac{F_{|S|}(t)}{F_{|S'|}(t)} \sum_{J\in\Gamma, n_J>0} x_{\Sigma_J}(t) = \frac{F_{|S|}(t)}{F_{|S'|}(t)} x_{\Sigma^{split}}(t).$$

(5): We continue to use the notation in the proof of (4). Then

$$x_{(\Sigma')^{split}}(t) = \frac{F_{|S|}(t)}{F_{|S'|}(t)} \sum_{J\in\Gamma} n_J x_{\Sigma_J}(t) \leq \frac{F_{|S|}(t)}{F_{|S'|}(t)} \deg(f) \sum_{J\in\Gamma} x_{\Sigma_J}(t) = \frac{F_{|S|}(t)}{F_{|S'|}(t)} \deg(f) x_\Sigma(t),$$

where the inequality follows from the fact that $n_J = |(\mathcal{G}/\mathcal{H})^{\langle j\rangle}| \leq |\mathcal{G}/\mathcal{H}| = \deg(f)$ for each $J \in \Gamma$. Now, taking the limit superior and using (6) and [P97, Prop. B.8], we get

$$\delta^u_{S'}(\Sigma') = \delta^u_{S'}((\Sigma')^{split}) \leq \deg(f)\delta^u_S(\Sigma),$$

as desired. □

Remark. The above proof shows that the bound $1 - \frac{1}{\deg(f)}$ in Lemma 7 (4) is best possible if and only if $f : S' \to S$ is Galois.

Let $\rho : \pi_1(S) \to \mathrm{GL}_r(\mathbb{Q}_\ell)$ be a continuous homomorphism. For each closed, conjugacy-invariant subset $\Phi \subset \rho(\pi_1(S))$, let Σ_Φ denote the set of all $s \in |S|$ such that $\rho(F_s) \in \Phi$. Conversely, for each $\Sigma \subset |S|$, let $\Phi_\Sigma \subset \rho(\pi_1(S))$ denote the union of the conjugacy classes of the $\rho(F_s)$, $s \in \Sigma$. (Be aware that, in general, Φ_Σ is not closed in $\rho(\pi_1(S))$). Let $G \subset \mathrm{GL}_{r,\mathbb{Q}_\ell}$ denote the Zariski-closure of $\rho(\pi_1(S))$.

Lemma 8. *Let $\Phi \subset \rho(\pi_1(S))$ be a closed, conjugacy-invariant subset and $\Sigma \subset |S|$ a subset. Then,*
(1) *One has: $\delta_S(\Sigma_\Phi) = 0$ ($\Leftrightarrow \delta^u_S(\Sigma_\Phi) = 0$) $\Leftrightarrow \Phi \subset \rho(\pi_1(S))$ has empty interior. In particular, if $\delta^u_S(\Sigma_\Phi) > 0$, Φ generates topologically an open subgroup of $\rho(\pi_1(S))$.*
(2) *If $\delta^u_S(\Sigma) > 1 - \frac{1}{|\pi_0(G)|}$ then Φ_Σ is Zariski-dense in G.*
(3) *If $\delta^u_S(\Sigma) = 1$ then Φ_Σ is dense in $\rho(\pi_1(S))$ (and, in particular, Zariski-dense in G).*

Proof. For (1), see [S12, Thm. 6.11] (the proof of which works as it is in our setting replacing naive density with Dirichlet density). For (2), let $X \subset G$ denote the Zariski-closure of Φ_Σ, and $\Phi := \rho(\pi_1(S)) \cap X(\mathbb{Q}_\ell) \subset \rho(\pi_1(S))$. For each subset $J \subset \pi_0(G)$, write $G_J \subset G$ (resp. $X_J \subset X$, resp. $\rho(\pi_1(S))_J \subset \rho(\pi_1(S))$, resp. $\Phi_J \subset \Phi$) for the inverse image of J in G (resp. X, resp. $\rho(\pi_1(S))$, resp. Φ). For $j \in \pi_0(G)$, write $(-)_j$ instead of $(-)_{\{j\}}$. When $J \subset \pi_0(G)$ is conjugacy-invariant, write $|S|_J := |S|_{\rho(\pi_1(S))_J}$ that is $|S|_J$ is the set of closed points $s \in |S|$ such that the image of $\rho(F_s)$ in $\pi_0(G)$ lies in J. Now, suppose Φ_Σ is not Zariski-dense in G or, equivalently, $X \subsetneq G$. Then there exists at least one $j_0 \in \pi_0(G)$ such that $X_{j_0} \subsetneq G_{j_0}$. Let $J_0 \subset \pi_0(G)$ denote the conjugacy class of j_0. As X is conjugacy-invariant, for each $j \in J_0$, one has $X_j \subsetneq G_j$, hence $\Phi_j \subset \rho(\pi_1(S))$ is a closed analytic subset with no interior point ([S12, Cor. 5.10 and Prop. 5.12]). Thus, $\Phi_{J_0} \subset \rho(\pi_1(S))$ is a conjugacy-invariant closed analytic subset with no interior point. So, by (1), $\delta^u_S(\Sigma_{\Phi_{J_0}}) = 0$. Let Γ denote the set of conjugacy classes of $\pi_0(G)$. Then

$$\Sigma \subset \Sigma_\Phi = \sqcup_{J\in\Gamma} \Sigma_{\Phi_J} \subset \Sigma_{\Phi_{J_0}} \sqcup \sqcup_{J\in\Gamma\setminus\{J_0\}} |S|_J,$$

hence $\delta^u_S(\Sigma) \leq 1 - \frac{|J_0|}{|\pi_0(G)|} \leq 1 - \frac{1}{|\pi_0(G)|}$ by the Cebotarev density theorem. This contradicts the assumption $\delta^u_S(\Sigma) > 1 - \frac{1}{|\pi_0(G)|}$. (3) follows, by a projective limit argument, from the Cebotarev density theorem [P97, Thm. B.9]. □

5. Proof of Proposition 3

5.1. We first make the following observation, which is used in the proof of Proposition 3 but works over arbitrary fields and might be of independent interest. For a prime ℓ, write $\mathfrak{l} = 2^2$ if $\ell = 2$ and $\mathfrak{l} = \ell$ otherwise.

Lemma 9. *Let Π be a profinite group, let $W_1, W_2 \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\Pi)$ and $f : W_1 \to W_2$ a morphism in $\mathrm{aMod}_{\mathbb{Q}_\ell}(\Pi)$. Assume there exists a Π-stable $\mathbb{Z}_\ell$-lattice $H_i \subset W_i$ such that Π acts trivially on $H_i/\mathfrak{l}H_i$, $i = 1, 2$. Then $f : W_1 \to W_2$ is a morphism in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\Pi)$.*

Proof. Let $\varphi \in \Pi$. By assumption there exists $m \geq 1$ such that the following diagram commutes:

$$(*) \qquad \begin{array}{ccc} W_1 & \xrightarrow{f} & W_2 \\ {\scriptstyle \varphi^m_{W_1}}\downarrow & & \downarrow{\scriptstyle \varphi^m_{W_2}} \\ W_1 & \xrightarrow{f} & W_2 \end{array}$$

Still by assumption, for $i=1,2$, $\varphi_{W_i}\in Id+\mathfrak{l}\mathrm{End}_{\mathbb{Z}_\ell}(H_i)$ so that $\varphi_{W_i}^m\in Id+\mathfrak{l}\ell^{v_\ell(m)}\mathrm{End}_{\mathbb{Z}_\ell}(H_i)$. Hence φ_{W_i} coincides with the convergent analytic series $\exp(\frac{1}{m}\log(\varphi_{W_i}^m))$. The conclusion then follows from the commutativity of $(*)$ and the continuity of f. □

In particular,

Corollary 10. *Let k be a field of characteristic $p\geq 0$ and A_i a g_i-dimensional abelian variety over k, $i=1,2$. If there exists a prime $\ell\neq p$ such that $A_i[\mathfrak{l}](\overline{k})=A_i[\mathfrak{l}](k)$, $i=1,2$ then every $\overline{k}$-morphism of abelian varieties $A_{1,\overline{k}}\to A_{2,\overline{k}}$ is defined over k. In general, there exists an integer m_g depending only on $g:=\max(g_1,g_2)$ such that every $\overline{k}$-morphism of abelian varieties $A_{1,\overline{k}}\to A_{2,\overline{k}}$ is defined over an extension of k with degree dividing m_g.*

Proof. By standard descent arguments, one may reduce the problem to the case where k is a finitely generated field. Then the first part of the assertion follows from Lemma 9 applied to $W_i:=V_\ell(A_i)$ and $H_i:=T_\ell(A_i)$, $i=1,2$, and Fact 1.3.2. The second part of the assertion follows from the first part taking m_g to be the lcm of $|\mathrm{GL}_{2g}(\mathbb{F}_3)|$, $|\mathrm{GL}_{2g}(\mathbb{F}_5)|$. □

5.2. **Proof of Proposition 3 (1).** Without loss of generality we may assume $V,W_1,...,W_r$ are semisimple $\pi_1(S)$-modules. Let $\Sigma\subset|S|$ denote the set of all $s\in|S|$ for which there exists $1\leq i_s\leq r$ such that $(s^*W_{i_s})^{ss}$ and $(s^*V)^{ss}$ are almost-isomorphic. Let $\rho:\pi_1(S)\to\mathrm{GL}_{\mathbb{Q}_\ell}(V)\times\mathrm{GL}_{\mathbb{Q}_\ell}(W_1)\times\cdots\times\mathrm{GL}_{\mathbb{Q}_\ell}(W_r)$ denote the representation corresponding to $V\oplus W_1\oplus\cdots\oplus W_r$; recall that $G\subset\mathrm{GL}_V\times\mathrm{GL}_{W_1}\times\cdots\times\mathrm{GL}_{W_r}$ denotes the Zariski-closure of $\rho(\pi_1(S))$ and that we assume $\delta_S^u(\Sigma)>1-\frac{1}{|\pi_0(G)|}$.

For $i=1,\dots,r$ let $Y_i\subset\mathrm{GL}_{\mathbb{Q}_\ell}(V)\times\mathrm{GL}_{\mathbb{Q}_\ell}(W_1)\times\cdots\times\mathrm{GL}_{\mathbb{Q}_\ell}(W_r)$ denote the closed subscheme of all $(g,g_1,\dots,g_r)$ such that $\chi_{W_i}(g_i)=\chi_V(g)$, where χ_V, χ_{W_i} denote the characteristic polynomial maps. As $\pi_1(S)$ is compact, there exist $\pi_1(S)$-stable $\mathbb{Z}_\ell$-lattices $H_i\subset W_i$, $i=1,\dots,r$ and $H\subset V$. Set $d:=\dim_{\mathbb{Q}_\ell}(V)$ and let N denote the lcm of the orders of the elements in $\mathrm{GL}_d(\mathbb{Z}/\mathfrak{l})$. Write $\varphi_N:G\to G,\ g\mapsto g^N$. From Lemma 9, we have

$$(G(\mathbb{Q}_\ell)\supset\rho(\pi_1(S))\supset)\Phi_\Sigma\subset\bigcup_{1\leq i\leq r}\varphi_N^{-1}(Y_i)(\mathbb{Q}_\ell)$$

and from Lemma 8 (2) , Φ_Σ is Zariski-dense in G. In particular, $G\subset\cup_{1\leq i\leq r}\varphi_N^{-1}(Y_i)$. So there exists $1\leq i\leq r$ such that $G^\circ\subset\varphi_N^{-1}(Y_i)$. But since $\varphi_N:G^\circ\to G^\circ$ is étale at 1 (observe that $\mathrm{Lie}(\varphi_N)$ is the multiplication-by-N map on $\mathrm{Lie}(G)$), it is dominant. As $\varphi_N:G^\circ\to G^\circ$ is dominant and $G^\circ\subset\varphi_N^{-1}(Y_i)$, one has $G^\circ\subset Y_i$. In particular, $\rho(U)\subset Y_i(\mathbb{Q}_\ell)$, where $U:=\ker(\pi_1(S)\to\pi_0(G))$. Since U is open in $\pi_1(S)$, $W_i|_U$,$V|_U$ are again semisimple in $\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$ and, by definition of Y_i, the characteristic polynomials of $\rho(u)$ acting on W_i and V coincide for every $u\in U$. This shows that $W_i|_U$, $V|_U$ are isomorphic in $\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$, hence W_i, V are isomorphic in $\mathrm{aMod}_{\mathbb{Q}_\ell}(\pi_1(S))$.

5.3. **Proof of Proposition 3 (2).** As $V|_{\pi_1(\overline{S})}$ is semisimple in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(\overline{S}))$, one has $V|_{\pi_1(\overline{S})}\simeq V^{ss}|_{\pi_1(\overline{S})}$. By Proposition 3 (1), there exists $1\leq i\leq r$ such that W_i^{ss} and V^{ss} are almost isomorphic. Thus, $V|_{\pi_1(\overline{S})}$ is almost isomorphic to $W_i^{ss}|_{\pi_1(\overline{S})}$, hence almost trivial.

5.4. **Remark.** Following the notation of Subsection 5.2, let T be a maximal torus of G°, C the centralizer of T in G (that is a Cartan subgroup of G), J the (injective) image of $\pi_0(C)$ in $\pi_0(G)$, and $\tilde{J}:=\cup_{h\in\pi_0(G)}hJh^{-1}$. Then, in the assumption of Proposition 3, the bound '$>1-\frac{1}{|\pi_0(G)|}$' can be weakened to '$>1-\frac{|\tilde{J}|}{|\pi_0(G)|}$'. (In particular, in the case where the map $\pi_0(C)\to\pi_0(G)$ is surjective, the bound can be weakened to '>0'.)

Indeed, assume $\delta^u(\Sigma)>1-\frac{|\tilde{J}|}{|\pi_0(G)|}$. Then, as in the proof of Lemma 8, we can show that there exists $j\in\tilde{J}$ such that $X_j=G_j$, hence, in the notation of Subsection 5.2, there exists $1\leq i\leq r$ such that $G_j\subset\varphi_N^{-1}(Y_i)$. We may assume $j\in J$ (replacing T by a conjugate) and $j^N=1$ (replacing N by a multiple if necessary). Then $\varphi_N:G\to G$ restricts to $\varphi_N:G_j\to G_1=G^\circ$. We claim that $\varphi_N:G_j\to G^\circ$

is dominant. To show this, consider the following commutative diagram:

$$\begin{array}{ccc} G^\circ \times C_j & \longrightarrow & G^\circ \times T \\ \big\downarrow & & \big\downarrow \\ G_j & \longrightarrow & G^\circ, \end{array}$$

where C_j is the inverse image of $j \in J$ $(\xleftarrow{\sim}\pi_0(C))$ in C, the vertical maps are $(x, y) \mapsto xyx^{-1}$, the upper horizontal map is $Id \times \varphi_N : (x, y) \mapsto (x, y^N)$ and the lower horizontal map is $\varphi_N : y \mapsto y^N$. As G° is reductive, the right vertical arrow is dominant. So, to show that the lower horizontal map is dominant, it suffices to show that the upper horizontal map is dominant or, equivalently, $\varphi_N : C_j \to T,\ y \mapsto y^N$ is dominant. To show this, it suffices to prove that $\varphi_{N,\overline{\mathbb{Q}}_\ell} : C_{j,\overline{\mathbb{Q}}_\ell} \to T_{\overline{\mathbb{Q}}_\ell}$ is dominant. As $j \in J$, we may take $\gamma \in C(\overline{\mathbb{Q}}_\ell)$ whose image in $\pi_0(G)$ is j. Then one has $\varphi_{N,\overline{\mathbb{Q}}_\ell} : C_{j,\overline{\mathbb{Q}}_\ell} = \gamma T_{\overline{\mathbb{Q}}_\ell} \to T_{\overline{\mathbb{Q}}_\ell}$, $\gamma t \mapsto (\gamma t)^N = \gamma^N t^N$ (as $\gamma \in C(\overline{\mathbb{Q}}_\ell)$ centralizes $T_{\overline{\mathbb{Q}}_\ell}$) , which is clearly dominant. As $\varphi_N : G_j \to G^\circ$ is dominant and $G_j \subset \varphi_N^{-1}(Y_i)$, one has $G^\circ \subset Y_i$. As in Subsection 5.2, this shows that $W_i|_U$ and $V|_U$ are isomorphic in $\mathrm{Mod}_{\mathbb{Q}_\ell}(U)$, where $U := \ker(\pi_1(S) \to \pi_0(G))$.

For the time being, we do not know if the bound can be weakened to '> 0' in general.

6. Ghosts

6.1. **Notation and convention.**

6.1.1. Given a field Q, for every monic $P \in Q[T]$ with roots $\alpha_1, \dots, \alpha_r$ in an algebraic closure $\overline{Q}$ of Q and integer $m \geq 1$, we write

$$P^{(m)}(T) := \prod_{1 \leq i \leq r} (T - \alpha_i^m) \in Q[T].$$

When $P(0) \neq 0$, let $A(P) \subset \overline{Q}^\times$ denote the subgroup generated by the $\alpha_1, \dots, \alpha_r \in \overline{Q}^\times$. Let m_P denote the order of the torsion subgroup of $A(P)$. Then for every $n \geq 1$, $A(P^{(m_P n)})$ is torsion-free. For $P_1, P_2 \in Q[T]$ such that $A(P_1 P_2) \subset \overline{Q}^\times$ is torsion-free and for every integer $m \geq 1$ we have

(1) $gcd(P_1^{(m)}, P_2^{(m)}) = gcd(P_1, P_2)^{(m)}$.
(2) $P_1 | P_2 \iff P_1^{(m)} | P_2^{(m)}$.

Given $V \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))$, write $P_{F,V} \in \mathbb{Q}_\ell[T]$ for the characteristic polynomial of the geometric Frobenius F acting on V. Recall that $V \mapsto P_{F,V}$ induces a bijective correspondence between the isomorphism classes of semisimple $V \in \mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(k))$ of $\mathbb{Q}_\ell$-dimension r and the degree-r monic polynomials $P \in \mathbb{Z}_\ell[T]$ such that $P(0) \in \mathbb{Z}_\ell^\times$ (equivalently, monic polynomials $P \in \mathbb{Z}_\ell[T]$ with roots in $\overline{\mathbb{Z}}_\ell^\times$).

6.1.2. From now on, we fix a $\mathbb{Q}$-rational compatible family V_ℓ, $\ell \neq p$ in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$, pointwise pure of weight $w \in \mathbb{Z}$. Note that the purity assumption implies that the Zariski-closure of the image of $\pi_1(\overline{S})$ acting on V_ℓ is a semisimple algebraic group ([D80, 1.3.9 and 3.4.1 (iii)]); in particular $V_\ell|_{\pi_1(\overline{S})}$ is a semisimple $\pi_1(\overline{S})$-module.

Fix $\ell \neq p$ and let $\rho_\ell : \pi_1(S) \to \mathrm{GL}(V_\ell)$ denote the corresponding representation. We retain the notation of Theorem 6 for $\overline{G}_\ell$, $\overline{T}_\ell$; let also $G_\ell \subset \mathrm{GL}_{V_\ell}$ denote the Zariski-closure of the image of $\pi_1(S)$ acting on V_ℓ.

For every $s \in |S|$, $(s^* V_\ell)^{ss}$ is uniquely determined by $P_s := P_{F_s, s^* V_\ell}$. Recall that by definition $P_{F_s, s^* V_\ell}$ is in $\mathbb{Q}[T]$ and independent of $\ell \neq p$, which justifies the notation. In particular, $A(P_s) = A(P_{F_s, s^* V_\ell})$ is independent of $\ell \neq p$; see Subsection 6.2.1 for the geometric interpretation of $A(P_s)$.

The remaining part of Section 6 is devoted to the proofs of Proposition 5 and Theorem 6.

6.2. **Preliminary reductions and observations.**

6.2.1. One may replace S by a connected étale cover freely. (Note that the assumption about the upper density in Theorem 6 (2) is preserved by Lemma 7 (4).) In particular, after replacing S by a connected étale cover which is independent of ℓ, one may assume that

(1) $\overline{G}_\ell$ is connected for every ℓ ([LaP95, Prop. 2.2]);
(2) $\langle A(P_s),\ s \in |S|\rangle \subset \overline{\mathbb{Q}}^\times$ is torsion-free;
(3) G_ℓ is connected for every ℓ.

We explain (2). Let $\ell' := 3$ if $p \neq 3$ and $\ell' = 5$ if $p = 3$. Choose a $\pi_1(S)$-invariant $\mathbb{Z}_{\ell'}$-lattice $H \subset V_{\ell'}$. Then it is enough to replace S with the connected étale cover corresponding to the kernel of $\pi_1(S)$ acting on H/ℓ'. Indeed, then for every $s \in |S|$ the roots of P_s lie in $1 + \ell'\overline{\mathbb{Z}}_{\ell'}$ hence the subgroup $\langle A(P_s),\ s \in |S|\rangle \subset 1 + \ell'\overline{\mathbb{Z}}_{\ell'}$ is torsion-free as well.

We explain (3). Let $\Theta_{\ell,s} \subset GL_{V_\ell}$ denote the algebraic envelope of the semisimple part $\rho_\ell(F_s)^{ss}$ in the multiplicative Jordan decomposition of $\rho_\ell(F_s)$ (that is the smallest algebraic subgroup of GL_{V_ℓ} containing $\rho_\ell(F_s)^{ss}$ or, equivalently, the Zariski-closure of the abstract subgroup generated by $\rho_\ell(F_s)^{ss}$) and let $T_{\ell,s} := \Theta^\circ_{\ell,s} \subset \Theta_{\ell,s}$ denote its neutral component. Since $\rho_\ell(F_s)^{ss}$ is semisimple, $\Theta_{\ell,s}$ is of multiplicative type and fixing a diagonalization $\overline{\mathbb{Q}}_\ell$-basis of $V_\ell \otimes \overline{\mathbb{Q}}_\ell$ for $\rho_\ell(F_s)^{ss}$ yields a canonical identification $A(P_s) \tilde{\to} X^*(\Theta_{\ell,s}) := \mathrm{Hom}(\Theta_{\ell,s,\overline{\mathbb{Q}}_\ell}, \mathbb{G}_{m,\overline{\mathbb{Q}}_\ell})$. As a result, the torsion subgroup of $A(P_s)$ identifies with the dual of the group of connected component $\pi_0(\Theta_{\ell,s})$ of $\Theta_{\ell,s}$. Thus, by the Cebotarev density theorem, if the set of $s \in |S|$ such that $A(P_s)$ is torsion-free has density 1 ('net' in the terminology of [S81, p. 16]) then G_ℓ is connected for every $\ell \neq p$ (since $A(P_s)$ is independent of $\ell \neq p$)[7]. In particular, (2) implies (3).

Following Serre [S81], we call $T_{\ell,s}$ the Frobenius torus attached to $s \in |S|$. Since P_s is in $\mathbb{Q}[T]$, $T_{\ell,s}$ is defined over $\mathbb{Q}$. Assuming 6.2.1 (2), Serre showed ([S81, Cor. p.13]; see also [LaP92, §7]) that the set of all $s \in |S|$ such that $T_{\ell,s}$ is a maximal torus in G_ℓ has density 1. In general (see Lemma 7 (5)), the set of all $s \in |S|$ such that $T_{\ell,s}$ is a maximal torus in G_ℓ has upper density $\geq \frac{1}{|\pi_0(G_\ell)|} > 0$. Since the reductive rank of G_ℓ is independent of ℓ ([S81, Thm. p.6]; see also [LaP92, (6.13)]), the property that $T_{\ell,s}$ is a maximal torus in G_ℓ is independent of ℓ as well.

6.2.2. From now on, we omit the subscript $(-)_\ell$ from our notation for ρ, V, G, $\overline{G}$ *etc.* Replacing V by its $\pi_1(S)$-semisimplification V^{ss} affects neither $P_{\rho(\pi),V}$ $(\pi \in \pi_1(S))$, $(s^*V)^{ss}$ $(s \in |S|)$ nor $V^{\overline{T}}$. Furthermore, since $V|_{\pi_1(\overline{S})}$ is semisimple in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(\overline{S}))$, $V^{ss}|_{\pi_1(\overline{S})}$ is isomorphic to $V|_{\pi_1(\overline{S})}$ in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(\overline{S}))$. So, from now on, we assume V is semisimple in $\mathrm{Mod}_{\mathbb{Q}_\ell}(\pi_1(S))$. In particular, G is connected reductive by assumption. Note that, then, since $\overline{G}$ is connected semisimple, normal in G and $G/\overline{G}$ is abelian, $\overline{G}$ is the derived subgroup of G.

6.3. **Proof of Proposition 5.**

6.3.1. *Construction of the ghost and independence of $\overline{T}$.*

6.3.1.1. Consider the following diagram

$$\begin{array}{ccccccccc} & & & & & \pi_1(S)/\pi_1(\overline{S}) \simeq \pi_1(k) & & \\ & & & \overset{\tilde{\rho}}{\swarrow} & & \downarrow \bar{\rho} & & \\ 1 \longrightarrow \mu(\mathbb{Q}_\ell) & \longrightarrow & Z(G)^\circ(\mathbb{Q}_\ell) & & \longrightarrow & (G/\overline{G})(\mathbb{Q}_\ell) & \longrightarrow & \mathrm{H}^1(\mathbb{Q}_\ell, \mu(\overline{\mathbb{Q}}_\ell)), \end{array}$$

where the lower exact row is the one induced by the short exact sequence of (commutative) algebraic groups

$$1 \to \mu \to Z(G)^\circ \to G/\overline{G} \to 1.$$

By definition, $\mu = Z(G)^\circ \cap \overline{G} \subset Z(\overline{G})$. But since over an algebraically closed field of characteristic 0, there are only finitely many isomorphism classes of connected semisimple algebraic groups of bounded rank and these are independent of the base field (they are in bijection with root data), μ is finite of order

[7] Actually, the converse is also true - see [S81, PS to 1st letter, 2nd letter].

bounded from above by a constant C which only depends on the $\mathbb{Q}_\ell$-dimension of V (hence, in particular, is independent of ℓ). So $\mathrm{H}^1(\mathbb{Q}_\ell, \mu(\overline{\mathbb{Q}}_\ell))$ is killed by $C!$. In particular, after replacing k by its degree-$C!$ field extension, we may assume that $\mathrm{im}(\overline{\rho})$ is contained in the image of $Z(G)^\circ(\mathbb{Q}_\ell) \to (G/\overline{G})(\mathbb{Q}_\ell)$ for every prime $\ell \neq p$. Since $\pi_1(k) \simeq \hat{\mathbb{Z}}$, $\overline{\rho}$ lifts to $\tilde{\rho} : \pi_1(k) \to Z(G)^\circ(\mathbb{Q}_\ell)$. This defines a semisimple action of $\pi_1(k)$ on $V^{\overline{T}}$ *via* $\pi_1(k) \xrightarrow{\tilde{\rho}} Z(G)^\circ(\mathbb{Q}_\ell) \to (Nor_G(\overline{T})/\overline{T})(\mathbb{Q}_\ell) \to \mathrm{GL}(V^{\overline{T}})$.

6.3.1.2. This action is independent of

- The lift $\tilde{\rho} : \pi_1(k) \to Z(G)^\circ(\mathbb{Q}_\ell)$. This follows from $\mu \subset Z(\overline{G}) \subset C_{\overline{G}}(\overline{T}) = \overline{T}$, where the last equality comes from the fact that $\overline{G}$ is connected semisimple,
- The choice of the maximal torus $\overline{T}' \subset \overline{G}$. More precisely, if $\overline{T}' \subset \overline{G}$ is another maximal torus, there exists $g \in \overline{G}(\overline{\mathbb{Q}}_\ell)$ such that $\overline{T}'_{\overline{\mathbb{Q}}_\ell} = g\overline{T}_{\overline{\mathbb{Q}}_\ell}g^{-1}$ and translation by g induces a $Z(G)^\circ_{\overline{\mathbb{Q}}_\ell}$-equivariant isomorphism $g\cdot : V^{\overline{T}} \otimes \overline{\mathbb{Q}}_\ell \tilde{\to} V^{\overline{T}'} \otimes \overline{\mathbb{Q}}_\ell$. In particular, $V^{\overline{T}} \otimes \overline{\mathbb{Q}}_\ell \simeq V^{\overline{T}'} \otimes \overline{\mathbb{Q}}_\ell$ hence $V^{\overline{T}} \simeq V^{\overline{T}'}$ as $\pi_1(k)$-modules.

Definition 11. *We call $V^{\overline{T}}$ equipped with this structure of almost $\pi_1(k)$-module, the ghost attached to V and denote it by $\Psi(V)$.*

6.3.2. *End of the proof of Proposition 5.* Let $T \subset G$ be a maximal torus containing $\overline{T}$. From 6.2.1 (2), for every $s \in |S|$, $T_s = \Theta_s$ hence there exists $g \in G(\overline{\mathbb{Q}}_\ell)$ such that $gT_{s,\overline{\mathbb{Q}}_\ell}g^{-1} \subset T_{\overline{\mathbb{Q}}_\ell}$. Since $g\rho(F_s)^{ss}g^{-1} \in T(\overline{\mathbb{Q}}_\ell)$ and $\tilde{\rho}(F)^{n_s} \in Z(G)^\circ(\mathbb{Q}_\ell) \subset T(\overline{\mathbb{Q}}_\ell)$ have the same image in

$$(G/\overline{G})(\overline{\mathbb{Q}}_\ell) \simeq (T/\overline{T})(\overline{\mathbb{Q}}_\ell) \simeq T(\overline{\mathbb{Q}}_\ell)/\overline{T}(\overline{\mathbb{Q}}_\ell)$$

(the first isomorphism comes from $\overline{T} \subset \overline{G} \cap T \subset C_{\overline{G}}(\overline{T}) = \overline{T}$), there exists $\bar{t} \in \overline{T}(\overline{\mathbb{Q}}_\ell)$ such that $g\rho(F_s)^{ss}g^{-1} = \tilde{\rho}(F)^{n_s}\bar{t}$. In particular, $g\rho(F_s)^{ss}g^{-1}$ and $\tilde{\rho}(F)^{n_s}$ coincide on $V^{\overline{T}} \otimes \overline{\mathbb{Q}}_\ell$. This shows Proposition 5.

6.4. **Functoriality.** The ghost enjoys natural functoriality properties. Let $C(S, \mathbb{Q}_\ell)$ denote the category of finite-dimensional $\mathbb{Q}$-rational semisimple $\mathbb{Q}_\ell$-representations of $\pi_1(S)$ which are pointwise pure. Then every $V \in C(S, \mathbb{Q}_\ell)$ automatically lies in a $\mathbb{Q}$-rational compatible family (see Subsection 6.6). Let G be a reductive algebraic group over $\mathbb{Q}_\ell$, $\psi : \pi_1(S) \to G(\mathbb{Q}_\ell)$ a representation whose image is Zariski-dense in G, $\overline{G}$ the Zariski-closure of the image of $\pi_1(\overline{S})$ in G, and $\overline{T}$ a maximal torus of $\overline{G}$. Let $C(S, \mathbb{Q}_\ell)_\psi$ denote the full subcategory of $C(S, \mathbb{Q}_\ell)$ consisting of $V \in C(S, \mathbb{Q}_\ell)$ such that the action of $\pi_1(S)$ on V factors through an (a unique) action of G on V. Then $V \mapsto \Psi_{\overline{T}}(V) := V^{\overline{T}}$ defines a natural additive exact functor $\Psi_{\overline{T}}(-) : C(S, \mathbb{Q}_\ell)_\psi \to \mathrm{aMod}_{\mathbb{Q}_\ell}(\pi_1(k))$. The exactness follows from the fact that $\overline{T}$ is linearly reductive.

- The functor $V \mapsto \Psi_{\overline{T}}(V)$ is functorial with respect to inverse images. More precisely, let Y be a smooth, separated, geometrically connected scheme of finite type over k and $f : Y \to S$ a k-morphism; write again $f : \pi_1(Y) \to \pi_1(S)$ for the corresponding morphism of profinite groups. Let $\overline{G}_Y \subset G_Y \subset G$ denote the Zariski-closures of the images of $\pi_1(\overline{Y})$ and $\pi_1(Y)$ under ψ, and $\psi_Y : \pi_1(Y) \to G_Y(\mathbb{Q}_\ell)$ the representation induced by ψ. Choose maximal tori T, $\overline{T}$, T_Y, $\overline{T}_Y$ of G, $\overline{G}$, G_Y, $\overline{G}_Y$, respectively, with $\overline{T}_Y \subset \overline{T} \subset T$ and $\overline{T}_Y \subset T_Y \subset T$. Assume that G_Y is reductive. Then one has a functor $f^* : C(S, \mathbb{Q}_\ell)_\psi \to C(Y, \mathbb{Q}_\ell)_{\psi_Y}$ and a morphism of functors $\Psi_{\overline{T}}(-) \hookrightarrow \Psi_{\overline{T}_Y}(f^*-)$. The latter is constructed as follows. Let $V \in C(S, \mathbb{Q}_\ell)_\psi$. Then the natural inclusion $V^{\overline{T}} \hookrightarrow V^{\overline{T}_Y}$ is $T_Y/\overline{T}_Y$-equivariant and induces a morphism of almost $\pi_1(k)$-modules $\Psi_{\overline{T}}(V) \hookrightarrow \Psi_{\overline{T}_Y}(f^*V)$.

  For general $f : Y \to S$, there is no hope that $\Psi_{\overline{T}}(-) \hookrightarrow \Psi_{\overline{T}_Y}(f^*-)$ be an isomorphism, but ad-hoc Lefschetz-type theorems show one can always construct a smooth geometrically connected curve Y over k and a morphism $f : Y \to S$ such that $\Psi_{\overline{T}}(-) \hookrightarrow \Psi_{\overline{T}_Y}(f^*-)$ is an isomorphism.

- Let G' be another reductive algebraic group over $\mathbb{Q}_\ell$, $\psi' : \pi_1(S) \to G'(\mathbb{Q}_\ell)$ a representation whose image is Zariski-dense in G', $\overline{G}'$ the Zariski-closure of the image of $\pi_1(\overline{S})$ in G', and $\overline{T}'$ a maximal torus of $\overline{G}'$. Let $h : G' \to G$ be a morphism of algebraic groups over $\mathbb{Q}_\ell$, such that $h \circ \psi' = \psi$ and $h(\overline{T}') = \overline{T}$. Then one has $C(S, \mathbb{Q}_\ell)_\psi \subset C(S, \mathbb{Q}_\ell)_{\psi'}$ and $\Psi_{\overline{T}'}(-)|_{C(S,\mathbb{Q}_\ell)_\psi} = \Psi_{\overline{T}}(-)$. Thus, if we fix a maximal pro-torus $\overline{\mathcal{T}}$ of the geometric part $\overline{\mathcal{G}}$ of the pro-reductive completion $\mathcal{G}$ (that is the projective limit of (G, ψ) as above) of $\pi_1(S)$, we obtain an additive exact functor $\Psi_{\overline{\mathcal{T}}}(-) : C(S, \mathbb{Q}_\ell) \to \mathrm{aMod}_{\mathbb{Q}_\ell}(\pi_1(k))$. Although

$\Psi_{\overline{\mathcal{T}}}(-)$ may depend on the choice of $\overline{\mathcal{T}}$, the composite $\Psi_{\overline{\mathcal{T}}}(-)_{\overline{\mathbb{Q}}_\ell} : C(S, \mathbb{Q}_\ell) \to \mathrm{aMod}_{\overline{\mathbb{Q}}_\ell}(\pi_1(k))$ of $\Psi_{\overline{\mathcal{T}}}(-)$ with $\mathrm{aMod}_{\mathbb{Q}_\ell}(\pi_1(k)) \to \mathrm{aMod}_{\overline{\mathbb{Q}}_\ell}(\pi_1(k))$, $V \mapsto V \otimes \overline{\mathbb{Q}}_\ell$ is independent of $\overline{\mathcal{T}}$ up to natural isomorphism. (Here, $\mathrm{aMod}_{\overline{\mathbb{Q}}_\ell}(-)$ is defined similarly to $\mathrm{aMod}_{\mathbb{Q}_\ell}(-)$, with $\mathbb{Q}_\ell$ being replaced by $\overline{\mathbb{Q}}_\ell$.) Indeed, this follows from the fact that in the proof of independence of $\overline{T}$ in Subection 6.3.1, the element $g \in \overline{G}^\circ(\overline{\mathbb{Q}}_\ell)$ such that $\overline{T}'_{\overline{\mathbb{Q}}_\ell} = g\overline{T}_{\overline{\mathbb{Q}}_\ell}g^{-1}$ is unique up to right multiplication by elements of $\overline{T}(\overline{\mathbb{Q}}_\ell)$ (as the centralizer of the maximal torus $\overline{T}$ in the connected semisimple algebraic group $\overline{G}$ coincides with $\overline{T}$ itself), hence the isomorphism $g\cdot : V^{\overline{T}} \otimes \overline{\mathbb{Q}}_\ell \tilde{\to} V^{\overline{T}'} \otimes \overline{\mathbb{Q}}_\ell$ is uniquely determined, independently of the choice of g.

- It would be interesting to study the functorial behavior of $V \mapsto \Psi(V)$ with respect to other operations, in particular higher direct images by smooth proper morphism. More generally one could ask for an extension of $V \mapsto \Psi(V)$ to constructible ℓ-adic sheaves, a derived version of it *etc.*

6.5. **Proof of Theorem 6.** For a subset $\Sigma \subset |S|$ and an integer $n \geq 1$, write $\Sigma(\leq n) \subset \Sigma$ (resp. $\Sigma(\geq n) \subset \Sigma$) for the set of all $s \in \Sigma$ with $n_s \leq n$ (resp. $n_s \geq n$).

We will deduce Theorem 6 from Claim 12 below, which roughly asserts that up to normalization $P = P_{F, V^{\overline{T}}}$ can be identified with the gcd of the $P_s = P_{F_s, s^*V}$, $s \in \Sigma(\leq N_\Sigma)$ for $\Sigma \subset |S|$ any subset of upper density 1 and N_Σ large enough, depending on Σ.

6.5.1. Let $\Sigma \subset |S|$ be an infinite subset and let n_Σ denote the smallest degree of a closed point in Σ. For each integer $n \geq n_\Sigma$, let $P_{\Sigma,n} \in \mathbb{Q}[T]$ denote the gcd of the $P_s^{(\frac{n!}{n_s})}$, $s \in \Sigma(\leq n)$ and $\tilde{P}_{\Sigma,n}$ the gcd of the $P_s^{(\frac{(n+1)!}{n_s})} = P_s^{(\frac{n!}{n_s})(n+1)}$, $s \in \Sigma(\leq n)$. Then $P_{\Sigma,n+1}$ and $P_{\Sigma,n}^{(n+1)}$ both divide $\tilde{P}_{\Sigma,n}$. But, from 6.1.1 (1) and 6.2.1 (2)[8], one has $P_{\Sigma,n}^{(n+1)} = \tilde{P}_{\Sigma,n}$. In particular, $P_{\Sigma,n+1} | P_{\Sigma,n}^{(n+1)}$, $n \geq 1$. So, the degree of the $P_{\Sigma,n}$, $n \geq n_\Sigma$ is decreasing, hence stabilizes for $n \geq N_\Sigma$ that is $P_{\Sigma,n} = P_{\Sigma,N_\Sigma}^{(\frac{n!}{N_\Sigma!})}$ for $n \geq N_\Sigma$. Then by 6.1.1 (2) and 6.2.1 (2)[8],

(1) $P_{\Sigma,N_\Sigma}^{(n_s)} | P_s^{(N_\Sigma!)}$, $s \in \Sigma$

(and even $P_{\Sigma,N_\Sigma} | P_s^{(\frac{N_\Sigma!}{n_s})}$, $s \in \Sigma(\leq N_\Sigma)$, by definition).

Claim 12. *Assume Σ has upper density* 1*. Then* $P^{(N_\Sigma!)} = P_{\Sigma,N_\Sigma}$.

6.5.2. **Proof of Claim 12 $\Rightarrow$ Theorem 6.** To prove Theorem 6 (2), choose any (infinite) subset $\Sigma \subset |S|$ of upper density 1 (*e.g.* $\Sigma = |S|$). From Claim 12, $P^{(N_\Sigma!)}$ is in $\mathbb{Q}[T]$ and independent of ℓ - that is Theorem 6 (2) holds after possibly replacing k with $k_{N_\Sigma!}$.

To prove Theorem 6 (1), let $\Sigma \subset |S|$ denote the set of all $s \in |S|$ such that $W|_{\pi_1(s)}$ is almost a submodule of $(s^*V)^{ss}$. Let $P_W \in \mathbb{Q}_\ell[T]$ denote the characteristic polynomial of $F \in \pi_1(k)$ acting on W. Since $\Sigma(\leq N_\Sigma)$ is finite there exists $n \geq 1$ such that for every $s \in \Sigma(\leq N_\Sigma)$, $P_W^{(n_s n)} | P_s^{(n)}$, so that

$$P_W^{(N_\Sigma! n)} = P_W^{(n_s n)(\frac{N_\Sigma!}{n_s})} | P_s^{(n)(\frac{N_\Sigma!}{n_s})} = P_s^{(\frac{N_\Sigma!}{n_s})(n)}.$$

This shows $P_W^{(N_\Sigma! n)} | P_{\Sigma,N_\Sigma}^{(n)}$ hence, from Claim 12, $P_W^{(N_\Sigma! n)} | P^{(N_\Sigma! n)}$. As both $V^{\overline{T}}$ and W are semisimple, this shows that $W|_{\pi_1(k_{N_\Sigma! n})}$ is a $\pi_1(k_{N_\Sigma! n})$-submodule of $V^{\overline{T}}|_{\pi_1(k_{N_\Sigma! n})}$.

Remark. The strategy of trying to prove that P is in $\mathbb{Q}[T]$ and independent of ℓ by writing it down as a gcd of characteristic polynomials of Frobenius that are in $\mathbb{Q}[T]$ and independent of ℓ is reminiscent of [D80, (4.5)] and [K94].

6.5.3. **Proof of Claim 12.**

[8] Here, we use the full strength of 6.2.1 (2).

6.5.3.1. Before diving into the proof of Claim 12, which is a bit technical, we sketch the idea. For simplicity, assume $N_\Sigma = 1$ and there is a $\pi_1(S)$-invariant $\mathbb{Z}_\ell$-lattice $H \subset V$ such that $\rho(\pi_1(S))$ acts trivially on H/ℓ. Since $P|P_{\Sigma,1}$ by Proposition 5, one has to show $r := \deg(P_{\Sigma,1}) \leq \deg(P) = \dim(V^{\overline{T}})$. Set $\delta := \dim(V)$. The fact that $\overline{G}$ is semisimple is only used to ensure that $\rho(\pi_1(\overline{S})) \cap \overline{T}(\mathbb{Z}_\ell)$ is open in $\overline{T}(\mathbb{Z}_\ell) := \overline{T}(\mathbb{Q}_\ell) \cap GL(H)$. So, again for simplicity, assume $G = T$ is a (split) torus and that, after fixing a suitable $\mathbb{Q}_\ell$-basis of V, it is contained in the diagonal torus $D = \mathbb{G}^\delta_{m,\mathbb{Q}_\ell} \subset GL_V \simeq \mathrm{GL}_{\delta,\mathbb{Q}_\ell}$ and $\rho(\pi_1(\overline{S})) \subset \overline{T}(\mathbb{Z}_\ell)$ is open in $\overline{T}(\mathbb{Z}_\ell)$. Write $B := P_{\Sigma,1} = (T-\beta_1)\cdots(T-\beta_r)$. By assumption and 6.5.1 (1), for every $s \in \Sigma$, $B^{(n_s)}|P_s$. Using that Σ has upper density 1 and up to permuting the coordinates, one reduces to the case where $\rho(\pi_1(S)) \subset (1+\ell\mathbb{Z}_\ell)^{\delta-r} \times (\beta_1,\ldots,\beta_r)^{\mathbb{Z}_\ell}$. Let $\overline{T}'$ denote the projection of $\overline{T}$ onto the last r coordinates. Since by assumption $\rho(\pi_1(\overline{S})) \subset \overline{T}(\mathbb{Z}_\ell)$ is open in $\overline{T}(\mathbb{Z}_\ell)$, the image $\overline{U}$ of $\rho(\pi_1(\overline{S}))$ in $\overline{T}'(\mathbb{Z}_\ell)$ is open. As $\overline{U}$ is also contained in $(\beta_1,\ldots,\beta_r)^{\mathbb{Z}_\ell}$ by construction, it has (ℓ-adic analytic) dimension 0 or 1. If $\overline{U}$ has dimension 0, $\overline{T}' = 1$ and $r \leq \dim(V^{\overline{T}})$ follows. If $\overline{U}$ has dimension 1, one easily shows (using that $\rho(\pi_1(S)) \subset (1+\ell\mathbb{Z}_\ell)^\delta$ and that V is pointwise pure (of weight $\neq 0$, after twisting V if necessary)) that $\beta_i = \beta_1$, $i = 1,\ldots,r$. Thus, twisting V by the character corresponding to β_1^{-1} (which does not affect $\overline{T}$ nor the degree of B), one is reduced to the case where $\overline{U}$ has dimension 0.

We now turn to the detailed argument.

6.5.3.2. By Proposition 5, one has $P^{(n_s)}|P_s$, $s \in |S|$, hence $P^{(n!)}|P_{\Sigma,n}$, $n \geq n_\Sigma$. Thus, to prove Claim 12 it is enough to show $\deg(P_{\Sigma,N_\Sigma}) = \deg(P) (= \dim(V^{\overline{T}}))$. Note that this last assertion is unchanged if N_Σ is replaced by any integer $\geq N_\Sigma$.

6.5.3.3. Fix $s_0 \in |S|$ such that $T := T_{\ell,s_0}$ is a maximal torus in G_ℓ (See Subsection 6.2.1). As $\dim(V^{\overline{T}})$ is independent of $\overline{T}$, we may assume $\overline{T} \subset T$. As $\delta^u(\Sigma) = 1 > \frac{1}{2}$, by Lemma 7 (3) there exist $s_1,\ldots,s_t \in \Sigma$ such that the gcd of $n_{s_1},\ldots,n_{s_t}$ is 1 or, equivalently, $m_1n_{s_1}+\cdots+m_tn_{s_t} = 1$ for some $m_1,\ldots,m_t \in \mathbb{Z}$. Write $d := \deg(P_{\Sigma,N_\Sigma})$. As $\dim(V^{\overline{T}})$ is independent of ℓ by [Chi04, Thm. 1.6] (which relies on [L02]) and d is independent of ℓ by definition, we may choose ℓ freely. In particular, we may assume that $\ell \neq 2$, that $P_{s_1},\ldots,P_{s_t} \in \mathbb{Z}_\ell[T]$ and that ℓ splits completely in the compositum of the splitting fields of $P_{s_0}, P_{s_1},\ldots,P_{s_t}$ (from the Cebotarev density theorem, the set of such primes has positive density). In particular we may assume that T is split over $\mathbb{Q}_\ell$ and that the roots $\alpha_1,\ldots,\alpha_d$ (counted with multiplicities) of P_{Σ,N_Σ} are in $(\mathbb{Z}_\ell^\times)^{N_\Sigma!}$. (Indeed, by 6.5.1 (1), $P_{\Sigma,N_\Sigma}^{(n_{s_i})}|P_{s_i}^{(N_\Sigma!)}$ for $i = 1,\ldots,r$, hence, for each $\alpha \in \{\alpha_1,\ldots,\alpha_d\}$, there exists a root γ_i of P_{s_i} such that $\alpha^{n_{s_i}} = \gamma_i^{N_\Sigma!}$. By the choice of ℓ one has $\gamma_i \in \mathbb{Q}_\ell$. In fact, one has even $\gamma_i \in \mathbb{Z}_\ell^\times$, since $P_{s_i} \in \mathbb{Z}_\ell[T]$ and $P_{s_i}(0) \in \mathbb{Q} \cap \pm(|k|^{\frac{1}{2}})^{\mathbb{Z}} \subset \mathbb{Z}_\ell^\times$. Now, one has $\alpha = (\gamma_1^{m_1}\cdots\gamma_t^{m_t})^{N_\Sigma!} \in (\mathbb{Z}_\ell^\times)^{N_\Sigma!}$.) Up to increasing N_Σ (depending on ℓ), we may assume $(\ell-1)|N_\Sigma$ and then $\alpha_1,\ldots,\alpha_d \in (1+\ell\mathbb{Z}_\ell) \cap (\mathbb{Z}_\ell^\times)^{N_\Sigma!} = (1+\ell\mathbb{Z}_\ell)^{N_\Sigma!}$.

6.5.3.4. As described in Subsection 6.5.3.1, we need to twist V by a character (twice) in the proof of Claim 12. We review below the effect of this operation with respect to the various assumptions we made in Subsections 6.2.1, 6.5.1 and 6.5.3.3. Let $\alpha \in \mathbb{Z}_\ell^\times$ and write $\chi_\alpha : \pi_1(S) \to \mathbb{Z}_\ell^\times$ for the composite of $a_S : \pi_1(S) \twoheadrightarrow \pi_1(k)$ and the character $\pi_1(k) \to \mathbb{Z}_\ell^\times$ defined by $F \mapsto \alpha$. We set $V(\alpha) := V$ on which $\pi_1(S)$ acts by $\rho \cdot \chi_\alpha$, and write $\rho(\alpha) : \pi_1(S) \to \mathrm{GL}(V)$ for the corresponding representation. We write $G(\alpha), \overline{G}(\alpha), T(\alpha), \overline{T}(\alpha), P_s(\alpha), P(\alpha), P_{\Sigma,n}(\alpha), \tilde{P}_{\Sigma,n}(\alpha)$ for $G, \overline{G}, T, \overline{T}, P_s, P, P_{\Sigma,n}, \tilde{P}_{\Sigma,n}$ defined by replacing (V,ρ) with $(V(\alpha),\rho(\alpha))$. For a monic polynomial $Q(T) = (T-\gamma_1)\cdots(T-\gamma_t) \in \overline{\mathbb{Q}}_\ell[T]$, we set $Q[\alpha](T) := (T-\alpha\gamma_1)\cdots(T-\alpha\gamma_t)$. Then
- In general, $\rho(\alpha)$ may neither be $\mathbb{Q}$-rational nor lie in a compatible family.
- If $\alpha \in \pm p^{\mathbb{Z}}$, then $\rho(\alpha)$ is $\mathbb{Q}$-rational, pointwise pure of weight $w + 2\log_{|k|}|\alpha|$ and lies in a compatible family of ℓ'-adic representations of $\pi_1(S)$ ($\ell' \neq p$).
- $\overline{G}(\alpha) = \overline{G}$, $\overline{T}(\alpha) = \overline{T}$, and $V(\alpha)^{\overline{T}(\alpha)} = V^{\overline{T}}$. In particular, 6.2.1 (1) holds for $\overline{G}(\alpha)$.
- $P_s(\alpha) = P_s[\alpha^{n_s}]$ and 6.2.1 (2) may not hold for $P_s(\alpha)$.
- 6.2.1 (3) may not hold for $G(\alpha)$.
- Even without 6.2.1 (3), $V(\alpha)^{\overline{T}(\alpha)}$ is equipped naturally with a structure of $\pi_1(k)$-module by twisting that of $V^{\overline{T}}$ by α. In particular, $P(\alpha) = P[\alpha]$.
- $P_{\Sigma,n}(\alpha) = P_{\Sigma,n}[\alpha^{n!}]$ and $\tilde{P}_{\Sigma,n}(\alpha) = \tilde{P}_{\Sigma,n}[\alpha^{(n+1)!}]$.

- Even without 6.2.1 (2), one has $P_{\Sigma,n}(\alpha)^{(n+1)} = \tilde{P}_{\Sigma,n}(\alpha)$, $P_{\Sigma,n+1}(\alpha)|P_{\Sigma,n}(\alpha)^{(n+1)}$ for $n \geq 1$ and $P_{\Sigma,n}(\alpha) = P_{\Sigma,N_\Sigma}(\alpha)^{(\frac{n!}{N_\Sigma!})}$ for $n \geq N_\Sigma$. Also, 6.5.1 (1) holds for $P_{\Sigma,N_\Sigma}(\alpha)$ and $P_s(\alpha)$.
- We may assume that $T(\alpha)$ is split over $\mathbb{Q}_\ell$. Indeed, write $\tilde{\rho}_\alpha = \rho \oplus \chi_\alpha$ for the representation of $\pi_1(S)$ corresponding to $V \oplus \mathbb{Q}_\ell(\alpha)$ and $\tilde{G}(\alpha)$ for the Zariski-closure of the image of $\pi_1(S)$ under $\tilde{\rho}_\alpha$. Then one has the inclusion morphism $i : \tilde{G}(\alpha) \hookrightarrow G \times \mathbb{G}_m$ (with projection morphisms $p_G : \tilde{G}(\alpha) \twoheadrightarrow G$ and $p_{\mathbb{G}_m} : \tilde{G}(\alpha) \to \mathbb{G}_m$) and the multiplication morphism $m : \tilde{G}(\alpha) \twoheadrightarrow G(\alpha)$. Set $N := \ker(p_G)$, which can be viewed as an algebraic subgroup of $\mathbb{G}_m$ through $p_{\mathbb{G}_m}$, hence is either $\mathbb{G}_m$ itself or finite (étale). Now, we may take a maximal torus $\tilde{T}(\alpha)$ of $\tilde{G}(\alpha)$ contained in $p_G^{-1}(T)$. As $p_G^{-1}(T)$ is an extension of T by N, we can show that either $\tilde{T}(\alpha) \simeq T \times \mathbb{G}_m$ or $p_G : \tilde{T}(\alpha) \twoheadrightarrow T$ has finite kernel. In both cases $\tilde{T}(\alpha)$ is split over $\mathbb{Q}_\ell$, hence so is $T(\alpha) := m(\tilde{T}(\alpha))$. (Note that for s_0 in Subsection 6.5.3.3, the roots of $P_{s_0}(\alpha) = P_{s_0}[\alpha^{n_{s_0}}]$ are in $\mathbb{Q}_\ell$, hence the Frobenius torus $T_{\ell,s_0}(\alpha)$ of $G(\alpha)$ at s_0 is split over $\mathbb{Q}_\ell$. However, $T_{\ell,s_0}(\alpha)$ may not be maximal in general.)
- For $s_1, \ldots, s_t$ in Subsection 6.5.3.3, $P_{s_1}(\alpha) = P_{s_1}[\alpha^{n_{s_1}}], \ldots, P_{s_t}(\alpha) = P_{s_t}[\alpha^{n_{s_t}}]$ are in $\mathbb{Z}_\ell[T]$ and their roots are in $\mathbb{Z}_\ell^\times$. The roots of $P_{\Sigma,N_\Sigma}(\alpha) = P_{\Sigma,N_\Sigma}[\alpha^{N_\Sigma!}]$ are in $(\mathbb{Z}_\ell^\times)^{N_\Sigma!}$ and (up to increasing N_Σ) even in $(1+\ell\mathbb{Z}_\ell)^{N_\Sigma!}$.

In summary, Claim 12 for (V,ρ) is equivalent to Claim 12 for $(V(\alpha),\rho(\alpha))$, and to prove the latter, we may (resp. may not) assume 6.2.1 (1), various $\mathbb{Q}_\ell$-splitness properties (resp. $\mathbb{Q}_\ell$-rationality, compatibility, (6.2.1 (2), 6.2.1 (3)). If $\alpha \in \pm p^{\mathbb{Z}}$, then $\rho(\alpha)$ is $\mathbb{Q}$-rational, pointwise pure of weight $w + 2\log_{|k|}|\alpha|$ and lies in a compatible family.

6.5.3.5. For an integer $r \geq 0$, let $\mathcal{P}_r \tilde{\to} \mathbb{A}^{r-1} \times \mathbb{G}_m$ ($\mathcal{P}_0 = \mathrm{Spec}(\mathbb{Z})$) denote the scheme of degree r monic polynomials with invertible constant term. Let δ denote the $\mathbb{Q}_\ell$-dimension of V, and $H \subset V$ a $\pi_1(S)$-invariant lattice. For $B \in \mathbb{Q}_\ell[T]$ monic of degree $r \leq \delta$ and with roots (counted with multiplicities) $\beta_1, \ldots, \beta_r$ in $1+\ell\mathbb{Z}_\ell$, consider the ℓ-adic analytic morphism

$$\begin{array}{llll} \phi_B : & \mathbb{Z}_\ell \times \mathcal{P}_{\delta-r}(\mathbb{Z}_\ell) & \to & \mathcal{P}_\delta(\mathbb{Z}_\ell) \\ & (\lambda, Q(T)) & \to & Q(T)(T-\beta_1^\lambda)\ldots(T-\beta_r^\lambda) \end{array}$$

Then, one has the following commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{GL}(V) & \xrightarrow{\chi_{N_\Sigma!}} & \mathcal{P}_\delta(\mathbb{Q}_\ell) & & \\ & & \uparrow & & \uparrow & & \\ \pi_1(S) & \xrightarrow{\rho} & \mathrm{GL}(H) & \xrightarrow{\chi_{N_\Sigma!}} & \mathcal{P}_\delta(\mathbb{Z}_\ell) & \xleftarrow{\phi_B} & \mathbb{Z}_\ell \times \mathcal{P}_{\delta-r}(\mathbb{Z}_\ell) \end{array}$$

where $\chi_{N_\Sigma!}$ is the map sending $g \in \mathrm{GL}(V)$ with characteristic polynomial $\chi(g)$ to $\chi_{N_\Sigma!}(g) := \chi(g^{N_\Sigma!})$.

6.5.3.6. Claim 12 will follow from Claim 13 below applied to $B = P_{\Sigma,N_\Sigma}$.

Claim 13. *Let $B \in \mathbb{Q}_\ell[T]$ be a monic polynomial of degree $r \leq \delta$ and with roots $\beta_1, \ldots, \beta_r$ in $1+\ell\mathbb{Z}_\ell$. Then,*
(1) If $B|P_{\Sigma,N_\Sigma}$, then $\rho(\pi_1(S)) \subset \chi_{N_\Sigma!}^{-1}(\mathrm{im}(\phi_B))$.
(2) If $P^{(N_\Sigma!)}|B|P_{\Sigma,N_\Sigma}$, then $P^{(N_\Sigma!)} = B$.

Proof. (1): For every $s \in \Sigma$, $B^{(n_s)}|P_{\Sigma,N_\Sigma}^{(n_s)}|P_s^{(N_\Sigma!)}$ by 6.5.1 (1). This shows that for every $s \in \Sigma$, $\chi_{N_\Sigma!}(\rho(F_s))$ lies in the image of ϕ_B. As $\mathrm{im}(\phi_B)$ is compact and Σ has upper density 1, $\rho(\pi_1(S)) \subset \chi_{N_\Sigma!}^{-1}(\mathrm{im}(\phi_B))$ by Lemma 8 (3) hence the conclusion follows.

(2): Since $P^{(N_\Sigma!)}|B$, it is enough to show that $r := \deg(B) \leq \deg(P^{(N_\Sigma!)}) = \dim(V^{\overline{T}})$. Up to replacing V by the Tate twist $V(|k|^{-n})$ ($= V(n)$ in the usual notation) and B by $B(|k|^{-n}) := B[|k|^{-N_\Sigma!n}]$ for a suitable $n \in \mathbb{Z}$, one may assume that $w \neq 0$ (see Subsection 6.5.3.4). Since T is split over $\mathbb{Q}_\ell$, fixing a suitable $\mathbb{Q}_\ell$-basis $e_1, \ldots, e_\delta$ of V, we may assume T is contained in the diagonal torus $D = \mathbb{G}_{m,\mathbb{Q}_\ell}^\delta \subset \mathrm{GL}_V \simeq \mathrm{GL}_{\delta,\mathbb{Q}_\ell}$. Set $H := \mathbb{Z}_\ell e_1 + \cdots + \mathbb{Z}_\ell e_\delta \subset V$. For an algebraic subgroup $J \subset \mathrm{GL}_V$, write $J(\mathbb{Z}_\ell)$ for $J(\mathbb{Q}_\ell) \cap \mathrm{GL}(H) \subset \mathrm{GL}(V)$. Since $\varphi_{N_\Sigma!} : \rho(\pi_1(S)) \to \rho(\pi_1(S))$, $g \mapsto g^{N_\Sigma!}$ is a local homeomorphism at 1, (1) implies that there exists an open subgroup $\Pi \subset \rho(\pi_1(S))$ such that $\chi(\Pi) \subset \mathrm{im}(\phi_B)$ and $\Pi \cap T(\mathbb{Z}_\ell) \subset (1+\ell\mathbb{Z}_\ell)^\delta \subset D(\mathbb{Q}_\ell)$. Let I denote the set of injective maps $\{1, \ldots, r\} \hookrightarrow \{1, \ldots, \delta\}$. For every $\sigma \in I$, consider the ℓ-adic analytic subgroup $D[\sigma] \subset D(\mathbb{Z}_\ell)$ whose elements are the $(x_1, \dots, x_\delta) \in D(\mathbb{Z}_\ell)$ with $x_{\sigma(i)} = \beta_i^\lambda$, $\lambda \in \mathbb{Z}_\ell$ for $i = 1, \dots, r$ and $x_i \in 1 + \ell\mathbb{Z}_\ell$ arbitrary for $i \in \{1, \dots, \delta\} \setminus \mathrm{im}(\sigma)$. As $\chi(\Pi) \subset \mathrm{im}(\phi_B)$, we have

$$\Pi \cap T(\mathbb{Z}_\ell) \subset \bigcup_{\sigma \in I} D[\sigma].$$

As I is finite, there exists at least one $\sigma \in I$ such that $\Pi \cap T(\mathbb{Z}_\ell) \cap D[\sigma]$ is an open subgroup of $\Pi \cap T(\mathbb{Z}_\ell)$. So, again, up to replacing Π by a smaller open subgroup, we may assume that $\Pi \cap T(\mathbb{Z}_\ell) \subset D[\sigma]$. Write $\overline{\Pi} := \rho(\pi_1(\overline{S})) \cap \Pi$.

Let $p_\sigma : D \twoheadrightarrow \mathbb{G}^r_{m,\mathbb{Q}_\ell}$ denote the projection onto the coordinates $\sigma(1), \dots, \sigma(r)$ and let $\overline{T}_\sigma$, T_σ denote the image of $\overline{T}$, T by the induced morphism of algebraic groups $\mu_\sigma : T \hookrightarrow D \overset{p_\sigma}{\twoheadrightarrow} \mathbb{G}^r_m$. By construction, we have

$$\overline{U} := \mu_\sigma(\overline{\Pi} \cap \overline{T}(\mathbb{Z}_\ell)) \subset U := \mu_\sigma(\Pi \cap T(\mathbb{Z}_\ell)) \subset (\beta_1, \dots, \beta_r)^{\mathbb{Z}_\ell}.$$

As $\overline{G}$ is semisimple, $\rho(\pi_1(\overline{S}))$ is open in $\overline{G}(\mathbb{Z}_\ell)$ [S66, §1, Cor.] hence $\overline{\Pi} \cap \overline{T}(\mathbb{Z}_\ell)$ is open in $\overline{T}(\mathbb{Z}_\ell)$ and $\overline{U}$ is open in $\overline{T}_\sigma(\mathbb{Z}_\ell)$.

So only two cases can occur:

a) $\overline{U} = 1$ that is $\overline{T}_\sigma = 1$ hence, after a permutation of the diagonal entries, $\overline{\Pi} \cap \overline{T}(\mathbb{Z}_\ell) \subset \{1\}^r \times (1 + \ell\mathbb{Z}_\ell)^{\delta - r} \subset D(\mathbb{Q}_\ell) = \mathbb{G}_m(\mathbb{Q}_\ell)^\delta$ thus $\overline{T} \subset \{1\}^r \times \mathbb{G}_m^{\delta - r} \subset D = \mathbb{G}_m^\delta$. In that case $r \leq \dim(V^{\overline{T}})$.

b) $\overline{U}$ has ℓ-adic analytic dimension 1 that is $\overline{T}_\sigma \simeq \mathbb{G}_{m,\mathbb{Q}_\ell}$ and the embedding $\overline{T}_\sigma \hookrightarrow \mathbb{G}^r_{m,\mathbb{Q}_\ell}$ is given by a cocharacter $(n_1, \dots, n_r) \in X_*(\mathbb{G}^r_{m,\mathbb{Q}_\ell}) \simeq \mathbb{Z}^r$ with $\gcd(n_1, \dots, n_r) = 1$. Again after a permutation of the diagonal entries, we may assume $0 = v_\ell(n_1) \leq v_\ell(n_i)$ for every $i = 1, \dots, r$. Then every element of $\overline{U}$ can be written in the form

$$(\zeta, \zeta^{\frac{n_2}{n_1}}, \dots, \zeta^{\frac{n_r}{n_1}})$$

for some $\zeta \in 1 + \ell\mathbb{Z}_\ell$. On the other hand, as $\overline{U} \subset (\beta_1, \dots, \beta_r)^{\mathbb{Z}_\ell}$ is open, there exists an integer $n \gg 0$ such that $(\beta_1^{\ell^n}, \dots, \beta_r^{\ell^n}) \in \overline{U}$ hence

$$\beta_i^{\ell^n} = \beta_1^{\ell^n \frac{n_i}{n_1}},\ i = 1, \dots, r.$$

As $B | P_{\Sigma, N_\Sigma} | P_s^{(\frac{N_\Sigma!}{n_s})}$ for $s \in \Sigma (\leq N_\Sigma)$, one has $|\beta_i| = |k|^{\frac{N_\Sigma! w}{2}}$, $i = 1, \dots, r$. As $w \neq 0$, this forces $n_1 = \dots = n_r$. Hence, $B = (T - \beta_1)^r$. As $B | P_{\Sigma, N_\Sigma}$, one has $\beta_1 \in (1 + \ell\mathbb{Z}_\ell)^{N_\Sigma!}$ (see Subsection 6.5.3.3). So take an (a unique) element $\alpha \in 1 + \ell\mathbb{Z}_\ell$ such that $\alpha^{N_\Sigma!} = \beta_1^{-1}$. Now, replace V with $V(\alpha)$ and B with $B(\alpha) := B[\alpha^{N_\Sigma!}] = B[\beta_1^{-1}]$ (see Subsection 6.5.3.4). Then $\beta_i = 1$, $i = 1, \dots, r$ and the argument above shows that, again, $r \leq \dim(V^{\overline{T}})$.

This concludes the proof of Claim 13. □

To prove Claim 12, set $B := P_{\Sigma, N_\Sigma} \in \mathbb{Q}[T] \subset \mathbb{Q}_\ell[T]$. As shown in Subsection 6.5.3.2, one has $P^{(N_\Sigma!)} | B$. Now, Claim 13 (2) implies Claim 12.

6.6. **A concluding remark.** Both because our starting point was a question of motivic nature and for simplicity, we stated Theorem 6 for a pure $\mathbb{Q}$-rational compatible family of lisse $\mathbb{Q}_\ell$-sheaves on S. But Theorem 6 and its proof extend[9] to pure E-rational compatible families of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on S. Recall that an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_\ell$ with finite determinant automatically lies in an E-rational compatible family $\mathcal{F}_\ell$, $\ell \neq p$ pure of weight 0 for some number field E. For S a curve, this is [L02, Thm. VII.6];

[9] More precisely, the only possible issue could come from our use of the fact (applied to $\rho(\pi_1(\overline{S}))$) that if a closed subgroup of the group of $\mathbb{Q}_\ell$-points of a semisimple algebraic group over $\mathbb{Q}_\ell$ is Zariski-dense, it is ℓ-adically dense ([S66, §1, Cor.]). This is *a priori* no longer true over a finite extension of $\mathbb{Q}_\ell$. Still, combining [Chi03, Main Thm. on p.66] (for irreducible sheaves with finite determinant) and the companion conjecture (to reduce the general case to the case treated in [Chi03]), one may always assume that for a pure E-rational compatible family of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on S there exists a finite extension F of E independent of ℓ, such that for every prime λ of F above $\ell \neq p$, the $\overline{\mathbb{Q}}_\ell$-sheaf comes from an F_λ-sheaf. Thus, it suffices to choose the prime ℓ with various splitting properties in Subsection 6.5.3.4, together with another splitting condition $F_\lambda = \mathbb{Q}_\ell$ and the condition (excluding only finitely many primes) that the (Weil) F_λ-sheaf is actually étale.

for higher dimensional S, see [D12], [Dr12]. When S is a curve, the Langlands correspondence for GL_r provides a bijective correspondance $\mathcal{F}_\ell \longleftrightarrow \pi$ between irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}_\ell$ with finite determinant and cuspidal automorphic representations π of $\mathrm{GL}_r(\mathbb{A})$, unramified on S and whose central character is of finite order. This bijection is characterized by the fact that the local factors of $\mathcal{F}_\ell$ and π coincide. Since ghosts can be read out on the local factors, one may ask for an automorphic interpretation of ghosts.

7. Abelian schemes

We now turn back to the original question of Rössler and Szamuely, namely the validity of **G-I** for a finite field k. So let $A \to S$ be an abelian scheme. Fix a prime $\ell \neq p$, set $V := V_\ell(A_{\overline{\eta}})$ and let $\rho : \pi_1(S) \to GL(V)$ denote the corresponding representation. Write G and $\overline{G}$ for the Zariski-closure of $\rho(\pi_1(S))$ and $\rho(\pi_1(\overline{S}))$ in GL_V respectively. Fix a maximal torus $\overline{T}$ of $\overline{G}$. Recall from the semisimplicity part of Fact 1.3.2 that G is reductive.

7.1. Ghost abelian varieties.

Recall that we can reformulate **G-I** as **R-I"** in terms of the ghost $\Psi(V)$ of $V = V_\ell(A_{\overline{\eta}})$.

Lemma 14. *Let k be a finite field and, for $i = 1, 2$, let A_i be an abelian variety over k with characteristic polynomial of Frobenius $\chi_i := P_{F,V_\ell(A_i)} \in \mathbb{Q}[T]$. Then $\chi :=\gcd(\chi_1, \chi_2) \in \mathbb{Q}[T]$ is the characteristic polynomial of Frobenius $\chi_{1,2}$ of the largest common k-isogeny factor $A_{1,2}$ of A_1, A_2.*

Proof. Let $\mathcal{A}(k)$ denote the set of all k-simple abelian varieties and let $\sim_k$ the k-isogeny equivalence relation on $\mathcal{A}(k)$. From Honda-Tate theory (*e.g.* [O08]), the map which associates to an abelian variety over k a root of the characteristic polynomial of its geometric Frobenius induces a bijection

$$\mathcal{A}(k)/\sim_k \tilde{\to} \mathcal{W}(|k|)/\sim,$$

where $\mathcal{W}(|k|)$ denotes the set of Weil $|k|$-numbers of weight -1 and $\sim$ the $\pi_1(\mathbb{Q})$-conjugacy equivalence relation on $\mathcal{W}(|k|)$. Furthermore, for $w \in \mathcal{W}(|k|)/\sim$ with minimal polynomial $\Pi_w \in \mathbb{Q}[T]$ over $\mathbb{Q}$, there exists a unique $n_w \in \mathbb{Z}_{\geq 1}$ such that $\Pi_w^{n_w} = P_{F,V_\ell(A_w)} =: \chi_w$, where $A_w \in \mathcal{A}(k)$ is an abelian variety corresponding to w, which we fix once for all. Thus, writing

$$A_i \sim_k \prod_{A \in \mathcal{A}(k)/\sim_k} A^{m_i(A)}$$

we get, explicitly

$$\chi_i = \prod_{w \in \mathcal{W}(|k|)/\sim} \chi_w^{m_i(A_w)} = \prod_{w \in \mathcal{W}(|k|)/\sim} \Pi_w^{n_w m_i(A_w)},\ i = 1, 2$$

hence

$$\chi = \prod_{w \in \mathcal{W}(|k|)/\sim} \Pi_w^{\min(n_w m_1(A_w), n_w m_2(A_w))} = \prod_{w \in \mathcal{W}(|k|)/\sim} \chi_w^{\min(m_1(A_w), m_2(A_w))} = \chi_{1,2}.$$

□

Corollary 15. *After possibly replacing k with a finite field extension which is independent of ℓ, there exists an abelian variety $\mathfrak{A}$ over k such that $V_\ell(\mathfrak{A}_{\overline{k}})$ and $\Psi(V)$ are almost isomorphic. Furthermore, for every $\Sigma \subset |S|$ of upper density 1, $\mathfrak{A}_{\overline{k}}$ is the largest common $\overline{k}$-isogeny factor of the $A_{\overline{s}}$, $s \in \Sigma$.*

We call $\mathfrak{A}$ the *ghost abelian variety* attached to $A \to S$ and denote it by $\Psi(A)$.

Proof. From Lemma 7 (4), one can freely replace S with a connected étale cover $f : S' \to S$ hence assume that 6.2.1 (1), 6.2.1 (2) hold and that $N_\Sigma = 1$. Thus, with the notation and from the claim in Subsection 6.5.1, $P = P_{\Sigma,1}$ with, by definition, $P_{\Sigma,1}$ the gcd of the $P_s (= P_{F_s,V_\ell(A_{\overline{s}})})$, $s \in \Sigma(\leq 1)$. So the assertion follows from Lemma 14 with $\mathfrak{A}$ the largest common k-isogeny factor of the A_s, $s \in \Sigma(\leq 1)$. □

Corollary 16. *Let $\mathfrak{A}$ be an abelian variety over $\overline{k}$. Assume $\mathfrak{A}$ is a common $\overline{k}$-isogeny factor of the $A_{\overline{s}}$ for $s \in \Sigma$ and $\Sigma \subset |S|$ of upper density 1. Then $\mathfrak{A}$ is a common $\overline{k}$-isogeny factor of the $A_{\overline{s}}$ for $s \in |S|$.*

7.2. Relation with Zarhin's microweights conjecture.

7.2.1. Up to replacing k with a finite field extension, assume $\Psi(A)$ is defined over k. Let $(A_\eta)_0 \subset A_\eta$ denote the largest weakly $\overline{k}$-isotrivial abelian subvariety of A_η (see [CT12a, §2.1]) so that, up to possibly replacing further k with a finite field extension, there exists an abelian variety $\mathfrak{A}_0$ over k such that $\mathfrak{A}_0 \times_k \overline{\eta}$ and $(A_\eta)_0 \times_\eta \overline{\eta}$ are $\overline{k(\eta)}$-isogenous and $(A_\eta)_0 \subset A_\eta$ is the largest abelian subvariety with this property. From [CT12a, Prop. 2.3 (1) $\Leftrightarrow$ (4)], $V_\ell(\mathfrak{A}_{0,\overline{k}})$ is almost isomorphic to the canonical[10] almost $\pi_1(k)$-module $V^{\overline{G}^\circ}$. In particular, $\mathfrak{A}_0$ is a $\overline{k}$-isogeny factor of $\Psi(A)$, which corresponds to the almost embedding of almost $\pi_1(k)$-modules $V^{\overline{G}^\circ} \hookrightarrow V^{\overline{T}} = \Psi(V)$. This shows the following assertions[11] are equivalent.

(i) **R-I"** for V: $V^{\overline{G}^\circ} = V^{\overline{T}}$ (namely, $\overline{G}$ acting on V has no non-trivial zero weight);
(ii) $\Psi(A)$ is $\overline{k}$-isogenous to $\mathfrak{A}_0$;
(iii) $\Psi(A) \times_k \eta$ is $\overline{k(\eta)}$-isogenous to $(A_\eta)_0$.

7.2.2. Given a reductive group H over a field Q of characteristic 0 and a representation W, consider the following properties:

- $M(H,W)$: the (non-zero) highest weights in W of each simple factor of the root system of H are microweights [Z85, 1.1.2].
- $Z(H,W)$: the simple factors of the root system of H have no non-trivial zero weight in W.

By definition of microweights, $M(H,W)) \Rightarrow Z(H,W)$ and since the simple factors of the root system of a normal subgroup N of H are simple factors of the root system of H, $M(H,W) \Rightarrow M(N,W)$ and $Z(H,V) \Rightarrow Z(N,W)$.

Conjecture 17. (Zarhin's microweights conjecture - [Z85, Conj. 0.4]) *Let F be a finitely generated field of characteristic $p \geq 0$ and $\mathfrak{A}$ an abelian variety over F. Fix a prime $\ell \neq p$ and let H denote the Zariski-closure of the image of $\pi_1(F)$ acting on $W = V_\ell(\mathfrak{A}_{\overline{F}})$. Then $M(H,W)$ holds (and each simple factor of the root system of H has type A, B, C or D).*

As S is normal, the canonical morphism $\pi_1(\eta) \to \pi_1(S)$ is surjective so that G can also be regarded as the image of $\pi_1(\eta)$ acting on $V = V_\ell(A_{\overline{\eta}})$. In particular, Conjecture 17 for A_η over $k(\eta)$ is equivalent to $M(G,V)$ hence implies $Z(\overline{G},V)$ or, equivalently, Condition (i) in 7.2.1. To sum it up:

Corollary 18. *The following assertions are equivalent:*

(1) $M(G,V)$;
(2) Zarhin's microweights conjecture (Conjecture 17) for A_η over $k(\eta)$,

and they imply the following equivalent assertions[11]:

(3) $Z(\overline{G},V)$;
*(4) **R-I"** for V.*

Though Conjecture 17 fails in general (see 7.3), it is known to hold in a wide range of cases as we explain below. In those cases, Rössler and Szamuely's original question thus has a positive anwser.

7.2.3. One says that $A \to S$ admits a 'curve-lift to characteristic 0' if there exists a field K finitely generated over k, a smooth, separated, geometrically connected curve C over K and a morphism of k-schemes $C \to S$ such that

- (i) The image of $\pi_1(C_{\overline{K}})$ acting on V_ℓ *via* $\pi_1(C_{\overline{K}}) \to \pi_1(S_{\overline{k}})$ is open in the image of $\pi_1(S_{\overline{k}})$.
- (ii) There exists a complete discrete valuation ring R of mixed characteristic with residue field K, a smooth, projective and geometrically connected curve $\mathcal{C}^{cpt}$ over R, a divisor $\mathcal{D} \hookrightarrow \mathcal{C}^{cpt}$, finite étale over R and an abelian scheme $\mathcal{A} \to \mathcal{C} := \mathcal{C}^{cpt} \setminus \mathcal{D}$ such that, if r denotes the closed point of Spec(R), the pullback of $\mathcal{A} \to \mathcal{C}$ to r identifies with $A \times_S C \to C$.

Corollary 19. *Assume that one of the following conditions holds.*

[10] Namely, every continuous section $\sigma : \pi_1(k)(\simeq \widehat{\mathbb{Z}}) \to \pi_1(S)$ of the structural morphism $\pi_1(S) \twoheadrightarrow \pi_1(k)$ gives rise to a continuous action of $\pi_1(k)$ on $V^{\overline{G}^\circ}$ through $\pi_1(k) \xrightarrow{\sigma} \pi_1(S) \to G$. This action depends on σ but if $\sigma' : \pi_1(k)(\simeq \widehat{\mathbb{Z}}) \to \pi_1(S)$ is another section, the first cohomology class $[\gamma \mapsto \sigma(\gamma)\sigma'(\gamma)^{-1}] \in H^1(k, \pi_0(\overline{G}))$ vanishes after replacing k by its degree-$|\pi_0(\overline{G})|! \cdot |\pi_0(\overline{G})|$ field extension).

[11] Recall that one also has **G-I** for $A \to S \Leftrightarrow$ **R-I** for $V := V_\ell(A_{\overline{\eta}}) \Leftrightarrow$**R-I'** for $V := V_\ell(A_{\overline{\eta}}) \Leftrightarrow$ **R-I"** for $V := V_\ell(A_{\overline{\eta}})$, and that **G-I** for $A \to S$ implies a positive answer to Rössler and Szamuely's original question for $A \to S$.

(1) $A \to S$ is generically ordinary;
(2) $A \to S$ admits a 'curve-lift to characteristic 0'.
Then $M(\overline{G}, V)$ (hence Conjecture 17 for $A \to S$) holds. In particular, if $A_{\overline{\eta}}$ contains no non-trivial weakly $\overline{k}$-isotrivial abelian subvariety (equivalently, if $\dim(V^{\overline{G}^\circ}) = 0$), the $A_{\overline{s}}$, $s \in \Sigma$ have no non-trivial common $\overline{k}$-isogeny factor.

Proof. Since S is normal, (1) follows from Conjecture 17 for ordinary abelian varieties [Z85, Cor. 4.2.1] applied to A_η over $k(\eta)$. Let us prove (2). The notion of 'curve-lift to characteristic 0' ensures that we can compare the image of $\pi_1(S_{\overline{k}})$ acting on V with a normal subgroup of the ℓ-adic monodromy group of an abelian variety over a number field *via* the specialization theory of pro-ℓ étale fundamental group hence deduce $M(\overline{G}, V)$ for $A \to S$ from the microweights conjecture for abelian varieties over number fields [P98, Cor (5.11)]. More precisely, after replacing $A \to S$ with $A \times_S C \to C$, we may assume $S = C$. Let η denote the generic point of C and let ζ denote the generic point of $\mathrm{Spec}(R)$. After replacing $\mathcal{C}$ by a connected étale cover, we may assume that $\mathcal{A}[\ell] \simeq (\mathbb{Z}/\ell)^{2g}_{\mathcal{C}}$ is constant. As all the objects are of finite type over R, there exists an integral regular scheme T with generic point ξ and of finite type over a number field F, a smooth, projective, geometrically connected morphism $\mathcal{C}_T \to T$, a divisor $\mathcal{D}_T \hookrightarrow \mathcal{C}_T^{cpt}$ finite étale over T, an abelian scheme $\mathcal{A}_T \to \mathcal{C}_T := \mathcal{C}_T^{cpt} \setminus \mathcal{D}_T$, an isomorphism $\mathcal{A}_T[\ell] \tilde{\to} (\mathbb{Z}/\ell)^{2g}_{\mathcal{C}_T}$ and an embedding $k(\xi) \hookrightarrow k(\zeta)$ such that the pullback of $(\mathcal{A}_T \to \mathcal{C}_T \hookrightarrow \mathcal{C}_T^{cpt}, \mathcal{A}_T[\ell] \tilde{\to} (\mathbb{Z}/\ell)^{2g}_{\mathcal{C}_T})$ to ζ *via* $\zeta \to \xi \to T$ identifies with $(\mathcal{A}_\zeta \to \mathcal{C}_\zeta \hookrightarrow \mathcal{C}_\zeta^{cpt}, \mathcal{A}_\zeta[\ell] \tilde{\to} (\mathbb{Z}/\ell)^{2g}_{\mathcal{C}_\zeta})$ over ζ. Let η_T denote the generic point of $\mathcal{C}_T$. By the same argument as in Subsection 2.1, one can find a closed point $x \in \mathcal{C}_T$ such that the neutral components of the Zariski-closures G_x of $\pi_1(x)$ acting on $V_\ell(\mathcal{A}_{T,\overline{x}})$ and G_T of $\pi_1(\mathcal{C}_T)$ acting on $V_\ell(\mathcal{A}_{T,\overline{\eta}_T}) \simeq V_\ell(\mathcal{A}_{T,\overline{x}})$ coincide. Diagram (7.2.3.1) below summarizes the situation:

(7.2.3.1)
$$\begin{array}{ccccccccccc}
\mathcal{A}_{T,x} & \longrightarrow & \mathcal{A}_T & \longleftarrow & \mathcal{A}_{T,\xi} & \longleftarrow & \mathcal{A}_\zeta & \longrightarrow & \mathcal{A} & \longleftarrow & A = \mathcal{A}_r \\
\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
\mathrm{Spec}(k(x)) & \xrightarrow{x} & \mathcal{C}_T & \longleftarrow & \mathcal{C}_{T,\xi} & \longleftarrow & \mathcal{C}_\zeta & \longrightarrow & \mathcal{C} & \longleftarrow & C = \mathcal{C}_r \\
\downarrow & & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
\mathrm{Spec}(F) & \longleftarrow & T & \xleftarrow{\xi} & \mathrm{Spec}(k(\xi)) & \longleftarrow & \mathrm{Spec}(k(\zeta)) & \xrightarrow{\zeta} & \mathrm{Spec}(R) & \xleftarrow{r} & \mathrm{Spec}(K = k(r))
\end{array}$$

Since for each abelian scheme appearing in Diagram (7.2.3.1) the ℓ-torsion is constant, the action of the étale fundamental group of the base on the ℓ-adic Tate module of the geometric generic fiber factors through its pro-ℓ completion. At the level of pro-ℓ completion, the theory of specialization of pro-ℓ fundamental group (*e.g.* [OrV00, Thm. 4.4]) provides the following (iso)morphisms

(7.2.3.2)
$$\begin{array}{ccccccc}
 & \pi_1(\mathcal{C}_{T,\overline{\xi}})^{(\ell)} & \xrightarrow{\simeq} & \pi_1(\mathcal{C}_{\overline{\zeta}})^{(\ell)} & & \xrightarrow{\simeq} & \pi_1(\mathcal{C}_{\overline{r}})^{(\ell)} \\
 & \downarrow & \swarrow & & \searrow & & \swarrow \\
\pi_1(x)^{(\ell)} \longrightarrow & \pi_1(\mathcal{C}_T)^{(\ell)} & & \longleftarrow & & \pi_1(\mathcal{C})^{(\ell)} &
\end{array}$$
.

From the microweights conjecture over number field $M(G_x, V)$ holds hence, as $G_x^\circ = G_T^\circ$, $M(G_T, V)$ holds. Let also $G_{T,\xi}$, $\overline{G}_{T,\xi}$ denote respectively the Zariski-closures of the images of $\pi_1(\mathcal{C}_{T,\xi})$ and $\pi_1(\mathcal{C}_{T,\overline{\xi}})$ acting on V. As $\overline{G}_{T,\xi}$ is normal in $G_{T,\xi}$ and $G_{T,\xi}^\circ = G_T^\circ$, $M(\overline{G}_{T,\xi}, V)$ holds, whence the assertion since, from (7.2.3.2), $\overline{G}_{T,\xi} = \overline{G}$. □

7.3. **Counter-examples to G-I.** The following examples of abelian schemes $A \to S$ for which $Z(\overline{G}, V)$ fails (hence for which Rössler and Szamuely's original question has a negative answer) were communicated to us by Oliver Bültel.

7.3.1. (Bültel - [B05, Thm.1.2], [B13, Lemma B.3], [B22]). Using the theory of integral models of Shimura varieties of PEL type attached to well-chosen unitary groups (admitting more than one sympletic representation with Hodge weight in $\{(-1, 0), (0, 1)\}$) and subtle deformation-theoretic arguments, Bültel showed that, up to isogeny, *every* faithful representation of any given semisimple algebraic group appears as a direct factor of the geometric monodromy of the generic fiber of an abelian scheme ([B05, Thm.1.2], [B13, Lemma B.3]). This gives a systematic way to construct abelian schemes for which $Z(\overline{G}, V)$ fails.

The detailed treatment of the following examples can be found in [B22]:

(1) An abelian scheme $A \to S$ with $A_{\overline{\eta}}$ a 6-dimensional simple abelian variety of type III in Albert's classification, whose geometric monodromy $\overline{G}$ is, over $\overline{\mathbb{Q}}_\ell$, a product of two copies of $SO(3)$ and whose ghost is a 2-dimensional supersingular abelian variety.
(2) An abelian scheme $A \to S$ with $A_{\overline{\eta}}$ a $2r$-dimensional simple abelian variety of type IV in Albert's classification (with $2r > 14$), whose geometric monodromy $\overline{G}$ is, over $\overline{\mathbb{Q}}_\ell$, a product of $s \geq 1$ copies of G_2 and whose ghost is an r-dimensional non-supersingular abelian variety.

7.3.2. (Katz - [K90, 9.1], [K04]). Locally constant $\overline{\mathbb{Q}}_\ell$-local systems which are built as (naïve) Fourier transform of elementary $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ with finite monodromy on $\mathbb{A}^1_k$ naturally appear as direct factors (over $\overline{\mathbb{Q}}_\ell$) of the generic ℓ-adic Tate module of an abelian scheme $A \to S$ (namely the connected component of the Picard scheme of a relative smooth projective curve built out from the cover trivializing the elementary $\mathbb{Q}_\ell$-sheaf) for $S \subset \mathbb{G}_{m,\mathbb{F}_p}$ a non-empty open subscheme. For details of this fact (which is probably well-known to experts but for which we could not find references in the existing litterature), see Appendix A. This applies in particular to the examples analyzed by Katz in [K90, Thm. 9.1.1], [K04, Thm. (3.2), (4.12)] which provides for each odd integer n and prime $p \gg_n 0$ (if n is itself a prime number, one can take $p \geq 2n+1$) examples of abelian schemes $A \to S$ whose geometric monodromy $\overline{G}$ has, over $\overline{\mathbb{Q}}_\ell$, a simple factor H_n such that the H_n-representation V has an isotypical component which is a direct sum of the quasi-minuscule H_n-representation V_n of H_n with:
- If $n \neq 7$, $H_n = SO(n)$ and V_n is the standard (vector) n-dimensional representation of $SO(n)$;
- If $n = 7$, $H_7 = G_2$ and V_7 is the standard 7-dimensional representation of G_2.
As a quasi-minuscule representation has non trivial zero weights, these abelian schemes have a non trivial ghost.

7.4. **Restrictions on** $\Psi(A)$**.** We show that if none of the simple $\overline{k(\eta)}$-isogeny factors of $A_{\overline{\eta}}$ is of type IV in Albert's classification nor weakly $\overline{k}$-isotrivial then $\Psi(A)$ is supersingular. More precisely, we say that $P \in \mathbb{Q}[T]$ is *supersingular (of weight w with respect to $|k|$)* if $P^{(m)} = (T - |k|^{\frac{wm}{2}})^{\deg(P)}$ for some $m \geq 1$. If $P(0) \neq 0$ and the subgroup $A(P) \subset \overline{\mathbb{Q}}^\times$ generated by the roots of P is torsion free, $P \in \mathbb{Q}[T]$ is supersingular if and only if $P^{(2)} = (T - |k|^w)^{\deg(P)}$.

Proposition 20. *Assume that none of the simple $\overline{k(\eta)}$-isogeny factors of $A_{\overline{\eta}}$ is of type IV in Albert's classification nor weakly $\overline{k}$-isotrivial. Then the characteristic polynomial of the Frobenius $F \in \pi_1(k)$ acting on $\Psi(V_\ell(A_{\overline{\eta}}))$ (equivalently $\Psi(A)$) is supersingular.*

Proof. Assume A is S-isogenous to $A_1 \times \cdots \times A_s$ with $A_i \to S$ an abelian scheme such that $A_{i,\overline{\eta}}$ is simple, $i = 1, \dots, s$. We use a subscript $(-)_i$ for the invariants V, $\overline{G}$, G *etc.* attached to $A_i \to S$. The decompositions

$$V^{\overline{T}} = \bigoplus_{1\leq i\leq s} V_i^{\overline{T}} = \bigoplus_{1\leq i\leq s} V_i^{\overline{T}_i}, \quad V^{\overline{G}^\circ} = \bigoplus_{1\leq i\leq s} V_i^{\overline{G}^\circ} = \bigoplus_{1\leq i\leq s} V_i^{\overline{G}_i^\circ},$$

show that to prove Proposition 20 one may assume $A_{\overline{\eta}}$ is not weakly $\overline{k}$-isotrivial and simple of type I, II or III. But as V is pure of Frobenius weight $w \neq 0$, $Z(G)$ contains the homotheties torus $\mathbb{G}_m$ so, by Lemma 21 below, $Z(G)^\circ = \mathbb{G}_m$, which forces supersingularity of $\Psi(A)$ (recall that, by construction, the action of $\pi_1(k)$ on $V^{\overline{T}}$ is through $Z(G)^\circ$). □

Lemma 21. *If $A_{\overline{\eta}}$ is simple and $E := \mathrm{End}_{\overline{\eta}}(A_{\overline{\eta}}) \otimes \mathbb{Q}$ is of type I, II, III then* $\dim(Z(G)) = 1$.

Proof. We recall first (part of) Albert's classification of finite dimensional simple $\mathbb{Q}$-algebras E endowed with a positive definite anti-involution $(-)^\dagger : E \tilde{\to} E$. See [Mu70, IV.21, Thm. 2]. Write $F := Z(E) \supset F_0 := F^\dagger$.

Type	Description
$\mathrm{I}(e_0)$	$F_0 = F = E$ is a totally real field
$\mathrm{II}(e_0)$ or $\mathrm{III}(e_0)$	$F_0 = F$ is a totally real field
$\mathrm{IV}(e_0, d)$	F is a CM field with maximal totally real subfield F_0.

As the scheme of endomorphisms $Sch/_\eta \to Ab$, $T \to \mathrm{End}_{AbSch/_T}(A_T)$ is representable by a commutative étale group scheme over η, $\mathrm{End}_{\overline{\eta}}(A_{\overline{\eta}}) = \mathrm{End}_{\eta^{sep}}(A_{\eta^{sep}})$. As $\mathrm{End}_{\overline{\eta}}(A_{\overline{\eta}}) \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(A_{\overline{\eta}}))$ $(\ell \neq p)$, the action of $\pi_1(\eta)$ on $\mathrm{End}_{\overline{\eta}}(A_{\overline{\eta}})$ factors through $\pi_1(\eta) \twoheadrightarrow \pi_1(S)$. Now, since $\mathrm{End}_{\overline{\eta}}(A_{\overline{\eta}})$ is a finitely generated $\mathbb{Z}$-module, after possibly replacing again S by a connected étale cover we may assume that $E = \mathrm{End}_\eta(A_\eta)\otimes\mathbb{Q}$. Fix a polarization on A_η. This defines a $\pi_1(S)$-equivariant non-degenerate alternating form $\phi: V\otimes V\to \mathbb{Q}_\ell(1)$ and a Rosati involution $(-)^\dagger$ on E. Since ϕ is G-equivariant and G contains the homotheties (since V is pure of weight $-1\neq 0$), there exists a surjective homomorphism $\mu: G\twoheadrightarrow \mathbb{G}_{m,\mathbb{Q}_\ell}$ such that $\phi(gv,gw)=\mu(g)\phi(v,w)$, $v,w\in V$, $g\in G$. Write $H:=\ker(\mu)\subset G$. Then G° is generated by H° and the homotheties (hence, in particular, $\mathrm{End}_{G^\circ}(V)=\mathrm{End}_{H^\circ}(V)$). So it is enough to show that H° is semisimple or, equivalently, $Z(H^\circ)$ is finite. By Fact 1.3.2

$$E_\ell := E\otimes\mathbb{Q}_\ell = \mathrm{End}_G(V)=\mathrm{End}_{H^\circ}(V)$$

hence $Z(H^\circ)=H^\circ\cap E_\ell^\times\subset Z(E_\ell)^\times = T^F_{\mathbb{Q}_\ell}$, where we regard $E_\ell^\times$ and $Z(E_\ell)^\times$ as algebraic subgroups of GL_V and write $T^F:=\mathrm{Res}_{F|\mathbb{Q}}(\mathbb{G}_{m,F})$. Still by definition of H,

$$\phi(v,w)=\phi(hv,hw)=\phi(v,h^\dagger hw),\ v,w\in V,\ h\in H$$

hence $h^\dagger h=1$, $h\in H$. But in type I, II, III, $(-)^\dagger$ induces the identity on $F=F_0$ hence $Z(H^\circ)\subset \ker((-)^2: T^F_{\mathbb{Q}_\ell}\to T^F_{\mathbb{Q}_\ell})$ is finite. □

Remark. Example 7.3.1 (2) shows that if $A_{\overline{\eta}}$ has simple factors of type IV, its ghost can have a non-supersingular factor.

Appendix A. Abelian schemes with non-trivial ghosts *via* ℓ-adic Fourier transform

The aim of this Appendix is to elucidate the fact - implicitly claimed by Bültel on [B13, p.8], that the (naïve) Fourier transform of a rank-1 elementary $\overline{\mathbb{Q}}_\ell$-sheaf with finite monodromy appears as the direct factor of the $\overline{\mathbb{Q}}_\ell$-Tate module on an abelian scheme. This fact is probably well-known to experts but we could not find a suitable reference for it in the existing literature. The precise statement is Theorem 22 below.

For details about ℓ-adic Fourier transform, see [KL85], [K90], [KiW01]. Fix a non-trivial character $\psi=\mathbb{F}_p\hookrightarrow\overline{\mathbb{Q}}_\ell^\times$. Let $\pi_1(\mathbb{A}^1_k)\twoheadrightarrow k$ denote the quotient of $\pi_1(\mathbb{A}^1_k)$ corresponding to the Lang isogeny $\phi:\mathbb{A}^1_k\to\mathbb{A}^1_k$, $x\mapsto x-x^{|k|}$. The resulting quotient $\pi_1(\mathbb{A}^1_k)\twoheadrightarrow k\overset{\psi\circ\mathrm{tr}_{k|\mathbb{F}_p}}{\to}\overline{\mathbb{Q}}_\ell^\times$ gives rise to a rank-1 $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}_\psi$ on $\mathbb{A}^1_k$. For every k-morphism $f: X\to\mathbb{A}^1_k$ of k-schemes, write $\mathcal{L}_{\psi(f)}:=f^*\mathcal{L}_\psi$. Write $p_1,p_2:\mathbb{A}^1_k\times\mathbb{A}^1_k\to\mathbb{A}^1_k$ for the 1st and 2nd projections respectively and $m:\mathbb{A}^1_k\times\mathbb{A}^1_k\to\mathbb{A}^1_k$, $(x,y)\mapsto xy$ for the product. With these notation, define the (naïve) Fourier transform (in degree 1) on the category $Sh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell)$ of constructible $\overline{\mathbb{Q}}_\ell$-sheaves as:

$$\begin{array}{cccc} F_\psi: & Sh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell) & \to & Sh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell) \\ & \mathcal{F} & \mapsto & R^1p_{2!}(p_1^*\mathcal{F}\otimes\mathcal{L}_{\psi(m)}) \end{array},$$

This naïve definition is only well-behaved on the full subcategory $ESh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell)\subset Sh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell)$ of elementary constructible $\overline{\mathbb{Q}}_\ell$-sheaves namely, those $\mathcal{F}\in Sh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell)$ such that:

(Elem-1) $\mathrm{H}^0_c(\mathbb{A}^1_{\overline{k}},\mathcal{F})=0$;

(Elem-2) For every $a\in\overline{k}$, $\mathrm{H}^2_c(\mathbb{A}^1_{\overline{k}},\mathcal{F}\otimes\mathcal{L}_{\psi(t\mapsto at)})=0$.

With these notation and terminology, one gets

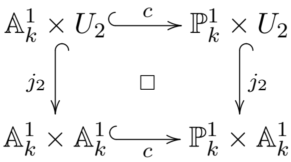
Theorem 22. *Let $\mathcal{F}\in ESh_c(\mathbb{A}^1_k,\overline{\mathbb{Q}}_\ell)$. Assume there exists a non-empty open subscheme $j_1: U_1\hookrightarrow\mathbb{A}^1_k$ such that $j_1^*\mathcal{F}$ has finite monodromy (equivalently, there exists a connected étale cover $\alpha_1: V_1\to U_1$ such that $\alpha_1^*j_1^*\mathcal{F}$ is constant). Then there exists a non-empty open subscheme $j_2: U_2\hookrightarrow\mathbb{A}^1_k$ and an abelian scheme $\pi: A\to U_2$ such that $j_2^*F_\psi(\mathcal{F})$ is a $\overline{\mathbb{Q}}_\ell$-local system and appears as a direct factor of $R^1\pi_*\overline{\mathbb{Q}}_{\ell,A}$.*

Remark 23. The rough strategy, which was suggested to us by Oliver Bültel, is to exhibit $\pi: A\to U_2$ as the connected component of the Picard scheme of a smooth proper relative curve $\overline{Z}_2\to U_2$ built out from $\alpha_1: V_1\to U_1$. In particular, the construction is explicit and one can determine U_2, the dimension of A *etc.*

Proof. The rough strategy, which was suggested to us by Oliver Bültel, is to exhibit $\pi: A\to U_2$ as the connected component of the Picard scheme of a smooth proper relative curve $\overline{Z}_2\to U_2$ built out from $\alpha_1: V_1\to U_1$.

For simplicity, assume $\mathcal{F}$ has $\overline{\mathbb{Q}}_\ell$-rank 1.

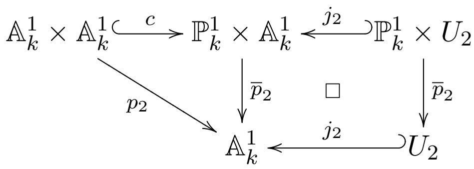
- Let $c : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ denote the canonical compactification and $j : \mathbb{G}_{m,k} \hookrightarrow \mathbb{A}^1_k$ the canonical inclusion. Let $\alpha_1 : X_1 \to \mathbb{A}^1_k$ (resp. $\overline{\alpha}_1 : \overline{X}_1 \to \mathbb{P}^1_k$) denote the normalization of $\mathbb{A}^1_k$ in $V_1 \overset{\alpha_1}{\to} U_1 \overset{j_1}{\hookrightarrow} \mathbb{A}^1_k$ (resp. of $\mathbb{P}^1_k$ in $V_1 \overset{\alpha_1}{\to} U_1 \overset{cj_1}{\hookrightarrow} \mathbb{P}^1_k$).
- As $j_1^*\mathcal{F}$ has finite monodromy, it is in particular pure of weight 0 and as $\mathcal{F}$ is elementary, there exists a non-empty open subscheme $j_2 : U_2 \hookrightarrow \mathbb{A}^1_k$ such that $j_2^* F_\psi(\mathcal{F})$ is a $\overline{\mathbb{Q}}_\ell$-local system which is pure of weight 1 (*e.g.* [K90, Thm. 7.3.8 (5)]). Up to shrinking U_2, one may furthermore assume $j_2 : U_2 \hookrightarrow \mathbb{G}_{mk}$.
- With the notation in the following Cartesian diagram



one has a functor isomorphism $Sh_c(\mathbb{A}^1_k \times \mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell) \to Sh_c(\mathbb{P}_k \times U_2, \overline{\mathbb{Q}}_\ell)$,

$$c_! j_2^* \simeq j_2^* c_!$$

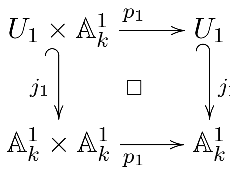
while by proper base-change and with the notation in the following diagram



one has a functor isomorphism $Sh_c(\mathbb{A}^1_k \times \mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell) \to Sh_c(U_2, \overline{\mathbb{Q}}_\ell)$,

$$j_2^* R^1 p_{2!} = j_2^* R^1 \overline{p}_{2*} c_! \simeq R^1 \overline{p}_{2*} j_2^* c_! \simeq R^1 \overline{p}_{2*} c_! j_2^*.$$

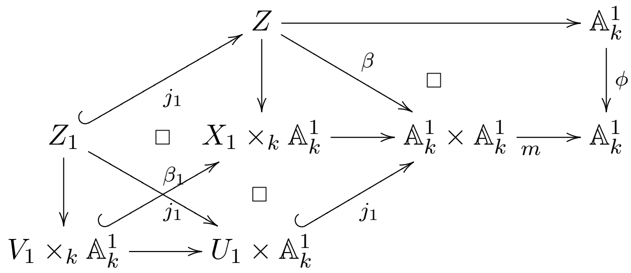
Set $\mathcal{G} := p_1^*\mathcal{F} \otimes \mathcal{L}_{\psi(m)}$. One thus has $j_2^* F_\psi(\mathcal{F}) = j_2^* R^1 p_{2!}\mathcal{G} \simeq R^1 \overline{p}_{2*} c_! j_2^* \mathcal{G}$.
- With the notation in the following diagram



and by (Elem-1), one has $\mathcal{F} \hookrightarrow j_{1*}j_1^*\mathcal{F}$. Hence, by left-exactness of p_1^*, $p_1^*\mathcal{F} \hookrightarrow p_1^* j_{1*} j_1^* \mathcal{F}$. By smooth base change $p_1^* j_{1*} j_1^* \mathcal{F} \simeq j_{1*} p_1^* j_1^* \mathcal{F}$ hence $p_1^*\mathcal{F} \hookrightarrow j_{1*} j_1^* p_1^* \mathcal{F}$. On the other hand, as $\mathcal{L}_{\psi(m)}$ is a local system, $\mathcal{L}_{\psi(m)} \tilde{\to} j_{1*} j_1^* \mathcal{L}_{\psi(m)}$. This shows

$$\mathcal{G} \hookrightarrow j_{1*} j_1^* p_1^* \mathcal{F} \otimes j_{1*} j_1^* \mathcal{L}_{\psi(m)} \hookrightarrow j_{1*}(j_1^* p_1^* \mathcal{F} \otimes j_1^* \mathcal{L}_{\psi(m)}) \tilde{\to} j_{1*} j_1^* \mathcal{G}.$$

With the notation in the following diagram



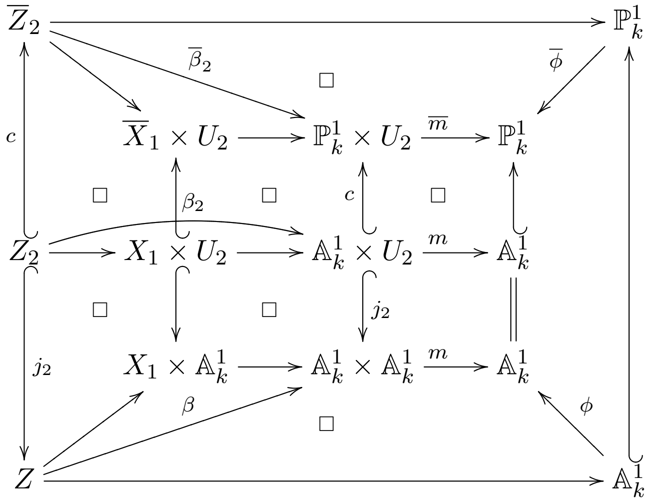
One has, also by assumption, $\beta_1^* j_1^* \mathcal{G} \simeq \overline{\mathbb{Q}}_{\ell, Z_1}$ hence $j_1^*\mathcal{G} \hookrightarrow \beta_{1*}\beta_1^* j_1^* \mathcal{G} \simeq \beta_{1*}\overline{\mathbb{Q}}_{\ell, Z_1}$. Whence, by left-exactness of j_{1*}

$$\mathcal{G} \hookrightarrow j_{1*} j_1^* \mathcal{G} \hookrightarrow j_{1*}\beta_{1*}\overline{\mathbb{Q}}_{\ell, Z_1} \simeq \beta_* j_{1*} \overline{\mathbb{Q}}_{\ell, Z_1} \simeq \beta_* \overline{\mathbb{Q}}_{\ell, Z}.$$

Consider the canonical compactification diagram (recall that $U_2 \hookrightarrow \mathbb{G}_{mk}$)

$$\begin{array}{ccccc} \mathbb{A}^1_k \times \mathbb{A}^1_k & \stackrel{j_2}{\hookleftarrow} & \mathbb{A}^1_k \times U_2 & \stackrel{j_2}{\hookrightarrow} & \mathbb{P}^1_k \times U_2 \\ \downarrow m & & \downarrow & & \downarrow \overline{m} \\ \mathbb{A}^1_k & = & \mathbb{A}^1_k & \stackrel{c}{\hookrightarrow} & \mathbb{P}^1_k \end{array}$$

Let $\overline{\phi} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ denote the normalization of $\mathbb{P}^1_k$ in $\mathbb{A}^1_k \stackrel{\phi}{\to} \mathbb{A}^1_k \stackrel{c}{\hookrightarrow} \mathbb{P}^1_k$ and consider the base-change diagram



By proper (or smooth) base-change $j_2^*\beta_*\overline{\mathbb{Q}}_{\ell,Z} \simeq \beta_{2*}j_2^*\overline{\mathbb{Q}}_{\ell,Z} \simeq \beta_{2*}\overline{\mathbb{Q}}_{\ell,Z_2}$ hence, by (left-) exactness of j_2^*,

$$j_2^*\mathcal{G} \hookrightarrow j_2^*\beta_*\overline{\mathbb{Q}}_{\ell,Z} \simeq \beta_{2*}\overline{\mathbb{Q}}_{\ell,Z_2}.$$

As a result, by left-exactness of $c_!$, $c_!j_2^*\mathcal{G} \hookrightarrow c_!\beta_{2*}\overline{\mathbb{Q}}_{\ell,Z_2}$. Composing with the embedding $c_!\beta_{2*}\overline{\mathbb{Q}}_{\ell,Z_2} \hookrightarrow c_*\beta_{2*}\overline{\mathbb{Q}}_{\ell,Z_2} \simeq \overline{\beta}_{2*}c_*\overline{\mathbb{Q}}_{\ell,Z_2} \simeq \overline{\beta}_{2*}\overline{\mathbb{Q}}_{\ell,\overline{Z}_2}$, one eventually gets

$$c_!j_2^*\mathcal{G} \hookrightarrow \overline{\beta}_{2*}\overline{\mathbb{Q}}_{\ell,\overline{Z}_2}.$$

- Let $\mathcal{Q}$ denote the cokernel of $c_!j_2^*\mathcal{G} \hookrightarrow \overline{\beta}_{2*}\overline{\mathbb{Q}}_{\ell,\overline{Z}_2}$. As $\overline{\beta}_{2*}\overline{\mathbb{Q}}_{\ell,\overline{Z}_2}$ is pure of weight 0, $\mathcal{Q}$ is also pure of weight 0. The cohomological exact sequence for $\overline{p}_{2*}$ yields the exact sequence

$$\overline{p}_{2*}\mathcal{Q} \to j_2^*F_\psi(\mathcal{F}) \simeq R^1\overline{p}_{2*}c_!j_2^*\mathcal{G} \to R^1\overline{p}_{2*}\overline{\beta}_{2*}\overline{\mathbb{Q}}_{\ell,\overline{Z}_2} \simeq R^1(\overline{p}_2\overline{\beta}_2)_*\overline{\mathbb{Q}}_{\ell,\overline{Z}_2},$$

where the last isomorphism is by exactness of $\overline{\beta}_{2*}$. By [D80], $\overline{p}_{2*}\mathcal{Q}$ is mixed of weights ≤ 0 while $R^1(\overline{p}_2\overline{\beta}_2)_*\overline{\mathbb{Q}}_{\ell,\overline{Z}_2}$ is pure of weight 1. As we fixed $j_2 : U_2 \hookrightarrow \mathbb{G}_{m,k}$ in such a way that $j_2^*F_\psi(\mathcal{F})$ is a $\overline{\mathbb{Q}}_\ell$-local system which is pure of weight 1, this shows $j_2^*F_\psi(\mathcal{F}) \hookrightarrow R^1(\overline{p}_2\overline{\beta}_2)_*\overline{\mathbb{Q}}_{\ell,\overline{Z}_2}$ and Theorem 22 holds with $\pi : A := Pic^\circ_{\overline{Z}_2|U_2} \to U_2$ the connected component of the relative Picard scheme of the proper smooth morphism $\overline{p}_2\overline{\beta}_2 : \overline{Z}_2 \to U_2$.

□

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