Equidistribution and Uniformity in Families of Curves

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Arithmetic of Shimura varieties over global fields
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My plan

(0) recollections from Ziyang's talk

(1) uniform Bogomolov conjecture

(2) equidistribution in families of abelian varieties

(3) proof of the uniform Bogomolov conjecture (following the approach of Ullmo and Zhang, 1998)

(4) lunch

Disclaimer: Talk does not represent the present state of research!

Except for (2), I will restrict to the case of (relative) curves.

All the results here have been generalized to higher-dimensional subvarieties (joint work with Tangli Ge and Ziyang Gao).

(4) is not actually part of the talk!

(0) recollections from Ziyang's talk

$\mathbb{K}$: a number field
C: a proper, smooth $\mathbb{K}$-algebraic curve of genus $g \geq 2$
q: an arbitrary $\mathbb{K}$-rational base point on C
$\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$: the Jacobian of C
$\iota : \mathrm{Jac}(C) \hookrightarrow \mathbb{P}^N$: an embedding associated with prin. pol.

Naive Weil height:
For $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(\mathbb{Q})$, $x_i \in \mathbb{Z}$, $\gcd(x_0, \ldots, x_N) = 1$,

$$h(p) = \log \max\{|x_0|, \ldots, |x_N|\}.$$

Néron-Tate height:

$$\widehat{h}_{NT}(x) = \lim_{k\to\infty} \frac{h((\iota \circ [2^k])(x))}{4^k} \in [0, \infty) \quad \text{for all } x \in \mathrm{Jac}(C)(\overline{\mathbb{Q}}).$$

$h(C)$: "modular height" of C (i.e., any (Weil) height on $\mathscr{M}_g$)

Notation: $c_i(\cdots)$ is a positive constant depending on data $(\cdots)$.

$\exists$ improvement of Faltings' Theorem, né Mordell conjecture:

Dimitrov–Gao–Habegger (2020):

$$\#C(\mathbb{K}) \leq c_1(g, [K:\mathbb{Q}])^{r+1}$$

where $r = \mathrm{rk}(\mathrm{Jac}(C)(\mathbb{K}))$.

Underlying main (new) ingredient (gap principle):

Dimitrov–Gao–Habegger (2020): Assume that $h(C) \geq c_2(g)$. Then,

$$\#\left\{p \in C(\overline{\mathbb{Q}}) \mid \widehat{h}_{NT}(p-q) \leq c_3(g) \cdot h(C)\right\} \leq c_4(g).$$

(1) The uniform Bogomolov conjecture

To overcome the restriction $h(C) < c_2(g)$, one can use

Theorem (K., (2021)), uniform Bogomolov conjecture:

$$\#\left\{p \in C(\overline{\mathbb{Q}}) \mid \widehat{h}_{NT}(p-q) \leq c_5(g)\right\} \leq c_6(g).$$

(1) David-Philippon (2007): $\mathrm{Jac}(C) = E^g$, E an elliptic curve.
(2) DeMarco-Krieger-Ye (2020): a family of genus 2 curves
(3) Wilms (2021): analogue for function fields, independently
(4) Yuan (2021, upcoming): approach without equidistribution

A standard specialization argument yields the following:

Corollary (K., (2021)), uniform Manin-Mumford conjecture:
C: a smooth algebraic curve over $\mathbb{C}$ of genus $g \geq 2$

$$\#\left\{p \in C(\mathbb{C}) \mid (p-q) \in \mathrm{Tors}(\mathrm{Jac}(C))\right\} \leq c_7(g).$$

Using the uniform Bogomolov conjecture, we obtain the following:

Theorem (Dimitrov–Gao–Habegger, 2020 + K., 2021):

$$\#C(\mathbb{K}) \leq c_8(g)^{r+1},\ r = \mathrm{rk}(\mathrm{Jac}(C)(\mathbb{K})).$$

Recall the "classical" version of the Bogomolov conjecture.

Theorem (Ullmo + Zhang, 1998):
A abelian variety
$C \subseteq A$ an irreducible algebraic curve
Assume that C is not a connected component of an algebraic subgroup of A. Then,

$$\exists c_9(C) > 0 : \{x \in C(\overline{\mathbb{Q}}) \mid \widehat{h}_{NT}(x) < c_9(C)\} < c_{10}(C).$$

Ullmo and Zhang used Arakelov geometry for the proof, namely the **equidistribution theorem of Szpiro–Ullmo–Zhang (1997)**.

(2) equidistribution in families of ab. var.

$\mathbb{K} \subseteq \mathbb{C}$: a number field

S: a $\mathbb{K}$-variety

$(\pi : A \to S)$: an abelian scheme

$(\iota : A \hookrightarrow \mathbb{P}^N)$: a projective immersion

H: the polarization of A induced by ι

For sufficiently small opens $\Delta \subseteq S(\mathbb{C})$, $\exists$ real-analytic map

$$b : A(\mathbb{C})|_\Delta \to (\mathbb{R}/\mathbb{Z})^{2g} \text{ (a \textbf{Betti map})}$$

that restricts to a group isomorphism on each fiber.

H induces a symplectic structure on $\mathbb{R}^{2g}$: $\exists$ symplectic basis x_i, y_i

$\omega := \sum_{i=1}^{g} dx_i \wedge dy_i$ on $(\mathbb{R}/\mathbb{Z})^{2g}$

$\beta|_\Delta := b^*\omega$: **Betti form** on $A(\mathbb{C})|_\Delta$ (sm. semi.-pos. $(1,1)$-form)

$\beta|_\Delta$ does not dependent on $b \rightsquigarrow$ a $(1,1)$-form β on $A(\mathbb{C})$

$\mathbb{K}$, S, $(\pi : A \to S)$: as above

$X \subseteq A$: an irreducible subvariety

X is called **non-degenerate** if $\beta|_{X(\mathbb{C})}^{\wedge \dim(X)} \neq 0$.

Recall from Ziyang's talk:

Proposition: The following two assertions are equivalent:

(1) X is non-degenerate.

(2) For each integer $n \geq 2$, the subvariety X has the **Habegger property** (Crelle, 2013) with respect to $[n] : A \to A$, i.e.

$$\deg([n^k]_*(X)) \geq c_{11} \cdot \deg([n^k] : A \to A)^{\dim(X)} = c_{11} n^{2k \dim(X)}$$

for some $c_{11} = c_{11}(X, [n]) > 0$.

The Habegger property can be also defined in arithmetic dynamics (as in Silverman's textbook, not the parallel summer school...).

Theorem (K., 2021):

Assume that $X \subset A$ satisfies the Habegger property, and define the

$$\textbf{equilibrium measure } \mu = \frac{\beta|_X^{\wedge \dim(X)}}{\int_{X(\mathbb{C})} \beta^{\wedge \dim(X)}}.$$

Then, for each sequence $(x_i)_{i \in \mathbb{N}} \in X(\overline{\mathbb{Q}})^{\mathbb{N}}$ such that

(1) $\widehat{h}_{NT}(x_i) \to 0$ $(i \to \infty)$, and

(2) $(x_i)_{i \in \mathbb{N}}$ converges to the generic point of X (in Zar. top.),

and every $f \in \mathscr{C}_c^0(X(\mathbb{C}))$, we have

$$\frac{1}{\# \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \cdot x_i} \sum_{y \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \cdot x_i} f(y) \xrightarrow[i \to \infty]{} \int_{X(\mathbb{C})} f \mu.$$

(1) Szpiro–Ullmo–Zhang (1997): $\dim(S) = 0$

(2) DeMarco–Mavraki (2020): fibered products of elliptic surfaces

(3) Yuan–Zhang (2021): generalization

Sketch of Proof:

$\overline{X}_k$: Zariski closure of $X_k = \iota([n^k](X))$ in $\mathbb{P}^N_{\mathscr{O}_K}$
$\overline{\mathscr{O}}(1)$: line bundle $\mathscr{O}(1)$ on $\mathbb{P}^N_{\mathscr{O}_K}$ with Fubini-Study metrics at $\infty_{\mathbb{K}}$
$h_{\overline{\mathscr{O}}(1)}$: **Arakelov height** associated with $\overline{\mathscr{O}}(1)$

Idea: Apply Szpiro–Ullmo–Zhang–Yuan equidistribution theorem to the sequence $((\iota \circ [n^k])(x_i))_{i \in \mathbb{N}}$ on $\overline{X}_k$ for $k \gg 1$.

Main problem: Need uniform control on

$$\left| \widehat{h}_{NT}(x_i) - \frac{h_{\overline{\mathscr{O}}(1)}((\iota \circ [n^k])(x_i))}{n^{2k}} \right| \longrightarrow 0 \ (k \to \infty).$$

Zarhin–Manin (1972): $\cdots \leq c_{12}(\pi) \cdot \max\{1, h_S(\pi(x_i))\} \cdot n^{-2k}$

Dimitrov–Gao–Habegger (2020): Since X satisfies the H. property,

$$h_S(\pi(x_i)) \leq c_{13}(X, S) \cdot \max\{1, \widehat{h}_{NT}(x_i)\}$$

for $i \gg 1$.

(3) proof of the uniform Bogomolov conjecture

Let $g \geq 2, n \geq 3$ be integers.

$\mathscr{C}_{g,n} \to \mathscr{M}_{g,n}$: universal family of smooth algebraic curves of genus g with Jacobi structure of level n

$\pi : A \to \mathscr{C}_{g,n}$: family of ab. var. with $\pi^{-1}(q \in C) = \mathrm{Jac}(C)$

$Y \subseteq A$: subvar. such that $Y \cap \pi^{-1}(q \in C) = (C - q) \subseteq \mathrm{Jac}(C)$

Y will be degenerate if $2\dim(Y) > 2g$ (i.e., always).

For each subvar. $X \subseteq A$, define the n-fold fibered products

$$X^{[n]} := X \times_{\mathscr{C}_{g,n}} \cdots \times_{\mathscr{C}_{g,n}} X.$$

For simplicity, we prove a slightly weaker version theorem.
For each $q \in \mathscr{C}_{g,n}(\overline{\mathbb{Q}})$, we write Y_q for the curve $\pi^{-1}(q)$.

Theorem (K., 2021):
There exists a dense open subset $U \subseteq \mathscr{C}_{g,n}$ such that we have

$$\left\{x \in Y_q(\overline{\mathbb{Q}}) \mid \widehat{h}_{NT}(x) \leq c_{14}(g)\right\} \leq c_{15}(g)$$

for every $q \in U(\overline{\mathbb{Q}})$.

Proof: (0) For $m \geq 2$ and $m' \geq 1$, define the "Faltings–Zhang map"

$$\begin{array}{rccc} \Delta_0 : & A^{[mm']} & \longrightarrow & A^{[(m-1)m']}, \\ & (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) & \longmapsto & (\underline{x}_1 - \underline{x}_2, \underline{x}_2 - \underline{x}_3, \ldots, \underline{x}_{m-1} - \underline{x}_m). \end{array}$$

We actually work with

$$\Delta = (\Delta_0, \mathrm{id}_{A^{[m']}}) : A^{[mm']} \times A^{[m']} \longrightarrow A^{[(m-1)m']} \times A^{[m']}.$$

Proof (Cont.): (i) All varieties involved have the Habegger property.
Ziyang's talk: $Y^{[m']}$ is non-degenerate if $m' \gg 1$.
$\Longrightarrow Y_1 := Y^{[mm']} \times Y^{[m']}$ is non-degenerate if $m' \gg 1$.
Similarly,

$$Y_2 := \Delta(Y^{[mm']} \times Y^{[m']}) = \Delta_0(Y^{[mm']}) \times Y^{[m']}$$

is non-degenerate if $m' \gg 1$.

(ii) η: the generic point of $\mathscr{C}_{g,n}$
If $m \gg 1$, then

$$\Delta|_{Y_{1,\eta}} : Y_{1,\eta} \to Y_{2,\eta}$$

is generically of degree 1.

Proof (Cont.): (iii) Write $\beta^{[k]}$ for the Betti form on $\pi^{[k]} : A^{[k]} \to \mathscr{C}_{g,n}$.

μ_1: equilibrium measure on $Y_1 = Y^{[mm']} \times Y^{[m']} = Y^{[(m+1)m']}$
μ_2: equilibrium measure on $Y_2 = \Delta(Y^{[mm']} \times Y^{[m']})$

With $d = \dim(Y_1) = \dim(Y_2)$, we have ($\propto$ = pos. proportional)

$$\mu_1 \propto (\beta^{[(m+1)m']}|_{Y_1})^{\wedge d}$$

and

$$\mu_2 \propto (\beta^{[mm']}|_{Y_2})^{\wedge d}.$$

As $d\Delta_0$ annihilates the diagonal of $Y^{[mm']}$, we have $(\Delta^*\mu_2)_p = 0$ for every point

$$p = (\underline{x}, \ldots, \underline{x}) \in Y_1$$

where $\underline{x} \in Y^{[m'],\mathrm{sm}}(\mathbb{C})$.

Choose a point $\underline{x} \in Y^{[m'],\mathrm{sm}}(\mathbb{C})$ with $\beta_{\underline{x}}^{[m']} \neq 0$. Then, $\mu_{1,p} \neq 0$.

Conclusion: $\mu_1 \neq \Delta^*\mu_2$ on $Y_1(\mathbb{C})$.

Proof (Cont.): (iv): By (iii), there exists some

$$f_1 \in \mathscr{C}_c^0(Y_1(\mathbb{C}))$$

such that

$$\left| \int_{Y_1(\mathbb{C})} f_1 \mu_1 - \int_{Y_1(\mathbb{C})} f_1 \Delta^* \mu_2 \right| > c_{16}.$$

As $\Delta|_{Y_{1,\eta}}$ is generically of degree 1, we can also assume that $f_1 = f_2 \circ \Delta$ for some

$$f_2 \in \mathscr{C}_c^0(Y_2(\mathbb{C})).$$

By equidistribution, $\exists$ a subvariety $Z_1 \subsetneq Y_1$ such that

$$\left| \int_{Y_1(\mathbb{C})} f_1 \mu_1 - \frac{1}{\# \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \cdot \underline{x}} \sum_{z \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \cdot \underline{x}} f_1(z) \right| < c_{16}/2$$

or

$$\widehat{h}_{NT}(x_1) + \cdots + \widehat{h}_{NT}(x_{(m+1)m'}) \geq c_{17}$$

for each $\underline{x} = (x_1, \ldots, x_{(m+1)m'}) \in (Y_1 \setminus Z_1)(\overline{\mathbb{Q}})$.
Similarly, $\exists$ subvariety $Z_2 \subsetneq Y_2$ such that ...

Proof (Cont.): (v): Set

$$\mathscr{S}_q = \{x \in Y_q(\overline{\mathbb{Q}}) \mid \widehat{h}_{NT}(x) < c_{14}(g)\}$$

for each $q \in \mathscr{C}_{g,n}(\overline{\mathbb{Q}})$. We have to prove $\#\mathscr{S}_q \leq c_{15}(g)$!

Claim 1. If $c_{14}(g) \ll 1$, $(\mathscr{S}_q)^{(m+1)m'} \subseteq Z(\overline{\mathbb{Q}})$, $Z := Z_1 \cup \Delta^{-1}(Z_2)$.

Proof of Claim 1.

Assume $\exists \underline{x} \in (\mathscr{S}_q)^{(m+1)m'} \setminus Z(\overline{\mathbb{Q}})$.

Then, (iv) yields

$$\left| \int_{Y_1(\mathbb{C})} f_1 \mu_1 - \int_{Y_2(\mathbb{C})} f_2 \mu_2 \right| < c_{16}.$$

But this means

$$\left| \int_{Y_1(\mathbb{C})} f_1 \mu_1 - \int_{Y_1(\mathbb{C})} f_1 \Delta^* \mu_2 \right| < c_{16}.$$

This contradicts the choice of f_1! Hence, $(\mathscr{S}_q)^{(m+1)m'} \subseteq Z(\overline{\mathbb{Q}})$.

□ Claim 1

Proof (Cont.): (vi) We conclude using an elementary observation.

Claim 2. Let

(1) $C \subset \mathbb{P}^N_{\mathbb{C}}$ a projective curve,

(2) $W \subset (\mathbb{P}^N_{\mathbb{C}})^M$ a Zariski-closed subset such that $C^M \not\subset W$,

(3) $\mathcal{S} \subset C(\mathbb{C})$ a set of points such that $\mathcal{S}^M \subset W(\mathbb{C})$.

Then,

$$\#\mathcal{S} \leq c_{18}(M, \deg_{\mathcal{O}(1)}(C), \deg_{\mathcal{O}(1)}(W)).$$

Proof of Claim 2. Easy. □ Claim 2

There exists a dense, open subset $U \subseteq \mathscr{C}_{g,n}$ such that we can apply Claim 2 with

$$M = (m+1)m',\ C = Y_q,\ W = Z|_q,\ \mathcal{S} = \mathcal{S}_q\ (q \in U).$$

The second alternative of Claim 2 cannot hold because of Claim 1. The first alternative yields a cardinality bound. □ Theorem

Buon appetito!