On uniform boundedness of Brauer groups
(joint with François Charles)

Conference on Algebraic Geometry and Number Theory
on the occasion of Jean-Louis Colliot-Thélène's 70th birthday
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Brauer groups and the Tate conjecture for divisors

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k : finitely generated field of characteristic $p \geqslant 0$
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Lemma (Colliot-Thélène) (Relation with the Tate conjecture for divisors)
$p \neq \ell$: prime. The following assertions are equivalent

- *(1)* $c_1 : \mathrm{Pic}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \twoheadrightarrow \varinjlim_{U \subset \pi_1(k) \text{ open}} H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^U$;
- *(2)* $Br(X_{\overline{k}})^U[\ell^\infty]$ *is finite for every open subgroup* $U \subset \pi_1(k)$.

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$$\sup\{|Br(X_{\bar{s}})^{\pi_1(s)}[\ell^\infty]| \mid s \in S(\leqslant d)\} < +\infty,$$

where $S(\leqslant d) := \{s \in |S| \mid [k(s) : k] \leqslant d\}$

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- Relax the condition on the ℓ-adic Tate conjecture for divisors

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Lemma *Assume $p = 0$ and $s \in S_\ell^{gen}$ for some ℓ. Then one has canonical π_1-equivariant isomorphisms*

$$NS(X_{\overline{\eta}}) \tilde{\to} NS(X_{\overline{s}}), \; Br(X_{\overline{\eta}}) \tilde{\to} Br(X_{\overline{s}})$$

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$+$ $Br(X_{\overline{s}})$, $Br(X_{\overline{\eta}})$ torsion

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S smooth variety over k, $f: X \to S$ smooth, proper

Assume X_s, $s \in |S|$ satisfies the ℓ-adic Tate conjecture for divisors

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Conjecture 3 *$p = 0$, $Lie(\rho_\ell(\pi_1(S_{\overline{k}})))^{ab} = 0$ + ??*
$\Rightarrow (S \backslash S_\ell^{gen})(k) \subset S$ not Zar-dense and $\exists U \subset \rho_\ell(\pi_1(S))$ open s.t.

$$U \subset \rho_\ell(\pi_1(s)),\ s \in S^{gen}(k)$$

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  Comparison of various categories of p-adic coefficients

Bol'shoye Spasibo

Bahut Dhanyavaad

Xie Xie

Merci beaucoup

GRAZIE MILLE

Thank you very much

Vielen Dank

Arigato Gozaimasu

Nagyon Köszönöm

Muchas Gracias