Introduction

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Conj (Mordell ~1910).

k number field, Y smooth proper curve/k

If $g_Y \geq 2$, $|Y(k)| < +\infty$.

§. Previous approaches.

Faltings (~1983)   Kim (~2005

Faltings: Shafarevich $\underset{\text{Parshin's trick}}{\Longrightarrow}$ Mordell

$S \in |\mathrm{Spec}(\mathcal{O}_k)|$ finite

$$|\mathcal{M}_g(\mathcal{O}_{k,S})| < +\infty$$

$$\Big\Uparrow \text{Torelli}$$

$$|\mathcal{A}_g(\mathcal{O}_{k,S})| < +\infty.$$

Existence of Kodaira-Parshin family.

G finite group with $Z(G) = \{1\}$.

$$\begin{tikzcd} & \mathcal{M}_g \\ \mathcal{H}_{G,Y} \arrow[ur] \arrow[d, "s"] & \end{tikzcd}$$

$$\mathcal{H}_{G,Y} : Sch/k \longrightarrow Sets$$

$$S \longmapsto \left\{ \begin{tikzcd} Z \arrow[r, "f"] & Y \times_k S \arrow[r, "p"] & S \\ S \arrow[u] \arrow[r] & Y \times_k S \setminus s(S) \arrow[u, hook] & \end{tikzcd} \; \text{finite},\ g_Z = ? \right\}$$

Y

étale galois with $\mathrm{Aut}(f) \simeq G$.

- $\mathcal{H}_{G,Y}$ is representable by $\mathcal{H}_{G,Y} = Y' \xrightarrow{\pi} Y$ (finite étale cover)

$S \subset |\mathrm{Spec}(\mathcal{O}_k)|$ finite containing:

- bad reductions of Y
- factor of $|G|$.

$\exists\, k'/k$ finite,

$$\begin{tikzcd} S' \arrow[r, hook] \arrow[d] & \mathrm{Spec}\, \mathcal{O}_{k'} \arrow[d] \\ S \arrow[r, hook] & \mathrm{Spec}\, \mathcal{O}_k \end{tikzcd}$$

$$\begin{tikzcd} & Y'(k') \arrow[dl] \arrow[d, "\pi"] \arrow[r, phantom, "\sim"] & \mathcal{Y}(\mathcal{O}_{k',S'}) \arrow[r] & \mathcal{M}_g(\mathcal{O}_{k',S'}) \\ \pi(Y'(k')) \arrow[r, phantom, "\supset"] & Y(k) & & \end{tikzcd}$$

finite, by Shafarevich

finite to our de Franchi;

only finitely many finite dominant morphisms between 2 sm curves of genus $\geq$ 2.

Core.

$$\begin{tikzcd} \mathcal{A}_g(\mathcal{O}_{k,S}) \arrow[r, "V_\ell"] \arrow[d, "\text{finite to 1}"] & \mathrm{Rep}_{\mathbb{Q}_\ell, 2g}(\pi_1(\mathcal{O}_{k,S})) \\ \mathcal{A}_g(\mathcal{O}_{k,S})/\text{isogeny} \arrow[ur, "\text{[Tate Conj 2]}"] & \end{tikzcd}$$

[semisimple Tate Conj 1.]

[pure of weight 1]

Lemma (Faltings) k field, $S \subset |\mathrm{Spec}\, \mathcal{O}_k|$ finite, $r \geq 1$, $w \in \mathbb{Z}$. Up to iso, $\exists$ only finitely many semisimple

$$\rho : \pi_1(\mathcal{O}_{k,S}) \longrightarrow GL_r(\mathbb{Q}_\ell)$$

pure of weight w.

Chabauty - Kim.

- p-adic cohomology $1 \to \pi_1(\cancel{Y}\, Y_{\bar k}) \to \pi_1(Y) \xrightarrow{} \pi_1(k) \to 1$ s_y
- anabelian

Conj (Grothendieck)

$$\boxed{Y(k) \xrightarrow{\sim} H^1\Big(\pi_1(k), \overset{\pi_1(Y_{\bar k})}{\cancel{\pi_1(X_{\bar k})}}\Big)}$$

$v \in \mathrm{Spec}(\mathcal{O}_{k,S})$

$$\begin{tikzcd}
\mathcal{Y}(\mathcal{O}_{k,S}) \arrow[r, hook] & \mathcal{Y}(\mathcal{O}_v) \arrow[r] & \mathcal{Y}(k(v)) \\
\mathcal{U}_{y_0,v} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \Omega_{y_0,v} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \bar{y}_0 \arrow[u]
\end{tikzcd}$$

~~Cha~~ Chabauty:

$$\Big[0 \to \mathbb{Q}_p \to H^1(Y_{\bar k} \setminus \{y, y_0\}, \mathbb{Q}_p) \to H^1(Y_{\bar k}, \mathbb{Q}_p) \to 0\Big]$$

$$\begin{tikzcd}
\mathcal{U}_v \arrow[r, hook, "j"] \arrow[d] & J_X(k) \arrow[r] \arrow[d] & H^1_f(k, H^1(Y_{\bar k}, \mathbb{Q}_p)^{(1)}) \arrow[d, "loc_v"] \\
\Omega_v \arrow[r, hook] & J_X(k_v) \simeq \mathbb{A}^g(k_v) \arrow[r] & H^1_f(k_v, H^1(Y_{\bar k}, \mathbb{Q}_p))
\end{tikzcd}$$

$\rightsquigarrow$

Chabauty (1941).

$$\boxed{\begin{aligned} &\text{rank } J_X(k) < \dim \mathbb{A}^g = g \\ &\Rightarrow |J_X(k) \cap \Omega_v| < +\infty. \end{aligned}}$$

Kim (2005)

$(H := H^1(Y_{\bar k}, \mathbb{Q}_p)(1))$

~~$H^1_f(k_v, H) = \mathrm{Ker} \to H^1(k_v,$~~

$$H^1_f(k_v, H) = \mathrm{Ker}\big(\underbrace{H^1(k_v, H) \longrightarrow H^1(k_v, H \otimes_{\mathbb{Q}_p} B_{cris})}_{\text{crystalline part}}\big)$$

$= H_f(k_v)$ affine space of dim g/k_v

$$H^1_f(k, H) = \mathrm{Ker}\Big(H^1(k, H) \longrightarrow \bigoplus_{w | p_v} H^1(k_w, H \otimes_{\mathbb{Q}_p} B_{cris}) \;\; \cancel{\oplus} \oplus \bigoplus_{w \nmid p_v} H^1(I_w, H)\Big)$$

crystalline / unramified

$\|$ $\nearrow$ affine space $/\mathbb{Q}_p$

$H_f(\mathbb{Q}_p)$

Facts

$$\begin{tikzcd}
\mathcal{U}_v \arrow[r, hook, "j_v"'] \arrow[d] & H_f(\mathbb{Q}_p) \\
\Omega_v \arrow[r, hook, "j_v"'] & H_{f,v}(k_v)
\end{tikzcd}$$

- j_v is ~~k_v and~~ k_v analytic with dense image

hence $\dim H_f < \dim H_{f,v} \Rightarrow |\mathcal{U}_v| < +\infty$

- rank $J(k) \le \dim H_f$.

NO GAIN!

~~H~~

$H = H^1(Y_{\bar k}, \mathbb{Q}_p) = \pi_1(\overset{Y_{\bar k}}{\cancel{X_{\bar k}}})^{ab} \otimes \mathbb{Q}_p$.

Kim. Replace $\pi_1(X_{\bar k})^{ab} \otimes \mathbb{Q}_p$ by larger (tannakian version) of quotients of ~~π~~

$\pi_1(\cancel{X}\, Y_{\bar k}) \otimes \mathbb{Q}_p$

(p-adic analytic with Z-dense image)

$$\begin{tikzcd}
\mathcal{U}_v \arrow[r, hook] \arrow[d] & \Omega_v \arrow[d] \arrow[r, "alb"] & \pi_{dR}/F^0\pi_{dR} \\
H_f(\mathbb{Q}_p) \arrow[r, hook] & H_{f,v}(k_v) \arrow[ur, "D_{cris}"']
\end{tikzcd}$$

$/\mathbb{Q}_p$ $\quad$ $/k_v$

alg varieties

algebraic

To show: $\dim H_f < \dim \pi_{dR}/F^0_{\pi_{dR}}$.

L-V. By product of both approaches.

$$\begin{tikzcd}
y \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & {[0 \to \mathbb{Q}_p \to H^1(Y_{\bar k} \setminus \{y, y_0\}, \mathbb{Q}_p) \to H^1(Y_{\bar k}, \mathbb{Q}_p) \to 0]} \\
U_v \arrow[r, hook] \arrow[d, "\int"] & H_f(\mathbb{Q}_p) \arrow[d] \\
\Omega_v \arrow[r, hook] & H_{f,v}(k_v)
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] & Y' \arrow[r, "\pi"] & Y
\end{tikzcd}$$

Jacobian of universal curve

KP family

⤳ Replace the H_f's by algebraic varieties (V) easier to control.

s.t. $U_v \xrightarrow{\text{finite-to-1}} V$

§ Strategy of the proof of Mordell by L-V.

1. Specific K-P families.

$$G = \Gamma_q = \mathbb{F}_q \rtimes \mathbb{F}_q^\times = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} ; a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\} \subset GL_2(\mathbb{F}_q).$$

"abelian-by-finite" family

$$\begin{tikzcd}
X_q \arrow[r] & Y'_q \arrow[r] & Y
\end{tikzcd}$$

polarizable abelian scheme

finite étale cover.

satisfying

(KP1) "big monodromy". $\forall y \in Y(\mathbb{C})$,

$$H_B(X_y, \mathbb{Q}) = \bigoplus_{y'/y} \left(H_B(Y_{y'}, \mathbb{Q}), \langle\, , \rangle \right)^{Zar}$$

symplectic pairing

$$\left(\prod_{y'|y} Sp(H_{B,y'}, \langle\, , \rangle) \right) \subset Im\left(\pi_1^{top}(Y(\mathbb{C}))\right)$$

(KP2) $\forall y \in Y(k)$, ~~$\pi^{-1}(y)$~~

$$\pi^{-1}(y) \longleftrightarrow \left\{ \pi_1(Y_{\bar k} \setminus \{y\}) \twoheadrightarrow \Gamma_q \right\} / \Gamma_q\text{-conj}.$$

$$\begin{tikzcd}
\pi^{-1}(y) \arrow[loop below, "\pi_1(k)"] & \pi_1(Y_{\bar k} \setminus \{y\}) \arrow[r, two heads] \arrow[d] & \Gamma_q \\
& \pi_1(Y_{\bar k}) \arrow[ur, "\text{not factoring}"'] &
\end{tikzcd}$$

(KP3) $X_q \to Y'_q$ of relative dim $d_q = \frac{q-1}{2}(2g-1)$.

2. $S \subset |Spec(\mathcal{O}_k)|$ finite set.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_v \arrow[l] & & \\
Y' \arrow[r] \arrow[d, "\pi"'] & \mathcal{Y}' \arrow[d] & \mathcal{Y}'_v \arrow[l] & & \\
Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] & \mathcal{Y}_v \arrow[l] \arrow[d] & & \\
k \arrow[r] & \mathcal{O}_{k,S} & \mathcal{O}_v \arrow[l] & & Y_v \arrow[ull] \arrow[dl] \\
& & & k_v \arrow[ul] & \\
& & k(v) \arrow[uu, dash] & Y_{v,0} \arrow[l] \arrow[uuul] &
\end{tikzcd}$$

$v \in |Spec(\mathcal{O}_{k,S})|$.

$v|p$ unramified.

~~Galois~~

$$Gal(k_v/\mathbb{Q}_p) \simeq Gal(k(v)/\mathbb{F}_p)$$
$$\sigma \longmapsto Frob.$$

Later, q, v will be chosen with specific properties.

3. Rough strategy.

(work more generally for $X \to Y$ smooth proper).

Construct $\mathcal{F}/k$ alg var, $\forall y_0 \in Y(k)$.

$$\begin{tikzcd}
\Omega_v = \Omega_{y_0,v} \arrow[r, "F_v"] & \mathcal{F}(k_v) \\
U_v = U_{y_0,v} \arrow[u, phantom, "\cup"] \arrow[ur] &
\end{tikzcd}$$

p-adic analytic

$$\dim F_v(\Omega_v)^{Zar} > \dim F_v(U_v)^{Zar}$$

Using various coh theories for X_y.

$$\begin{tikzcd}
\text{Betti}/\mathbb{C} \arrow[d, leftrightarrow] & \text{de Rham}/k \arrow[dl] \arrow[dr] & & \mathbb{Q}_p\text{-étale coh}/k \arrow[r] & \mathbb{Q}_p\text{-étale coh}/k_v \arrow[d] \\
\text{de Rham}/\mathbb{C} & & \text{de Rham}/k_v \arrow[rr, leftrightarrow] & & \text{crystalline}/k_v
\end{tikzcd}$$

$$\begin{tikzcd}
 & & y \arrow[d, phantom, "\cap" rotate=90] \arrow[r] & H^1(X_y, \mathbb{Q}_p) \circlearrowleft \pi_1(\mathcal{O}_{k,S}) \arrow[d, phantom, "\Downarrow"] \\
 & y_0 \in Y(k) \arrow[r, squiggly] & U_{y_0,v} \arrow[dll, squiggly] \arrow[d, hook] \arrow[r] & Rep^i_{\mathbb{Q}_p,r}(\pi_1(\mathcal{O}_{k,S\cup\{p\}})) \arrow[d, "loc_v"] \\
H^i_{dR}(X_y/k) \arrow[d, phantom, "\cap"] & y_v\ y_y & \in \Omega_{y_0,v} \arrow[d] \arrow[r] & Rep^i_{\mathbb{Q}_p,r}(\pi_1(k_v)) \arrow[dl, "D_{cris} = (- \otimes_{\mathbb{Q}_p} B_{cris})^{\pi_1(k_v)}"] \\
\text{Filt-Mod}^r_k \arrow[d] & & \text{Filt Mod}^r_{k_v} \arrow[d, phantom, "\Downarrow"] & \varphi \cdot \sigma\text{-linear} \arrow[d, phantom, "\circlearrowright"] \\
\text{Filt-Mod}_\mathbb{C} & & H^1_{dR}(X_{y_v}/k_v) \arrow[r, "\sim"] & H^1_{cris}(X_{y_0,v,0}/W(k/v)) \otimes k_v \\
H^i_{dR}(X_{y_\mathbb{C}}/\mathbb{C}) \arrow[d, leftarrow, "\otimes_\mathbb{Q} \mathbb{C}"] & & & \\
H^i_B(X_{y_\mathbb{C}}, \mathbb{Q}_p) & & &
\end{tikzcd}$$

Can compare de-Rham coh of nearby fibres ~~that~~ thanks to GM connection.

$$\begin{tikzcd}
H^i_{dR}(X_{y,v}/k_v) \arrow[dr, "\sim"] & \\
 & H^i(X_{y,v,0}/W(k(v)) \otimes k_v. \\
H^i_{dR}(X_{y_0,v}/k_v) \arrow[uu, "\nabla_{GM}", "\wr"'] \arrow[ur, "\sim"] &
\end{tikzcd}$$

Similarly on $\mathbb{C}$-side of picture.

point. Gauss-Manin connection is defined$/k$.

$\mathcal{F}$: flag variety defined by Hodge filtration on $H^i_{dR}(X_y/k)$.

$\Omega_v \xrightarrow{F_v} \mathcal{F}(k_v)$ v-adic periods map

$\Omega_\mathbb{C} \xrightarrow{F_\mathbb{C}} \mathcal{F}(\mathbb{C})$ complex analytic periods map.

$\rightsquigarrow$ can compare

$$\dim F_v(\Omega_v)^{Zar} \geq \dim F_\mathbb{C}(\Omega_\mathbb{C})^{Zar}.$$

$$\begin{tikzcd}
F_\mathbb{C} : \Omega_\mathbb{C} \arrow[r] \arrow[d, hook] & \mathcal{F}(\mathbb{C}) \arrow[d, phantom, "\circlearrowleft"] \\
Y(\mathbb{C}) \arrow[d, leftrightarrow] & \pi_1(Y(\mathbb{C})) \\
Y(\mathbb{C})^{an} \arrow[uur] & D
\end{tikzcd}$$

$$\dim F_\mathbb{C}(\Omega_\mathbb{C})^{Zar} \geq \dim G F_\mathbb{C}(y_0)$$

$G = Z$-closure of $Im(\pi_1(Y(\mathbb{C})) \to H^1_B(X_{y_\mathbb{C}}, \mathbb{Q}))$

G is controlled.

To show. $\dim F_v(U_v)^{Zar} < \dim(G F_\mathbb{C}(y_0))$.

4. More detailled strategy for Mordell.

~~$\mathcal{X}_y$~~ ~~X_y~~ $\mathcal{Y}'_v \longrightarrow \mathcal{Y}_v \rightsquigarrow$ ~~G_v~~ $\mathcal{O}_v$ ~~H~~.

$\forall y_v \in \Omega_v$,

$$\mathcal{Y}_{v,y_v} \longleftrightarrow \mathcal{Y}_{v,y_v,0} \longleftrightarrow \mathcal{Y}_{v,y_0,v}.$$

$$\begin{tikzcd}
 & y'_v \mid y_v & \\
y \in U_v & (y',v') \mid (y,v) & (y'_0,v') \mid (y_0,v) \arrow[l, "identification"']
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] H^i_{dR}(X_{y_0,v}/k_v) \arrow[d, equal] & = \bigoplus_{(y_0',v')|(y_0,v)} \Big(H^i_{dR}(X_{y_0',v}/k(y_0')) \arrow[d, equal] & , \langle , \rangle \Big) \\ V_v^{dR} & V^{dR}_{(y_0',v')} & \end{tikzcd}$$

symplectic pairing arising from polarization and compatible with Hodge filtration.

$$E_v = \prod_{(y_0',v')|(y_0,v)} k(y_0')$$

$\curvearrowright$ Hodge filtration $F^2 = 0 \subset F^1 \subset F^0 = V^{dR}_{(y_0',v')}$

$\Uparrow$

Lagrangian (maximal isotropic subspace)

$$\begin{tikzcd} \Omega_v \arrow[r, "F_v"] \arrow[d] & \mathcal{F}(k_v) \\ \mathcal{H}_v(k_v) \arrow[ur, hook] & \end{tikzcd}$$

$$\rightsquigarrow F_v : \Omega_v \longrightarrow \mathcal{F}(k_v)$$

$$\mathcal{H}_v(k_v) \simeq \cancel{\bigoplus} \prod_{(y_0',v')|(y_0,v)} \mathcal{H}_{(y_0',v')}(k_v)$$

$$\mathcal{H}_v = \mathrm{Res}_{E_v/k_v} LGr(V_v^{dR}, \langle , \rangle)$$

$$\mathcal{H}_{(y_0',v_0')} = \mathrm{Res}_{\overset{k(y_0')_{v'}}{\cancel{(k(y_0')_{v'})}}/k_v} LGr(V^{dR}_{(y_0',v')}, \langle , \rangle).$$

a) $Sp(V, \langle , \rangle) \circlearrowright LGr(V, \langle , \rangle)$ transitive

$\overset{(KP1)}{\Longrightarrow} F_v : \Omega_v \longrightarrow \mathcal{H}_v(k_v)$ has Z-dense image.

Rmq. $\dim \mathcal{H}_{(y_0',v')} = [k(y_0')_{v'}, k_v] \cdot \frac{d(d+1)}{2}$

$(2d = \dim V^{dR}_{(y_0',v')})$

b) $\forall$ ~~y~~ $y'|y \in U_{0,v}$, $[k(y'), k] < +\infty$, $k(y')/k$ unramified outside S

$\Rightarrow$ $\underbrace{\text{finitely many possibilities for } k(y')}$ for w in $k(y')/v$.

$\forall (y_0', v')|(y_0, v)$

R = system of rep of iso classes of

$$H^1(X_{\bar{y}}, \mathbb{Q}_p) \circlearrowleft \pi_1(k(y')) \longleftarrow \pi_1(k(y')_{v'})$$

$$\begin{tikzcd}[row sep=small] (y',v') \arrow[d, leftrightarrow] & \big| \, (y,v) \arrow[d, phantom, sloped, "\in"] \\ (y_0',v') & U_{y_0,v} \end{tikzcd}$$

Then $F_{(y_0',v')}(U_v) \subset \bigcup_{\rho \in R_{(y_0',v')}} Z(\varphi) : F_{(y_0',v')}(D_{cris}(\rho))$

$$Z(\varphi) := \left\{ \rho \in \mathrm{Aut}_{k(y_0')_{v'}}(V^{dR}_{(y_0',v')}) ; \begin{array}{l} \rho \in R_{(y_0',v')}, \\ \rho \circ \varphi = \varphi \circ \rho \end{array} \right\}$$

$$\subset Z(\varphi^{[k_v : \mathbb{Q}_p]})$$

Lemma. $\dim Z(\varphi^{[k_v:\mathbb{Q}_p]}) \leq \underline{4d^2} = \dim(GL(V^{dR}_{(y_0',v')}))$

Assume

① $4d^2 < [k(y_0')_{v'} : k_v] \cdot \frac{d(d+1)}{2}$

$\Uparrow$

$[k(y_0')_{v_0'} : k_v] \geq 8.$

$LHS \geq \dim Z(\varphi^{[k_v:\mathbb{Q}_p]}) \sim \dim F_{(y_0',v')}(U_v)^{Zar}$

$RHS \leq \dim F_{(y_0',v')}(\Omega_v)$

② $\forall y \in U_{y_0,v}$, $\exists (y',v')|(y,v)$ s.t.

$$\begin{tikzcd}[row sep=small] (y',v') \arrow[d, leftrightarrow] \\ (y_0',v') \end{tikzcd}$$

- $R_{(y',v')}$ ~~f~~ is finite
- $[k(y')_{v'}, k_v] \geq 8$.

$\Uparrow$ Faltings' lemma

$\pi_1(k(y')) \circlearrowright H^1(X_y, \mathbb{Q}_p)$ is semi-simple

$$Y(k)^{**} := \left\{ y \in Y(k) ;\ \exists (y',v') \mid (y,v) \text{ s.t. } \begin{array}{l} (i)\ [k(y')_{v'} : k_v] \geq 8 \\ (ii)\ H^1(X_{y'}, \mathbb{Q}_p) \circlearrowleft \pi_1(k(y')) \text{ simple} \\ \quad \circlearrowright \text{ ~~simple~~} \qquad (\text{not semisimple}) \\ \quad \pi_1(Y, y') \end{array} \right\}$$

$U^{**}_{y_0,v} = \Omega_v \cap Y(k)^{**}$, $|U^{**}_{y_0,v}| < +\infty$.

<u>Last step.</u> To show: $|U_v \setminus U^{**}_{y_0,v}| < +\infty$.

<u>Rough idea</u>: Assume $y \in Y(k)$ is s.t.

(i) hold but not (ii)

$$\begin{tikzcd} \exists\, 0 \subsetneq W \subsetneq H^1(X_y, \mathbb{Q}_p) \arrow[dd] & & \\ \pi_1(k(y)) \arrow[u, phantom, "\circlearrowright"] & \pi_1(k(y')_{v'}) \arrow[l] & \\ 0 \subsetneq W^{dR} \subsetneq V^{dR}_{(y',v')} & & \end{tikzcd}$$

<u>Fact.</u> If there are enough $(z',v') \mid (y,v)$ satisfies (i), then $\dim W^{dR} \cap F^1 W^{dR}_{(y',v')} \geq \frac{1}{2} \dim W^{dR}$

(provided v is friendly, a bit of p-adic Hodge theory)

+ locus of Lagrangian subspaces $L \subset V^{dR}_{(y',v')}$ s.t. $0 \subsetneq W \subsetneq V^{dR}_{(y',v')}$ φ-invariant with

$\dim L^{\varphi} \cap W \geq \frac{1}{2} \dim W$, $L \cap W$ is a strictly closed subvar of $\mathcal{H}_{(y'_0,v')}$

"Enough" is formalized by size

$\pi_1(k) \circlearrowright M$ finite ~~un~~ unramified at v.

$$\text{size}(M, <u)_v = \frac{|\{u \in M;\ |v \cdot u| < u\}|}{|M|}$$

$Y(k)^{*}_v = \{ y \in Y(k) ;\ \text{size}(\pi^{-1}(y), <8)_v < \frac{1}{d+1}$

ⓐ $\forall$ odd prime q, friendly place v, $y_0 \in Y(k)$

$$|U^{*}_{y_0,v} \setminus U^{*}_{y_0,v} \cap U^{**}_{y_0,v}| < +\infty$$

ⓑ $\exists$ odd prime q, friendly v

s.t. $Y(k) = Y(k)^{*}_v$.

Kodaira-Parshin family

François Charles

§1. Parshin's trick for Mordell.

Goal. Shafarevich $\Rightarrow$ Mordell

(finiteness of curves over $\mathcal{O}_{K,S}$ of genus g)

$K = \#$ field

S finite set of places (containing primes with 2)

$\mathcal{Y}/\mathcal{O}_{K,S}$ smooth proper curve.

$Y = \mathcal{Y}_K$. $Y(K) \xleftarrow{\sim} \mathcal{Y}(\mathcal{O}_{K,S})$ $g(Y) \geq 2$.

Let $P \in Y(K)$

Goal. construct $Y' \xrightarrow{\pi} Y$ finite étale outside P. Y' has a smooth proper model over $\mathcal{O}_{K,S}$ (ramified at P)

Given such a Y', we (almost) recover P because the set of ~~Y~~ dominant $Y' \to Y$ is finite.

Sketch of construction. (Assume $Y(k) \neq \emptyset$)

Let J be $\mathrm{Jac}(Y)$ (good reduction over $\mathcal{O}_{K,S}$)

$$\begin{tikzcd} J \arrow[d, "\varphi = {[2]}"] \\ J \end{tikzcd}$$

$\tilde{Y} = \varphi^{-1}(Y)$ étale over Y, $\tilde{Y}$ geom connected of degree 2^{2g}

$P \in Y(k)$

$D = \varphi^{-1}(P)$ is a divisor of degree 2^{2g}

Up to replacing K by an extension of bounded degree, ~~un r~~ unramified over $\mathcal{O}_{K,S}$,

$$\mathcal{O}(D) \simeq \mathcal{L}^{\otimes 2} \quad (\text{for some } \mathcal{L})$$

(~~f^n inc contains~~ involves finitely many # field)

$\rightsquigarrow \exists\, Y' \to \tilde{Y}$ double cover ramified ~~exceptionally~~ exactly at D.

Then $Y' \to Y$ is ramified exactly above P.

§2. Hurwitz spaces (§7.3)

K field of char 0, ~~X~~ Y/K smooth proper curve.

G a finite group with trivial centre.

We want to consider

$$\left\{ Z \xrightarrow{f} Y \;\middle|\; \begin{array}{l} f \text{ is finite étale galois outside a single} \\ K\text{-point of } Y, \text{ branched over it} \end{array},\ + G \xrightarrow{\sim} \mathrm{Aut}(Z/Y) \right\} / \simeq$$

Hurwitz spaces ~~shot~~ should represent the corresponding functor. $\mathcal{H}_G$

Fact. $\mathcal{H}_G$ exists as a DM stack, with projective coarse moduli space.

[Ref: Romagny, Wewers]

To check whether ~~$\mathcal{H}_G$~~ $\mathcal{H}_G$ is actually a scheme, we need to compute automorphisms of $\{Z \xrightarrow{f} Y, \psi : G \xrightarrow{\sim} \mathrm{Aut}(Z/Y)\}$

Automorphisms of $Z \xrightarrow{f} Y$ are given by $g \in G$, act by conjugation on ψ.

Since $Z(G) = \{1\}$, no automorphisms, so we actually get a proj scheme.

Get * universal curve $Z \longrightarrow \mathcal{H}_G$

* natural map $\mathcal{H}_G \xrightarrow{b} Y$

$(Z \to Y) \longmapsto$ branched point.

$$\begin{tikzcd}
G \arrow[r, phantom, "\circlearrowright"] & Z \arrow[d, "f"] \arrow[dd, bend left, "\pi"] \\
\text{relatively Galois with group } G & Y \times \mathcal{H}_G \arrow[d, "p_2"'] \\
& \mathcal{H}_G
\end{tikzcd}$$

f is branched exactly above the graph of G.

Fact. $\mathcal{H}_G \longrightarrow Y$ is finite étale. $(\mathcal{H} \neq \phi)$

(pass to $\mathbb{C}$, topol argument + Riemann existence thm)

§3 Kodaira-Parshin family

$G = \mathrm{aff}(q) = \mathbb{F}_q \rtimes \mathbb{F}_q^{\times} = \{x \mapsto ax+b;\ a \in \mathbb{F}_q^{\times}, b \in \mathbb{F}_q\}$

q prime, $\# G = q(q-1)$, $Z(G)$ trivial

Set $Y_q' = \mathcal{H}_G$ Hurwitz space

$Y_q' \longrightarrow Y$ étale finite.

Let $C \longrightarrow Y_q'$ be the universal curve

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & Y_q' \\
Y \times Y_q' \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Get

$$\begin{tikzcd}
J(Y) \times Y_q' \arrow[rr] \arrow[dr, "\text{ab sch}"'] & & J(C/Y_q') \arrow[dl, "\text{ab sch}"] \arrow[r, phantom, "\circlearrowleft"] & G \\
& Y_q' & &
\end{tikzcd}$$

$\mathbb{F}_q^{\times}$ stabilizer of a pt

Consider morphisms of abelian schemes / Y_q'

$J(Y) \times Y_q' \longrightarrow J(C/Y_q)^G \longrightarrow J(C/Y_q)^{\mathbb{F}_q^{\times}} \hookrightarrow J(C/Y$

Set $Z = \mathrm{Coker}\left(J(C/Y_q')^G \longrightarrow J(C/Y_q')^{\mathbb{F}_q^{\times}}\right)$

Def. The Kodaira-Parshin family associated to q is the abelian-by-finite family

$Z_q \longrightarrow Y_q' \xrightarrow{\pi} Y$.

Alternative construction

G acts on $\mathbb{F}_q$.

Galois theory: G-sets give an int cover

$$\begin{tikzcd}
{|G|} \arrow[d] & g \arrow[d, mapsto] \\
\mathbb{F}_q & gx
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[rr, "\deg q(q-1)"] \arrow[dr] & & Y \times Y_q' \\
& \tilde{C} \arrow[ur, "\deg q"'] &
\end{tikzcd}$$

Then $Z = \mathrm{Coker}\left(J(Y) \times Y_q' \xrightarrow{\text{torsion kernel}} J(\tilde{C}/Y_q')\right)$

Compute the dim of Z?

Consider a single fiber $\tilde{C}_{y'} \xrightarrow{\text{deg } q.\ \text{ramified above one point } P} Y$ monodromie $= q$-cycle.

RH $\tilde{g} = g(\tilde{C}_{y'})$, $2\tilde{g} - 2 = q(2g-2) + q - 1$.

$\tilde{g} = gq - \frac{1}{2}(q-1)$.

Relative dim of Z is

$$\tilde{g} - g = \frac{(q-1)(2g-1)}{2}.$$

§4. How to use ~~a~~ the KP family.

Fact. (that we will prove)

The KP family has full monodromy

i.e., $Z \longrightarrow Y$, then the Z-closure of $\pi_1(Y,y)$ in $H^1(Z_y, \mathbb{Q})$ contains the product of the symplectic groups (coming from canonical polarization) of the connected components of Z_y.

Then * find a "good" model of $Z_q \rightarrow Y'_q \longrightarrow Y/\mathcal{O}_{K,S}$

* for suitable q

* for a suitable place v of K, $v \notin S$

Two arguments.

5.3. $\left\{ y \in Y(k),\ \mathrm{size}_v(\pi^{-1}(y), \leq 8) < \frac{1}{d+1} \right\}$

(prop of these elements in the $\pi_1(*)$-set $\pi^{-1}(y)$ that have orbit of $\# \leq 8$ under Frob_v)

(rel dim of Z/Y'_q.)

is finite.

5.4. If $y \in Y(k)$, $\mathrm{size}_v(\pi^{-1}(y), \leq 8) < \frac{1}{d+1}$.

Mordell conjecture after Faltings.

Bruno Klingler.

References

- ·[F] Faltings
- [W] Chap5 in Book
- ·[D] Deligne Bourbaki (Tate & Shafarevich)
- [Sz] Szpiro Bourbaki

Thm (4.10) (Mordell conjecture)

K number field, C/K (smooth proj geom connected) curve, $g(C) \geq 2$.

Then $C(K)$ is a finite set.

① Pre-Faltings. Shafarevich $\Rightarrow$ Mordell.

we replace points by geom objects.

Prop (3.7) (Kodaira-Parshin).

K, S finite set of finite places of K, (containing all $v|2$).

C/K curve of $g \geq 1$, good reduction outside S.

$P \in C(K) \rightsquigarrow$ $\exists K_P$ finite extension of K

$C_P/K_P \xrightarrow{\varphi_P} C/K_P$ finite s.t.

(0) $[K_P : K]$, $g(C_P)$ are bounded in terms of g only

($[K_P : K] = 2^{2g}$, $g(C_P) = 2^{2g-1}(4g-3)+1$

(1) K_P unramified outside S

(2) C_P has good reduction outside places of K_P above S.

(3) φ_P ramified exactly at P.

$\rightsquigarrow$ Shafarevich type problem.

B regular scheme ($\mathrm{Spec}\,\mathcal{O} - S$)

F a type of alg var (e.g. F = curve of genus g)

$Ш(B, F)$ = set of B-iso classes of $f: X \to B$ with fibers of ~~a~~ type ~~of~~ F.

In our case,

$Ш(\mathrm{Spec}\,\mathcal{O} - S$, curves of genus g)

Exp. $Ш(\mathrm{Spec}\,\mathcal{O} - S$, n points) is finite

extensions K'/K of deg n, unramified outside S.

Thm (4.4) (Shafarevich conj 62, Faltings)

$Ш(\mathrm{Spec}\,\mathcal{O} - S$, curves of genus ≥ 2) is finite

$\Leftrightarrow$ $\exists$ only finitely many iso classes of curves of ~~gen~~ genus g, ($g \geq 2$) over K with good reduction outside S.

Prop (Parshin)

Thm (4.4) $\Rightarrow$ Thm (4.10)

Proof. By Exp & (0), (1), the collection of fields K_P appearing in the K-P construction is finite.

K'/K finite containing all K_P.

- C of genus g with good reduction outside S.

WLOG, S contains all $v|2$.

Kodaira-Parshin

$$C(K) \xrightarrow{\text{with finite fiber}} Ш(\mathrm{Spec}\,\mathcal{O}'-S', g')$$

$$P \longmapsto C_P.$$

Shafarevech conj: RHS is finite. □

§2. Passing to Abelian varieties.

Jacobians.

$C/\mathbb{C}$ $J(C) = \mathrm{Pic}^0(C)$

$J(C)$ = connected component of identity of

$$\mathrm{Pic}\,C = H^1(C, \mathcal{O}_C^\times) = H^1(C, \mathbb{G}_m)$$

In family, $f: X \to B$ (f qqp)

$P_{X/B}: (Sch/B)^\circ \longrightarrow Ab$

$T \longmapsto \mathrm{Pic}(X_T) = H^1_{Z}(X \times_B T, \mathbb{G}_m)$

not representable; not a sheaf for Z-topo.

Take (U_α) Zariski cover of B, L/X line bundle,

$L|_{X \times_B U_\alpha}$ trivial, L not trivial in general.

If $\cdot\ f_*(\mathcal{O}_{X \times_B T}) = \mathcal{O}_T$, $\forall\, T \in Sch/B$

$\cdot\ f$ has a section $\varepsilon: B \to X$

(e.g. if $B = \mathrm{Spec}\,K$, $X(K) \neq \phi$).

$\forall\, T \in Sch/B$,

$$\mathrm{Pic}_{X/B}(T) = \frac{\mathrm{Pic}(X_T)}{pr_T^* \mathrm{Pic}(T)}$$

(is a sheaf)

If $B = \mathrm{Spec}\,\mathcal{O}$ or K, $f: X \to B$ is a proj curve

relative Picard scheme $\mathrm{Pic}_{X/B} = R^1 f_* \mathcal{O}_X^\times$

relative Jacobson $J(X/B)$ gp scheme$/B$.

Lemma. C a curve$/K$, B scheme of dim 1 with function field K.
If $P \in B$, C has good reduction at P then $J(C)$ has good reduction at P.

Jacobian embedding. C/K, $P \in C(K)$.

$j: C \longrightarrow \mathrm{Jac}(C)$ s.t.

$$j(K): C(K) \longrightarrow \mathrm{Jac}(C)(K)$$

$$Q \longmapsto Q - P.$$

Polarization.

Def. A Ab var$/K$. A polarization of A is an ample line bundle $\mathcal{L}/A$ over K up to alg equiv.

$$\frac{c_1(\mathcal{L})^g}{g!} =: \chi(\mathcal{L})$$

A principle polarisation is one $\mathcal{L}$ of $\chi = 1$.

Prop. C/K, $P \in C(K)$

$$\Theta = \underbrace{j(C) + \cdots + j(C)}_{g-1}$$ is an ample divisor on $J(C)$ of $\chi = 1$.

Thm (3.1). $K = \#$ field, S, $g, n > 0$

The set of iso classes of Ab var$/K$ of dim g with polarisation of deg n and good reduction outside S is finite

($\Leftrightarrow$ $\text{Ш}\left(\operatorname{Spec}\mathcal{O}_K - S, \begin{array}{l}\text{Ab var of dim } g \text{ with}\\ \text{polarization of degree } n\end{array}\right)$ is finite).

Thm(3.1) $\Rightarrow$ Thm(4.4)

§3. Height.

Arakelov version.

K # field, L rk 1 v.s$/K$.

We suppose L endowed with family of norms

$\forall v$ place of K, $\|\lambda \ell\|_v = \|\lambda\|_v \|\ell\|$

$\lambda \in K$, $\ell \in L$

$\forall \ell \in L \setminus \{0\}$, $\|\ell\|_v = 1$ for almost all v.

Product formula $\Rightarrow$ $\prod_v \|\ell\|_v$ indep of ℓ.

Def. $\mathcal{L} = (L, (\|\cdot\|_\sigma))$

$\deg \mathcal{L} := -\log \prod_v \|\ell\|_v$.

~~$\mathcal{O} = \mathcal{O}_K$ $L_\mathcal{O}$ an $\mathcal{O}$ - form~~

$\mathcal{O} = \mathcal{O}_K$. $L_\mathcal{O}$ an $\mathcal{O}$ - form of L

$L_\mathcal{O}$ endows L with $\|\cdot\|_v$, $\forall v$ finite

$L_v \xrightarrow{\sim} K_v$ and transport $\|\cdot\|_v$ on K_v

$L_{\mathcal{O}_v} \longrightarrow \mathcal{O}_{K_v}$ $\quad \#(L_\mathcal{O}/\mathcal{O}_\ell) = \prod_{v \text{ finite}} \|\ell\|_v^{-1}$.

- If we have for each $\sigma : K \longrightarrow \mathbb{C}$,

$\langle \, , \rangle : L \otimes_\sigma \mathbb{C} \otimes_\mathbb{C} \overline{L \otimes_\sigma \mathbb{C}} \longrightarrow K \otimes_\sigma \mathbb{C}$

hermitian

$\|\ell\|_\sigma = \langle \ell, \ell \rangle_\sigma^{\frac{\varepsilon_\sigma}{2}}$ where $\varepsilon_\sigma = 1$ $K_\sigma = \mathbb{R}$
$\varepsilon_\sigma = 2$ $K_\sigma = \mathbb{C}$.

$$\deg \mathcal{L} = \log \frac{\#(L/\mathcal{O}\ell)}{\prod_\sigma \|\ell\|_\sigma}.$$

$X/\mathcal{O} \hookrightarrow \mathbb{P}^N_\mathcal{O}$

$f : X \longrightarrow \operatorname{Spec}\mathcal{O}$, $\mathcal{L}$

$P \in X(K) \longleftrightarrow \operatorname{Spec}\mathcal{O} \xrightarrow{s_P} X \overset{i}{\hookrightarrow} \mathbb{P}^N_\mathcal{O}$

$h(P) = \deg\left(~~s_P \circ f~~ (i \circ s_P)^* \mathcal{O}_{\mathbb{P}^N}(1) \right)$,

The iso classes of $\mathcal{L}$ is a group $\operatorname{Pic}_c(\mathcal{O})$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}^{r_1+r_2}/\log \mathcal{O}^\times \arrow[r] & \operatorname{Pic}_c \mathcal{O} \arrow[r] \arrow[d, "\deg"] & cl\,\mathcal{O} \arrow[r] & 0. \\ & & \mathbb{R} & & \end{tikzcd}$$

additive.

- Euler - Poincaré

$$\chi'(\mathcal{L}) = -\log \frac{2^r \operatorname{vol}(\mathbb{R}^n/L)}{2^{r_1} \pi^{r_2}}$$

where $\mathbb{R}^n = \bigoplus L \otimes_\sigma K_\sigma$.

Exp. $\chi'(\mathcal{O}) = r_2 \log \frac{\pi}{2} - \frac{1}{2} \log D$.

- $H^0(\mathcal{L}) = \{ s \in L \,/\, \|s\|_\sigma \leq 1, \forall \sigma \}$

$H^0(\mathcal{O}) = \{0\} \cup \mu(\mathcal{O})$

$$\chi'(\mathcal{L}) = \log \lim_{t \to +\infty} \left(2^{-n} \# H^0(\mathcal{L}_t) / t^n \right)$$

$\mathcal{L}_t = \mathcal{L}$ metric at ∞ is multiplied by t^{-1}.

Lemma. (0) $\chi'(\mathcal{L}) = \deg\mathcal{L} + \chi'(\mathcal{O})$

(1) $\deg\mathcal{L} < 0 \Rightarrow H^0(\mathcal{L}) = 0$.

(2) $\deg\mathcal{L} = 0,\ H^0(\mathcal{L}) \neq 0 \Rightarrow \mathcal{L} = \mathcal{O}$

(3) (Minkowski) If $\deg\mathcal{L} \geq -\chi'(\mathcal{O})$ then $H^0(\mathcal{L}) \neq 0$.

- A/K Abelian var of dim g

$\rightsquigarrow \omega(A) := \Lambda^g(\mathrm{Lie}\,A)^\vee$ rk 1 v.s/K

Recall A/K admits a unique Néron model

$A_\mathcal{O} \longrightarrow \mathrm{Spec}\,\mathcal{O}$ with universal property

$\forall R/\mathcal{O}$ sm, $A_\mathcal{O}(R) \xrightarrow{\sim} A(R_K)$

$\omega(A)_\mathcal{O} = \Lambda^g(\mathrm{Lie}\,A_\mathcal{O})^\vee$

is an $\mathcal{O}$-form of $\omega(A)$

$\Rightarrow$ get $\|\cdot\|_v$ on $\omega(A)$ for v finite

- $\sigma : K \hookrightarrow \mathbb{C}$

$\omega(A)_\sigma : \omega(A) \otimes_\sigma \mathbb{C} = H^{g,0}(\sigma A)$

$$\langle\alpha, \alpha\rangle_\sigma := \frac{1}{(2\pi)^g}\int \alpha\wedge\bar{\alpha}$$

$$h(A) := \frac{\deg\omega(A)}{[K:\mathbb{Q}]}.$$

Exp. $K = \mathbb{Q}$, α one of the generators of $\omega(A)_\mathbb{Z}$

$$h(A) = -\frac{1}{2}\log\frac{1}{(2\pi)^g}\int_{A(\mathbb{C})}\alpha\wedge\bar{\alpha}.$$

Generalization.

- $G \underset{e}{\overset{a}{\rightleftarrows}} \mathrm{Spec}\,\mathcal{O}$ flat gp scheme of $\underset{\text{rel}}{\wedge}$ dim g.

- $\omega(G) := e^*H^{-g}(\underbrace{Ra^!\mathcal{O}_{\mathrm{Spec}\,\mathcal{O}}}_{\text{relative dualizing sheaf}})$

is an invertible sheaf on $\mathrm{Spec}\,\mathcal{O}$

In fact $Ra^!\mathcal{O}_{\mathrm{Spec}\,\mathcal{O}} = a^*\omega(G)[g]$.

If the generic fiber of G is proper over K, (i.e., finite exts of a finite gp by an Ab var) integration on $(\sigma G)(\mathbb{C})$ gives a metrized structure on $\omega(G)$ and so $\deg\omega(G)$ is well defined

§4. Faltings proof of Thm 3.1

Step 1.

Thm (3.8) (Prop 1.3 Deligne).

K # field, $g \geq 1$, $n \geq 1$, $h \geq 0$. There are only finitely many iso classes of polarised $(A, \theta)/K$ of dim g, degree of pola n, $h(A) \leq h$.

Step 2. Understanding the height in isogeny classes. (indep of Step 1)

Def. • A flat commutative gp scheme $/\mathrm{Spec}\,\mathcal{O}_K$ is <u>semi-abelian</u> if its fibres are ext of ab var by tori

- A/K is of <u>semi-stable reduction</u> if its canonical Néron model $A^0_\mathcal{O}$ is semi-abelian.

($\Leftrightarrow$ A/K is the generic fiber of some semi-ab sch$/\mathrm{Spec}$

Thm. (Grothendieck).

If for some $n \geq 3$, the n-div points are in $A(K)$, then A_K has semi-stable reduction.

Thm 3.5 (= Thm 2.4 Deligne)

A/K ab with semi-stable reduction.

There exists a finite set E of ~~primes~~ prime numbers s.t. if $\varphi : A'/_K \longrightarrow A/_K$ is an isogeny, with $(\deg \varphi, E) = 1$, then

$$h(A') = h(A).$$

- If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ exact sequence of commutative gp sch $/\operatorname{Spec} \mathcal{O}$

$$\omega(A) \simeq \omega(A') \otimes \omega(A'')$$

(metrized world)

Hence $\deg \omega(A) = \deg \omega(A') + \deg \omega(A')$

Let $\varphi : A \longrightarrow B$ iso with kernel H

$H_{\mathcal{O}}$ = submatic classes of H in $A_{\mathcal{O}}^{0}$

If A semi-stable reduction, then

$$0 \to H_{\mathcal{O}} \to A_{\mathcal{O}}^{0} \to B_{\mathcal{O}}^{0} \to 0$$

is exact.

$$\Rightarrow \cancel{h(A) =} \; h(B) = h(A) - \frac{1}{[K:\mathbb{Q}]} \deg \omega(H_{\mathcal{O}})$$

The ~~existens~~ existence of E s.t. for $(\deg \varphi, E) = 1$, $\deg \omega(H_{\mathcal{O}}) = 0$ follows from a result of Raynaud on finite gp sch.

- Similarly, working with l-divisible groups rather than finite $\mathbb{Z}/l\mathbb{Z}$-modules

$$A_{l^\infty}(\overline{K}) \subset A(\overline{K})$$
$$\cup$$

W a l-divisible subgp of $A_{l^\infty}(\overline{K})$ stable by $\mathrm{Gal}(\overline{K}/K)$.

$W \Longleftrightarrow T_l(W) \subset T_l(A)$

with $T_l(A)/T_l(W)$ has no l-torsion

W_{l^n} image of W in ~~A_{l^∞}~~ A_{l^n}.

$B(n) := A/W_{l^n}$ $\quad \varphi_n : A \to B(n)$

Thm (2.6 Deligne / 2.7 Schappacher).

For $n \gg 0$, $h(B(n))$ is indep of n.

Step 3. Tate conj $\Longleftarrow$ Thm 2.7 + Thm 3.8.

Thm (Tate's conj)

1). The action of G_K on $V_l(A) = T_l(A) \otimes \mathbb{Q}_l$ is semisimple

2) $\mathrm{Hom}_K(A, B) \otimes \cancel{\mathbb{Z}} \, \mathbb{Z}_l \xrightarrow{\sim} \mathrm{Hom}_{G_K}(T_l(A), T_l(B))$

Cor. TFAE

i) $A \underset{\text{isogeny}}{\simeq} B \, / K$

ii) $V_l(A) \underset{G_K}{\simeq} V_l(B)$.

Idea. For A PPAV,

W totally isotropic

$\Longrightarrow B(n)$ is still PPAV.

$2.7 + 3.8 \Rightarrow n >> 0$, $B(n)$ are K-iso

$v_n : B(n) \xrightarrow{\sim} B(n_0)$

$$\begin{tikzcd}
A \arrow[r, "\varphi_n"] \arrow[dr, "\ell^n"'] & B(n) \arrow[d, "\lambda_n"] \\
& A
\end{tikzcd}$$

$u_n = \lambda_n \circ v_n \circ \lambda_n^{-1}$

$\in \operatorname{End}_K(A) \otimes \mathbb{Q}_\ell$

$u = \lim_n u_n \in \operatorname{End}_K(A) \otimes \mathbb{Q}_\ell$

Then $T_\ell(n)(T_\ell(A) \otimes \mathbb{Q}_\ell) = W$.

Thm (2.7 Deligne).

$\hat{\mathbb{Z}}[G_K] \xrightarrow{\rho} \operatorname{End}_{\hat{\mathbb{Z}}}(T_\ell(A))$

Then $\rho(\hat{\mathbb{Z}}[G_K])$ ~~has~~ is of finite index in the commutant of End(A).

Cor (2.8 Deligne) K # field.

An isogeny class of Ab var contains only finitely many iso classes.

Proof. A' isog $A \Leftrightarrow \forall \ell$, $T_\ell(A') \subset V_\ell(A)$ lattice stable by G_K

$T_\ell(A') = T_\ell(A)$ for almost all ℓ.

A' isom to $A \Leftrightarrow \exists b \in \operatorname{End}(A) \otimes \mathbb{Q}$ s.t. $\forall \ell$, $bT_\ell(A) = T_\ell(A')$

As $\mathbb{Q}_p[G_K]$ has image in $\operatorname{End}(V_\ell(A))$ equal to the commutant of $\operatorname{End} A \otimes \mathbb{Q}_\ell$, it follows that $(\operatorname{End} A \otimes \mathbb{Q}_\ell)^*$ has only finitely many orbits in the space of lattices $T_\ell(A') \subset V_\ell(A)$ stable under G_K

$G/\mathbb{Q}$:= multiplicative group of the $\mathbb{Q}$-alg $\operatorname{End} A \otimes \mathbb{Q}$.

$G(\mathbb{A}^f)$ acts on $(T_\ell(A'))_\ell$

- By what we proved: only finitely many orbits
- For ~~a set~~ each orbit, :

iso classes $\longleftrightarrow \underbrace{G(\mathbb{Q}) \backslash G(\mathbb{A}^f) / K}_{\text{finite set}}$

K a compact open in $G(\mathbb{A}^f)$

Step 4. (finiteness for isogeny classes)

Thm 2.8 [W] $K, S, g \geq 1$.

$\exists$ only finitely many isogeny classes of Ab var$/K$ of dim g and with good reduction.

"Proof" $A' \underset{\text{isog}}{\sim} A \overset{\text{Tate}}{\Longleftrightarrow} V_\ell(A') \underset{G_K}{\simeq} V_\ell(A)$

Lemma 2.7 (2.3 L-V)

Fix integers $w, d \geq 0$, K, S as above. Up to conjugation, $\exists$ only finitely many ss rep $\rho : G_K \to GL_d(\mathbb{Q}_\ell)$ s.t.

(a) ρ is unramified outside S

(b) ρ is pure $\boxed{\text{of weight } -w}$ integral

(i.e., $\forall v \notin S$, the eigenvalues of Fr are alg integers ~~numbers~~, all of whose conjugates ~~has~~ have $\mathbb{C}$-absolute value $(Nv)^{\frac{w}{2}}$.

Come from

Thm (3.1 Deligne) K, S, l, d, w fixed

- $\exists$ a finite set T of places of K, $T \cap S = \phi$

  $\rho : G_K \longrightarrow GL_d(\mathbb{Q}_l)$ semisimple

  unramified outside S is uniquely determined

  by the traces $\mathrm{Tr}\,\rho(F_v)$, $v \in T$.

Periods.

Salim Tayou

General setting

K # field, Y/K smooth var

$\pi: X \to Y$ smooth proj morphism.

$\iota: K \hookrightarrow \mathbb{C}$, S set of finite places.

$\pi: \mathcal{X} \to \mathcal{Y}$ over $\mathcal{O} = \mathcal{O}_S$ smooth proj.

• $\mathcal{H}^q = R^q\pi_*\Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$, $q \geq 0$, $\mathcal{O}_{\mathcal{Y}}$ locally free.

$\nabla: \mathcal{H}^q \longrightarrow \mathcal{H}^q \otimes \Omega^1_{\mathcal{Y}/\mathcal{O}}$ integrable.

Fix $y_0 \in \mathcal{Y}(\mathcal{O})$, $U = \{y \in \mathcal{Y}(\mathcal{O});\ y \equiv y_0 \bmod v\}$

v place unramified over p

§1. Complex period map

Let $X^{an} \to Y^{an}$, Y^{an} connected.

$\mathcal{H}^q$ $\mathcal{O}_{Y^{an}}$ locally free

$\Updownarrow$

local system on Y^{an} $R^q\pi_*\underline{\mathbb{C}}_{X^{an}} = H^q$.

H^q ~~flat~~ sheaf of ~~co~~ flat local sections of $\mathcal{H}^q$.

$\Leftrightarrow$ Representations

$$\mu: \pi_1(Y^{an}, y_0) \longrightarrow GL(\underline{H^q_{Sing}(X_{y_0}, \mathbb{C})})$$
$$H^q_{Sing}(X_{y_0}, \mathbb{C}) \simeq H^q_{dR}(X_{y_0}, \mathbb{C})$$

$$\underset{\nabla}{\mathcal{H}^q} \simeq (H^q \otimes \underset{1\otimes d}{\mathcal{O}_{Y^{an}}})$$

Let $y_0 \in \Omega_{\mathbb{C}} \subseteq Y^{an}$ contractible neighborhood.

$\rightsquigarrow H^q|_{\Omega^{an}_{\mathbb{C}}} \simeq H^q(X_{y_0}, \mathbb{C})_{\Omega_{\mathbb{C}}}$

( ) local system.

Let $y \in \Omega_{\mathbb{C}}$.

$\mathcal{X}(y_0, y)$.

$$\begin{tikzcd}
H^q_{dR}(X_{y_0}, \mathbb{C}) \arrow[rr] \arrow[dr, "\simeq"'] & & H^q_{dR}(X_y, \mathbb{C}) \arrow[dl, "\simeq"] \\
& H^q_{Sing}(X_{y_0}, \mathbb{C}) &
\end{tikzcd}$$

$(s_0 \longmapsto (\nabla s \text{ section}, \nabla s = 0) \longmapsto (s(y))$

(s)

Filtration. For $i \geq 0$, $\mathcal{F}^i\mathcal{H}^q = \operatorname{Im}(R^q\pi_*\Omega^{\bullet \geq i}_{X/Y} \to R^q\pi_*\Omega^\bullet_{X/Y})$

$\mathcal{O}_{Y^{an}}$ - locally free.

At each ~~fibre~~ point $y \in Y^{an}$, $F^i\mathcal{H}^q$ induces a Hodge filtration on $H^q(X^{an}_y, \mathbb{C})$ with constant Hodge numbers.

Let $\mathcal{F}l/K$ be the flag variety $/K$ parametrizing

$$V_0 \supseteq V_1 \supseteq \cdots \supseteq V_q,\quad V_0 = H^q_{dR}(X_{y_0}, K)$$
$$\dim_K V_i = \dim_K F^iH^q_{dR}(X_{y_0}, K)$$

Local period map.

$$\Phi_{\mathbb{C}}: \Omega_{\mathbb{C}} \longrightarrow \mathcal{F}l^{an}(\mathbb{C})$$
$$y \longmapsto (F^0\mathcal{H}^q_y, \ldots, F^q\mathcal{H}^q_y)$$

① $\Phi_{\mathbb{C}}$ is holomorphic (Griffiths).

$$\begin{tikzcd}
\tilde{Y}^{an} \arrow[r] \arrow[d] & \mathcal{F}l^{an}(\mathbb{C}) \\
\Omega_{\mathbb{C}} \subseteq Y^{an} \arrow[ur, "\Phi_{\mathbb{C}}"] &
\end{tikzcd}$$

universal cover

$$\mu: \pi_1(Y^{an}, y_0) \longrightarrow GL(V_{\mathbb{C}})$$
$$\Gamma := \overline{\mu(\pi_1(Y^{an}, y_0))}^{Zar}$$

Prop. $\Gamma \Phi_{\mathbb{C}}(y_0) \subseteq \overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}^{Zar} \subseteq Fl^{an}(\mathbb{C})$.

$\rightsquigarrow \pi_1(Y^{an}, y_0)\cdot y_0 \subseteq \tilde{Y}^{an}$

$$\begin{tikzcd}
\Rightarrow \pi_1(Y^{an}, y_0)y_0 \arrow[dr, bend right] \arrow[rr, phantom, "\subseteq"] & & \overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}^{Zar} \\
& \Gamma\cdot y_0 \arrow[ur] &
\end{tikzcd}$$

§2. p-adic periods

$\underset{\substack{\downarrow \\ p}}{v}$ unramified place of K

$\mathcal{O}_{(v)} \hookrightarrow \mathcal{O}_v$, $K_v = \mathrm{Frac}\,\mathcal{O}_v$, k residue field.

Fix $y_0 \in \mathcal{Y}(\mathcal{O}_{(v)})$, $\bar{y}_0 \in \mathcal{Y}(k)$

2.1. Cohomology of nearby fibres.

If $y \in \mathcal{Y}(\mathcal{O}_v)$, $y \equiv y_0 \mod v$

$$X(y_0, y):\ H^q_{dR}(X_{y_0}, K_v) \longrightarrow H^q_{dR}(X_y, K_v)$$

$$s_0 \longmapsto \left| \begin{array}{l} s \text{ section of } \mathcal{H}^q_{(v)} \\ \nabla(s)=0 \end{array} \right. \longmapsto s(y)$$

Dwork $p>2$
Berthelot $p=2$

$$s_0 \longmapsto \sum_{\substack{\underline{m}=(m_1,\dots,m_r) \\ |z_i|_v<|p|_v^{\frac{1}{p-1}}}} \frac{\underline{z}^{\underline{m}}}{\underline{m}!}\left(\nabla\left(\frac{\partial}{\partial \underline{z}}\right)^{\underline{m}} s_0\right)(y)$$

$\downarrow$ in some sense.

2.2. Crystalline nature of CM.

Z/k scheme.

* ~~Co~~ Crystalline site $/\mathcal{O}_v$

Objects $(U \hookrightarrow T, \delta)$

$U \overset{o}{\hookrightarrow} Z$

$U \hookrightarrow T$ closed by ideal J.

$\delta = (\delta_n)_n$ ~~pove~~ power divided structure

$\delta_n\ J \longrightarrow \mathcal{O}_J$ $\left(\text{“}\delta_n(x) = \frac{x^n}{n!}\text{”}\right)$

Morphisms. $(U \hookrightarrow T, \delta) \longrightarrow (U' \hookrightarrow T', \delta')$

is
$$\begin{tikzcd}
U \arrow[r, hook, "o"] \arrow[d] & U' \arrow[d] \\
T \arrow[r] & T'
\end{tikzcd}$$
with compatible condition ~~bet~~ between δ, δ'.

Topology A covering $\{(U_i \hookrightarrow T_i, \delta_i)\} \to (U \hookrightarrow T, \delta)$

s.t. $T_i \overset{o}{\hookrightarrow} T$, $T = \bigcup_i T_i$.

Denote $(Z/\mathcal{O}_v)_{cris}$ crystalline site~~s~~ of $Z/\mathcal{O}_v$.

= sheaves of the crystalline site.

Exp. Structural crystalline sheaf $\mathcal{O}_{Z/\mathcal{O}_v, cris}$

$\mathcal{O}_{Z/\mathcal{O}_v, cris}(U \hookrightarrow T, \delta) = \mathcal{O}_T(T)$

Functoriality $g: Z' \to Z/k$,

$\rightsquigarrow g_*: (Z'/\mathcal{O}_v)_{cris} \longrightarrow (Z/\mathcal{O}_v)_{cris}$.

Crystals F is a sheaf on $Cris(Z/\mathcal{O}_v)$

F is a crystal of $(\mathcal{O}_{Z/\mathcal{O}_v})_{cris}$-modules

① For any morphism $u: (U' \hookrightarrow T', \delta') \to (U'' \hookrightarrow T'', \delta)$

$u^* F(U, T, \delta) \xrightarrow{\sim} F(U', T', \delta')$ isom

② F is a coherent $\mathcal{O}_{Z/\mathcal{O}_v, cris}$ module.

Exp. ① $\mathcal{O}_{Z/\mathcal{O}_v}$ is a crystal

$\pi: X_0 \to Y_0$ smooth proper,

$\pi_{cris,*}: (X_0/\mathcal{O}_v)_{cris} \longrightarrow (Y_0/\mathcal{O}_v)_{cris}$

$q \geq 0$, $R^q \pi_{cris,*} \mathcal{O}_{X_0/\mathcal{O}_v, cris}$ ~~no~~ is not crystal, but crystal up to isogeny, i.e.,

$\exists\ \mathcal{E}$ crystal and a map $i: \mathcal{E} \longrightarrow R^q \pi_{cris,*} \mathcal{O}_{X_0/\mathcal{O}_v, cris}$

with kernel and cokernel p-torsion.

$$\begin{tikzcd}
\mathcal{X}_0 \arrow[r, hook] \arrow[d] & \hat{\mathcal{X}}_0 \arrow[d] \\
\mathcal{Y}_0 \arrow[r, hook] \arrow[d] & \hat{\mathcal{Y}}_0 \arrow[d] \\
\mathrm{Spec}\,K \arrow[r, hook] & \cancel{\mathrm{Spec}}\,\mathrm{Spf}(\mathcal{O}_v)
\end{tikzcd}$$

~~Et~~

$\mathcal{E}(\hat{\mathcal{Y}}_0)$ is a $\mathcal{O}_{\hat{\mathcal{Y}}_0}$-locally free sheaf with a connection and F Frobenius operator.

$$\begin{tikzcd}
y_0 \arrow[r, hook, "i_0"] & \hat{\mathcal{Y}}_0 \\
y \arrow[ur, hook, "i"'] &
\end{tikzcd}
\qquad
\begin{aligned}
i^*\mathcal{E} &\simeq H^q_{cris}(\mathcal{X}_{y}, K_v) \\
i_0^*\mathcal{E} &\simeq H^q_{cris}(\mathcal{X}_{y_0}, K_v).
\end{aligned}$$

Berthelot - Ogus

$$(\mathcal{E}(\hat{\mathcal{Y}}_0), \nabla) \simeq (R^q\hat{f}_*\Omega^\bullet_{\hat{\mathcal{X}}_0/\hat{\mathcal{Y}}_0}, \nabla_{GM}).$$

2.3. Berthelot-Raynaud rigidification functor.

Let Rig_{K_v} be category of rigid analytic space $/K_v$

Objects. $(X, \mathcal{O}_X, \tau)$, τ admissible subsets of X. stable by finite intersection.

$\phi, X \in \tau$.

~~① $\phi, X \in \tau$~~,

- $\forall\, \mathcal{U} \in \tau$, has a set of admissible coverings $cov(\mathcal{U})$.

  ① $(\mathcal{U}) \in cov(\mathcal{U})$

  ② ~~$V \subseteq W$~~, $V \subseteq \mathcal{U}$, $(\mathcal{U}_i)_i \in cov(\mathcal{U})$,

  $(\mathcal{U}_i \cap V)_i \in cov(\mathcal{U})$

  ③ $\mathcal{U} \in \tau$, $\begin{cases} (\mathcal{U}_i)_i \in cov(\mathcal{U}) \\ ~~(\mathcal{U})_j~~ \in (\mathcal{U}_{i,j})_j \in cov(\mathcal{U}_i) \end{cases}$

  $\Rightarrow (\mathcal{U}_{i,j})_{i,j} \in cov(\mathcal{U}_i)$.

- X has a covering $(\mathcal{U}_i)_i \in cov(X)$ s.t. $\forall\, \mathcal{U}_i$ is an <u>affinoid domain</u>.

- Affinoid domains $r \geq 1$

$$K_v\{T\} = \{f = \sum_\alpha a_\alpha T_1^{\alpha_1} \cdots T_r^{\alpha_r},\ |a_\alpha| \xrightarrow[|\alpha| \to \infty]{} 0\}$$

↳ Banach algebra

⇝ Tate algebra $K_v\{T\}/I$, I closed, finitely generated.

$\mathcal{U}_i$ = maximal spectrum of $K_v\{T\}/I$.

$\mathcal{O}_{\mathcal{U}_i}(\mathcal{U}_i) = K_v\{T\}/I$.

Morphisms. morphisms of locally ringed spaces preserving admissibles.

Let $FS/\mathcal{O}_v$ (formal scheme) $\mathfrak{X}$ locally noetherian formal scheme $/\mathrm{Spf}\,\mathcal{O}_v$

$\mathfrak{X}_{red}$ locally of f.t. $/\mathrm{Spec}\,k$.

Berthelot-Ogus

- $rig: FS/\mathcal{O}_v \longrightarrow Rig_{K_v}$

$$(\hat{\mathcal{Y}}_0, \hat{\mathcal{O}}_{\mathcal{Y}_0}) \longmapsto (\mathcal{Y}_0^{rig}, \hat{\mathcal{O}}_{\mathcal{Y}_0})$$

$$\mathcal{E}(\hat{\mathcal{Y}}_0) \longrightarrow \mathcal{E}^{rig}(\hat{\mathcal{Y}}_0).$$

<u>sp</u> : $(\mathcal{Y}_0^{rig}, \mathcal{O}_{\mathcal{Y}_0}^{rig}) \longrightarrow (\hat{\mathcal{Y}}_0, \hat{\mathcal{O}}_{\mathcal{Y}_0})$

$$\mathcal{O}_v\{t_1, \cdots, t_r\}[[X_1, \cdots, X_m]] \longmapsto \{(t_1, \cdots, t_r) \in K_v^r,\ |t_i|_v \leq 1\} \times \{(x_1, \cdots, x_m) \in K_v^m,\ |x_i| < 1\}$$

$$\Omega_v = Sp^{-1}(\bar{y}_0) = \{(z_1, \cdots, z_n) \in K^n,\ |z_i|_v < 1\}.$$

Dwork-Katz

$$(\mathcal{E}(\hat{\mathcal{Y}}_0)^{rig}/\Omega_v^{rig})^{\nabla=0} \longrightarrow \mathcal{E}(\hat{\mathcal{Y}}_0)^{rig}_y/\Omega_v.$$

$\forall\, y \in \Omega_v$.

$$\begin{tikzcd}
H^q_{dR}(X_{y_0}, K_v) \arrow[d, "GM"] \arrow[r] & \left(\mathcal{E}(y_0)/\Omega_v^{rig}\right)^{\nabla} \simeq H^q_{cris}(X_{y_0}, K_v) \arrow[dl] \\
H^q_{dR}(X_y, K_v) &
\end{tikzcd}$$

$\Phi_v : \Omega_v^{rig} \longrightarrow \mathcal{F}l^{rig}(K_v)$

$y \longmapsto$ Hodge filtration on $H^q_{dR}(X_y, \otimes K_v)$

§3. p-adic vs. complex.

$\mathcal{F}l \hookrightarrow \mathbb{P}^N$

$\Phi_{\mathbb{C}} : \Omega_{\mathbb{C}} \longrightarrow \mathcal{F}l^{an}(\mathbb{C}) \longrightarrow \mathbb{P}^N$

$\Phi_v : \Omega_v \longrightarrow \mathcal{F}l^{rig}(K_v) \longrightarrow \mathbb{P}^N$

$(z_1, \cdots, z_m) \longmapsto [f_1, \cdots, f_N]$

$f_i \in K[\![z_1, \cdots, z_m]\!]$.

Prop. $\exists Z \hookrightarrow \mathcal{F}l/K$ closed subscheme s.t.

$$\overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}^{Zar} = Z^{an}(\mathbb{C})$$

$$\overline{\Phi_v(\Omega_v)}^{Zar} = Z^{rig}\cancel{(K_v)}.$$

Proof. I ideal of $K[X_0, \cdots, X_N]$ generated by homogeneous polynomials Q s.t. $Q(f_0, \cdots, f_N) = 0$.

$P \in \mathbb{C}[X_0, \cdots, X_N]$ s.t. $P(f_0, \cdots, f_N) = 0$.

$\sum a_I f_0^{k_0} \cdots f_N^{k_N}$

$$\rightsquigarrow \sum_{(t_1, \cdots, t_m)} F(a_I) z_1^{t_1} \cdots z_m^{t_m} = 0$$

$\Rightarrow F(a_I) = 0, \ \forall I = (t_1, \cdots, t_m)$

$\Rightarrow P$ is a $\mathbb{C}$-linear combinaition of K-solution of I. $\square$

Cor. ① $\dim_{K_v} \overline{\Phi_v(\Omega_v)}^{Zar} = \dim_{\mathbb{C}} \overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}^{Zar}$

$\geq \dim_{\mathbb{C}}(\Gamma \Phi_{\mathbb{C}}(y_0))$.

② If $\mathcal{F}l_v^{bad} \subseteq \mathcal{F}l_v^{rig}$ Z-closed,

$\dim \mathcal{F}l_v^{bad} < \dim_{\mathbb{C}}(\Gamma \cdot \Phi_{\mathbb{C}}^{an}(y_0))$

then $\Phi_v^{-1}(\mathcal{F}l_v^{bad})$ is a proper K_v-analytic subset of Ω_v.

§4. Towards Mordell.

$\mathcal{O} = \mathcal{O}_S$. $y_0 \in \mathcal{Y}(\mathcal{O})$ v place unramified

$\mathcal{U} = \{ y \in \mathcal{Y}(\mathcal{O}) ; y \equiv y_0 \bmod v \}$

$\rho_y : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}_{K_v}(\underbrace{H^q_{\acute{e}t}(X_{y,\overline{K}}, \mathbb{Q}_p)}_{V_{\mathbb{Q}_p}})$

~~Rx~~ $\rho_y / \mathrm{Gal}(\overline{K}_v/K_v)$ is a crystalline representation

B_{cris}/K_v
- Banach algebra
- Filtration
- Frobenius
- G_K-action

$B_{cris} \otimes_{\mathbb{Q}_p} V_{\mathbb{Q}_p}) = D_{cris}(V_{\mathbb{Q}_p})$

$\dim_{K_v} D_{cris}(V_{\mathbb{Q}_p}) \leq \dim_{\mathbb{Q}_p} V_{\mathbb{Q}_p}$

equality $\Longrightarrow V_{\mathbb{Q}_p}$ is crystalline.

X_y has good reduction at v

$\Longrightarrow \rho_y$ is crystalline at v (Fontaine)

Fontaine's functor.

Font: G_{K_v} $\mathbb{Q}_p$-crystalline repn $\xrightarrow[\text{fully faithful}]{} \overset{\text{filtered } \varphi\text{-module}}{\mathcal{FL}}$

(W, ϕ, F)

- F decreasing filtration on W
- W K_v-vector space
- ϕ is a σ-linear morphism
  $\sigma : K_v \to K_v$ lift of Frobenius.

$$\mathrm{Font}(\rho_y) = \left(H^q_{\mathrm{cris}}(X_y, K_v),\ \phi,\ F^\bullet \right)$$

indep of y.

Hodge filtration on $H^q_{dR}(X_y, K_v)$

Prop. $X \to Y$ proper smooth$/K$.

Assume that $\dim_{K_v} \big(\underbrace{Z(\phi^{[K_v:\mathbb{Q}_p]})}_{\text{algebraic group}}\big) < \dim_{\mathbb{C}} T\Phi_{\mathbb{C}}(y_0)$

Then the set

$\{ y \in Y_0(\mathbb{Q}) ;\ y \equiv y_0 \bmod v,\ \rho_y \text{ semi-simple} \}$

is a proper K_v-analytic subset of K_v.

Proof. $y \in U$, $\rho_y \in \{\rho_i\}_{\text{finite}}$

$\rightsquigarrow (V, \phi, \mathrm{Fil}(y)) \in \left\{ \begin{matrix} \text{finite set of} \\ \text{isom classes} \end{matrix} \right\}$

$\Phi_v(U) \in \bigcup_i \underset{\text{centralizer}}{Z(\phi_v)} \cdot \Phi_v(y_0) \subseteq$

$\subseteq Z(\phi_v^{[K_v:\mathbb{Q}_p]})$ of $\dim < \dim_{\mathbb{C}} T\Phi_{\mathbb{C}}(y_0)$

$\Rightarrow U$ is contained in a proper K_v analytic subset of Ω_v. $\square$

The S-unit équation

Olivier Wittenberg

§1. Statement

Thm. (Siegel, Mahler, Lang).

K # field, S finite set of places. $\mathcal{O}_S$,

$\{t \in \mathcal{O}_S^* ;\ 1-t \in \mathcal{O}_S^*\}$ is finite.

Rmq. This is $\mathcal{Y}(\mathcal{O}_S)$, $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}_S} \setminus \{0,1,\infty\}$.

More general thm of same authors:

$\mathcal{Y}$ affine $\mathcal{O}_S$-scheme of finite type,

$Y = \mathcal{Y} \otimes_{\mathcal{O}_S} K$ smooth hyperbolic curve

$\Longrightarrow \mathcal{Y}(\mathcal{O}_S)$ finite.

§2. Modified Legendre family

$E_z : y^2 = x(x-1)(x-z)$.

Fix $m \geq 1$. $t \in k$ char $k = 0$.

Let $X_t = E_{t^{\frac{1}{m}}}$ ell curve over $k[z]/(z^m - t)$ viewed as a k-scheme.

$$X_t(\mathbb{C}) = \coprod_{z^m = t} E_z(\mathbb{C})$$

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[rr, bend left, "\text{modified Legendre family}"] \arrow[d, bend right, "\text{relative ell curve } y^2 = x(x-1)(x-z)"'] & \mathcal{Y}' \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{Y} \arrow[d, phantom, "\shortparallel"] \\ {} & \mathbb{P}^1_{\mathbb{Z}_S} \setminus \{0, \mu_m, \infty\} & \mathbb{P}^1_{\mathbb{Z}_S} \setminus \{0,1,\infty\} \end{tikzcd}$$

relative ell curve $y^2 = x(x-1)(x-z)$.

$X = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 1. $t_0 \in Y(\mathbb{C})$. The Z-closure of the image of $\pi_1(Y(\mathbb{C}), t_0) \longrightarrow GL(H^1(X_{t_0}(\mathbb{C}), \mathbb{Q}))$

$$GL(H^1(X_{t_0}(\mathbb{C}), \mathbb{Q})) = GL\Big(\bigoplus_{z^m = t_0} H^1(E_z(\mathbb{C}), \mathbb{Q})\Big)$$

contains $\prod_{z^m = t_0} SL(H^1(E_z(\mathbb{C}), \mathbb{Q}))$

Lemma 2. L number field, $p > 2$ unramified in L.

The set $\left\{ z \in L ;\ \begin{array}{l} z \text{ and } 1-z \text{ are units above } p \\ \text{and } V_p(E_z) \text{ is not simple} \\ (\text{as } \mathbb{Q}_p[G_L]\text{-module}) \end{array} \right\}$

is finite.

§3. Proof of Theorem

Fix K, S

Q0. finiteness of $\overset{\mathcal{U}_0}{\overset{\shortparallel}{\{t \in \mathcal{O}_S^* ;\ 1-t \in \mathcal{O}_S^*\}}}$?

Let $m = |\mu_{2^\infty}(K)|$. By enlarging K, S, we may assume $m \geq 8$ and $S \supset$ places above 2, infinite places, places ramified/$\mathbb{Q}$, and all the places above the trace to $\mathbb{Q}$ of a place of S.

Reduce to

Q1. Finiteness of

$\mathcal{U}_1 = \{t \in \mathcal{U}_0 ;\ t \notin (K^*)^2\}$?

Proof. This is enough.

$t \in \mathcal{U}_0$, $\sqrt{t} \in K \Rightarrow \sqrt{t} \in \mathcal{U}_0$

$1 - t = (1+\sqrt{t})(1-\sqrt{t})$

$t\in\mathcal{U}_0,\ t^{\frac{1}{m}}\notin K \Rightarrow$ ~~$t\in\mathcal{U}_1$ or $\mathcal{U}_1^2$~~
$t\in\mathcal{U}_1\cup\mathcal{U}_1^2\cup\mathcal{U}_1^4\cup\cdots\cup\mathcal{U}_1^{\frac{m}{2}}$

$t\in\mathcal{U}_0,\ t^{\frac{1}{m}}\in K \Rightarrow t\in\mathcal{U}_1^m$

since $t=\left(t^{\frac{1}{m}}\right)^m=(\underbrace{\zeta t^{\frac{1}{m}}}_{\in\mathcal{U}_0})^m$ for ~~set~~ primitive m-th unit ζ.

$\zeta\notin(K^\times)^2\Rightarrow$ one of $t^{\frac{1}{m}},\ \zeta t^{\frac{1}{m}}$ is in $\mathcal{U}_1$.

$\Rightarrow \mathcal{U}_0\subset\mathcal{U}_1\cup\mathcal{U}_1^2\cup\cdots\cup\mathcal{U}_1^m$. $\square$

For $t\in\mathcal{U}_1$, $K(t^{\frac{1}{m}})/K$ cyclic of degree m, unramified outside S.

$\rightsquigarrow$ Reduction to

Q2. Finiteness of $\mathcal{U}_2=\{(t,\iota);\ t\in\mathcal{U}_1,\ K(t^{\frac{1}{m}})\overset{\iota}{\cong}L\}$
for a fixed cyclic L/K?

From ~~kr~~ now on, fix L/K cyclic of deg m.
Fix v place of K, $v\notin S$, inert in L,
p trace of v to $\mathbb{Q}$.
Fix $(t_0,\iota_0)\in\mathcal{U}_2$.

Q3. Finiteness of

$$\mathcal{U}_3=\{(t,\iota)\in\mathcal{U}_2;\ \underbrace{(t,\iota)\equiv(t_0,\iota_0)\bmod v}_{\substack{\shortparallel\\ t\equiv t_0 \bmod v}}\}\ ?$$

$$\begin{tikzcd}
K_v(t^{\frac{1}{m}}) \arrow[rr, "\sim", "\text{canonical}"'] & & K_v(t_0^{\frac{1}{m}}) \\
& L_v \arrow[ul, "\iota", "\wr"'] \arrow[ur, "\iota_0"', "\wr"] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{U}_3 \arrow[r] & \left\{\begin{array}{l}\text{2-dim } \cancel{\text{ve}}\ \text{v.s.}/\mathbb{Q}_p\\ \cancel{+\text{canonic}}+\text{continuous action of } G_L\\ \text{unramified outside of } S,\ \text{pure of wt}1\\ \text{integral}\end{array}\right\}/\text{iso} \\
(t,\iota) \arrow[r, mapsto] & H^1_{et}((E_{t^{\frac{1}{m}}})_{\bar{L}},\mathbb{Q}_p)
\end{tikzcd}$$

$$\begin{array}{ccc}
\cap & & \cup \\
\mathcal{U}_3\setminus\{\text{finite set}\} & \xrightarrow[\text{Lemma 2}]{} & \underbrace{\{\text{semi-simple ones}\}}_{\text{finite by Faltings' Lemma.}}
\end{array}$$

Q4. Finiteness of

$\mathcal{U}_4=\{(t,\iota)\in\mathcal{U}_3;\ H^1_{et}((E_{t^{\frac{1}{m}}})_{\bar{L}},\mathbb{Q}_p)$ is iso to $V\}$

for some $\mathbb{Q}_p[G_L]$-module V?

$$\begin{tikzcd}
& \left\{\begin{array}{l}\text{2-dim v.s.}/\mathbb{Q}_p\\ +\text{cont action of } G_{L_v}\\ \text{that is crystalline}\end{array}\right\} \arrow[r, "D_{cris}", "\text{fully-faithful}"'] & \left\{\begin{array}{l}\text{2-dim v.s. } D/L_v\\ +\text{Frob}: D\to D \text{ bijective}\\ \text{semi-linear w.r.t. Frob}_{L_v}\\ +\text{descending filtration}\end{array}\right\} \\
\mathcal{U}_4 \arrow[ur] & H^1_{et}((E_{t^{\frac{1}{m}}})_{\bar{L}},\mathbb{Q}_p) \arrow[r, mapsto] & H^1_{dR}(E_{t^{\frac{1}{m}}}/L_v) \\
(t,\iota) \arrow[ur, mapsto] & &
\end{tikzcd}$$

$$H^1_{dR}(E_{t^{\frac{1}{m}}}/L_v) = H^1_{dR}(E_{t^{\frac{1}{m}}}/K_v)$$
$$L_v \circlearrowright H_{dR}(X_t/K_v)$$

L_v acts through $L_v\xrightarrow{\sim}K_v(t_0^{\frac{1}{m}})\xrightarrow{\sim}K_v(t^{\frac{1}{m}})$
It's independent of L.

$$\mathcal{U}_5 := \mathrm{Im}(\mathcal{U}_4\longrightarrow \mathcal{Y}(\mathcal{O}_S))$$
$$(t,\iota)\longmapsto t.$$

Q5. Finiteness of $\mathcal{U}_5$?

For $t\in\mathcal{U}_5$, the $H^1_{dR}(X_t/K_v)$ viewed as L_v-v.s. + Frobenius (semilinear w.r.t. L_v) + filtration (L_v-linear) are all isomorphic.

$\Omega_v = \{t \in \mathcal{Y}(\mathcal{O}_v) ;\ t \equiv t_0 \bmod v\}$

Local trivialization of GM connection

$$\begin{tikzcd}
\rightsquigarrow H^1_{dR}(X_t/K_v) \arrow[r, "GM", "\sim"'] \arrow[dr, dash, "\wr"] & H^1_{dR}(X_{t_0}/K_v) \\
& H^1_{cris}(X_{t_0 \bmod v}/W(\mathbb{F}_v)) \otimes K_v
\end{tikzcd}$$

compatible with Frob.

$$\begin{tikzcd}
H^0_{dR}(X_t/K_v) \arrow[r, "GM", "\sim"'] \arrow[d, phantom, "\parallel"] & H^0_{dR}(X_{t_0}/K_v) \arrow[d, phantom, "\parallel"] \\
K_v(t^{\frac{1}{m}}) \arrow[r, "\sim", "canonical"'] & K_v(t_0^{\frac{1}{m}})
\end{tikzcd}$$

$\Rightarrow H^0_{dR} \subset H^1_{dR}$

GM compatible with cup-product

$\Rightarrow$ GM is L_v-linear.

$\mathcal{H}/K = \left\{ \begin{matrix} m\text{-dim linear subspaces in} \\ H^1_{dR}(X_{t_0}/K) \end{matrix} \right\} \simeq Gr(m, 2m)$

$$\begin{tikzcd}
\text{period map } \Phi_v : \Omega_v \supset U_5 \arrow[r] \arrow[dr] & \mathcal{H}(K_v) \arrow[d, phantom, "\cup"] \\
& \mathcal{H}^0(K_v) \arrow[d, phantom, "\parallel"] \\
& \{L_v\text{-linear subspaces}\}
\end{tikzcd}$$

$\Rightarrow \Phi_v(U_5) \subset$ one orbit of

$$\begin{tikzcd}
& Z = \left\{ \begin{matrix} L_v\text{-linear automorphism of } H^1_{dR}(X_{t_0}/K_v) \\ \text{which commute with } \phi_{Frob} \end{matrix} \right\} \arrow[dl, phantom, "\ni"] \arrow[d, "\circlearrowright"] \\
\mathbb{Q}_p\text{-points of an alg group}/\mathbb{Q}_p & \mathcal{H}^0(K_v)
\end{tikzcd}$$

Q. Φ_v^{-1}(one orbit) is finite?

$Z(\phi) \subset Z(\phi^{[K_v:\mathbb{Q}_p]})$ K_v-pts of an alg gp$/K_v$.

Want. Φ_v^{-1}(orbit of $Z(\phi^{[K_v:\mathbb{Q}_p]})$) is a proper K_v-analytic subset of Ω_v.

Enough: $\underbrace{\dim Z(\phi^{[K_v:\mathbb{Q}_p]})}_{(I)} < \underbrace{\dim_{K_v}\left(Z\text{-closure of } \Phi_v(\Omega_v)\right)}_{(II)}$

(I) Apply

Lemma 2.10 [L.V].

$E/F = L_v/K_v$ cyclic field extension, $\sigma \in Gal(E/F)$ generator, V v.s.$/E$. $(V = H^1_{dR}(X_{t_0}/K_v))$

$\phi : V \to V$ σ-semilinear automorphism
$(\phi^{[K_v:\mathbb{Q}_p]})$

$Z(\phi) = \{f \in End_E(V),\ \phi f = f \phi\}$ F-v.s ~~e.v.~~

Then $\dim_F Z(\phi) = \dim_E Z(\phi^{[E:F]})$
$\leq (\dim_E V)^2$.

get (I) ≤ 4.

Proof. (vector space$/F$) $=$ $\left(\begin{matrix}\text{vector space}/E \\ + \text{semilinear action of } Gal(E/F)\end{matrix}\right)$

$Z(\phi^{[E:F]}) \circlearrowleft$ semilinear action of $Gal(E/F)$
by $\sigma \cdot f = \phi \circ f \circ \phi^{-1}$

$Z(\phi^{[E:F]}) = E \otimes_F Z(\phi^{[E:F]})^{Gal(E/F)} = E \otimes_F Z(\phi)$.

$\square$

(II) $\dim_{K_v}\left(\overline{\Phi_v(\Omega_v)}^{Zar}\right)$ $\quad \Phi_v(\Omega_v) \subset \mathcal{H}^0(K_v)$

$\subset \mathcal{H}(K_v)$

$\geq \dim_{\mathbb{C}}$ (Z-closure of the orbit of $\Phi_{\mathbb{C}}(t_0)$ under $\pi_1(Y(\mathbb{C}), t_0)$ on $\mathcal{H}(\mathbb{C})$)

$\mathcal{H}(\mathbb{C}) = \{$ m-dim subspaces in $H^1_{dR}(X_{t_0}/\mathbb{C}) = \bigoplus_{z^m = t_0} H^1_{dR}(E_{t^{\frac{1}{m}}}, \mathbb{C})$ $\}$

$$\cup\!\!\!\!\! \mid$$

$$\underbrace{\Phi_{\mathbb{C}}(t_0) \in \prod_{z^m = t_0} \mathbb{P}(H^1_{dR}(E_{t^{\frac{1}{m}}}, \mathbb{C}))}$$

$\subset$ Z-closure of orbit by Lemma 1.

has dim $m \geq 8$.

(modulo Lemma 1 & 2) $\square$

§4. Proof of Lemma 1.

$\Gamma \subset GL(H^1(X_{t_0}(\mathbb{C}), \mathbb{Q}))$ Z-closure of $\pi_1(Y(\mathbb{C}), t_0)$

$\qquad \parallel$

$GL(\bigoplus_{z^m = t_0} H^1(E_{t^{\frac{1}{m}}}(\mathbb{C}), \mathbb{Q}))$

Want: $\Gamma \supset \prod SL_2$

(i) Γ transitively permutes the summands (look at $t=0$)

$$\begin{tikzcd} & X \arrow[d] & \\ \mathbb{P}^1 \supset & Y' \arrow[d] & z \arrow[d] \\ \mathbb{P}^1 \supset & Y & z^m \end{tikzcd}$$

(ii) $\Gamma \cap \prod_{z^m = t_0} SL(H^1(E_{t^{\frac{1}{m}}}(\mathbb{C}), \mathbb{Q})) \xrightarrow{pr_z} SL(H^1)$

$\forall z$.

Recall. Z-closure of global monodromy of usual Legendre family in SL_2.

Apply this to $\pi_1(Y'(\mathbb{C}), t_0') \Longrightarrow$ (ii).

(iii) $\Gamma \supset (1, \dots, 1, u, 1, \dots, 1)$ u unipotent in SL_2 (one of the factors)

look at monodromy around $t = 1$.

Now we prove $\Gamma \supset \prod_z SL(H^1)$

$V = Lie(\Gamma \cap \prod SL_2) \subset (\mathfrak{sl}_2)^m$.

(ii) $\Rightarrow V \xrightarrow{pr_i} \mathfrak{sl}_2, \ \forall i$.

Hence $V \cap (\{0\} \times \dots \times \{0\} \times \mathfrak{sl}_2 \times \{0\} \times \dots \times \{0\})$

is an ideal of $\mathfrak{sl}_2$, which is non-zero as positive dimensional by (iii).

Hence this is $\mathfrak{sl}_2$, i.e.,

$V \supset \{0\} \times \dots \times \{0\} \times \mathfrak{sl}_2 \times \{0\} \times \dots \times \{0\}$.

(i) $\Rightarrow V = (\mathfrak{sl}_2)^m$ ✓.

$\square$

§5. Proof of Lemma 2.

Preliminaries.

Lemma 2.8 of [L-V] over $\mathbb{Q}$.

> $\rho : G_{\mathbb{Q}} \longrightarrow \mathbb{Q}_p^*$ unramified outside a finite set of places $S \ni p$, pure of weight w outside S, Hodge-Tate at p.
>
> Then the Hodge-Tate weight of ρ at p is $\frac{w}{2}$.

Convention. $\mathbb{Q}_p(1)$ has Hodge-Tate weight -1 (pure of weight 1)

Recall. K_v p-adic field, $\rho : G_{K_v} \to \mathbb{Q}_p^\times$.

ρ is Hodge - Tate $\Leftrightarrow$ ρ is de Rham

$\Leftrightarrow$ ρ is Tate twist of a finitely ramified ~~chara~~ character (i.e., of a character ρ' s.t. $\rho'(I_K)$ - finite)

Proof of 2.8.

By ~~a~~ Tate twisting, we assume the HT weight at p is 0 (i.e., $\rho(I_{\mathbb{Q}_p})$ finite)

$$\begin{tikzcd}
\mathbb{I}_{\mathbb{Q}} \arrow[r, "\cancel{res}", "rec"'] & G_{\mathbb{Q}}^{ab} \arrow[r, "\rho"] & \mathbb{Q}_p^\times \\
\mathbb{Q}_v^* \arrow[u, "\cancel{\rho\circ res}"] \arrow[r, "rec"] & G_{\mathbb{Q}_v}^{ab} \arrow[u] &
\end{tikzcd}$$

ideles

$(\rho\circ res)\left(\prod_{v\in S} \mathbb{Z}_v^*\right)$ is finite

(at p by assumption, at $v\neq p$ trivial)

$(\rho\circ res)\left(\prod_{v\notin S} \mathbb{Z}_v^*\right) = 1$.

$q\notin S$ prime

$$\underset{\overset{\parallel}{0}\ \text{global rec law}}{(\rho\circ rec)(q,\dots,q)} = \overbrace{(\rho\circ rec)(1,\dots,1,\overset{\text{at } q}{q},1,\dots,1)}^{\rho(\mathrm{Frob}_q)} + \underbrace{(\rho\circ rec)(q,\dots,q,\underset{\text{at } q}{1},q,\dots,q)}_{\text{of finite order}}$$

so $w=0$.

$\square$

Lemma 2.10/$\mathbb{Q}$ L number field, ~~$\rho : G_L$~~

$\rho : G_L \to Gl_n(\mathbb{Q}_p)$ Hodge-Tate at all $v|p$ unramified outside S pure of weight w

Then $$\sum_{v|p}[L_v:\mathbb{Q}_p]\underbrace{t_H(\rho|_{G_{L_v}})}_{\text{y-coordinates of endpoint of Hodge polygone}} = [L:\mathbb{Q}]\frac{nw}{2}$$

Proof. $\det \mathrm{Ind}_{G_L}^{G_{\mathbb{Q}}}\rho : G_{\mathbb{Q}} \to \mathbb{Q}_p^\times$ unramified outside S', pure of weight $wn[L:\mathbb{Q}]$

Lemma 2.8 $\Rightarrow$ HT weight is $\frac{wn[L:\mathbb{Q}]}{2}$

Also $= t_H(\det \mathrm{Ind}_{G_L}^{G_{\mathbb{Q}}}\rho|_{G_{\mathbb{Q}_p}}) = t_H(\mathrm{Ind}_{G_L}^{G_{\mathbb{Q}}}\rho|_{G_{\mathbb{Q}_v}})$

$= \sum_{v|p} t_H(\rho|_{G_{L_v}})[L_v:\mathbb{Q}_p]$.

$\square$

Proof of Lemma 2. Fix L, p,

Enough to show finiteness of

$\left\{ z\in L \,;\, z, 1-z \text{ units ~~of~~ above } p,\ V_p(E_z) \text{ is not simple as } \mathbb{Q}_p[G_L]\text{-module and s.t. } z\equiv z_0 \bmod v,\ \forall v|p \right\}$.

for a fixed z_0 as on the

Same set-up

$\rightsquigarrow H^1_{dR}(E_z/L_v) \overset{GM}{\simeq} H^1_{dR}(E_{z_0}/L_v)$ $\forall v|p$

$\Omega_v = \{z\in \mathcal{O}_v^*,\ 1-z\in\mathcal{O}_v^* ~~p~~\}$

$\Phi_v : \Omega_v \to \mathcal{H}(L_v)$ not constant. K_v-analytic

Claim. If $z \in L$, z, $1-z$ units above p, and if $V_p(E_z)$ is not simple, then $\exists\, v | p$,

$$\Phi_v(z) = \Phi_v(z_0).$$

Proof of claim.

Let $W \subset \overset{H^1((E_z)_{\bar{L}}, \mathbb{Q}_p)}{\cancel{V_p(E_z)}}$ be a 1-dim subspace stable under G_L. At each $v|p$, Hodge-Tate of weight 0 or 1. Lemma 2.10 $\Rightarrow$ one is $\neq 0$.

Fix such v.

$$D_{cris}(W) \subset H^1_{dR}(E_z/L_v).$$

Should have $D_{cris}(W) = F^1 H^1_{dR}(E_z/L_v)$

$$\left.\begin{array}{l} \text{But } W \text{ dim } 1. \\ \text{HT weight of } W = 1 \end{array}\right\} \Rightarrow \text{Frob}^{[L_v:\mathbb{Q}_p]} \text{ has eigenvalue of slope 1 on } D_{cris}(W).$$

Conclusion $F^1 H^1_{dR}(E_z/L_v)$ is the slope 1 Frob eigenspace of H^1_{dR}.

$\Longrightarrow$ Claim. $\square$

Beginning of the proof of Lawrence-Venkatesh.

Anna Cadoret

$X_q \longrightarrow Y_q' \longrightarrow Y$

k number field, q prime,

Y smooth proper of genus $g \geq 2$ $/k$,

S ~~family of~~ finite set of places of k s.t.

$X_q \longrightarrow Y_q' \xrightarrow{\pi_q} Y$ has good reduction $/\mathcal{O}_{k,S}$

$v \in |\mathrm{Spec}(\mathcal{O}_{k,S})|$ unramified.

$$Y(k)_v^{**} = \left\{ y \in Y(k) \;\middle|\; \begin{array}{l} \exists\, (y',v') \mid (y,v),\ \text{s.t.} \\ \text{(i) } [k(y')_{v'} : k_v] \geq 8 \\ \text{(ii) } H^1(X_{y'}, \mathbb{Q}_p) \circlearrowleft \pi_1(k(y')) \\ \quad \text{is simple} \end{array} \right\}$$

$\forall y_0 \in Y(k)$, $U_{y_0,v} = \{ y \in Y(k) ; y \equiv y_0 (v) \}$

$U_{y_0,v} \cap Y(k)_v^{**}$ is finite.

<u>Last step</u>. Show $U_{y_0,v} \setminus U_{y_0,v} \cap Y(k)_v^{**}$ is finite.

1. $\forall$ odd prime q, $\forall v$ as above friendly, the set $U_{y_0,v}^* \setminus U_{y_0,v}^* \cap U_{y_0,v}^{**}$ is finite.

$$Y(k)_v^* = \left\{ y \in Y(k) \;\middle|\; \mathrm{size}(\pi_q^{-1}(y) < 8)_v < \frac{1}{d_q + 1} \right\}$$

$d_q = \deg \pi_q$.

2. $\left.\begin{array}{l} \exists \text{ odd prime } q, \\ \exists \text{ friendly } v \text{ as above} \end{array}\right\}$ s.t. $Y(k) = Y(k)_v^n$.

q prime $\quad \Gamma_q = \mathbb{F}_q \rtimes \mathbb{F}_q^\times \quad Z(\Gamma_q) = 1$

$$\begin{tikzcd} 1 \arrow[r] & \Gamma_q^{der} \arrow[r] \arrow[d, equal] & \Gamma_q \arrow[r] & \Gamma_q^{ab} \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{F}_q & & \mathbb{F}_q^\times & \end{tikzcd}$$

$k/\mathbb{Q}$ number field, Galois,

Y smooth proper geom connected curve $/k$, of genus $g \geq 2$.

$Y_q' \xrightarrow{\pi_q} Y$ étale cover

(KP 2) $\forall y \in Y(k)$,

$$\pi_q^{-1}(y) \longleftrightarrow \left\{ \begin{array}{c} \pi_1(Y_{\bar{k}} \setminus \{y\}) \twoheadrightarrow \Gamma_q \\ \text{not factorising through } \pi_1(Y_{\bar{k}}) \end{array} \right\} \Big/ \Gamma_q \text{ conj}$$

(KP 1) $\deg \pi_q = d_q = \frac{q-1}{2}(2g-1)$.

[LV, Prop 5.4] $n \geq 1$ integer.

$\exists$ odd prime p

$\exists$ v friendly as above s.t.

$\forall y \in Y(k)$, $\mathrm{size}(\pi_q^{-1}(y) < n)_v < \frac{1}{d_q + 1}$.

Friendly places

$$\begin{tikzcd} C \arrow[r, phantom, "\overset{\text{index } 2}{\triangleleft}"] & C^+ \arrow[r, phantom, "\triangleleft"] \arrow[d, equal] & \pi_1(\mathbb{Q}) \\ & \text{conjugacy classes by complex conjugation} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Q} \subset & E_k^+ \subset \arrow[d, equal] & E_k \subset k \arrow[d, equal] \\ & \bar{\mathbb{Q}}^{\pi_1(k)C^+} & \bar{\mathbb{Q}}^{\pi_1(k)C} \end{tikzcd}$$

maximal totally real subfield.

$v \in |\mathrm{Spec}\, \mathcal{O}_{k,S}|$ is friendly iff

- if $E_k^+ = E_k$, v is unramified
- if $E_k^+ \subsetneq E_k$, and $v|_{E_k} / v|_{E_k^+}$ is inert.

Size. $\pi_1(k) \circlearrowright M$ finite.

$$\begin{tikzcd} \pi_1(k) & \\ \pi_1(k_{nr}) \arrow[u] \arrow[r] & \pi_1(k(v)) \arrow[u, phantom, "\circlearrowleft"] \\ & \sigma_v \end{tikzcd}$$

$$(M<n)_v = \{\mu \in M \mid |\sigma_v \mu| < n\}$$

$$\text{size}(M<n)_v = \frac{|(M<n)_v|}{|M|}$$

Rmq. $M \to N$ $\pi_1(k)$-equivariant which is ~~finite to~~ $\overset{}{\underset{(a:1)}{}}$ 1 between $\pi_1(k)$ finite sets.

$$\text{size}(M<n)_v \leq \text{size}(N<n)_v$$

A/. Estimate $\text{size}(\pi_q^{-1}(y)<n)_v$

have to understand $\sigma_v \circlearrowright \pi_q^{-1}(y)$

$$Sp \cdot \pi_1(Y_{\bar{k}} \setminus \{y\}) \twoheadrightarrow \Gamma_q$$

Riemann existence Thm.

$$\pi_1(Y_{\bar{k}} \setminus \{y\}) = \widehat{\langle c_1, \dots, c_g, c_1', \dots, c_g', \gamma \mid [c_1, c_1'] \cdots [c_g, c_g'] \gamma = 1 \rangle}$$

$$\pi_q^{-1}(y) \hookrightarrow \left\{ \begin{array}{l} (c_1, \dots, c_g, c_1', \dots, c_g') \in \Gamma_q \\ \langle c_1, \dots, c_g, c_1', \dots, c_g' \rangle = \Gamma_q, \\ [c_1, c_1'] \cdots [c_g, c_g'] \neq 1 \end{array} \right\} = U.$$

$$U \xrightarrow[\Gamma_q \to \Gamma_q^{ab}]{} \!\!\!\!\twoheadrightarrow V \left(= \left\{ \begin{array}{l} a_1, \dots, a_g, a_1', \dots, a_g' \in \mathbb{F}_q^\times \\ \langle a_1, \dots, a_g, a_1', \dots, a_g' \rangle = \mathbb{F}_q^\times \end{array} \right\} \right)$$

$$\begin{tikzcd} (\pi_1(Y_{\bar{k}} \setminus \{y\}) \twoheadrightarrow \Gamma_q) \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & (\pi_1(Y_{\bar{k}} \setminus \{y\}) \twoheadrightarrow \Gamma_q \twoheadrightarrow \Gamma_q^{ab}) \arrow[d, phantom, "\cap"] \\ \text{Hom}(\pi_1(Y_{\bar{k}} \setminus \{y\}), \Gamma_q) \arrow[r] & \text{Hom}(\pi_1(Y_{\bar{k}} \setminus \{y\}), \Gamma_q^{ab}) \arrow[d, phantom, "\shortparallel"] \\ & \text{Hom}(\pi_1(Y_{\bar{k}}), \mathbb{Z}/q-1) = H^1(Y_{\bar{k}}, \mathbb{Z}/q-1) \end{tikzcd}$$

1) The map $U \to V$ is N to 1, with

$$N = q^{2g-1}(q-1).$$

$$P : \Gamma_q^{2g} \longrightarrow (\Gamma_q^{ab})^{2g}$$

$$\left(\begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix} \right) \longmapsto (a_i)$$

$$(\underline{a}, \underline{a}')$$

$$P^{-1}(\underline{a}, \underline{a}') \simeq \mathbb{F}_q^{2g} \ni (\underline{c}, \underline{c}')$$

If $\langle \underline{c}, \underline{c}' \rangle = \Gamma_q$,

$[c_1, c_1'] \cdots [c_g, c_g'] \neq 1$

if $(\underline{a}, \underline{a}') \in V$,

$$P^{-1}(\underline{a}, \underline{a}') \cap U$$

$$\overset{\psi}{(\underline{c}, \underline{c}')} \qquad \prod [c_i, c_i'] \neq 1.$$

$$\shortparallel$$

$$\begin{pmatrix} 1 & \sum_{1 \leq i \leq g} (1-a_i) b_i - (1-a_i') b_i' \\ 0 & 1 \end{pmatrix}$$

$$N = |\mathbb{F}_q^{2g} \setminus \mathbb{F}_q^{2g-1}|.$$

Reduce to show.

$$\text{size}(V<u)_v < \frac{1}{d_q+1}$$

$$\text{VI}$$

$$\text{size}(U<u)_v.$$

$$V = \{(a_1, \dots, a_g, a_1', \dots, a_g') \in \mathbb{F}_q^\times, \langle \underline{a}, \underline{a}' \rangle = \mathbb{F}_q^\times\}$$

$$|V| \geq \frac{(q-1)^{2g}}{2}$$

Lemma. $V_{N,r} = \{(a_1, \dots, a_r) \in \mathbb{Z}/N^r ; \langle a_1, \dots, a_r \rangle = \mathbb{Z}/N\}$

$$|V_{N,r}| = N^r \prod_{p \mid N} (1 - p^{-r}) \geq \frac{N^r}{2}.$$

$V \subset H^1(Y_{\bar{k}}, \mathbb{F}_q^\times) =: H_q$.

$(V<n)_v \subset (H_q < n)_v$

$$\mathrm{size}(V<n)_v \le \frac{\mathrm{size}(H_q<n)_v}{(q-1)^{2g}}.$$

$$< \frac{1}{d_q+1}.$$

2). Estimate $|(H_q < n)_v|$.

$\langle , \rangle : H_q \times H_q \longrightarrow \mu_{p-1}^{\vee}$

$\pi_1(k)$-equivariant perfect pairing.

$\langle \sigma_v x, \sigma_v y \rangle = \pi_v^{-1} \langle x, y \rangle$

$\pi_v = |k(v)| = p_v^a, \quad a \le [k:\mathbb{Q}]$.

$$(H_q, <n)_v \subset \bigcup_{1\le i\le n} \underset{\underset{x,\,y}{\rotatebox{90}{$\in$}}}{Ker(\sigma_v^i - Id)}$$

$\langle x, y \rangle = \langle \sigma_v^i x, \sigma_v^i y \rangle$

$= \pi_v^{-i} \langle x, y \rangle$

$\rightsquigarrow (1-\pi_v^{-i}) \langle x, y \rangle = 0$.

Assume $\forall r$ odd prime, $r \mid q-1$.

ord of π_v in $\mathbb{F}_r^\times > u$ where $r \nmid \pi_v^i - 1$, $1 \le i \le u$.

then $(1-\pi_v^i)\langle x, y\rangle = 0 \iff 2^\gamma \langle x, y \rangle = 0$

where γ = multiplicity of 2 in $q-1$.

Now apply to $2^\gamma Ker(\sigma_v^i - Id)$.

<u>Lemma.</u> $\langle , \rangle : A \otimes A \longrightarrow \mathbb{Q}/\mathbb{Z}$ non-degenerate pairing, A finite as group, $B \subset A$ sbgp s.t. $\langle B, B \rangle = 0$. Then $|B| \le \sqrt{|A|}$.

$\Rightarrow |Ker(\sigma_v^i - Id)| \le 2^{2\gamma g}(q-1)^g$

$$|(H_q, <n)_v| \le \sum_{i=1}^{n} 2^{2\gamma g}(q-1)^g$$

$$= n 2^{2\gamma g}(q-1)^g$$

<u>Conclusion.</u>

$$\mathrm{size}(V<n)_v \le \frac{n 2^{2\gamma g+1}(q-1)^g}{(q-1)^{2g}} \le \frac{1}{d_q+1}$$

$$\iff n < \frac{(q-1)^g}{2^{2g\gamma}((q-1)(2g-1)+1)}.$$

<u>What we want.</u>

v unramified place, not in S friendly,

$\pi_v = |k(v)| = p_v^a, \quad a \le [K:\mathbb{Q}]$

q odd prime

3) $$n < \frac{(q-1)^g}{2^{2g\gamma}((q-1)(2g-1)+1)}$$

$\forall r$ odd prime $\mid q-1$

π_v has order $> n$ in $\mathbb{F}_r^\times$.

Fix q satisfying 3) +

1) $q-1 = 2^\gamma \prod_{\substack{r\ \text{prime} \\ > u[k:\mathbb{Q}]}} r^{\alpha_r}$ $\quad \gamma \le \gamma_1$ fixed,

2) $k/\mathbb{Q}$ & $\mathbb{Q}(\zeta_{q-1})/\mathbb{Q}$ are linearly disjoint.

$(q-1, \operatorname{disc}(k/\mathbb{Q})) = 1 \Rightarrow$

By Dirichlet thm on primes in authentic progressions,

$\exists\ \infty$ many primes q satisfying 1) & 2).
hence satisfying 3).

~~&~~
Choose v using Čebotarev density,

$$\begin{tikzcd}
\operatorname{Gal}(k(\zeta_{q-1})/\mathbb{Q}) \arrow[r, phantom, "\simeq"] & \operatorname{Gal}(k/\mathbb{Q}) \arrow[r, phantom, "\times"] & \operatorname{Gal}(\mathbb{Q}(\zeta_{q-1})/\mathbb{Q}) \arrow[dd] \\
\operatorname{Gal}(E_k/E_k^+) & \operatorname{Gal}(k/E_k^+) \arrow[l, two heads] \arrow[u, phantom, "\cup"] & \\
 & & \prod\limits_{\substack{r \text{ odd prime} \\ r \mid q-1}} \operatorname{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})
\end{tikzcd}$$

Take $a \in \operatorname{Gal}(\mathbb{Q}(\zeta_{q-1})/\mathbb{Q})$ lifting a generator of $\prod\limits_{\substack{r \text{ odd} \\ r \mid q-1}} \operatorname{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$.

then any unramified v in $k(\zeta_{q-1})/$~~$\mathbb{Q}k$~~$\mathbb{Q}$
~~mapping~~ whose Frob ~~may~~ induce $(\tilde{\sigma}, a)$
will work

Proof of Prop 5.3 [LV]

Lucia Mocz.

Recall. K #field of group G_K

Y/K curve of genus ≥ 2.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ Y' \arrow[r] \arrow[d, "\pi"] & \mathcal{Y}' \arrow[d, "\pi"] \\ Y \arrow[r] & \mathcal{Y} \end{tikzcd}$$

abelian-by-finite family with smooth model

- π finite étale
- $X \to Y'$ polarized ab sch fixed of dim d
- $S \subseteq M_K$ finite set, $\mathcal{O} = \mathcal{O}_S$
- Assume π has full monodromy.

- $v \in M_K \setminus S$ friendly:
  - unram/$\mathbb{Q}$
  - $$\begin{tikzcd} K \arrow[d, dash] & v \arrow[d, dash] & \\ E\ (\text{max CM}) \arrow[d, dash] & w \arrow[d, dash] & \\ E^+\ (\text{tot real}) & w & \text{inert} \end{tikzcd}$$

- $\forall\ G_K$-set E

size of Frob orbit.

$$\mathrm{size}_v(E) = \frac{\#\{x \in E;\ |\mathrm{Frob}_x| < 8\}}{\# E}$$

Goal To prove

Prop 5.3 $Y(K)^* := \{ y \in Y(K);\ \mathrm{size}(\pi^{-1}(y)) < \frac{1}{d+1} \}$ is finite

largest value that technique applies.

Naive approach (incorrect)

$Y' = Y$, $X \to Y$, v friendly: attach

$$\rho_{y,v}\ \text{~~Rin~~} := \underset{\overset{\circlearrowleft}{\phi}}{H^1_{cris}(X_y)} \simeq H^1_{dR}(X_y/K_v)^{\circlearrowleft \mathrm{Fil}}$$

Fontaine: equiv to data of filtered $\mathbb{Q}$-mods.

Assumption.

* $\tilde{\forall} y \in Y$, $\rho_{y,v}$ comes from ρ_y that is (semi)simple

Faltings' Lemma

$\Longrightarrow \forall\ \rho_{y,v} \simeq \rho'$ for some $\rho' \in \Sigma$ where Σ is a finite set of crystalline rep.

Want. $\forall\ \rho' \in \Sigma$, finitely many $y \in Y(K)$ s.t. $\rho_{y,v} \simeq \rho'$.

Use period map

$$\Phi_v : \Omega_v \longrightarrow \mathcal{H}_v$$

~ (Lagrangian) Grassmannian parametrizing filtered (ϕ)-mods

$$y \longmapsto \left(\mathrm{Fil}^1 H_{dR}(X_y/K_v) \subset H_{dR}(X_y/K_v)\right)$$

$\forall$ two points, $y, y' \in \Omega_v$, $\rho_{y,v} \cong \rho_{y',v}$ if they are in the same $Z(\phi)$-orbit in $\mathcal{H}_v$.

But $\dim_{K_v}(\mathcal{H}_v) = \frac{d(d+1)}{2}$

$\dim_{K_v}(Z(\phi)) \leq 4d^2$.

No way to compare these two.

Corrections

1. Use ~~fin~~ abelian-by-finite families with full monodromy to enlarge dim of $\mathcal{H}_v$ (Lemma 6.2) [Need: full monodromy ~~but~~
2. Loci of "bad" points accounting for failure of simplicity $\rightsquigarrow$ proper closed subvar of $\mathcal{H}_v$ (Lemma 4.1) [Need crucially: points of bounded size]

Geometry of ~~finite-by~~ abelian-by finite families

$y \in Y(K)$, $\pi^{-1}(y)$ finite scheme

- $E_y = \mathcal{O}_{\pi^{-1}(y)}$ étale K-alg $E_y \simeq \prod_{\pi(y')=y} K(y')$

- $X_y = X \times_{Y'} \underbrace{Y' \times_Y y}_{E_y} \simeq \coprod_{\pi(y')=y} X_{y'}$ $\rightarrow$ polarized ab var of dim $d / K(y')$

- cup product is E_y-linear

$$H^1_{dR}(X_y, K_v) \times H^1_{dR}(X_y, K_v) \longrightarrow E_y$$

Fix $y_0 \in Y(K)$ $\quad E_0 := E_{y_0}$

$$E_{0,v} := E_{y_0} \otimes_K K_v \simeq \prod_{(y',w)} K(y')_w .$$

$\forall y \in \Omega_v := \{ y \in Y(K) \,;\, y \equiv y_0 \bmod v \}$

$$\begin{tikzcd}
\bigoplus_{\tau} E_{y,w} \arrow[r] \arrow[d, phantom, "\wr"] & \bigoplus_{\tau} E_{y_0,v} \arrow[d, phantom, "\wr"] \\
H^1_{dR}(X_y/K_v) \arrow[r, "GM", "\sim"'] \arrow[d, "\wr"] & H^1_{dR}(X_{y_0}/K_v) = \cancel{V} V_v \arrow[d, "\wr"] \\
\prod_{(y',w)} H^1_{dR}(X_{y'}/K(y')_w) \arrow[r, "\prod GM", "\sim"'] \arrow[d, phantom, "\wr"] & \prod_{(y_0',w)} H^1_{dR}(X_{y_0'}/K(y_0')_w) \arrow[d, phantom, "\wr"] \\
\bigoplus_{\tau} \bigoplus_{(y',w)} K(y')_w \arrow[r] & \bigoplus_{\tau} \bigoplus_{(y_0',w')} K(y_0')_w
\end{tikzcd}$$

In particular,

$$\begin{tikzcd}
{\begin{matrix}(y',w) \\ (y,v)\end{matrix}} \arrow[r, leftrightarrow, "GM"] & {\begin{matrix}(y_0',w_0) \\ (y_0,v)\end{matrix}}
\end{tikzcd}$$

~~If~~ if $\cdot\; K(y')_w \xrightarrow[GM]{\sim} K(y_0')_w$

$\cdot\; H^1_{dR}(X_{y'}/K(y')_w) \xrightarrow[GM]{\sim} H^1_{dR}(X_{y_0'}/K(y_0')_w)$

Def. $\mathcal{G}_v = \mathrm{Res}^{E_{0,v}}_{K_v} Gr(V_v, d)$

$= \left\{ F \subseteq V_v \,;\, \begin{matrix} \cdot F \text{ free as } E_{y_0,v}\text{-mod} \\ \cdot \operatorname{rk} F = d \end{matrix} \right\}$

$\mathcal{H}_v = \mathrm{Res}^{E_{0,v}}_{K_v} LGr(V_v, \omega)$

$= \left\{ F \subseteq V_v \,;\, \begin{matrix} \text{same as above} \\ + \omega(F) = 0 \end{matrix} \right\}.$

Have decompositions

$$\mathcal{G}_v \simeq \prod_{(y_0',w)} \mathrm{Res}^{E_{0,v}}_{K_v} Gr(V_{y_0',w}, d) = \prod_{(y_0',w)} \mathcal{G}_{(y_0',w)}$$

$$\mathcal{H}_v = \prod_{(y_0',w)} \mathcal{H}_{(y_0',w)}$$

Period map.

$$\Phi_v : \Omega_v \longrightarrow \mathcal{H}_v \xrightarrow{pr} \mathcal{H}_{(y_0',w)}$$

$$\begin{tikzcd}
y \arrow[r, mapsto] & \mathrm{Fil}^1 H^1_{dR}(X_{y'})
\end{tikzcd}$$

$\qquad\qquad \cap\, GM$

$\qquad\qquad \cancel{V_v} V_{(y_0',w)}$

Main lemmas.

Lemma ~~4.1~~ 6.1 (Generic simplicity)

$\exists$ finite $F \subseteq \Omega_v \cap Y(K)^*$ s.t. $\forall y \in (\Omega_v \cap Y(K)^*) \setminus$

$\exists (y',w) \mid (y, v)$ s.t.

1) $[K(y')_w : K_v] \geq 8$

2) $\rho_{y'}$ is simple as $G_{K(y')}$-repn where

$\rho_{y'}$ is $G_{K(y')} \curvearrowright H^1_{dR}((X_{y'})_{\overline{K(y')}}, \mathbb{Q}_p)$.

Granting Lemma ~~4.1~~ 6.1 + fact that only finitely many $K(y')$ show up

Faltings' Lemma $\Rightarrow$ Each pair $(K(y')_w, \rho_{y'}|_{K(y')_w})$ is iso to some $(K_v', \rho_v') \in \Sigma$, where Σ is a finite set of pairs

- $K_v'/\mathbb{Q}_p$ finite unramified
- ρ_v' crystalline representation

Lemma 6.2 Fix $\cdot$ K_v'/K_v ~~wo~~ unram s.t. $[K_v', K_v] \geq 8$

$\cdot$ $\rho_{v'}$ a $2d$-dim crystalline repn of $G_{K_{v'}}$.

Then there are finitely many $y \in \Omega_v \cap Y(K)^*$ s.t. $\exists (y', w)$ for which

$\cdot$ $(K(y')_w, \rho_{y',w}) \underline{\simeq} (K_v', \rho_v')$

Lemma 6.1 + Lemma 6.2 + ~~Lemma~~ Faltings' Lemma $\Rightarrow$ Prop 5.3.

Proof of 6.2. $(y', w) \xleftrightarrow{GM} (y_0', w_0)$

$$\Rightarrow \rho_{y',v} \longleftrightarrow \Big(\underset{\substack{\circlearrowleft \\ \Phi_v}}{V_{y_0',w_0}},\ \underset{\substack{\cap \\ \mathcal{H}_{y_0',w_0}}}{\mathrm{Fil}^1(H^1_{dR}(X_{y'}))} \underset{GM}{\subseteq} V_{y_0',w_0}\Big)$$

$$Z(\Phi_v) := \{f \in \mathrm{End}_{K(y_0')_{\cancel{K_v w_0}\, w_0}}(V_{y_0',w_0}) ; f\circ\phi = \phi\circ f\}$$

$\forall$ two points in $\mathcal{H}_{y_0',w_0}$ arise from isom crys reps, if they are in the same $Z(\Phi_v)$-orbit. in $\mathcal{G}_{y_0',w_0}$.

Lemma 2.1 $\Rightarrow$

$(\dim_{K(y_0')_{w_0}} V_{y_0',w_0})^2$ ~~Q~~ $= 4d^2$

$\vee\!\!/$

$\dim_{K_v}(Z(\Phi_v))$

Lemma 3.2. + full monodromy

$\Rightarrow$ Φ_v has Z-dense image in $\mathcal{H}_{(y_0',w_0)}$

$\Rightarrow$ By computation,

$\dim_{K_v}\mathcal{H}_{y_0',w_0} = [K(y_0')_{w_0}, K_v] \cdot \frac{d(d+1)}{2} \geq 4d(d+1) > 4d^2$,

$\Rightarrow$ $\Phi_v^{-1}(Z(\Phi_v))$ is a proper K_v-analytic subset of Ω_v.

$\Rightarrow$ finitely many $y \in \Omega_v \cap Y(K)^*$ for which ~~$\exists(y',w), (K(y')_w, \rho_{y',w}) \simeq (K_v'$,~~ $\exists(y', w)$ s.t. $\rho_{y',w}$ is in a prescribed $Z(\Phi_v)$-orbit.

$\square$

~~Primili~~

Preliminary on p-adic rep.

$\eta : G_K \longrightarrow \mathbb{Q}_p^\times$ continuous character unram only at finitely many places, HT at p.

By CFT,

$$\begin{tikzcd} \eta : \mathbb{A}_K^\times/K^\times \arrow[r] & \mathbb{Q}_p^\times \\ \eta_p : (K\otimes\mathbb{Q}_p)^\times \arrow[u] \arrow[r] & \mathbb{Q}_p^\times \arrow[u, phantom, "\wr"] \end{tikzcd}$$

- Tate $\Rightarrow$ η, η_p are loc alg, i.e., agree locally.
- ~~$\eta : \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \longrightarrow \mathbb{G}_{m,\mathbb{Q}_p}$~~

$\eta_p : \mathrm{Res}_{K\otimes\mathbb{Q}_p/\mathbb{Q}_p}(\mathbb{G}_m) \longrightarrow \mathbb{G}_{m,\mathbb{Q}_p}$

Lemma 2.8 v friendly place of K, $\forall \rho : G_K \to \mathbb{Q}_p^\times$ continuous character $\eta : G_K \to \mathbb{Q}_p^\times$ ramified only at finitely many places, HT at p, pure of weight w, locally at each $v|p$,

$$\eta^2\big|_{K_v^\times} = \underset{\text{finite}}{\chi} \circ Nm_{K_v/\mathbb{Q}_p}^{w}, \quad \text{i.e., } HT(\eta|_{K_v^\times}) = \frac{w}{2}.$$

. . .

Def. $y \in Y(K)^* \cap \Omega_v$ is bad if

$\forall (y', w)$ above (y, v) . $[K(y')_w : K_v] \geq 8$ implies that the repn $\rho_{y'}$ is not simple.

Sublemma (bad locus in $\mathcal{H}_v$).

Let $y \in Y(K)^* \cap \Omega_v$ be bad. Then there exists

- $(y', w) | (y, v)$ with $[K(y')_w : K_v] \geq 8$,
- non-zero Frob subspace $0 \neq W^{dR}_{y', w} \subsetneq H^1_{dR}(X_{y'}/K(y')_w)$

  s.t. $\dim_{K(y')_w} Fil^1 W^{dR}_{y', w} \geq \frac{1}{2} \dim_{K(y')_w} W^{dR}_{y', w}$,

  (gives proper/closed conditions requires: size $\leq \frac{1}{d+1}$).

Proof of sublemma.

$y' \in Y'$ s.t. $\pi(y') = y$, $[K(y')_w : K_v] \geq 8$ bad,

$$0 \neq W_{y'} \subsetneq \rho_{y'}\big|_{K(y')_w} \underset{\text{Fontaine}}{\rightsquigarrow} 0 \neq \underset{\Phi_w\text{-stable}}{W^{dR}_{y', w}} \subsetneq \underset{\circlearrowleft\, \Phi_w}{H^1_{dR}(X_{y'}/K(y')_w)}.$$

$W_{y'}$ preserves bilinear form

$$\Rightarrow \dim(W^{dR}_{y', w}) \leq d.$$

AFTSOC $\dim F^1 W^{dR}_{y', w} < \frac{1}{2} \dim W^{dR}_{y', w}$,

Imput from p-adic Hodge.

1). By Fontaine,

$$D_{cris}(Ind^{K_v}_{K(y')_w} W_{y'}) \simeq \bigoplus_{w|v} W^{dR}_{y', v}$$

$$\begin{array}{c} \shortparallel \\ \left(Ind^{K_v}_{K(y')_w} W_{y'} \otimes_{\mathbb{Q}_p} B_{cris}\right)^{G_{K_v}} \end{array}$$

filtered φ-module.

2). Lemma 2.10. Relation of H-T weights

$$\sum_{w|v} [K(y')_w : K_v] \underset{< \frac{1}{2} - \frac{1}{2d}}{\frac{\dim Fil^1 W^{dR}_{y', w}}{\dim W^{dR}_{y', w}}} = [K(y')_w : K_v] \frac{\dim Fil^1 D_{cris}(W_{y'})}{\dim D_{cris}(W_{y'})}$$

3) By Lemma 2.8, $\frac{\dim Fil^1 D_{cris}(W_{y'})}{\dim D_{cris}(W_{y'})} = \frac{1}{2}$.

Sum up all y' with $\pi(y') = y$

~~$[K(y')_w : K$~~

$$\sum_{[K(y')_w : K_v] \geq 8} [K(y')_w : K_v]\left(\frac{1}{2} - \frac{1}{2d}\right) + \sum_{\sim < 8} [K(y')_w : K_v]$$

$$\geq \frac{1}{2} \sum_{\pi(y') = y} [K(y') : K] = \frac{1}{2} \sum_{(y', w)} [K(y')_w : K_v].$$

Compare terms:

$$\frac{1}{2} \sum_{\sim < 8} [K(y')_w : K_v] \geq \frac{1}{2d} \sum_{\sim \geq 8} [K(y')_w : K_v]$$

$$\rightsquigarrow \text{size}_v(\pi^{-1}(y)) \geq \frac{1}{d+1}.$$

□

Def. $\mathcal{H}^{bad}_{(y_0',w)} \subseteq \mathcal{H}_{(y_0',w)}$

$\shortparallel$

$\{F \in \mathcal{H}_{(y_0',w)} ; \exists 0 \neq \underset{\Phi_v\text{-stable}}{W} \subsetneq V \text{ s.t. } \dim(F\cap W) \geq \frac{1}{2}\dim W\}$

By the sublemma, suffice to show $\exists$ finitely many $y \in Y(K)^* \cap \Omega_v$ for which Φ_v project to $\mathcal{H}^{bad}_{(y_0',w)}$.

Lemma 6.4. Let (V,w) be a symplectic v.s. over field of char 0, $\dim V = 2d$.

E: set of r-tuples $(F_1,\cdots,F_r) \in LGr(V,w)^r$ for which $\exists 0 \neq W \subsetneq V$ s.t.

$$\dim(F_j \cap W) \geq \frac{1}{2}\dim(W), \ \forall j.$$

If $r \geq 5$, E is contained in proper Z-closed subset of $LGr(V,w)^r$ (For application, $r \geq 8$)

Proof. E closed:

$$\begin{tikzcd} \tilde{E} \arrow[d] & \subseteq & G\cancel{L}Gr(V) \times LGr(V,w)^r = X \arrow[d, "pr"] \\ E & \subseteq & LGr(V,w)^r \end{tikzcd}$$

$\tilde{E} := \{(W,F_1,\cdots,F_r) \in X ; \dim(F_j \cap W) \geq \frac{1}{2}\dim W, \forall j\}$

upper-semi continuity $\Rightarrow \tilde{E}$ closed

$\Rightarrow E$ closed.

E proper: (i.e. find $(F_1,\cdots,F_r) \notin E$, $r=5$)

$e_1,\cdots,e_d,e_1',\cdots,e_d'$ symplectic basis for V

s.t. $\langle e_i, e_i' \rangle = -\langle e_i', e_i \rangle = 1$, others $=0$.

Consider:

$F_1 = \text{Span}(e_1,\cdots,e_d)$

$F_2 = \text{Span}(e_1',\cdots,e_d')$

$F_3 = \text{Span}(e_1+e_1',\cdots,e_d+e_d')$

$F_4 = \text{Span}(e_1+2e_1',\cdots,e_d+(d+1)e_d')$

Then $F_i \cap F_j = 0$, $V = F_i \oplus F_j$.

$\forall W$, $W = (W\cap F_i) \oplus (W \cap F_j)$.

isos

$$\begin{tikzcd} & F_3 \ \ e_i+e_i' \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ F_1 \ \ e_i \arrow[rr, "\sim", "\Phi_{12,3}"'] & & F_2 \ \ e_i' \end{tikzcd} \qquad \begin{tikzcd} & F_4 \ \ e_i+(i+1)e_i' \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ F_1 \ \ e_i \arrow[rr, "\sim", "\Phi_{12,4}"'] & & F_2 \ \ (i+1)e_i' \end{tikzcd}$$

~~$\rightsquigarrow$ autoisos : $F_3 \circlearrowleft \Phi_{12,3}$, $F_4 \circlearrowleft \Phi_{12,4}$.~~

$\Phi_{12,4}^{-1}\Phi_{12,3} : W \cap F_1 \xrightarrow{\sim} W \cap F_1$.

Finitely many $W \cap F_1$ satisfy stability under $\Phi_{12,4}^{-1}\Phi_{12,3}$, same as $W \cap F_2$.

$\rightsquigarrow$ Finitely many W.

$\forall W$, $\dim(F_i \cap W) \geq \frac{1}{2}\dim W$ gives a properly closed subset of $LGr(V)$

$\Rightarrow$ Choose F_5 s.t. this is not satisfied. $\square$

Lemma 6.3 (Application of 6.2).

There is a Z-open subset $A \subset \mathcal{H}_{(y_0',w_0)}$ s.t.

$$A \cap \mathcal{H}^{bad}_{(y_0',w_0)} = \phi$$

By Lemma 3.2 + full monodromy, $\exists$ finitely many $y \in \Omega_v \cap Y(K)^{\times}$ for which . . . . . .

Big Monodromy I.

Alexander Betts

Sources

[LV]

[FM] Farb + Margalit
A primer on mapping class groups.

Notations

- Y compact oriented surface of genus ≥ 2
- $y \in Y$
- $q \neq 2$ prime
- $\mathrm{Aff}(q) = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$

§1. Aff(q)-covers

Def. A (singly ramified) $\mathrm{Aff}(q)$-cover of Y is a compact surface Z with $\pi: Z \to Y$ s.t. 1) $Z|_{Y \setminus y} \longrightarrow Y \setminus y$ is a degree q covering map whose monodromy representation

$$\mathrm{Cov}: \pi_1(Y \setminus y) \longrightarrow S_q$$

has image $\mathrm{Aff}(q)$.

2) $Z \to Y$ not a covering map.

Rmks.

- Not Galois!
- $\{\mathrm{Aff}(q)\text{-cover } Z \to Y\}/_{\text{iso}}$

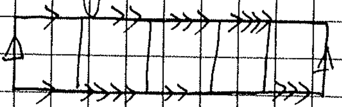
$$\longleftrightarrow \left\{ \begin{array}{l} \text{surjections } \pi_1(Y \setminus y) \to \mathrm{Aff}(q) \\ \text{s.t. } \mathrm{Cov}(\text{loop around } y) \neq 0 \end{array} \right\} \Big/ \text{conj.}$$

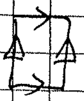
- ~~Cov~~ $\mathrm{Cov}(\text{loop around } y) \in \mathbb{F}_q^+ \setminus 0$ so acts as a q-cycle, so $\#\pi^{-1}(y) = 1$.

- ~~Imor~~ Isomorphic $\mathrm{Aff}(q)$-covers are uniquely iso

$Z(\mathrm{Aff}(q)) = \{1\}$.

Exp. Take $Y = T$ the torus, pick $g \in \mathbb{F}_q^\times$ a generator. Have a singly ~~covered~~ ramified $\mathrm{Aff}(q)$ cover given by

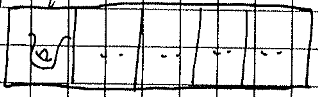




(glue top of ith square to bottom of gi-th)

unramified away from ~~torus~~ ~~cor~~ corner(s) of $\square$.

If $g \geq 2$, sew on a S_{g-1} in the middle of every square



Prop 8.5 "normal form"

$Z \to Y$ a singly ramified $\mathrm{Aff}(q)$-cover. Then we can write $Y = S_{g-1} \# T$

$$= S_{g-1} \setminus \text{disc} \bigcup_{\text{annulus}} T \setminus \text{disc}.$$

s.t.

- $Z|_{S_{g-1} \setminus \text{disc}} \longrightarrow S_{g-1} \setminus \text{disc}$ is a trivial ~~q-th cover~~ q-fold cover
- $Z|_{T \setminus \text{disc}} \longrightarrow T \setminus \text{disc}$ extends to a singly ramified $\mathrm{Aff}(q)$-cover of T.

Proof. Let Cov: $\pi_1(Y\setminus y) \twoheadrightarrow Aff(q)$ classify $Z \to Y$.

We will decompose Y as follows:

i) Cut Y along two parallel simple closed curves either side of y, s.t., if Y^1 is the piece of $Y\setminus$ curves not containing y, then

$$\pi_1(Y^1) \to \pi_1(Y\setminus y) \twoheadrightarrow Aff(q)$$

has image $\mathbb{F}_q^+$.

ii) Cut Y^1 along a simple curve between its boundary components $\rightsquigarrow Y^2$. s.t. ~~$\pi_1(Y^2)$~~

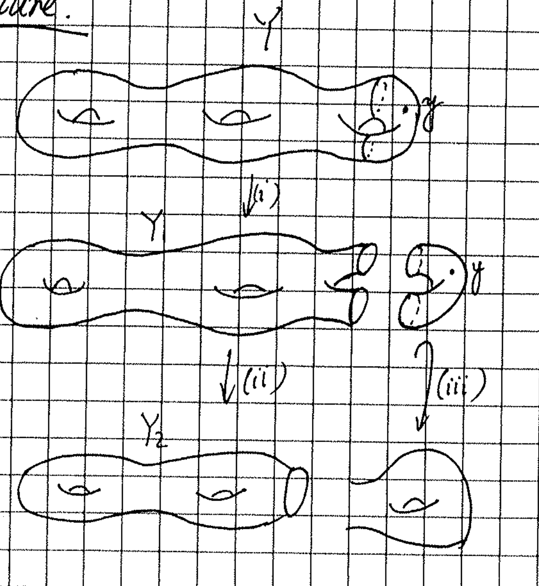
$$\pi_1(Y^2) \to \pi_1(Y\setminus y) \twoheadrightarrow Aff(q)$$

is ~~triv~~ trivial.

iii) ~~$Y^2 \setminus d$~~ $Y^2 \simeq S_{g-1}\setminus$ disc and $Z|_{Y^2} \to Y^2$ is trivial, a small nbhd of $Y\setminus Y^2$ is ~~iso~~ $\simeq T\setminus$ disc, A = annulus.

then $Z|_{T\setminus disc} \to T\setminus$ disc ~~is~~ extends to T

Picture.



For i), $\pi_1(Y\setminus y) \twoheadrightarrow Aff(q) \twoheadrightarrow \mathbb{F}_q^\times$

factor through

$$\begin{tikzcd} \pi_1(Y\setminus y)^{ab} \arrow[r, "\simeq"] & H_1(Y\setminus y, \mathbb{Z}) \arrow[r, "="] & H_1(Y,\mathbb{Z}) \arrow[r] & \mathbb{F}_q^\times \end{tikzcd}$$

PD This map is given by

$$\gamma \longmapsto g^{\gamma\cdot\alpha_1}$$

for some $\alpha_1 \in H_1(Y,\mathbb{Z})$ primitive (i.e., non-divisible)

α_1 can be represented by a simple closed curve through y. Shift left and right to find the curves α_1^+, α_1^-:

$\pi_1(Y^1) \to \pi_1(Y\setminus y) \twoheadrightarrow \mathbb{F}_q^\times$ is trivial

(+ image of α_1^+ is $\neq 0$.)

For ii), consider $\pi_1(Y^1) \xrightarrow{cov} \mathbb{F}_q^+$, factor through $H_1(Y^1,\mathbb{Z}) \longrightarrow \mathbb{F}_q^+$

PD $\Rightarrow$ This map is $\gamma \mapsto a(\gamma\cdot\alpha_2)$

with $a \in \mathbb{F}_q^+ \setminus 0$.

$\alpha_2 \in H_1(Y^1, \partial Y^1, \mathbb{Z})$ s.t.

$\partial\alpha_2 \in H_0(\partial Y^1, \mathbb{Z}) = \mathbb{Z}^2$ is $(1,-1)$

Lemma 8.4 $\Rightarrow$ α_2 can be represented by a simple curve ~~connection~~ connecting ∂Y^1.

□

Cor. Up to self-homeomorphism of $Y\setminus y$, $\exists!$ $Aff(q)$-cover

$$\begin{tikzcd} Z' \arrow[r, "\sim"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "\sim"] & Y \end{tikzcd}$$

§2. The topological Kodaira-Parshin curve family

There is a sequence

$$Z_q \xrightarrow[\text{relative surface}]{\text{proper fibration}} Y_q' \xrightarrow{\text{finite covering}} Y$$

with fibers

- $Y'_{q,y''} = \{Z \longrightarrow Y\}/iso$
- $Z_{q,y'} = \coprod_{Z \to y} Z$.

For a fixed y, $Z \longrightarrow Y$

$$H_1(Z, \mathbb{Q}) \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} H_1(Y, \mathbb{Q})$$

Define $H_1^{pr}(Z) = \mathrm{Ker}(\pi_*)$

$\rightsquigarrow H_1(Z, \mathbb{Q}) = \pi^* H_1(Y, \mathbb{Q}) \oplus H_1^{pr}(Z)$.

($\pi_*\pi^* = q$, and if $x \longmapsto \tilde{x}$ for projection, $H_1(Z, \mathbb{Q}) \longrightarrow H_1^{pr}(Z)$

then $\tilde{x} \cdot \tilde{y} = x \cdot y - \frac{1}{q} \pi_*(x) \cdot \pi_*(y)$)

$\rightsquigarrow$ monodromy action

$$\pi_1(Y, y) \xrightarrow{Mon} Sp\left(\bigoplus_{Z \to Y} H_1^{pr}(Z)\right)$$

(as action ~~p~~ respects pairing, π_*, etc.)

Thm 8.1 "big monodromy".

Z-closure of Im(Mon) contains $\prod_{Z \to Y} Sp(H_1^{pr}(Z))$

§3. Mapping class groups

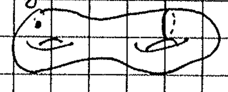
Def. $MCG = MCG(Y \setminus y) := \pi_0 \mathrm{Aut}^+(Y \setminus y)$ is the group of orientation-preserving automorphs of $Y \setminus y$ modulo the subgroup of automorphisms isotopic to identity.

(i.e., ~~is~~ autos that can be continuously deformed to the identity)

Exps. 1. Dehn twists

Take a simple closed curve $e \subseteq Y \setminus y$.

Dehn twist D_e of e is the auto of $Y \setminus y$ given by "cut along e, twist one ~~component~~ component by 360°, and stick back together.



For a tubular nbh, $S^1 \times I$,

$(\xi, t) \longmapsto (\xi \exp(2\pi i \cdot t), t)$.

2. Point-pushing maps

Given a loop $\gamma \in \pi_1(Y, y)$, $\exists$ an auto of $Y \setminus y$ given by dragging the puncture along ~~γ~~ γ

Given a contractible $\Omega \ni y$, $\forall y' \in \Omega$, $\exists$ a homeo

$$\overline{\Omega} \setminus y \xrightarrow{\sim} \overline{\Omega} \setminus y' \quad \text{fixing } \partial\Omega \text{ pointwise}$$

(unique up to ~~iso t~~ isotopy)

By choosing Ω to cover γ, we can composing all homeos $Y \setminus y' \xrightarrow{\sim}$ ~~Y~~ $Y \setminus y''$.

Thm (Birman ES)

The map $\pi_1(Y, y) \longrightarrow MCG(Y \setminus y)$ is injective, its image is normal and the quotient is $MCG(Y)$.

Extending the action to MCG.

Given $\alpha \in \mathrm{Aut}^+(Y\setminus y)$, $Z \longrightarrow Y$ $\mathrm{Aff}(q)$-cover, there is another $\mathrm{Aff}(q)$-cover $\alpha^*(Z)$ fitting into

$$\begin{tikzcd}
\alpha^*(Z) \arrow[r] \arrow[d] & Z \arrow[d] \\
Y \arrow[r, "\alpha", "\sim"'] & Y
\end{tikzcd}$$

- $\mathrm{Aut}^+(Y\setminus y) \circlearrowleft$ $\{Z \to Y$ ~~Affine~~ $\mathrm{Aff}(q)$-cover$\}/\mathrm{iso}$

  $\circlearrowleft \coprod_{Z\to Y} Z$

- $\mathrm{MCG}(Y\setminus y) \circlearrowleft \{Z \to Y\}/\mathrm{iso}$

  $\circlearrowleft H_1(\coprod_{Z\to Y} Z)$ preserves intersections and maps $\pi_* : H_1^{pr}(Z) \to H_1(Y)$

Check $\mathrm{MCG} \longrightarrow Sp(\bigoplus_{Z\to Y} H_1^{pr}(Z))$ extends the monodromy action of $\pi_1(Y, y)$

Action of Dehn twists.

Take a simple closed curve e on $Y\setminus y$

$\mathrm{MCG} \circlearrowleft H_1(Y\setminus y) = H_1(Y)$

$D_e \in$ ~~#~~ $\mathrm{Aut}(H_1(Y))$

Fact. [FM, §6.3.1]

$D_e : \gamma \longmapsto \gamma + (\gamma\cdot e)[e]$.

Let $(d_1, \dots, d_k)$ ~~by~~ be the cycle decomposition of $\mathrm{Cov}(e) \in \mathrm{Aff}(q)$ (Fixed $Z \to Y$)

(either $=(q)$ or $=(1, \underbrace{r, \dots, r}_{\frac{q-1}{r} \text{ times}})$ for $r \mid q-1$)

Choose M divisible by all d_i

$\pi^{-1}(e) = e_1 \sqcup \cdots \sqcup e_k$

with $e_i \to e$ is a d_i-to-one cover.

$\rightsquigarrow D_e^M$ acts on Z via $\prod_i D_{e_i}^{\frac{M}{d_i}}$.

Lemma 8.2 If e is ~~a non-s~~ non-separating, then $[e_1], \dots, [e_k]$ ~~or~~ $\in H_1(Z)$ are linearly independent.

Cor.
- $\mathrm{Im}(D_e^M - 1|_{H_1(Z)}) = \mathrm{Span}([e_1], \dots, [e_k])$
- $rk(D_e^M - 1|_{H_1^{pr}(Z)}) = k-1$
- $rk(D_e^M - 1|_{H_1^{pr}(Z)})$ determines the cycle decomposition of $\mathrm{Cov}(e)$.
- If $\mathrm{Cov}(e)$ maps to a generator of $\mathbb{F}_q^\times$ (cycle of type $(1, q-1)$), $\pi^{-1}(e) = e^+ \sqcup e^-$

  $e^+ \xrightarrow{\sim} e$

  $e^- \longrightarrow e$ $(q-1)$-to-1,

  $D_e^M|_{H_1^{pr}(Z)}$ is a ~~transaction~~ transvection in direction $\tilde{e} := [e^+]$.

Big Monodromy II.

Netan Dogra

$Y, Z \to Y$

$\pi_1(Y,y)_0 \xrightarrow{\prod\rho_i} \prod_{i=1}^{N} Sp(H_1^{pr}(Z_i))$

Thm 8.1

$$\overline{Im(\pi_1(Y,y)_0)}^{Z} = \prod_{i=1}^{N} Sp(H_1^{pr}(Z_i))$$

Birman exact sequence

$$1 \to \pi_1(Y,y) \to MCG(Y\setminus y) \to MCG(Y) \to 1.$$

$$\pi_1(Y,y)_0 \longrightarrow MCG(Y-y)_0$$

$\prod\rho_i \circlearrowright \qquad \circlearrowleft \prod\tilde{\rho}_i$

$\bigoplus H_1^{pr}(Z_i)$

$(V, \langle\, , \rangle)$

$T_v : w \longmapsto w + \langle w, v\rangle v$

Lemma 2.12 $G < Sp(V)^N$ s.t.

(1) $\forall i$, $\pi_i : G \to Sp(V)$ is surjective

(2) $\forall i \neq j$, $\exists g \in G$, $\pi_i(g)$ and $\pi_j(g)$ are unipotent & $\dim(V^{\pi_i(g)}) \neq \dim(V^{\pi_j(g)})$ $(*)$

Then $G = \prod_{i=1}^{N} Sp(V)$.

Lemma 2.14. $S \subset V$. Suppose $\forall v, w \in S$,

$\exists v_1, \dots, v_n \in S$, $v_1 = v$, $v_n = w$, and

$$\prod_{i=1}^{n-1} \langle v_i, v_{i+1}\rangle \neq 0. \qquad (**)$$

$\overline{\langle T_v ; v \in S\rangle}^{Zar} \ni T_v$, $\forall v \in \langle w ; w \in S\rangle$.

Strategy for 8.1.

(0) Prove $Im(\prod\rho_i)$ not central

(1) $(\tilde{\rho}_i)_i$ satisfies $(*)$

(2) $\exists$ transvection~~s~~ ~~T_v~~ T_v in $Im(\rho_i)$ for $v \in S$ ~~s.t.~~ satisfying $(**)$, and generating $H_1^{pr}(Z_i)$.

(0) follows from $Y \to \mathcal{A}_g$ non-constant.

Proof of 2.12

Show $\mathfrak{g} = \prod\mathfrak{sp}(V)$.

Goursat's lemma. L_1, L_2 Lie ~~to~~ algebras

$$\begin{tikzcd} & L_3 \subset L_1 \times L_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ L_1 & & L_2 \end{tikzcd}$$

sub Lie algebra s.t.

p_1 & p_2 surjectives, $I_j = Ker(p_j)$

~~$I_1 \triangleleft L_2 I_1 \triangleleft L_1$~~

$I_1 \triangleleft L_2$, $I_2 \triangleleft L_1$

Then image of L_3 in $L_1/I_2 \times L_2/I_1$ is the graph of an isomorphism.

$$\begin{tikzcd} & L_3/I_1+I_2 \arrow[dl, "\sim"'] \arrow[dr, "\sim"] \arrow[d, phantom, "G"] & \\ L_1/I_2 \arrow[rr, "\sim"] & {} & L_2/I_1 \end{tikzcd}$$

all automorphisms of $\mathfrak{sp}(V)$ come from conjugation

Induction on N. $\square$

Proof of 2.14 WLOG, $\#S = n < \infty$.

$n = 2$; $v_1, v_2 \in V$, $\langle v_1, v_2\rangle \neq 0$

$V' = \langle v_1, v_2\rangle$ $V'' = (V')^{\perp}$

$Sp(V') \subset Sp(V)$.

$\quad \shortparallel$

SL_2.

$T_{v_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $T_{v_2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

generates SL_2.

Induction.

$v_1, \dots, v_n$

$\langle T_{v_1}, \cdots, T_{v_n} \rangle^{Zar} \supset \langle T_v ; v \in \langle v_1, \cdots, v_{n-1} \rangle \rangle$

~~$\forall$~~ $w \in \langle v_1, \dots, v_{n-1} \rangle$

$T_v \in \langle T_{v_i}, i \leq n \rangle^{Zar}$, $v = w + v_n$.

$\square$

(1) Show for $Z_1 \to Y$, $Z_2 \to Y$ non-iso,

$\exists \gamma \in MCG(Y \setminus y)_0$ ~~s.t.~~ unipotent image in $Sp(H_1^{pr}(Z_i))$ $_{i=1,2}$

s.t. $\dim H_1^{pr}(Z_1)^{\gamma} \neq \dim H_1^{pr}(Z_2)^{\gamma}$.

How to final γ?

e simple closed curve on $Y \setminus y$,

$Cov_1(e)$ $Cov_2(e)$ $\in Sym(\mathbb{F}_q)$

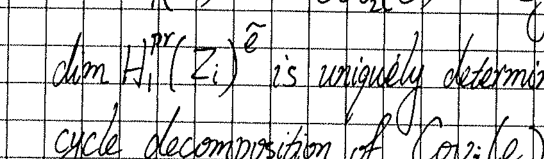
$\dim H_1^{pr}(Z_i)^{\tilde{e}}$ is uniquely determined by cycle decomposition of $Cov_i(e)$.

$$\begin{tikzcd} \pi_1(Y \setminus y) \arrow[r] & Aff(q) \\ {[e]} \arrow[u, phantom, "\in" rotate=90] & \end{tikzcd}$$

$f_1, f_2 : \pi_1(Y \setminus y) \longrightarrow \mathbb{F}_q^{\times}$

$Ker f_1 \neq Ker f_2$ ✓.

Suppose $Ker f_1 = Ker f_2$.



$g_1, g_2 : \pi_1(Y') \longrightarrow \mathbb{F}_q^{+}$.

If $Ker(g_1) \neq Ker(g_2)$

$\rightsquigarrow$ simple closed curve e on Y' s.t.

$g_1(e) = 0$, $g_2(e) \neq 0$

$\Rightarrow$ e is a q-cycle acting on $\mathbb{F}_q$ via $Z_2 \to Y$

is trivial acting on $\mathbb{F}_q$ via $Z_1 \to Y$.

✓

If $Ker(f_1) = Ker(f_2)$, $Ker(g_1) = Ker(g_2)$.

Z_1 & Z_2 have the same normal form decomposition, i.e., same decomposition of Y as $S_{g-1} \# T$ in normal form.

Choose loops $\beta_1, \beta_2 \in T - D \cup \{y\}$ generating $\pi_1(T - \{y\})$ s.t.

$Cov_1, Cov_2 : \pi_1(T - \{y\}) \longrightarrow Aff(q)$

with $Cov_1(\beta_1) = \begin{pmatrix} \alpha_1 & \\ & 1 \end{pmatrix}$

$Cov_2(\beta_1) = \begin{pmatrix} \alpha_2 & \\ & 1 \end{pmatrix}$

$Cov_i(\beta_2) = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$\rightsquigarrow Cov_1 \neq Cov_2 \Rightarrow \alpha_1 \neq \alpha_2$.

We can construct a word w in β_1, β_2 s.t. $Cov_1(w)$ and $Cov_2(w)$ have different cycle type

Fix $Z \to Y$, $V_f = H_1^{pr}(Z)$

(2) $S \subset V_f$ s.t. $T_v \in Im(MCG(Y-y)_0)$, $\forall v \in S$

S generates $H_1^{pr}(Z)$

$\forall v, w \in S$, $\exists v_1, \dots, v_n \in S$, $v_1 = v$, $v_n = w$

$\prod_{i=1}^{n-1} \langle v_i, v_{i+1} \rangle \neq 0$.

e on $Y - y$ is liftible if its image in $\mathbb{F}_q^{\times}$ is a generator.

Lemma 8.10. $\exists S$ set of liftable curves on $Y - y$

s.t. 1) S generates $H_1^{pr}(Z)$

2) $\forall v, w \in S$, $\exists v_1, \cdots, v_n \in S$, $v_1 = v$, $v_n = w$,

$$\prod_{i=1}^{n-1} \langle v_i, v_{i+1} \rangle \neq 0.$$

Lemma 8.11. $Z \to Y$ normal form $(T^0 = T - \overset{disc}{D})$

$\exists$ simple closed curves $\gamma_0, \cdots, \gamma_q$ in $T^0 - *$ satisfying:

1) $\exists \alpha \in \mathbb{F}_q^\times$ generator, $\forall j$,

$$\mathrm{Cov}(\gamma_j) \longmapsto \alpha$$

2) $\forall j$, $(\mathbb{F}_q)^{\gamma_j} = \{j\}$

3) γ^+ generates the homology of

$$Z \times T^0 = \widetilde{T}$$

modulo boundary

4) γ_j all pass through $p \in \partial \mathrm{Disc}$ with same orientation.

$8.11 \Rightarrow 8.10$:

Let W be a set of simple closed curves on S_{g-1} lifting the set of primitive elements of $H_1(S_{g-1})$ all passing through p.

$A(w, j)$ simple closed curve representing $\gamma_j \cdot w^{\pm 1}$, $\forall w \in W$, γ_j as in 8.11.

(avoid ~~into~~ self intersection?)

$\widetilde{A}(w, j)$ generate $H_1^{pr}(Z)$.

Remaining point

$w_1, j_1, w_2, j_2 \rightsquigarrow \exists w_3, j_3$ s.t.

$$\widetilde{A}(w_1, j_1) \cdot \widetilde{A}(w_3, j_3) \neq 0$$

$$\widetilde{A}(w_2, j_2) \cdot \widetilde{A}(w_3, j_3) \neq 0.$$

$$[\widetilde{A}(w_1, j_1)] \cdot [\widetilde{A}(w_2, j_2)]$$

$= \pm [\widetilde{w}_1] [\widetilde{w}_2] + [\widetilde{\gamma}_{j_1}] \cdot [\widetilde{\gamma}_{j_2}] \pm$ ~~$[w_1][w_2]$~~

~~$+(z_1 - \frac{1}{q})[\gamma_{j_1}][\gamma_{j_2}]$~~

$= \pm [w_1][w_2] + (z_1 - \frac{1}{q})[\gamma_{j_1}] \cdot [\gamma_{j_2}]$.

$\neq 0$ $(\ldots)$