GALOIS REPRESENTATIONS AND MOTIVES

M2 - SORBONNE UNIVERSITÉ, 2024-25

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k: field, $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ category of finite dimensional continuous $\mathbb{Q}_\ell$-representations of $\pi_1(k)$, that is

- Objects: pairs (V, ρ), with V a finite-dimensional $\mathbb{Q}_\ell$-vector space and $\rho : \pi_1(k) \to \mathrm{GL}(V)$ a continuous morphism of topological groups, where $\mathrm{GL}(V) \simeq \mathrm{GL}_r(\mathbb{Q}_\ell)$ is endowed with the ℓ-adic topology and $\pi_1(k)$ with the profinite topology.
  When no confusion can arise, we simply write $V := (V, \rho)$ and $\sigma \cdot v := \rho(\sigma)(v)$.
- Morphisms $(V_1, \rho_1) \xrightarrow{f} (V_2, \rho_2)$: Morphisms $V_1 \xrightarrow{f} V_2$ of $\mathbb{Q}_\ell$-vector spaces such that

$$f \circ \rho_1(\sigma) = \rho_2(\sigma) \circ f, \quad \sigma \in \pi_1(k).$$

It immediately follows from the compactness of $\pi_1(k)$ and the continuity of ρ that for every $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ the subgroup $\Pi := \rho(\pi_1(k)) \subset \mathrm{GL}(V)$ is a compact subgroup of $\mathrm{GL}(V)$. In particular, the following lemma applies.

em:Image **Lemma 2.1.** *Let V be a finite dimensional $\mathbb{Q}_\ell$-vector space. Then, for every compact subgroup $\Pi \subset \mathrm{GL}(V)$ and $\mathbb{Z}_\ell$-lattice $\Lambda \subset V$, there exists $g \in \mathrm{GL}(V)$ such that $g\Pi g^{-1}$ stabilizes Λ.*

Proof. As $\Lambda \subset V$ is open, the subgroup $S := Stab_\Pi(\Lambda) \subset \Pi$ is an open subgroup of Π (if $\lambda_1, \ldots, \lambda_r$ is a $\mathbb{Z}_\ell$-basis of Λ, S is the inverse image of the open subset $\Lambda^{\oplus r} \subset V^{\oplus r}$ of $V^{\oplus r}$ by the continuous map $\Pi \to V^{\oplus r}$, $\sigma \to (\sigma\lambda_1, \ldots, \sigma\lambda_r)$). As Π is compact $S \subset \Pi$ finite index (consider the open covering

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$\Pi = \bigsqcup_{\sigma\in\Pi/S} \sigma S)$. So that one gets a Π-stable $\mathbb{Z}_\ell$-lattice $\widetilde{\Lambda} := \sum_{\pi\in\Pi/S} \pi\Lambda$. But, then, for every $g \in \mathrm{GL}(V)$ such that $g\widetilde{\Lambda} = \Lambda$, $(g\Pi g^{-1})\Lambda = g\Pi\widetilde{\Lambda} \subset g\widetilde{\Lambda} = \Lambda$, as claimed. □

The category $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is very nice. It is a neutral Tannakian category over $\mathbb{Q}_\ell$. We are not going to give now the formal definition of a neutral Tannakian category over a field but let us stress that it means, in particular, that is it a $\mathbb{Q}_\ell$-linear abelian category which is endowed with a natural $\otimes$-structure:

$$(V_1, \rho_1) \otimes (V_2, \rho_2) = (V_1 \otimes_{\mathbb{Q}_\ell} V_2, \rho_1 \otimes_{\mathbb{Q}_\ell} \rho_2)$$

and which admits

- a trivial object: $\mathbb{I} = \mathbb{Q}_\ell$ endowed with the trivial action of $\pi_1(k)$;
- duals: $(V, \rho)^\vee = (V^\vee, \rho^\vee)$, where $\rho^\vee(\sigma)(f) = f \circ \rho(\sigma)^{-1}$;
- inner Hom: $\mathrm{Hom}((V_1, \rho_1), (V_2, \rho_2)) = (V_1, \rho_1)^\vee \otimes (V_2, \rho_2)$.

Furthermore, for every $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$, let $\langle V, \rho\rangle^\otimes \subset \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ denote the Tannakian subcategory generated by (V, ρ) that is the smallest full subcategory containing $\mathbb{I}$ and stable by subquotients, $\otimes$, $(-)^\vee$, $\otimes$ and let $G \subset \mathrm{GL}_V$ denote the Zariski-closure of the image $\Pi := \rho(\pi_1(k))) \subset \mathrm{GL}(V)$ in GL_V. Then the canonical forgetful functor

$$\omega : \langle V, \rho\rangle^\otimes \to \mathrm{Vect}_{/\mathbb{Q}_\ell}, \ \ (W, \rho_W) \mapsto W$$

factors through an equivalence of $\otimes$-categories

$$\begin{tikzcd} \langle V,\rho\rangle^\otimes \arrow[r, "\omega"] \arrow[d, "\simeq"'] & \mathrm{Vect}_{/\mathbb{Q}_\ell} \\ \mathrm{Rep}_{\mathbb{Q}_\ell}(G), \arrow[ur, "\mathrm{For}"'] & \end{tikzcd}$$

where $\mathrm{Rep}_{\mathbb{Q}_\ell}(G)$ denotes the category of $\mathbb{Q}_\ell$-rational algebraic representations of G.

2.1. **Localizations.** $k/\mathbb{Q}$: number field, $v \in \Sigma_k$, recall that for every $\bar{v} \in \Sigma_{\bar{k},v}$ one has a canonical isomorphism of profinite groups

$$\pi_1(k) \supset D_{\bar{v}}(\overline{k}|k) \overset{\alpha_{\bar{v}}}{\tilde{\to}} \pi_1(\hat{k}^v)$$

so that to every $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ one can attach a *local representation*

$$\rho_{\bar{v}} := \rho \circ \alpha_{\bar{v}}^{-1} : \pi_1(\hat{k}^v) \to \mathrm{GL}(V)$$

From the commutative diagram

$$\begin{tikzcd} D_{\sigma\bar{v}}(\overline{k}|k) \arrow[r, "\alpha_{\sigma\bar{v}}", "\simeq"'] \arrow[d, "\sigma-\sigma^{-1}"', "\simeq"] & \pi_1(\hat{k}_v) \\ \sigma D_{\bar{v}}(\overline{k}|k)\sigma^{-1} \arrow[ur, "\simeq", "\sigma\alpha_{\bar{v}}"'] & \end{tikzcd}$$

one gets $\rho_{\sigma\bar{v}} = \rho(\sigma)\rho_{\bar{v}}(-)\rho(\sigma)^{-1}$ *viz.* the Π-conjugacy class of $\rho_{\bar{v}}$ depends only on v and not on $\bar{v} \in \Sigma_{\bar{k},v}$; we will usually denote by $\rho_v : \pi_1(\hat{k}^v) \to \mathrm{GL}(V)$ an arbitrary representative $\rho_{\bar{v}}$ in this Π-conjugacy class.

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2.2. **Ramification.**

- $k/\mathbb{Q}_p$ p-adic field. One says that $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is *unramified* (resp. *semistable*) if $I_k \subset \ker(\rho)$ *viz* $\rho(I_k) = 1$ (resp. $\rho(I_k) \subset \mathrm{GL}_(V)$ is unipotent). One says that $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is *potentially unramified* (resp. *potentially semistable*) if there exists a finite field extension K/k such that $(V, \rho|_{\pi_1(K)})$ is *unramified* (resp. *semistable*).

  If $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is unramified, one has a canonical factorization

$$\begin{tikzcd} \pi_1(k) \arrow[r, "\rho"] \arrow[d] & \mathrm{GL}(V)\ , \\ \pi_1(k)/I_k \simeq \mathrm{Gal}(k^{\mathrm{ur}}|k) \simeq \pi_1(k^-) \arrow[ur, "\rho^-"'] & \end{tikzcd}$$

which enables to define the *characteristic polynomial of the (geometric) of* ρ

$$\chi_\rho(T) := \det(T - \overline{\rho}(F_{k^-})Id) \in \mathbb{Z}_\ell[T],$$

where F_{k^-} = is the inverse of the arithmetic Frobenius $\phi_{k^-} : x \to x^{|k^-|}$ pro-generating $\pi_1(k^-)$.

- $k/\mathbb{Q}$: number field, $v \in \Sigma_k$. One says that $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is *unramified* (resp. *semistable*, resp. *potentially unramified*, resp. *potentially semistable*) at v if one (equivalently every) localization $\rho_v : \pi_1(\hat{k}^v) \to \mathrm{GL}(V)$ of ρ at v is. If $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is unramified at v, the characteristic polynomial
$$\chi_{\rho,v}(T) := \chi_{\rho_v} \in \mathbb{Z}_\ell[T]$$
is independent of the choice of the localization $\rho_v : \pi_1(\hat{k}^v) \to \mathrm{GL}(V)$ of ρ at v and is called the the *characteristic polynomial of the (geometric) of* ρ *at* v.

We will write $U_{\rho,\nmid\ell} \subset \Sigma_{k,\nmid\ell}$ for the set of all $v \in \Sigma_{k,\nmid\ell}$ such that (V, ρ) is unramified at v. Given a subset $U \subset \Sigma_{k,\nmid\ell}$, let $\mathrm{Rep}^{U,ss}_{\mathbb{Q}_\ell}(\pi_1(k)) \subset \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ denote the full subcategory of all (V, ρ) which are semisimple with $U \subset U_{\rho,\nmid\ell}$. Then, if $\Sigma_k \setminus U$ is finite, we have seen as a consequence of the Cebotarev density theorem that the canonical map
$$\mathrm{Rep}^{U,ss}_{\mathbb{Q}_\ell}(\pi_1(k))/_\simeq \hookrightarrow \mathbb{Z}_\ell[T]^U, \quad (V, \rho) \mapsto (\chi_{\rho,v}(T))_{v \in U}$$
is injective.

The reason why we exclude the places dividing ℓ in defining U_ρ is that the ramification behaviour at places not dividing ℓ and at those dividing ℓ differs drastically; in the latter case, the notion of being unramified or semistable as defined above are not really pertinent or, rather, have to be significantly generalized to be pertinent. Let us describe more precisely what happens.

2.2.1. *Places not dividing* ℓ. $k/\mathbb{Q}_p$ p-adic field, $\ell \neq p$ prime. The following is a consequence of the structure of $\pi_1(k)$.

mistable **Lemma 2.2.** (Grothendieck) *Every object* $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ *is potentially semistable.*

Proof. From Lemma 2.1, one may assume there is a $\mathbb{Z}_\ell$-lattice $\Lambda \subset V$ such that $\Pi := \rho(\pi_1(k)) \subset \mathrm{GL}(\Lambda)$. Up to replacing k by the finite field extension K/k such that $\pi_1(K) = \ker(\pi_1(k) \xrightarrow{\rho} \mathrm{GL}(\Lambda) \twoheadrightarrow \mathrm{GL}(\Lambda/\ell))$, one may assume $\Pi \subset 1 + \ell\mathrm{End}(\Lambda)$, which is a *pro-*$\ell$ *group* so that $\rho : \pi_1(k) \to \mathrm{GL}(\Lambda)$ factors *via* the pro-ℓ completion $\pi_1(k) \twoheadrightarrow \pi_1(k)^{(\ell)} \simeq \mathbb{Z}_\ell(1) \rtimes \mathbb{Z}_\ell$ as $\rho : \mathbb{Z}_\ell(1) \rtimes \mathbb{Z}_\ell \to 1 + \ell\mathrm{End}(\Lambda) \subset \mathrm{GL}(\Lambda)$. We are thus to show that, if $t \in \mathbb{Z}_\ell(1)$ is a pro-ℓ generator, $\rho(t) \in 1 + \ell\mathrm{End}(\Lambda) \subset \mathrm{GL}(\Lambda)$ is unipotent or, equivalently, that $u := \log(\rho(t))$ (which is well defined as $\Pi \subset 1 + \ell\mathrm{End}(\Lambda)$) is nilpotent *viz* has characteristic polynomial $\chi_u(T) = T^r$. For every $\sigma \in \pi_1(k)$ one has

$$\rho(\sigma)\rho(t)\rho(\sigma)^{-1}) = \rho(\sigma t \sigma^{-1}) = \rho(t^{\chi_{\ell^\infty}(\sigma)}) = \rho(t)^{\chi_{\ell^\infty}}$$

so $\rho(\sigma)u\rho(\sigma)^{-1}) = \chi_{\ell^\infty}(\sigma)u$, which ensures $\chi_u(T) = \chi_{\chi_{\ell^\infty}(\sigma)}(T) \in \mathbb{Z}_\ell[T]$. Writing $\chi_u(T) = T^r + \sum_{0 \leq i \leq r} a_i(u)T^{r-i} \in \mathbb{Z}_\ell[t]$, this is equivalent to $a_i(\chi_{\ell^\infty}(\sigma)u) = \chi_{\ell^\infty}(\sigma)^i a_i(u)$. But as $k/\mathbb{Q}_p$ is a finite field extension, the image of the ℓ-adic cyclotomic character, $\chi_{\ell^\infty} : \pi_1(k) \to \mathbb{Z}_\ell^\times (\simeq \mathbb{Z}_\ell \times \mathbb{F}_\ell^\times)$ is infinite (and even open). In particular, one can always find $\sigma \in \pi_1(k)$ such that $\chi_{\ell^\infty}(\sigma)$ is not torsion hence $\chi_u(T) = T^r$. $\square$

2.2.2. *Places dividing* ℓ. $k/\mathbb{Q}_p$ p-adic field. When $\ell = p$, the proof of Lemma 2.2 completely fails and the situation is more complicated. This is the realm of p-adic Hodge theory. We give below a brief overview of it, mostly for cultural purpose as we will only resort to the notion of Hodge-Tate representation in the following.

2.2.2.1. *General formalism of* B*-admissible representations.* We will apply the formalism below with $G = \pi_1(k)$, $F = \mathbb{Q}_p$ and B one of Fontaine's period ring.

B: topological, commutative integral ring endowed with a continuous action of a topological group G, $E := B^G$, $F \subset E$ a closed subfield. Let $\mathrm{Rep}_B(G)$ denote the category of B-representation of G, that is

- Objects: free B-modules of finite rank M endowed with a continuous semilinear action of G, meaning that $g\cdot(b\mu+\mu')=(g\cdot b)(g\cdot\mu)+g\cdot\mu'$, $g\in G$, $b\in B$, $\mu,\mu'\in M$.
- Morphisms: Morphisms $M_1\xrightarrow{f}M_2$ of B-modules such that $f(g\cdot -)=g\cdot f(-)$, $g\in G$.

Consider the functor

$$D_B:\mathrm{Rep}_F(G)\overset{B\otimes_F -}{\to}\mathrm{Rep}_B(G)\overset{(-)^G}{\to}\mathrm{Mod}/_E,\ \ V\mapsto(B\otimes_F V)^G$$

and the morphism of B-modules

$$\alpha_V:B\otimes_E D_B(V)\to B\otimes_F V,\ \ b\otimes\delta\mapsto b\delta.$$

One says that B is (F,G)-regular if (i) $(E=)B^G=Frac(B)^G$ and (ii) for every $0\neq b\in B$ such that $Fb\subset B$ is G-stable, one has $b\in B^\times$.

dgeBasic **Lemma 2.3.** *Assume B is (F,G)-regular. Then,*

(1) For every $V\in\mathrm{Rep}_F(G)$, $\alpha_V:B\otimes_E D_B(V)\hookrightarrow B\otimes_F V$; in particular, $\dim_E(D_B(V))\leq\dim_F(V)<+\infty$. Furthermore, the following properties are equivalent.

(i)	$\dim_E(D_B(V))=\dim_F(V)$;
(ii)	$\alpha_V:B\otimes_E D_B(V)\tilde{\to}B\otimes_F V$ *is an isomorphism;*
(iii)	$B\otimes_F V\simeq B^{\oplus r}$ *in* $\mathrm{Rep}_B(G)$,

in which case one says that V is a B-admissible representation.

(2) Let $\mathrm{Rep}_F^B(G)\subset\mathrm{Rep}_F(G)$ denote the full subcategory of B-admissible representations. Then,

(a) $\mathrm{Rep}_F^B(G)\subset\mathrm{Rep}_F(G)$ is a Tannakian subcategory (viz is an abelian subcategory containing $\mathbb{I}$ and stable by subquotient, $\oplus$, $\otimes$, $(-)^\vee$).

(b) The restricted functor $D_B:\mathrm{Rep}_F^B(G)\to\mathrm{Vect}/_E$ is an exact, faithfull $\otimes$-functor.

Proof. We prove (1). For the injectivity of $\alpha_V:B\otimes_E D_B(V)\to B\otimes_F V$, from the canonical commutative diagram

$$\begin{tikzcd}
B\otimes_F V \arrow[rr, hook] & & Frac(B)\otimes_F V \\
B\otimes_E D_B(V) \arrow[u, "\alpha_V"] \arrow[r, hook] & B\otimes_E D_{Frac(B)}(V) \arrow[r, hook] & Frac(B)\otimes_E D_{Frac(B)}(V), \arrow[u]
\end{tikzcd}$$

and as $Frac(B)$ is again (F,G) regular, one may replace B by $Frac(B)$ hence assume B is a field. We are to prove that if $/delta_1,\dots,\delta_r\in D_B(V)$ are E-linearly independent then they are B-linearly independent. We argue by induction on r. If $r=1$, there is nothing to prove. If $r\geq 2$, les $b_1,\dots,b_r\in B$ not all zero, such that $\sum_{1\leq i\leq r}b_i\delta_i=0$. By induction, $b_i\neq 0$, $i=1,\dots,r$ hence, as B is a field, up to replacing b_i with b_i/b_1, $i=1,\dots,r$, oma $b_1=1$ and

$$\delta_1=-\sum_{2\leq i\leq r}b_i\delta_i.$$

Using that $g\cdot\delta_i=\delta_i$, $i=1,\dots,r$, one thus gets

$$-\sum_{2\leq i\leq r}b_i\delta_i=\delta_1=g\cdot\delta_1=-\sum_{2\leq i\leq r}gb_ig\cdot\delta_i=-\sum_{2\leq i\leq r}g(b_i)\delta_i.$$

hence

$$\sum_{2\leq i\leq r}(gb_i-b_i)\delta_i=0,$$

which, by induction, forces $b_i=gb_i$, $i=1,\dots,r$. As this holds for every $g\in G$, we thus get $b_i\in B^G=E\supset F$, $2=1,\dots,r$: a contradiction. For the second part of (1), we no longer assume B is a field (in that case, the equivalences are obvious!). the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are easy. We prove the implication (i) $\Rightarrow$ (ii). Let $v_1,\dots,v_r\in V$ a F-basis and $\delta_1,\dots,\delta_r\in D_B(V)$ a E-basis. As $1\otimes v_1,\dots,1\otimes v_r$ is a B-basis of $B\otimes_F V$ one can write

$$\delta_j=\sum_{1\leq i\leq r}b_{i,j}1\otimes v_i$$

for some $r\times r$-matrix $\Omega=(b_{i,j})_{1\leq i,j\leq r}$ with coefficients in B. The injectivity of α_V is equivalent to $b:=\det\Omega\neq 0$ and the surjectivity to $b\in B^\times$. Let $\psi:G\to F^\times$ be the character corresponding to the G-representation $\bigwedge_F^r V=Fv_1\wedge\cdots\wedge v_r$. As $D_B(V)$ is a free B-module of the same rank as $B\otimes_F V$, one has

$$\bigwedge_B^r\alpha_V:\bigwedge_B^r D_B(V)\hookrightarrow\bigwedge_B^r(B\otimes_F V)\simeq B\otimes_F\bigwedge_F^r V=Bv_1\wedge\cdots\wedge v_r$$

and $\delta_1\wedge\cdots\wedge\delta_r=bv_1\wedge\cdots\wedge v_r$. Then, for every $g\in G$,

$$\delta_1\wedge\cdots\wedge\delta_r=g\cdot\delta_1\wedge\cdots\wedge g\delta_r=g(b)g\cdot v_1\wedge\cdots\wedge v_r=gb\psi(g)v_1\wedge\cdots\wedge gv_r$$

hence $gb=\psi(g)^{-1}b$, $g\in G$, which, by the defining condition (ii) of (F,G)-regularity ensures $b\in B^\times$.

Deducing (2) from (1) is formal. For instance, let $0\to V'\to V\to V''\to 0$ be a short exact sequence in $\mathrm{Rep}_F(G)$ with $V\in\mathrm{Rep}_F(G)$. As $D_B:\mathrm{Rep}_F(G)\to\mathrm{Vect}/_E$ is left exact, one gets an exact sequence of E-vector spaces

SES1 (1) $$0\to D_B(V')\to D_B(V)\to D_B(V'')$$

As

$$\begin{aligned}\dim_E(D_B(V'))+\dim_E(D_B(V''))&\geq\dim_E(D_B(V))\\&=\dim_F(V)\\&=\dim_F(V')+\dim_F(V'')\geq\dim_E(D_B(V'))+\dim_E(D_B(V'')),\end{aligned}$$

one gets $\dim_E(D_B(V))=\dim_E(D_B(V'))+\dim_E(D_B(V''))$ hence (1) is a short exact sequence and, the inequalities $\dim_E(D_B(V'))\leq\dim_F V'$, $\dim_E(D_B(V''))\leq\dim_F V''$ are necessarily equalities that is $V',V''\in\mathrm{Rep}_F^B(G)$. This shows $\mathrm{Rep}_F^B(G)\subset\mathrm{Rep}_F(G)$ is stable under subobject and $D_B:\mathrm{Rep}_F^B(G)\to\mathrm{Vect}/_E$ is an exact. For the remaining part of the assertion see *e.g.* BrinonConrad [BrCo09, Thm. 5.2.1]. □

Fontaine

2.2.2.2. *Back to p-adic representations.* The main output of p-adic Hodge theory in the construction of various period rings B_τ which captures the significant behaviour of p-adic represenations arising from geometry. These period rings B_τ are topological, commutative integral rings endowed with a continuous action of $G=\pi_1(k)$ such that $E_\tau:=B_\tau^G\supset F:=\mathbb{Q}_p$; they are $(\mathbb{Q}_p,\pi_1(k))$-regular. The following table summarizes the most important of these period rings and related structures.

Related cohomology theory	τ	E_τ	Related categorical structure $\mathcal{C}_\tau$
Hodge-Tate	HT	k	Gr_k: $\mathbb{Z}$-graded finite dim. k-vector spaces $V=\oplus_{n\in\mathbb{Z}}V^n$
de Rham	dR	k	Fil_k: $\mathbb{Z}$-filtered finite dim. k-vector spaces $V=\cup_n F^nV\cdots\supset F^{n+1}V\supset F^nV\supset\cdots\cap_n F^nV=0$
Crystalline	cris	k_0	FilM_k^φ: Filtered φ-module over k *i.e.* $(V_0,\varphi,F^\bullet V)$ V_0: finite dim. k_0-vector spaces, $\varphi:V_0\tilde{\to}V_0$ σ-semilinear automorphism $(V,F^\bullet V)\in\mathrm{Fil}_k$.
Log-crystalline	st	k_0	$\mathrm{FilM}_k^{(\varphi,N)}$: Filtered φ-module over k *i.e.* $(V_0,\varphi,N,F^\bullet V)$ $(V_0,\varphi,F^\bullet V)\in\mathrm{FilM}_k^\varphi$ $N:V_0\to V_0$ k_0-linear nilpotent endomorphism such thta $N\varphi=p\varphi N$.

Here $\mathbb{Q}_p\subset k_0:=k\cap\mathbb{Q}_p^{\mathrm{ur}}\subset k$ is the maximal unramified subextension of $\mathbb{Q}_p$ contained in k and $\sigma:k_0\to k_0$ its (arithmetic) Frobenius.

For simplicity, write $\mathrm{Rep}_{\mathbb{Q}_p}^\tau(\pi_1(k)):=\mathrm{Rep}_{\mathbb{Q}_p}^{B_\tau}(\pi_1(k))$, $D_\tau:=D_{B_\tau}$ and call B_τ-admissible objects just τ object. In each case one has a factorization

$$\begin{tikzcd}\mathrm{Rep}_{\mathbb{Q}_p}^{B_\tau}(\pi_1(k)) \arrow[r, "D_\tau"] \arrow[d] & \mathrm{Vect}_{/E_\tau} \\ \mathcal{C}_\tau \arrow[ur, "\mathrm{For}"'] & \end{tikzcd}$$

As already observed, $D_\tau : \mathrm{Rep}^{B_\tau}_{\mathbb{Q}_p}(\pi_1(k)) \to \mathcal{C}_\tau$ is always exact and faithful. For $\tau = \mathrm{HT}, \mathrm{dR}$, $D_\tau : \mathrm{Rep}^{B_\tau}_{\mathbb{Q}_p}(\pi_1(k)) \to \mathcal{C}_\tau$ is not full but for $\tau = \mathrm{cris}, \mathrm{st}$, $D_\tau : \mathrm{Rep}^{B_\tau}_{\mathbb{Q}_p}(\pi_1(k)) \to \mathcal{C}_\tau$ is also full - hence induces an equivalence of categories onto their essential image. We even have explicit quasi-inverses:

$$\begin{aligned}
&\text{For } V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k)) \text{ and } D := D_{\mathrm{cris}}(V),\\
&\qquad\qquad V_{\mathrm{cris}}(D) = F^0(D \otimes_{k_0} B_{\mathrm{cris}})^{\varphi = Id};\\
&\text{For } V \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(\pi_1(k)) \text{ and } D := D_{\mathrm{st}}(V),\\
&\qquad\qquad V_{\mathrm{st}}(D) = F^0(D \otimes_{k_0} B_{\mathrm{st}})^{\varphi = Id, N=0}
\end{aligned}$$

Eventually, define the full subcategories $\mathrm{Rep}^{\mathrm{pcris}}_{\mathbb{Q}_p}(\pi_1(k)) \subset \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))$ (resp. $\mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(\pi_1(k)) \subset \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))$) of potentially cristalline representations (resp. semistable) as the one of those $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))$ which become cristalline (resp. semistable) after restricting to $\pi_1(K)$ for a finite field extension K/k. One has the following inclusions

$$\begin{tikzcd}
& \mathrm{Rep}^{\mathrm{pcris}}_{\mathbb{Q}_p}(\pi_1(k)) \arrow[dr, hook] & & & \\
\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k)) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(\pi_1(k)) \arrow[r, equal] & \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(k)) \arrow[r, hook] & \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(\pi_1(k)) \\
& \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(\pi_1(k)) \arrow[ur, hook] & & &
\end{tikzcd}$$

They are not difficult to check once one knows the construction of the rings B_τ, except for the inclusion $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(k)) \subset \mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(\pi_1(k))$, formerly called the p-adic monodromy conjecture and now a (difficult) theorem [**?**].

In the remaining part of these lectures, the only notion we will really handle is the one of Hodge-Tate representations, for which the corresponding period ring is easy to describe, namely:

$$B_{\mathrm{HT}} = \oplus_{i \in \mathbb{Z}} \mathbb{C}_k(i),$$

where $\mathbb{C}_k$ is the completion of $\overline{k}$ for the unique valuation $v_{\bar{k}} : \bar{k}^\times \to \mathbb{Q}$ extending the valuation $v_k : k^\times \to \mathbb{Z}$. As $\pi_1(k)$ acts continuously on $\overline{k}$ for the topology defined by v $v_{\bar{k}}$, its action extends by continuity on $\mathbb{C}_k$. Note that, as a field $\mathbb{C}_p := \mathbb{C}_{\mathbb{Q}_p} = \mathbb{C}_k$; here the subscript $-_k$ is to record the action of $\pi_1(k)$ (which is just the restriction of the one of $\pi_1(\mathbb{Q}_p)$ to $\pi_1(k)$). If $(-)_i$ denotes the degree ith component of an element in B_{HT}, multiplicative structure on B_{HT} is given by $(xy)_i = \sum_j x_j y_{i-j}$. By the universal property of $\mathbb{C}_k[T]$, there is a unique morphism of $\mathbb{C}_k$-algebras $\mathbb{C}_k[T] \to B_{H_T}$, $T \mapsto t$, where $t_i = \delta_{1,i}1$, and as $t \in B^\times_{HT}$ with inverse $(t^{-1})_i = \delta_{-1,i}$, $\mathbb{C}_k[T] \to B_{H_T}$ localizes as $\mathbb{C}_k[T, T^{-1}] \to B_{H_T}$, which is clearly an isomorphism of $\mathbb{Z}$-graded $\mathbb{C}_k$-algebras; in particular $\mathbb{C}_k^\times \times \mathbb{Z} \simeq \mathbb{C}_k[T, T^{-1}]^\times \tilde{\to} B^\times_{H_T}$ and $\mathbb{C}_k(T) = Frac(\mathbb{C}_k[T, T^{-1}]) \tilde{\to} Frac(B_{H_T})$. Letting $\pi_1(k)$ acts on T *via* $\chi_\ell : \pi_1(k) \to \mathbb{Z}_\ell^\times$, $\mathbb{C}_k[T, T^{-1}] \tilde{\to} B_{H_T}$ becomes $\pi_1(k)$-equivariant.

The following summarizes the main properties of $\mathbb{C}_k$.

Tate **Theorem 2.4.** (Tate [Ta67, §3]) *On a*

(1) $\mathbb{C}_k$ *is an algebraically closed field and for every closed subgroup* $\Pi \subset \pi_1(k)$, ${\mathbb{C}_k}^\Pi \subset \mathbb{C}_k$ *identifies with the completion of* $\overline{k}^\Pi$ *for* $v_{\bar{k}^\Pi} = v_{\bar{k}}|_{\bar{k}^\Pi}$*; in particular,* ${\mathbb{C}_k}^{\pi_1(k)} = k \subset \mathbb{C}_k$

(2) $\mathbb{C}_k(i)^{\pi_1(k)} = 0$ *if* $i \neq 0$*;*

(3) $\mathrm{H}^1(k, \mathbb{C}_k) \simeq k[\log(\chi_p)]$ *and* $\mathrm{H}^1(k, \mathbb{C}_k(i)) = 0$ *if* $i \neq 0$.

Corollary 2.5. *The ring* B_{HT} *is* $(\mathbb{Q}_p, \pi_1(k))$*-regular.*

Proof. Condition (i) immediately follows from Theorem 2.4, considering the embeddings

$$\mathbb{C}_k[T, T^{-1}] \subset Frac(\mathbb{C}_k[T, T^1]) = \mathbb{C}_k(T) \subset \mathbb{C}_k((T))$$

and taking $\pi_1(k)$-invariants. For condition (ii), let $x = \sum_i x_i t^i \in B_{\mathrm{HT}}$ such that $\mathbb{Q}_p x \subset B_{\mathrm{HT}}$ is $\pi_1(k)$-stable that is there exists a continuous character $\psi : \pi_1(k) \to \mathbb{Q}_p^\times$ such that

$$\sum_i \chi_\ell^i(\sigma)\sigma(x_i)t^i = \sigma x = \psi(\sigma)x = \sum_i \psi(\sigma)x_i t^i,\ \ \sigma \in \pi_1(k).$$

Assume there exists $i \neq j$ such that $x_i, x_j \neq 0$ so that

$$\psi(\sigma) = \frac{\chi_\ell^i(\sigma)\sigma(x_i)}{x_i} = \frac{\chi_\ell^j(\sigma)\sigma(x_j)}{x_j},\ \ \sigma \in \pi_1(k)$$

hence

$$= \chi_\ell^{i-j}(\sigma)\sigma(\frac{x_i}{x_j}) = \frac{x_i}{x_j},\ \ \sigma \in \pi_1(k)$$

that is $0 \neq \frac{x_i}{x_j} \in \mathbb{C}_k(i-j)^{\pi_1(k)}$, which contradicts Theorem 2.4 (2). □

Ex. $\mathbb{Q}_p(i) = (\mathbb{Q}_p, \chi_p^i) \in \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(\pi_1(k))$, $i \in \mathbb{Z}$.

The condition of being crystalline is the good analogue when $\ell = p$ of the condition of being unramified when $\ell \neq p$; we will see later why but one reason is that to every $(V,\rho) \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k))$ one can attach a characteristic polynomial of crystalline Frobenius as follows namely, writing $D_{\mathrm{cris}}(V) := (D_0, \varphi_0, F^\bullet)$, the characteristic polynomial $\chi_\rho(T) \in k_0[T]$ of the k_0-linear automorphism $\varphi^m : D_0 \tilde{\to} D_0$ (sometimes called the linearized crystalline Frobenius), where , $m := [k_0 : \mathbb{Q}_p]$.

So, if $k/\mathbb{Q}$ is a number field and $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$, write $U_{\rho,\ell} \subset \Sigma_{k,\ell}$ for the set of all $v \in \Sigma_{k,\ell}$ such that (V,ρ_v) is crystalline at v. Write $U_\rho = U_{\rho,\nmid\ell} \sqcup U_{\rho,\ell}$.

2.3. **The axiomatic.** We now review natural assumptions one usually imposes on ℓ-adic represention and explain why they are natural.

2.3.1. *The axiomatic.* Let $k/\mathbb{Q}$ a number field.

2.3.1.1. *Ramification.* Let $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$. For $v \in U_\rho$, let $\chi_{\rho,v} \in \mathbb{Q}_\ell[T]$ denote the corresponding characteristic polynomial of Frobenius as defined at the beginning of 2.2 if $v \nmid \ell$ and let $\chi_{\rho,v} \in (\hat{k}^v)_0[T]$ the characteristic polynomial of the crystalline Frobenius as defined at the end of 2.2.2.2 if $v \mid \ell$. Consider the following conditions on (V,ρ).

(AEU) *Almost Everywhere Unramified*: The subset $U_\rho \subset \Sigma_k$ is cofinite.

Let $Q \subset Q \subset \overline{\mathbb{Q}}$ be a subextension.

(R/$_Q$) *Q-rational*: There exists a cofinite (in Σ_k) subset $U \subset U_\rho \subset \Sigma_k$ such that $\chi_{\rho,v} \in Q[T]$, $v \in U$.

Let $q > 0, w$ be real numbers. One says that $\alpha \in \overline{\mathbb{Q}}$ is q-pure of weight w if for every embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\iota(\alpha)| = q^{\frac{w}{2}}$. Let $W \subset \mathbb{R}$ be a subset.

(P/$_W$) *Pure of weights in W*: There exists a cofinite (in Σ_k) subset $U \subset U_\rho \subset \Sigma_k$ such that $\chi_{\rho,v} \in \overline{\mathbb{Q}}[T]$ and its roots are $|k_v^-|$-pure of weight in W, $v \in U$.

When $W = \{w\}$ is a singleton, we will simply write (P/$_w$):=(P /$_W$) and will say that (V,ρ) is pure of weight w. If we do not want to specify the weight w, we will write (P/$_*$)

If $\tau =$ HT, dR, st, cris, pst, pcris.

(τ) τ: $(V,\rho_v) \in \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(\hat{k}^v))$, $v \in \Sigma_{k,\ell}$.

2.3.1.2. *Compatibility.* Let $Q \subset Q \subset \overline{\mathbb{Q}}$ be a subextension.

Let ℓ, ℓ' be primes. One says that $(V_\ell, \rho_\ell) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k)$ and $(V_{\ell'}, \rho_{\ell'}) \in \mathrm{Rep}_{\mathbb{Q}_{\ell'}}(\pi_1(k)$ are *étale Q-compatible*

$$(V_\ell, \rho_\ell) \sim_{Q,\mathrm{et}} (V_{\ell'}, \rho_{\ell'})$$

if (V_ℓ,ρ_ℓ), $(V_{\ell'},\rho_{\ell'})$ are both Q-rational and there exists a cofinite (in Σ_k) subset $U\subset U_{\rho_\ell}\cap U_{\rho_{\ell'}}\subset \Sigma_k$ such that, for every $v\in U[\frac{1}{\ell\ell'}]$, $\chi_{\rho_\ell,v}$, $\chi_{\rho'_\ell,v}$ lie in $Q[T]$ and $\chi_{\rho_\ell,v}=\chi_{\rho'_\ell,v}$. One says that (V_ℓ,ρ_ℓ), $(V_{\ell'},\rho_{\ell'})$ are *Q-compatible*

$$(V_\ell,\rho_\ell)\sim_Q (V_{\ell'},\rho_{\ell'})$$

if (V_ℓ,ρ_ℓ), $(V_{\ell'},\rho_{\ell'})$ are both Q-rational and there exists a cofinite (in Σ_k) subset $U\subset U_{\rho_\ell}\cap U_{\rho_{\ell'}}\subset \Sigma_k$ such that, for every $v\in U$, $\chi_{\rho_\ell,v}$, $\chi_{\rho'_\ell,v}$ lie in $Q[T]$ and $\chi_{\rho_\ell,v}=\chi_{\rho'_\ell,v}$. If we want to keep track of the cofinite subset U, we will write

$$(V_\ell,\rho_\ell)\sim_{Q,\mathrm{et},U} (V_{\ell'},\rho_{\ell'}),\ \ (V_\ell,\rho_\ell)\sim_{Q,U} (V_{\ell'},\rho_{\ell'}).$$

Note that $(V_\ell,\rho_\ell)\sim_{Q,\mathrm{et},U} (V_{\ell'},\rho_{\ell'})$ is equivalent to $(V_\ell,\rho_\ell)\sim_{Q,U[\frac{1}{\ell\ell'}]} (V_{\ell'},\rho_{\ell'})$.

Variants: One may ask (V_ℓ,ρ_ℓ), $(V_{\ell'},\rho_{\ell'})$ to be strictly étale Q-compatible (resp. Q-compatible) by requiring that $U=U_{\rho_\ell}\cap U_{\rho_{\ell'}}$.

Let $\mathcal{L}$ be a set (possibly infinite) of primes. For each $\ell\in\mathcal{L}$, let $(V_\ell,\rho_\ell)\in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k)$. Consider the following conditions on the family $(\underline{V},\underline{\rho})=(V_\ell,\rho_\ell)$, $\ell\in\mathcal{L}$

(RC/$_{Q,\mathrm{et}}$) *Étale Q-compatible*: There exists a cofinite (in Σ_k) subset $U\subset\cap_\ell U_{\rho_\ell}\subset\Sigma_k$ such that

$$(V_\ell,\rho_\ell)\sim_{Q,U[\frac{1}{\ell\ell'}]} (V_{\ell'},\rho_{\ell'}),\ \ \ell,\ell'\in\mathcal{L}.$$

(RC/$_Q$) *Q-compatible*: There exists a cofinite (in Σ_k) subset $U\subset\cap_\ell U_{\rho_\ell}\subset\Sigma_k$ such that

$$(V_\ell,\rho_\ell)\sim_{Q,U} (V_{\ell'},\rho_{\ell'}),\ \ \ell,\ell'\in\mathcal{L}.$$

Variants: One may ask $(\underline{V},\underline{\rho})$ to be strictly étale Q-compatible (resp. Q-compatible) by requiring that $U=\cap_\ell U_{\rho_\ell}$ and $(\underline{V},\underline{\rho})$ to be weakly étale Q-compatible (resp. Q-compatible) by requiring that

$$(V_\ell,\rho_\ell)\sim_{Q,\mathrm{et}} (V_{\ell'},\rho_{\ell'}),\ \ \ell,\ell'\in\mathcal{L}\ \ \text{(respectively }\ (V_\ell,\rho_\ell)\sim_{Q} (V_{\ell'},\rho_{\ell'}),\ \ \ell,\ell'\in\mathcal{L}).$$

2.3.1.3. *Semisimplicity.* Recall that $(V,\rho)\in\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is said to be simple if it is non-zero and does not admit any other non-zero subrepresentation than itself, and to be semisimple if it satisfies the following equivalent conditions:

(SS) (SS-1) Every extension $0\to V'\to V\to V''\to 0$ in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ splits in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$;

(SS-2) $V=\sum_{W\subset V\ \mathrm{simple}} W$;

(SS-3) There exists a (automatically unique up to isomorphism) decomposition

$$V\simeq\oplus_{1\leq i\leq r}V_i^{\oplus n_i}$$

in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ with $V_1,\dots,V_r$ simple, pairwise non-isomorphic in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$.

Consider also the condition:

(FSS) *Frobenius-semisimple*: There exists a cofinite (in Σ_k) subset $U\subset U_\rho\subset\Sigma_k$ such that $\rho_v(Fr_{k_v}^-)\in V$ is semisimple, $v\in U[\frac{1}{\ell}]$.

2.3.2. *Stability.* One may ask whether the various axioms we imposed are robust enough that is preserved by natural operations. Here is a summary. Les K/k be a finite field extension.

	Subquotients	**Extensions**	$\otimes$ **and** $(-)^\vee$	$(V,\rho)\Rightarrow(V,\rho\vert_{\pi_1(K)})$	$(V,\rho)\Leftarrow(V,\rho\vert_{\pi_1(K)})$
(AEU)	yes	**?**	yes	yes	yes
(HT)	yes	**no**	yes	**yes**	**yes**
(dR)	yes	**no**	yes	yes	yes
(st)	yes	**no**	yes	no	no
(crys)	yes	**no**	yes	no	no
(R/$_Q$)	no	yes	yes	yes	no
(P/$_*$)	yes	yes	yes	yes	yes
(RC/$_{Q,\mathrm{et}}$)	no	yes	yes	yes	no
(SS)	yes	no	**yes**	**yes**	**yes**
(FSS)	yes	no	yes	yes	yes

Let us explain the bold assertions.

(1) To show that none of (HT), (dR), (st), (crys) is stable by extension, it is enough to exhibit and extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p$ which is not Hodge-Tate. Consider the 2-dimensional $\mathbb{Q}_\ell$-vector space $V \simeq \mathbb{Q}_\ell e_1 \oplus \mathbb{Q}_\ell e_2$ equipped with $\sigma \cdot e_1 = e_1$, $\sigma \cdot e_2 = \log(\chi_\ell(\sigma))e_1 + e_2$, namely the matrix of $\rho(\sigma)$ in (e_1, e_2) is given by

$$\begin{pmatrix} 1 & \log(\chi_\ell(\sigma)) \\ 0 & 1 \end{pmatrix}, \quad \sigma \in \pi_1(k).$$

Then V sits into a short exact sequence

$$0 \to \mathbb{Q}_p e_1 \to V \to \mathbb{Q}_p \bar{e}_2 \to 1.$$

Fix $v \in \Sigma_{k,\ell}$ so that the above restricts to an extension of $\pi_1(\hat{k}^v)$-representation. Applying $D_{\mathrm{HT}}(-)$, one gets

$$0 \to \hat{k}^v e_1 \to D_{\mathrm{HT}}(V|_{\pi_1(\hat{k}^v)}) \to \hat{k}^v \bar{e}_2,$$

so that $\dim_{\hat{k}^v}(D_{\mathrm{HT}}(V|_{\pi_1(\hat{k}^v)})) = 2$ if and only if $\bar{e}_2$ lifts to $b \otimes e_1 + 1 \otimes e_2 \in B_{\mathrm{HT}} \otimes_{\mathbb{Q}_\ell} V$ with

$$\sigma \cdot (b \otimes e_1 + 1 \otimes e_2) = (\sigma(b) + \log(\chi_\ell(\sigma))) \otimes e_1 + 1 \otimes e_2 = b \otimes e_1 + 1 \otimes e_2, \sigma \in \pi_1(\hat{k}^v)).$$

Namely, $\log(\chi_\ell(\sigma)) = \sigma(b) - b$, $\sigma \in \pi_1(\hat{k}^v))$, that is $[\log(\chi_\ell)] = 0$ in $\mathrm{H}^1(\hat{k}^v, \mathbb{C}_k)(= \mathrm{H}^1(\hat{k}^v, B_{\mathrm{HT}}))$, contradicting Theorem 2.4 (3).

(2) The equivalence (V, ρ) satisfies (HT) $\Leftrightarrow$ $(V, \rho|_{\pi_1(K)})$ satisfies (HT) is a consequence of the following

Lemma 2.6. *$k/\mathbb{Q}_p$ p-adic field. Let K/k a finite field extension. For every $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))$ the canonical K-linear morphism*

$$\alpha_{K/k} : D_{\mathrm{HT}}(V) \otimes_k K \tilde{\to} D_{\mathrm{HT}}(V|_{\pi_1(K)})$$

is an isomorphism.

Proof. For a finite field extension K/k, write $D_{\mathrm{HT},K} := D_{\mathrm{HT}}(V|_{\pi_1(K)})$. If the assertion holds for finite Galois extensions K/k then it holds for arbitrary finite field extensions K/k. Indeed, if K'/k denotes the Galois closure of K/k, this follows from the canonical commutative diagram

$$\begin{tikzcd} (D_{\mathrm{HT},k} \otimes_k K) \otimes_K K' \arrow[r, "\alpha_{K/k} \otimes Id"] \arrow[d, "\simeq"] & D_{\mathrm{HT},K} \otimes_K K' \arrow[d, "\alpha_{K'/K}"'] \\ D_{\mathrm{HT},k} \otimes_k K' \arrow[r, "\alpha_{K'/k}"] & D_{\mathrm{HT},K'}. \end{tikzcd}$$

and the fact that $\alpha_{K/k} : D_{\mathrm{HT}}(V) \otimes_k K \to D_{\mathrm{HT},K}$ is an isomorphism if and only if $\alpha_{K/k} \otimes Id$: $(D_{\mathrm{HT}}(V) \otimes_k K) \otimes_K K' \to D_{\mathrm{HT},K} \otimes_K K'$ is. Assume now K/k is Galois. Then $\mathrm{Gal}(K|k)$ acts semilinearly on $D_{\mathrm{HT},K} = (B_{\mathrm{HT}} \otimes_{Q_p} V)^{\pi_1(K)}$ with

$${D_{\mathrm{HT},K}}^{\mathrm{Gal}(K|k)} = D_{\mathrm{HT},k}.$$

This reduces the proof to the following classical claim.

Claim. (Galois descent for vector spaces) *Let K/k be a finite field extension and V a finite dimensional K-vector space endowed with a semilinear action of $G := \mathrm{Gal}(K|k)$. Then the canonical morphism $\alpha : V^G \otimes_k K \tilde{\to} V$ is an isomorphism.*

Proof of the claim. For the injectivity, we are to show that if $v_1, \dots, v_r \in V^G$ are k-linearly independent then $v_1, \dots, v_r \in V$ are K-linearly independent. We argue by induction on r. If $r = 1$, there is nothing to prove. Assume $r \geq 2$ and let $v_1, \dots, v_r \in V^G$ k-linearly independent. If there exists $x_1, \dots, x_r \in K$ not all zero such that $\sum_{1 \leq i \leq r} x_i v_i = 0$. By induction, $x_i \neq 0$, $i = 1, \dots, r$ and, up to replacing x_i with x_i/x_1, $i = 1, \dots, r$, oma $x_1 = 1$. Then, for every $\sigma \in G$,

$$-\sum_{2 \leq i \leq r} x_i v_i = v_1 = \sigma \cdot v_1 = -\sum_{2 \leq i \leq r} \sigma(x_i) \sigma \cdot v_i = -\sum_{2 \leq i \leq r} \sigma(x_i) v_i,$$

whence

$$\sum_{2\leq i\leq r}(\sigma(x_i)-x_i)v_i=0.$$

By induction, $\sigma(x_i)=x_i$, $i=2,\ldots,r$. As this holds for every $\sigma\in G$, $x_i\in K^G=k$: a contradiction. For the surjectivity, as the morphism $\alpha:V^G\otimes_k K\to V$ is G-equivariant, $\mathrm{im}(\alpha)\subset V$ is a G-stable K-vector subspace so that $V/\mathrm{im}(\alpha)$ is also endowed with a G-semilinear action of G such that the canonical projection $p_\alpha:V\twoheadrightarrow V/\mathrm{im}(\alpha)$ is G-equivariant. Furthermore, for every $v\in V$, $Tr_G(v):=\sum_{\sigma\in G}\sigma v\in V^G\subset\mathrm{im}(\alpha)$ hence $Tr_G(p_\alpha(v))=p_\alpha(Tr_G(v))=0$. This forces $V/\mathrm{im}(\alpha)=0$ as, if there exists $v\in V$ such that $0\neq p_\alpha(v)=:\overline{v}$ then, there exists $x\in K$ such that $tr_G(x\overline{v})\neq 0$ as, otherwise, for every $x\in K$, $0=Tr_G(x\overline{v})=\sum_{\sigma\in G}\sigma(x)\sigma\cdot\overline{v}$ would imply $\sum_{\sigma\in G}\sigma(-)\sigma\cdot\overline{v}=0$ but then, by Dedekind Lemma[1], one would have $f(\sigma\cdot\overline{v})=0$, $f\in(V/\mathrm{im}(\alpha))^\vee$ (hence $\sigma\cdot\overline{v}=0$), $\sigma\in G$. □

(3) The fact that a tensor product of semisimple representations is again semisimple is non-trivial and uses that $\mathbb{Q}_\ell$ has characteristic 0. The quickest way to see it is by invoking that an algebraic group G over a field Q of characteristic 0 is reductive (that is has trivial unipotent radical $R_u(G)=1$) if and only if it is linearly reductive (that is all its Q-rational algebraic representations are semisimple). Indeed, let $(V_i,\rho_i)\in\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$, $i=1,2$ and set $(V,\rho):=(V_1,\rho_1)\oplus(V_2,\rho_2)$; let $G\subset\mathrm{GL}_V$ denote the Zariski-closure of $\Pi:=\rho(\pi_1(k))\subset\mathrm{GL}(V)$. If $(V_i,\rho_i)\in\mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$, $i=1,2$ then, clearly, $(V,\rho)\in\mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$ hence V is a faithful, semisimple representation of G. This imposes $R_u(G)=1$ as, otherwise, $R_u(G)\subset\mathrm{GL}_V$ would be a non-trivial unipotent subgroup hence $0\subsetneq V^{R_u(G)}\subsetneq G$ while, as $R_u(G)$ is normal in G, $V^{R_u(G)}\subset V$ is G-stable. As V is a semisimple G-representation, one would thus get a G-equivariant decomposition $V=V^{R_u(G)}\oplus W$. But, by construction, $W^{R_u(G)}=0$, which contradicts the fact that W is a faithful representation of $R_u(G)$. So G is reductive hence linearly reductive; in particular $V_1\otimes V_2$ is a semisimple $\mathbb{Q}_\ell$-algebraic representation of G. The assertion thus follows from the fact that $\rho_1\otimes\rho_2:\pi_1(k)\to\mathrm{GL}(V_1\otimes V_2)$ factors as

$$\pi_1(k)\xrightarrow{\rho}G\to\mathrm{GL}(V_1\otimes V_2)$$

and from the Zariski-density of Π in G.

(4) The equivalence (V,ρ) satisfies (SS) $\Leftrightarrow$ $(V,\rho|_{\pi_1(K)})$ satisfies (SS) is purely group-theoretical. The implication $\Rightarrow$ uses that $\mathbb{Q}_\ell$ has characteristic 0. More precisely, let Q be a field, V a finite-dimensional Q-vector space endowed with the action of a group G.

   (a) Let $U\subset G$ be a normal subgroup. If V is semisimple as a G-representation then V is also semisimple as a U-representation: Decomposing V as a direct sum of simple G-representations, oma V is a simple G-representation. Let $W\subset V$ be a simple U-subrepresentation. Then as V is a simple G-representation and $\sum_{g\in G}gW\subset V$ is G-stable (and non-zero), $\sum_{g\in G}gW=V$. It remains to observe that $gW\subset V$ is again a simple U-subrepresentation: U acts on gW *via* $u\cdot gw=g(g^{-1}ugw)$ and $g:W\tilde{\to}gW$ is a U-equivariant isomorphism.

   (b) Let $U\subset G$ be a finite index subgroup. Assume Q has characteristic $\nmid[G:U]$. If V is semisimple as a U-representation then V is also semisimple as a G-representation: Let $0\neq W\subset V$ be a G-subrepresentation - hence a U-subrepresentation. Write $V=W\oplus W'$ as a U-representation. Let $p_{W,W'}:V\to W$ denote the corresponding projection, which, by construction, is a U-equivariant morphism and observe that the averaged morphism

$$p:V\to W,\quad v\mapsto\frac{1}{[G:U]}\sum_{g\in G/U}gp_{W,W'}g^{-1}(v)$$

   is G-equivariant with $\mathrm{im}(p)=W$ and $p^2=p$. So that one gets $V=W\oplus\ker(p)$ as G-representations.

Semisimplification: One has a canonical semisimplification functor

$$(-)^{\mathrm{ss}}:\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))\to\mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$$

[1]Recall this asserts that if M is a $\mathbb{Z}$-module (or even just a commutative monoid) and K a field then the characters $M\to K^\times$ are K-linearly independent (regarded as elements of K^M).

constructed as follows. For $V \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$, let $S_0V \subset V$ denote the *socle* of V *i.e.* the sum of all simple subobjects $V' \subset V$ in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$. This is the largest semisimple subobject of V in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$; if V is non-zero, S_0V is also non-zero and $S_0V = V$ if and only if V is semisimple. One then defines inductively $V \supset S_{i+1}V \supset S_iV$ as the inverse image of $S_0(V/S_iV) \subset V/S_iV$ by the canonical projection $V \twoheadrightarrow V/S_iV$. The processus ends and one sets

$$V^{\mathrm{ss}} = \bigoplus_{i\geq 0} S_{i+1}V/S_iV.$$

By construction $V^{\mathrm{ss}} \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$ and is called the $\pi_1(k)$-*semisimplification* of V. The functoriality immediately follows from the construction since for every morphism $f : V_1 \to V_2$ in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ one has $f(S_0(V)) \subset S_0(V_2)$ hence, by induction $f(S_iV_1) \subset S_iV_2$, which, passing to quotients yields morphisms $f_i : S_{i+1}V_1/S_iV_1 \to S_{i+1}V_2/S_iV_2$ such that the square

$$\begin{tikzcd} S_{i+1}V_1 \arrow[r, "f|_{S_{i+1}V_1}"] \arrow[d] & S_{i+1}V_2 \arrow[d] \\ S_{i+1}V_1/S_iV_1 \arrow[r, "f_i"'] & S_{i+1}V_2/S_iV_2. \end{tikzcd}$$

One sets $f^{\mathrm{ss}} := \oplus_i f_i : V_1^{ss} \to V_2^{ss}$ and easily checks that $Id^{\mathrm{ss}} = Id$, $(gf)^{\mathrm{ss}} = g^{\mathrm{ss}}f^{\mathrm{ss}}$.

In terms of matrices, semisimplifying amounts to "forgetting" what is above the block diagonal. For $(V,\rho) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$ wiyth $\pi_1(k)$-semisimplication $(V^{\mathrm{ss}}, \rho^{\mathrm{ss}}) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$, setting $\Pi := \rho(\pi_1(k))$, $\Pi^{\mathrm{ss}} := \rho^{ss}(\pi_1(k)))$ one has a canonical short exact sequence of ℓ-adic Lie groups

$$1 \to R_u(\Pi) \to \Pi \to \Pi^{\mathrm{ss}} \to 1,$$

where $R_u(\Pi) \subset \mathrm{GL}(V)$ is a unipotent subgroup.

- If (P) is a property of $V \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ which is stable subquotients then (P) for V implies (P) for V^{ss}.
- Conversely, if (P) is a property of $V \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ which is stable extensions then (P) for V^{ss} implies (P) for V.

2.3.3. *Where does the axiomatic come from? $k/\mathbb{Q}$* number field. As already mentioned, the most important - and basically the only non-trivial - natural examples of ℓ-adic representations of $\pi_1(k)$ arise from the ℓ-adic cohomology of varieties over k. In the following, we will say that $(V,\rho) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is geometric if it appears as a subquotient of a representation of the form $\mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_\ell(j))$ for Y/k a smooth, proper variety over k. It follows from the general formalism of ℓ-adic cohomology (Weil cohomology) that the full subcategory $\mathrm{Rep}^{\mathrm{geo}}_{\mathbb{Q}_\ell}(\pi_1(k)) \subset \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ of geometric representations is a Tannakian subcategory.

If Y/k is smooth proper variety by spreading out, there exists a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$ and a smooth proper morphism $\mathcal{Y} \to U$ such that $Y \to \mathrm{spec}(k)$ fits into a Cartesian square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d] \\ \mathrm{spec}(k) \arrow[r, "\eta"] & U \end{tikzcd}$$

For $v \in |U|$, one gets

BC (2)
$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}_v \arrow[l] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}_v^- \arrow[l] \arrow[d] \\ \mathrm{spec}(k) \arrow[r, "\eta"] & U & \mathrm{spec}(\widehat{\mathcal{O}}_k^v) \arrow[l] & \mathrm{spec}(k_v^-) \arrow[l] \arrow[ll, bend left, "v"] \end{tikzcd}$$

2.3.3.1. *(AEU) and smooth proper base change.* The condition (AEU) for geometric representations is a special case of the following (deep!) theorem.

ProperBC

Theorem 2.3.4. (Smooth proper base change) *Let S be an integral scheme and $\mathcal{Y} \to S$ a smooth proper morphism. Then for every prime ℓ invertible on S and integers $n \geq 1$, $i \geq 0$, the étale sheaf $R^i f_* \mathbb{Z}/\ell^n$ is locally constant constructible (viz. representable by an étale cover of S) and, for every geometric point $\bar{s}$ on S, one as a canonical isomorphism*

$$(R^i f_* \mathbb{Z}/\ell^n)_{\bar{s}} \simeq \mathrm{H}^i(\mathcal{Y}_{\bar{s}}, \mathbb{Z}/\ell^n).$$

In terms of representations, Theorem 2.3.4 implies in particular that for every geometric points $\bar{s}_1$, $\bar{s}_2$ over points $s_1, s_2 \in S$, one has canonical (after choosing an isomorphism of fiber fuctors $\bar{s}_1 \tilde{\to} \bar{s}_2$) isomorphisms

$$\mathrm{H}^i(\mathcal{Y}_{\bar{s}_1}, \mathbb{Q}_\ell) \tilde{\to} \mathrm{H}^i(\mathcal{Y}_{\bar{s}_2}, \mathbb{Q}_\ell),$$

equivariant with respect to the canonical morphisms of profinite groups

$$\pi_1(s_1, \bar{s}_1) \to \pi_1(S, \bar{s}_1) \tilde{\to} \pi_1(S, \bar{s}_2) \leftarrow \pi_1(s_2, \bar{s}_2)$$

In our setting, we take $S = U$, $s = \eta, \eta_v$ or v and get:

$$\begin{tikzcd}
\mathrm{H}^i(\mathcal{Y}_{\bar{\eta}}, \mathbb{Q}_\ell) \arrow[r, dash, "\simeq"] & \mathrm{H}^i(\mathcal{Y}_{\bar{\eta}_v}, \mathbb{Q}_\ell) \arrow[r, dash, "\simeq"] & \mathrm{H}^i(\mathcal{Y}_{\bar{v}}, \mathbb{Q}_\ell) \\
\pi_1(U) \arrow[u, bend left] & \pi_1(\widehat{\mathcal{O}}_k^v) \arrow[l] \arrow[u, bend left] & \pi_1(v) \arrow[l, "\simeq"'] \arrow[u, bend left] \\
\pi_1(\eta) = \pi_1(k) \arrow[u] & \mathrm{Gal}(\widehat{k}_v^{\mathrm{ur}} | \widehat{k}^v) \arrow[u, two heads, "\simeq"] & \pi_1(k_v^-) \arrow[l, "\simeq"'] \arrow[u, "\simeq"] \\
D_{\bar{v}}(\overline{k}|k) \arrow[u, hook] & \pi_1(\eta_v) = \pi_1(\widehat{k}_v) \arrow[l, "\simeq"'] \arrow[u, two heads] &
\end{tikzcd}$$

Remark (Frobenii) Let κ be a finite field of characteristic $p > 0$ and Y/κ a variety. The arithmetic Frobenius $\varphi_\kappa : \bar{\kappa} \tilde{\to} \bar{\kappa}$, $x \mapsto x^{|\kappa|}$ induces a automorphism of scheme $\Phi_\kappa := \varphi_\kappa^\# : \mathrm{spec}(\kappa) \tilde{\to} \mathrm{spec}(\kappa)$. On $\overline{Y} := Y \times_\kappa \bar{\kappa}$ one has 3 Frobenii endomorphisms:

Name	Notation	Definition	Ex: $X = \mathbb{A}^1_\kappa = \mathrm{spec}(\kappa[T])$
Arithmetic Frobenius	$\Phi_{\overline{Y}}$	$Id \times \Phi_\kappa$	
$\lvert\kappa\rvert$-Absolute Frobenius	$Fr_{\overline{Y}}$		
Relative Frobenius	$Fr_{\overline{Y}\lvert Y}$	$Fr_Y \times Id$	

Recall that for a $\mathbb{F}_p$-scheme Y, the $q = p^r$-absolute Frobenius $F_X : X \to X$ is the identity on X_{zar} and $a \mapsto a^q$ on $\mathcal{O}_X$. The various Frobenii at stake fit into the following Cartesian diagram

$$\begin{tikzcd}
\overline{Y} \arrow[r, "Fr_{\overline{Y}|Y}"] \arrow[rr, bend left, "Fr_{\overline{Y}}"] \arrow[dr, bend right] & \overline{Y} \arrow[r, "\Phi_{\bar{Y}}", "\simeq"'] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{Y} \arrow[d] \\
& \mathrm{spec}(\bar{\kappa}) \arrow[r, "\Phi_\kappa", "\simeq"'] & \mathrm{spec}(\bar{\kappa})
\end{tikzcd}$$

One checks that the action $Fr^*_{\overline{Y}} : \mathrm{H}^i(Y_{\bar{\kappa}}, \mathbb{Q}_\ell) \to \mathrm{H}^i(Y_{\bar{\kappa}}, \mathbb{Q}_\ell)$ of $Fr_{\overline{Y}}$ on $\mathrm{H}^i(Y_{\bar{\kappa}}, \mathbb{Q}_\ell)$ induced by functoriality is the identity so that $Fr_{\overline{Y}|Y}$ acts as $\Phi_{\overline{Y}}^{-1}$ on $\mathrm{H}^i(Y_{\bar{\kappa}}, \mathbb{Q}_\ell)$. Due to its geometric nature, the Frobenius that most naturally appears when studying $\mathrm{H}^i(Y_{\bar{\kappa}}, \mathbb{Q}_\ell)$ is $Fr_{\overline{Y}|Y}$; this is why one often use the geometric Frobenius when the arithmetic one *e.g.* when defining characteristic polynomials of Frobenii *etc.*

Let $\Sigma_k(Y) \subset \Sigma_k$ denote the set of places of good reduction for Y/k that is the set of all $v \in \Sigma_k$ such that $Y_{\hat{k}^v}/\hat{k}^v$ fits into a Cartesian square

$$\begin{tikzcd} Y \arrow[d] & Y_{\hat{k}^v} \arrow[l] \arrow[r] \arrow[d] & \mathcal{Y}_{[v]} \arrow[d] \\ \mathrm{spec}(k) \arrow[ur, phantom, "\square"] & \mathrm{spec}(\hat{k}^v) \arrow[l] \arrow[r] \arrow[ur, phantom, "\square"] & \mathrm{spec}(\hat{\mathcal{O}}_k^v), \end{tikzcd}$$

with $\mathcal{Y}_{[v]} \to \mathrm{spec}(\hat{\mathcal{O}}_k^v)$ smooth, proper. The output of Theorem 2.3.4 is that for $V_\ell = \mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_\ell) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$, one has

$$U_{\rho,\nmid\ell} \supset \Sigma_k(Y)[\frac{1}{\ell}] \supset U[\frac{1}{\ell}].$$

One may ask whether $U_{\rho,\nmid\ell} = \Sigma_k(Y)[\frac{1}{\ell}]$. This does not hold in general, even taking into account all cohomological degrees namely considering $V = \oplus_{0\leq i\leq 2/\mathrm{rm}\ \dim(Y)}\mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_\ell)$. One heuristic reason is that in general Y/k does not have a privilegiate model over $\mathrm{Spec}(\mathcal{O}_k)$. There is one exception however - the one of abelian varieties, where such a model exists and is called the Néron model.

Reduction of abelian varieties More precisely, let Y/k be a smooth, separated scheme of finite type over k, a *Néron model for* for Y over $S := \mathrm{spec}(\mathcal{O}_k)$ is a smooth, separated scheme $\mathcal{Y} \to S$ of finite type with generic fiber Y and which is minimal in the following sense: for every smooth, separated scheme $\mathcal{Y}' \to S$ of finite type fini with generic fiber Y and for every k-morphism $u_k : Y \to Y$, there exists a unique morphism of S-schemes $u : \mathcal{Y}' \to \mathcal{Y}$ inducing u_k on the generic fibers. If a Néron model exists, it is automatically unique and if Y/k is a group scheme, its Néron model is automatically a group scheme over S.

Theorem 2.7. (BLR [BLR90]) *Every abelian variety Y/k admits a Néron model $\mathcal{Y} \to \mathrm{spec}(\mathcal{O}_k)$.*

Furthermore, for every $v \in \Sigma_k$, the special fiber $(\mathcal{Y}^\circ)_v$ of the neutral component $\mathcal{Y}^\circ$ of $\mathcal{Y}$ at $v \in \Sigma_k$ fits into a diagram of extensions

$$\begin{tikzcd} & 1 \arrow[d] & & & \\ & U_v \arrow[d] & & & \\ 1 \arrow[r] & \square_v \arrow[r] \arrow[d] & (\mathcal{Y}^\circ)_v \arrow[r] & A_v \arrow[r] & 1 \\ & T_v \arrow[d] & & & \\ & 1 & & & \end{tikzcd},$$

where A_v, T_v, U_v are respectively an abelian variety, a torus and a unipotent group scheme over the residue field k_v^-. One says that Y has *good reduction* (resp. *semistable reduction* , *bad reduction*) at $v \in \Sigma_k$ if $\square_v = 1$ (resp. $U_v = 1$, otherwise) and that Y has *potentially good reduction* (resp. *potentially semistable reduction*) at $v \in \Sigma_k$ if there exists a finite Galois extension K/k such that Y_K has good reduction (resp. semistable reduction) at every $w \in \Sigma_{K,v}$.

With these definitions,

STG **Theorem 2.8.**
(1) (SerreTate [ST68]) *For every $v \in \Sigma_{k,p}$, the following condition are equivalent*

(i) there exists a prime $\ell \neq p$ such that $\mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell)$ is unramified at v;
(i') for every prime $\ell \neq p$, $\mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell)$ is unramified at v;
(ii) Y/k has good reduction at v.

(2) ([SGA7]) *For every $v \in \Sigma_{k,p}$, the following condition are equivalent*

(i) there exists a prime $\ell \neq p$ such that $\mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell)$ is semistable at v;

(i') for every prime $\ell \neq p$, $\mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell)$ is semistable at v;

(ii) Y/k has semistable reduction at v.

Combining Theorem 2.8 (2) and Lemma 2.2, one obtains the following striking geometric corollary.

Corollary 2.9. *An abelian variety over a number field has everywhere good reduction.*

2.3.4.1. *Admissiblity and p-adic comparison theorems.* $k/\mathbb{Q}_p$ p-adic field. Consider the notation in the following Cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}_v \arrow[l] \arrow[d] \\ \mathrm{spec}(k) \arrow[r] & \mathrm{spec}(\mathcal{O}) & \mathrm{spec}(k^-) \arrow[l] \end{tikzcd}$$

To understand $V := \mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_p) \in \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))$, the basic strategy is to try and compare p-adic étale cohomology to one of four other p-adic cohomology theories $\mathrm{H}^i_\tau(Y)$, $\tau = \mathrm{HT}$, dR, crys , st. More precisely,

(1) Hodge-Tate cohomology:

$$\mathrm{H}^i_{\mathrm{HT}}(Y) = \oplus_{0 \leq j \leq i} \mathrm{H}^{i-j}(Y, \Omega^j_{Y|k})$$

with $\mathbb{Z}$-graduation $\mathrm{H}^i_{\mathrm{HT}}(Y)_j = \mathrm{H}^{i-j}(Y, \Omega^j_{Y|k})$, $j \in \mathbb{Z}$.

(2) de Rham cohomology: hypercohomology of the de Rham complex

$$\mathrm{H}^i_{\mathrm{dR}}(Y) := \mathbb{H}^i(Y, \Omega^\bullet_{Y|k})$$

with $\mathbb{Z}$-filtration $F^j\mathrm{H}^i_{\mathrm{dR}}(Y) = \mathrm{im}(\mathbb{H}^i(Y, \Omega^{\bullet \geq j}_{Y|k}) \to \mathbb{H}^i(Y, \Omega^\bullet_{Y|k}))$, $j \in \mathbb{Z}$ "Hodge filtration".

(3) if Y/k has good reduction with smooth proper model $\mathcal{Y}/\mathcal{O}$, crystalline cohomology: crystalline cohomology of the special fiber

$$\mathrm{H}^i_{\mathrm{cris}}(Y) := \mathrm{H}^i_{\mathrm{cris}}(\mathcal{Y}_v/\mathcal{O}_{k_0}) \otimes_{\mathcal{O}_{k_0}} k_0,$$

where $\mathcal{O}_{k_0} = W(k^-)$ denotes the Witt vectors of k^- or, equivalently, the ring of integers of the maximal unramfied extension $\mathbb{Q}_p \subset k_0 = k \cap \mathbb{Q}_p^{\mathrm{ur}} \subset k$ of $\mathbb{Q}_p$ in k. $\mathrm{H}^i_{\mathrm{cris}}(Y)$ is a finite k_0-vector space equipped with a σ-semilinear crystalline Frobenius arising by functoriality from the Frobenius $Fr_{\mathcal{Y}_v}$. The fact that $\mathrm{H}^i_{\mathrm{cris}}(Y) \otimes_{k_0} k$ carries a Hodge filtration is a consequence of Berthelot-Ogus comparison isomorphism with de Rham cohomology of the generic fiber

$$\mathrm{H}^i_{\mathrm{cris}}(Y) \otimes_{k_0} k \tilde{\to} \mathrm{H}^i_{\mathrm{dR}}(Y).$$

(4) if Y/k has semistable reduction with semistable proper model $\mathcal{Y}/\mathcal{O}$, semistable cohomology: log-crystalline cohomology of the special fiber

$$\mathrm{H}^i_{\mathrm{st}}(Y) := \mathrm{H}^i_{\mathrm{logcris}}(\mathcal{Y}_v/\mathcal{O}_{k_0}) \otimes_{\mathcal{O}_{k_0}} k_0,$$

where $\mathcal{O}_{k_0} = W(k^-)$ denotes the Witt vectors of k^- or, equivalently, the ring of integers of the maximal unramfied extension $\mathbb{Q}_p \subset k_0 = k \cap \mathbb{Q}_p^{\mathrm{ur}} \subset k$ of $\mathbb{Q}_p$ in k. $\mathrm{H}^i_{\mathrm{cris}}(Y)$ is a finite k_0-vector space equipped with a σ-semilinear crystalline Frobenius φ^{cris} arising by functoriality from the Frobenius $Fr_{\mathcal{Y}_v}$ and a k_0-linear nilpotent operator N such that $N\varphi^{\mathrm{cris}} = p\varphi^{\mathrm{cris}}N$. The fact that $\mathrm{H}^i_{\mathrm{st}}(Y) \otimes_{k_0} k$ carries a Hodge filtration is a consequence of Hyodo-Kato comparison isomorphism with de Rham cohomology of the generic fiber

$$\mathrm{H}^i_{\mathrm{st}}(Y) \otimes_{k_0} k \tilde{\to} \mathrm{H}^i_{\mathrm{dR}}(Y).$$

Here the condition that Y/k has semistable reduction means that the singularities of the special fiber are not too bad. More precisely, if $I(\mathcal{Y}_v)$ denotes the set of irreducible components of $\mathcal{Y}_v$ and for $I \subset I(\mathcal{Y}_v)$ one sets $\mathcal{Y}_v^I := \cap_{C \in I} C$ (scheme-theoretic intersection) then one requests that the elements of $I(\mathcal{Y}_v)$ be Cartier divisors and that for every $I \subset I(\mathcal{Y}_v)$, $\mathcal{Y}_v^I$ is geometrically reduced and smooth over k^-.

To proceed, for each of the above four cohomology theories τ, Fontaine has built a suitable period ring B_τ, which is $(\mathbb{Q}_p, \pi_1(k))$-regular and formulated the following unifying conjecture.

Conjecture 2.10. (Fontaine) *Under the conditions in the following table*

		HT	*dR*	*st*	*crys*
Y/k	*No condition*	$\vee$	$\vee$		
	Semistable reduction			$\vee$	
	Good reduction				$\vee$

$V \in \mathrm{Rep}^\tau_{\mathbb{Q}_p}(\pi_1(k))$ *and there is a functorial isomorphism* $D_\tau(V) \tilde{\to} \mathrm{H}^i(Y)_\tau \otimes_{E_\tau} B_\tau$.

This conjecture is now a theorem, due to Faltings and Tsuji (and with alternative proofs by Niziol, Bhatt, Beilinson). Noe that the case of crystalline comparison / good reduction justifies the comment made earlier that for $\ell = p$, the good analogue of being unramified is being crystalline.

2.3.4.2. *Purity, $\mathbb{Q}$-compatibility and the Weil conjectures.* The Weil conjectures follow from the so-called standard conjectures of Grothendieck and, actually, it is partly to prove the Weil conjectures that Grothendieck devlopped the formalism of pure motives and ℓ-adic cohomology. The Weil conjectures were first proved by Weil himself for abelian varieties ([We49], [M70, §21, Application II]) and in general by Deligne ([D74], [D80]) by geometric arguments going around Grothendieck's original strategy. Actually, using that ℓ-adic cohomology is a Weil cohomology (in particular satisfies the Lefschetz trace formula), it is not difficult to reduce the Weil conjectures to its core - the "Riemann hypothesis".

Let κ be a finite field of characterstic $p > 0$, Y/κ a smooth, proper variety of dimension d and $\ell \neq p$ a prime. Let $Fr_\kappa \in \pi_1(\kappa)$ denote the geometric Frobenius; recall that for every integer $i \geq 0$, it acts as $Fr^*_{Y_{\bar\kappa}|Y}$ on $\mathrm{H}^i(Y_{\bar\kappa}, \mathbb{Q}_\ell)$. Set $P_{i,\ell} := det(Id - Fr^*_\kappa T | \mathrm{H}^i(Y_{\bar\kappa}, \mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[T]$, $i = 0, \dots, 2d$. One wants to show that $P_i := P_{i,\ell}$ is in $\mathbb{Q}[T]$ and independent of ℓ. This follows from

Theorem 2.11. (Riemann hypothesis [D74], [D80]) *The eigenvalues of Fr_κ acting on $\mathrm{H}^i(Y_{\bar\kappa}, \mathbb{Q}_\ell)$ are $|\kappa|$-pure of weight i.*

Let us sketch how to deduce the $\mathbb{Q}$-compatibilty from Theorem 2.11. As already mentioned, it follows from the defining axioms of a Weil cohomology that ℓ-adic cohomology satisfies the Lefschetz trace formula (*e.g.* [?, §3.3.3]), which, in the case we are interested in, relate the zeta function of Y that is the power series

$$\sum_{x \in |XY_0|, deg(x)|n} deg(x) = |Y_0(\mathbb{F}_{q^n})| = \sum_i (-1)^i Tr(F^{n*} | H^i(Y, \mathbb{Q}_\ell)).$$

This shows that $Z(Y,T) \in \mathbb{Z}[[T]] \cap \mathbb{Q}_\ell(T)$. It is then a litlle (not that easy) exercise to check that $\mathbb{Q}[[T]] \cap \mathbb{Q}_\ell(T) \subset \mathbb{Q}(T)$.
But, now, writing $Z(Y,T) = \frac{N(T)}{D(T)}$ with $N, D \in \mathbb{Q}[T]$ coprime with positive constant terms, one gets

$$N(T) = P_{1,\ell}(T)P_{3,\ell}(T) \cdots P_{2d-1,\ell}(T),\ D(T) = P_{1,\ell}(T)P_{3,\ell}(T) \cdots P_{2d-1,\ell}(T).$$

From Theorem 2.11, if $K \subset \overline{\mathbb{Q}}_\ell$ is the decomposition field of $N(T)D(T)$, the set of roots of $P_{i,\ell}(T)$ is the subset of roots α of $N(T)D(T)$ with the property that for every complex embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\iota(\alpha)| = |\kappa|^{-\frac{i}{2}}$. As this set is $\mathrm{Gal}(K|\mathbb{Q})$-invariant and independant de ℓ, one gets $P_{i,\ell}$ lies in $\mathbb{Q}[T]$ and is independent of ℓ.

Using that crystalline cohomology is also a Weil cohomology, Katz and Messing [KM74] deduced from Theorem 2.11 that the characteristic polynomial of the linearized crystalline Frobenius acting on $\mathrm{H}^i_{\mathrm{cris}}(Y/W(\kappa))$ also lies in $\mathbb{Q}[T]$ and coincides with one of the geometric Frobenius acting on $\mathrm{H}^i(Y_{\bar\kappa}, \mathbb{Q}_\ell)$, $\ell \neq p$. As a

2.3.4.3. *Semisimplicity.* In general, one does not know that motivic representations are semisimple; it's one of the central conjecture - mostly motivated by the standard conjectures - in arithmetic geometry,

TateSS **Conjecture 2.12.** (Semisimplicity Conjecture) *Let k be finitely generated over its prime field. Then $\mathrm{Rep}^{\mathrm{geo}}_{\mathbb{Q}_\ell}(\pi_1(k)) \subset \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$.*

Conjecture 2.12 for $k/\mathbb{Q}$ number field implies that objects in $\mathrm{Rep}^{\mathrm{geo}}_{\mathbb{Q}_\ell}(\pi_1(k))$ are semisimple and Conjecture 2.12 for $k/\mathbb{F}_p$ a finite field, combined with Theorem 2.3.4 implies that objects in $\mathrm{Rep}^{\mathrm{geo}}_{\mathbb{Q}_\ell}(\pi_1(k))$ are Frobenius-semisimple.
Conjecture 2.12 is almost entirely open but one striking result towards it is the following celebrated theorem.

ltingsAV **Theorem 2.13.** (FaltingsRationalPoints [FW84]) *If Y/k is an abelian variety, then $\mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$.*

As $\mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k)) \subset \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ is a Tannakian subcategory, this implies $\langle \mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell)\rangle^{\otimes} \subset \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$; in particular, $\mathrm{H}^i(Y_{\bar{k}}, \mathbb{Q}_\ell(j)) = (\Lambda^i \mathrm{H}^1(Y_{\bar{k}}, \mathbb{Q}_\ell)(j) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$, $i \geq 0$, $j \in \mathbb{Z}$.

Independently of Conjecture 2.12, the reason why one has to impose axioms (FSS), (SS) is heuristic since the Q-rationality, purity and Q-compatibility axioms only capture information on the semisimple part of $\rho(F_{\bar{k_v}})$ through their characteristic polynomials $\chi_{\rho,v}$, $v \in U_\rho$ and these data determine the isomorphism class of (V, ρ) only if $(V, \rho) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$. Also, the group theory required to study ℓ-adic representations is significantly simpler and better understood in the semisimple case. Actually, condition (SS) is not so restrictive as one can always attach to an arbitrary representation $(V, \rho) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$ its semisimplification.

Summary

(AEU)	Smooth proper base-change (Grothendieck)
(τ)	comparison theorems between p-adic cohomologies (Faltings, +...)
$(\mathrm{RC}/_{Q,\mathrm{et}})$ and $(\mathrm{P}/_*)$	Riemann hypothesis (Deligne)
$(\mathrm{RC}/_Q)$ and $(\mathrm{P}/_*)$	Katz-Messing
(SS), (FSS)	??

3. ℓ-independence and the motivic philosophy

3.1. **First ℓ-independency results.** $k/\mathbb{Q}$ number field. Let $\mathcal{L}$ be a set of primes and, for each $\ell \in \mathcal{L}$ let $(V_\ell, \rho_\ell) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$; write $\Pi_\ell := \rho_\ell(\pi_1(k)) \subset \mathrm{GL}(V_\ell)$ and

$$G_\ell := \overline{\Pi}_\ell^{\mathrm{zar}} \subset \mathrm{GL}_{V_\ell}$$

for its Zariski-closure in GL_{V_ℓ}.

Assume the resulting family $(\underline{V}, \underline{\rho}) = (V_\ell, \rho_\ell)$, $\ell \in \mathcal{L}$ satisfies $(\mathrm{RC}/_{\mathbb{Q},\mathrm{et}})$. Let r denote the common $\mathbb{Q}_\ell$-dimension of V_ℓ. Here is a first basic example of ℓ-independency result.

p:CCrank **Proposition 3.1.** *Assume G_ℓ° is unipotent for one prime ℓ then G_ℓ° is unipotent for every prime ℓ. Furthermore, the kernel of $\epsilon_\ell : \pi_1(k) \to \pi_0(G_\ell)$ is independent of ℓ.*

Proof. Let $G_\ell \twoheadrightarrow G_\ell^{\mathrm{red}} := G_\ell / R_u(G_\ell)$ denote the maximal reductive quotient of G_ℓ; it identifies with the Zariski-closure of $\Pi_\ell^{\mathrm{ss}} := \rho_\ell^{\mathrm{ss}}(\pi_1(k)) \subset \mathrm{GL}_{V_\ell^{\mathrm{ss}}}$. As $R_u(G_\ell) \subset G_\ell^\circ$, one has $\pi_0(G_\ell) = \pi_0(G_\ell^{\mathrm{red}})$ and

$$\ker(\epsilon_\ell : \pi_1(k) \to \pi_0(G_\ell)) = \ker(\epsilon_\ell^{\mathrm{ss}} : \pi_1(k) \to \pi_0(G_\ell^{\mathrm{red}})).$$

Also G_ℓ° is unipotent if and only if $R_u(G_\ell) = G_\ell^\circ$ if and only if $G_\ell^{\mathrm{red}} = \pi_0(G_\ell^{\mathrm{red}})$ is finite. As a result, up to replacing (V_ℓ, ρ_ℓ) with its $\pi_1(k)$-semisimplification $(V_\ell^{\mathrm{ss}}, \rho_\ell^{\mathrm{ss}})$, oma $(V_\ell, \rho_\ell) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$, $\ell \in \mathcal{L}$. Note that $(\underline{V}^{\mathrm{ss}}, \underline{\rho}^{\mathrm{ss}}) = (V_\ell^{\mathrm{ss}}, \rho_\ell)^{\mathrm{ss}}$, $\ell \in \mathcal{L}$ still satisfies $(\mathrm{RC}/_{\mathbb{Q},\mathrm{et}})$. So, assuming $(V_\ell, \rho_\ell) \in \mathrm{Rep}^{\mathrm{ss}}_{\mathbb{Q}_\ell}(\pi_1(k))$, $\ell \in \mathcal{L}$, we are to prove that if G_ℓ is finite for one prime ℓ then G_ℓ is finite for every prime ℓ and that $\ker(\rho_\ell) \subset \pi_1(k)$ is independent of ℓ. Fix $\ell_0 \in \mathcal{L}$ such that G_{ℓ_0} is finite and let K/k denote the finite Galois extension corresponding to $\ker(\rho_{\ell_0}) \subset \pi_1(k)$. We are to show that $\pi_1(K) \subset \ker(\rho_\ell)$hat is $(V_\ell, \rho_\ell)|_{\pi_1(K)}$ is the trivial representation, $\ell \in \mathcal{L}$. But, this follows from the fact that, one the one hand,

$$(V_{\ell_0}, \rho_{\ell_0})|_{\pi_1(K)} \sim_{\mathbb{Q},\mathrm{et}} (V_\ell, \rho_\ell)|_{\pi_1(K)}$$

and, one the other hand

$$(V_{\ell_0}, \rho_{\ell_0}|_{\pi_1(K)} = \mathbb{Q}_{\ell_0}^{\oplus r}|_{\pi_1(K)} \sim_{\mathbb{Q},\mathrm{et}} \mathbb{Q}_\ell^{\oplus r}|_{\pi_1(K)},$$

where $\mathbb{Q}_\ell$ denote the trivial representation. As a result

$$(V_\ell, \rho_\ell)|_{\pi_1(K)} \sim_{\mathbb{Q},\mathrm{et}} \mathbb{Q}_\ell^{\oplus r}|_{\pi_1(K)}, \quad \ell \in \mathcal{L}$$

hence, as both $(V_\ell, \rho_\ell)|_{\pi_1(K)}$ and $\mathbb{Q}_\ell^{\oplus r}|_{\pi_1(K)}$ are semisimple, $(V_\ell, \rho_\ell)|_{\pi_1(K)} \simeq \mathbb{Q}_\ell^{\oplus r}|_{\pi_1(K)}$ by Cebotarev. $\square$

Recall that if Q is a field and G an algebraic group over Q, the reductive rank $\mathrm{rdrank}(G)$ of G is the common dimension of its maximal tori and that G is unipotent if and only if it has rank 0. Proposition 3.1 can be upgraded as follows.

m:CCrank **Theorem 3.2.** *The* (i) *kernel of* $\epsilon_\ell : \pi_1(k) \to \pi_0(G_\ell)$;
(ii) *reductive rank* $\mathrm{rdrank}(G_\ell)$,
are independent of ℓ.

The proof of Theorem 3.2 relies on the fact that both $\ker(\epsilon_\ell)$ and $\mathrm{rdrank}(G_\ell)$ are encoded on the image of $G_\ell \subset \mathrm{GL}_{V_\ell}$ *via* the characteristic polynomial map

$$\begin{array}{llll} \chi : & \mathrm{GL}_{r,\mathbb{Q}} & \to & \mathbb{A}_{\mathbb{Q}}^{r-1} \times \mathbb{G}_{m,\mathbb{Q}} \\ & g & \to & \chi(g) = (a_1(g), \ldots, a_r(g)), \end{array}$$

(where $\det(TI_r - g) = T^r + \sum_{1 \leq i \leq r} a_i(g) T^{r-i}$) and that this image is independent of ℓ.

3.1.1. *Preliminaries about characteristic polynomial map.* The restriction $\chi : \mathbb{G}_{m,\mathbb{Q}}^r \subset \mathrm{GL}_{r,\mathbb{Q}} \to \mathbb{A}_{\mathbb{Q}}^{r-1} \times \mathbb{G}_{m,\mathbb{Q}}$ identifies $\mathbb{A}_{\mathbb{Q}}^{r-1} \times \mathbb{G}_{m,\mathbb{Q}}$ with the quotient

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{Q}}^r \arrow[r, "\chi"] \arrow[d] & \mathbb{A}_{\mathbb{Q}}^{r-1} \times \mathbb{G}_{m,\mathbb{Q}} \\ \mathbb{G}_m^r/\mathcal{S}_r, \arrow[ur, "\simeq"] & \end{tikzcd}$$

(parametrizing GL_r-conjugacy classes of semisimple elements in GL_r). In particular, $\chi : \mathbb{G}_{m,\mathbb{Q}}^r \subset \mathrm{GL}_{r,\mathbb{Q}} \to \mathbb{A}_{\mathbb{Q}}^{r-1} \times \mathbb{G}_{m,\mathbb{Q}}$ is a finite morphism of degree $r!$.

Let now Q a field of characteristic 0 and $G \subset \mathrm{GL}_{r,Q}$ an algebraic subgroup. Then,

Rank **Lemma 3.3.** $\chi(G^\circ) \subset \mathbb{A}_Q^{r-1} \times \mathbb{G}_{m,Q}$ *is a closed subvariety, defined over* $\mathbb{Q}$ *and of dimension* $\mathrm{rdrank}(G)$.

Proof. Let $G^{\mathrm{ss}} \subset G$ denote the subset of semisimple elements in G. Using that for every $g \in G$, $\chi(g) = \chi(g^{\mathrm{ss}})$, where $g = g^{\mathrm{ss}} g^{\mathrm{u}} = g^{\mathrm{u}} g^{\mathrm{ss}}$ is the multiplicative Jordan decomposition of g in G (equivalently $\mathrm{GL}_{r,Q}$) and that, given a maximal torus $T \subset G$, every semisimple element in G is $G(\overline{Q})$-conjugate to an element in T, one has: for every $g \in G(\overline{Q})$,

$$\chi(G^\circ) = \chi(G^{\circ\mathrm{ss}}) = \chi(gTg^{-1}), \quad \gamma \in \mathrm{GL}_r(\overline{Q})$$

Furthermore there always exists $\gamma \in \mathrm{GL}_r(\overline{Q})$ such that $\gamma T \gamma^{-1} \subset \mathbb{G}_{m,\overline{Q}}$. So

$$\chi(G^\circ) = \chi|_{\mathbb{G}_{m,\overline{Q}}^r}(\gamma T \gamma^{-1}), \quad .$$

As $\chi|_{\mathbb{G}_{m,\overline{Q}}^r} : \mathbb{G}_{m,\overline{Q}}^r \to \mathbb{A}_{\overline{Q}}^{r-1} \times \mathbb{G}_{m,\overline{Q}}$ is

- finite, $\chi(G^\circ) \subset \mathbb{A}_{\overline{Q}}^{r-1} \times \mathbb{G}_{m,\overline{Q}}$ is a closed subvariety of dimension $\dim(gTg^{-1}) = \mathrm{rdrank}(G)$;
- defined over $\mathbb{Q}$ and that every subtorus of $\mathbb{G}_{m,\overline{Q}}^r$ is automatically defined over $\mathbb{Q}$, $\chi(G^\circ) \subset \mathbb{A}_{\overline{Q}}^{r-1} \times \mathbb{G}_{m,\overline{Q}}$ is defined over $\mathbb{Q}$.

The latter assertion uses the equivalence of categories

$$M \longleftrightarrow X^*(M_{\overline{Q}}) := \mathrm{Hom}_{\mathrm{GrAlg}/\overline{Q}}(M_{\overline{Q}}, \mathbb{G}_{m,\overline{Q}})$$

between groups of multiplicative types over Q and $\mathbb{Z}$-module of finite types endowed with a continuous action of $\pi_1(Q)$ plus the fact that, under this equivalence, $X^*(\mathbb{G}^r_{m,\overline{Q}}) = \mathbb{Z}^{\oplus r}$ endowed with the trivial action of $\pi_1(Q)$. □

CC **Lemma 3.4.** *For every $1 \neq C \in \pi_0(G)$ there exists $f_C \in \mathbb{Z}[a_1, \dots, a_r]$ such that $f_C = 0$ on $\chi(C)$ and $f_C \neq 0$ on $\chi(G^\circ)$; in particular, for every $C \in \pi_0(G)$, the Zariski-closure of $\chi(C)$ in $\mathbb{A}^{r-1}_Q \times \mathbb{G}_{m,Q}$ is equal to $\chi(G^\circ)$ (if and) only if $C = G^\circ$.*

Proof. Let $g \in G(Q) \setminus G^\circ(Q)$. Fix a faithful Q-linear representation $\phi : \pi_0(G) \to \mathrm{GL}(W)$ so that $\phi(g) \neq 1$. In particular, $\phi(g)$ has at least one eigenvalue $\lambda \neq 1$, which is automatically a primitive root of unity of order say n. As the tautological representation $V := Q^{\oplus r}$ of G is faithful, the Q-linear representation $G \twoheadrightarrow \pi_0(G) \to \mathrm{GL}(W)$ appears as a subquotient of a representation of the form $\oplus_{1\leq i\leq s} T^{m_i,n_i}(V)$. Let $\lambda_1, \dots, \lambda_r$ the eigenvalues of g acting on V (and counted with multiplicities). The characteristic polynomial of g acting on $T^{m,n}(V)$ can thus be written as

$$\prod_{\substack{1\leq i_1,\dots,i_m\leq r\\ 1\leq j_1,\dots,j_n\leq r}} (T - \frac{\lambda_{i_1}\cdots\lambda_{i_m}}{\lambda_{j_1}\cdots\lambda_{j_n}}) = \prod_{\substack{1\leq i_1,\dots,i_m\leq r\\ 1\leq j_1,\dots,j_{rn-n}\leq r}} (T - \frac{\lambda_{i_1}\cdots\lambda_{i_m}\lambda_{j_1}\cdots\lambda_{j_{rn-n}}}{a_r(g)^{N_{m,n}}}) = P_{m,n}(T, a_1(g), \dots, a_r(g), a_r(g)^{-1})$$

avec $P_{m,n}(T, a_1, \dots, a_r, a_r^{-1}) \in \mathbb{Z}[a_1, \dots, a_r, a_r^{-1}]$. Fix an integer $N \gg 0$ such that

$$\widetilde{P}_{m,n} := \prod_{\zeta\in\Phi_n(1)} P_{m,n}(\zeta, a_1, \dots, a_r, a_r^{-1})a_r^N \in \mathbb{Z}[a_1, \dots, a_r],$$

where $\Phi_n(1) \subset \overline{\mathbb{Z}}^\times$ denotes the set of primitive nth roots of 1. With these notation, the function $f = \prod_{1\leq i\leq s} P_{m_i,n_i} \in \mathbb{Z}[a_1, \dots, a_r]$ has the requested properties. □

3.1.2. *Proof of Theorem 3.2.*

3.1.2.1. *Proof of Theorem 3.2 (1).* From Proposition 3.1, one may assume Π_ℓ is infinite for all $\ell \in L$.

n:CCrank **Lemma 3.5.** *Let $\ell \in \mathcal{L}$. The following assertions are equivalent.*

(i) $G^\circ_\ell = G_\ell$;
(ii) *For every $f \in \mathbb{Z}[a_1, \dots, a_r]$, the set $\Sigma^f_k \subset \Sigma_k$ of all $v \in U_{\rho_\ell,\nmid\ell}$ such that $f \circ \chi_{\rho_{v,\ell}} = 0$ has density 0 or 1.*

Lemma 3.5 ⇒ Theorem 3.2 (1): Fix $\ell_0 \in L$ and set $K := \overline{k}^{\ker(\epsilon_{\ell_0})}$. Up to replacing $(\underline{V}, \underline{\rho})$ with $(\underline{V}, \underline{\rho}|_{\pi_1(K)})$, oma $G^\circ_{\ell_0} = G_{\ell_0}$ so that (i) - hence equivalently (ii) - of Lemma 3.5 holds for $\ell = \ell_0$. But as $(\underline{V}, \underline{\rho})$ satisfies (RC/$_{\mathbb{Q},\mathrm{et}}$), if (ii) of Lemma 3.5 holds for ℓ_0, (ii) - hence equivalently (i) - of Lemma 3.5 holds for every $\ell \in \mathcal{L}$; this shows $\ker(\epsilon_{\ell_0}) \subset \ker(\epsilon_\ell)$. By symmetry, $\ker(\epsilon_{\ell_0}) = \ker(\epsilon_\ell)$

Proof. of Lemma 3.5.

(i) ⇒ (ii). Wlog oma $\rho_\ell(\pi_1(k)) := \Pi_\ell \subset \mathrm{GL}_r(\mathbb{Z}_\ell)$. Assume $G_\ell = G^\circ_\ell$ and let $f \in \mathbb{Z}[a_1, \dots, a_r]$ such that $f(\chi(G_\ell)) \neq 0$. Write $V_f \subset M_{r,\mathbb{Z}}$ for the reduced closed subscheme defined by $f \circ \chi : M_{r,\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ and

$$X_f := V_f(\mathbb{Z}_\ell) \cap \Pi_\ell \subsetneq \Pi_\ell.$$

By construction X_f is stable under Π_ℓ-conjugacy. One has:

Claim. *$\partial X_f = X_f$ and $\mu(X_f) = 0$, where $\mu : \mathcal{B}_{\Pi_\ell} \to [0, 1]$ is the normalized Haar measure on Π_ℓ.*

In particular, by the profinite Cebotarev density theorem, the set $\Sigma^f_k \subset \Sigma_k$ of all $v \in \Sigma_k$ such that $\rho_{v,\ell}(Fr_{k_{\overline{v}}}) \in X_f$ has density 0.

Proof of the claim. As by construction X_f is closed in Π_ℓ, $\partial X_f = X_f$ if and only if X_f has empty interior in Π_ℓ. We argue by contradiction. Otherwise, X_f would contain an open subset of Π_ℓ, which we may always assume to be of the form gU for some $g \in \Pi_\ell$ and normal open subgroup $U \subset \Pi_\ell$. As G_ℓ is connected, one has $\overline{U}^{\text{zar}} = G_\ell$. Indeed,

$$G_\ell = \overline{\Pi}_\ell^{\text{zar}} = \overline{\bigsqcup_{\gamma \in \Pi_\ell/U} \gamma U}^{\text{zar}} = \bigcup_{\gamma \in \Pi_\ell/U} \overline{\gamma U}^{\text{zar}} = \bigcup_{\gamma \in \Pi_\ell/U} \gamma \overline{U}^{\text{zar}}$$

but as G_ℓ is smooth and connected hence irreducible, and $\Pi_\ell/U$is finite, there exists $\gamma \in G_\ell/U$ such that $G_\ell = \gamma\overline{\gamma U}^{\text{zar}} = \gamma\overline{U}^{\text{zar}}$ *i.e.* $G_\ell = \overline{U^{\text{zar}}}$. On the other hand, as $V_f \subset M_{r,\mathbb{Q}_\ell}$ is Zariski-closed, one gets

$$V_f \supset \overline{gU}^{\text{zar}} = g\overline{U}^{\text{zar}} = gG_\ell = G_\ell,$$

which contradicts the definition of f.
It remains to prove that $\mu(X_f) = 0$. Let $V_1, \dots, V_s$ denote the irreducible components of V_f. From $X_f = \bigcup_{1 \leq i \leq s} V_i(\mathbb{Z}_\ell) \cap G_\ell$, it is enough to prove that $\mu(V_i(\mathbb{Z}_\ell) \cap \Pi_\ell) = 0$, $i = 1, \dots, s$. Hence we may assume V_f is irreducible. We proceed in two steps.

(1) For every $g_1, \dots, g_s, g'_1, \dots, g'_s \in \Pi_\ell$ one has

$$\bigcup_{1 \leq i \leq s} g_i X_f g'_i \subsetneq \Pi_\ell.$$

We argue by induction on s. If $s = 1$ there is nothing to say. Assume $s \geq 2$. By induction hypothesis,

$$\bigcup_{1 \leq i \leq s-1} g_i X_f g'_i \subsetneq \Pi_\ell$$

so that X_f contains the non-empty open subset $g_s^{-1}(\Pi_\ell \setminus \bigcup_{1 \leq i \leq s-1} g_i X_f g'_i) g_s^{\prime -1}$, which contradicts the fact that X_f has empty interior in Π_ℓ.

(2) Let $V \hookrightarrow M_{r,\mathbb{Z}_\ell}$ be an irreducible closed subscheme. If $X := V(\mathbb{Z}_\ell) \cap \Pi_\ell$ satsifies (*), then $\mu(X) = 0$.

We argue by induction on the dimension d of $V_{\mathbb{Q}_\ell}$. If $d = 0$, there is nothing to say (recall Π_ℓ is assumed to be infinite). If $d \geq 1$, fix $x \in X$ and set $\gamma_0 = 1$. For every $\gamma_1 \in \Pi_\ell \setminus Xx^{-1}$ one has $\gamma_1 V \neq V$ (otherwise there would exist $g \in X$ such that $\gamma_1 x = g$, contradicting the choice of γ_1). Choose inductively

$$\gamma_{i+1} \in \Pi_\ell \setminus \bigcup_{0 \leq j \leq i} \gamma_j X x^{-1}.$$

By construction one has $\gamma_j V \neq \gamma_{i+1} V$, $j = 0, \dots, i$ (otherwise there would exist $g \in X$ such that $\gamma_{i+1} x = \gamma_j g$, contradicting the choice of γ_{i+1}). As V is irreducible, $\gamma_j V \cap \gamma_{i+1} V$ has dimension $\leq$dim$V - 1$. On the other hand, for every $g_1, \dots, g_s, g'_1, \dots, g'_s \in \Pi_\ell$ one has

$$\bigcup_{1 \leq k \leq s} g_k \gamma_j V \cap \gamma_{i+1} V \cap \Pi_\ell g'_k \subset \bigcup_{1 \leq k \leq s} (g_k \gamma_j) V \cap \Pi_\ell g'_k \subsetneq \Pi_\ell.$$

Hence, by induction hypothesis, $\mu(\gamma_j V \cap \gamma_{i+1} V \cap \Pi_\ell) = 0$. Set now $X_i := \gamma_i V \cap \Pi_\ell$ and $Y_i := \bigcup_{0 \leq j \leq i} X_j$. From the above $\mu(Y_i \cap X_{i+1}) = 0$ hence, using

$$Y_{i+1} = (Y_\setminus (Y_i \cap X_{i+1})) \sqcup (X_{i+1} \setminus (Y_i \cap X_{i+1})) \sqcup (Y_i \cap X_{i+1})),$$

one gets

$$\mu(Y_{i+1}) = \mu(Y_i) + \mu(X_{i+1}) = \sum_{0 \leq j \leq i+1} \mu(X_j) = (i+1)\mu(X),$$

where the last equality follows from the Π_ℓ-invariance of Haar measure. Taking $i \to +\infty$, one gets $\mu(X) = 0$.

(ii) $\Rightarrow$ (i). We argue by contradiction. Assume $G_\ell^\circ \subsetneq G_\ell$ and fix $1 \neq C \in \pi_0(G_\ell)$. From Lemma 3.5 there exists $f := f_C \in \mathbb{Z}[a_1, \dots, a_r]$ such that $f = 0$ on $\chi(C)$ and $f \neq 0$ on $\chi(G_\ell^\circ)$. Let $\pi_0(G_\ell)^f \subset \pi_0(G_\ell)$ denote the set of all $C \in \pi_0(G_\ell)$ such that $f = 0$ on $\chi(C)$ and write $a := |\pi_0(G_\ell)^f|$, $b := |\pi_0(G_\ell)| - a$. By assumption $1 < a < |\pi_0(G_\ell)|$. As $\pi_0(G_\ell)$ is abelian, it follows from Cebotarev density theorem and the Claim in the proof of (i) $\Rightarrow$ (ii) that the subset $\Sigma_k^f \subset \Sigma_k$ of all $v \in \Sigma_k$ such that

$$\epsilon_\ell : \pi_1(k) \xrightarrow{\rho_\ell} G(\mathbb{Q}_\ell) \twoheadrightarrow \pi_0(G_\ell)$$

is unramified at v with $\epsilon_{v,\ell}(\phi_{k_v^-}) \in \pi_0(G_\ell)^f$ has density $0 < \frac{a}{a+b} < 1$, contradicting (ii).

$\square$

3.1.2.2. *Proof of Theorem 3.2 (2).* According to Theorem 3.2 (1), after possibly replacing k with a finite field extension, which does not alter G_ℓ°, oma $G_\ell^\circ = G_\ell$, $\ell \in \mathcal{L}$ so that, from Lemma 3.3, $\mathrm{rdrank}(G_\ell) = \dim(\chi(G_\ell))$, $\ell \in \mathcal{L}$. It is thus enough to show that $\dim(\chi(G_\ell))$ is independent of $\ell \in \mathcal{L}$. From **??**, the subset

$$\Phi_\ell := \bigcup_{v \in U_{\rho_\ell, \nmid \ell}} \subset \Pi_\ell$$

of all Π_ℓ-conjugacy class of Frobenii $\rho_{v,\ell}(\phi_{k_v^-})$, $v \in U_{\rho_\ell, \nmid \ell}$ is ℓ-adic analytically dense in $G_\ell(\mathbb{Q}_\ell)$ - hence *a fortiori* Zariski-dense in G_ℓ; in particular,

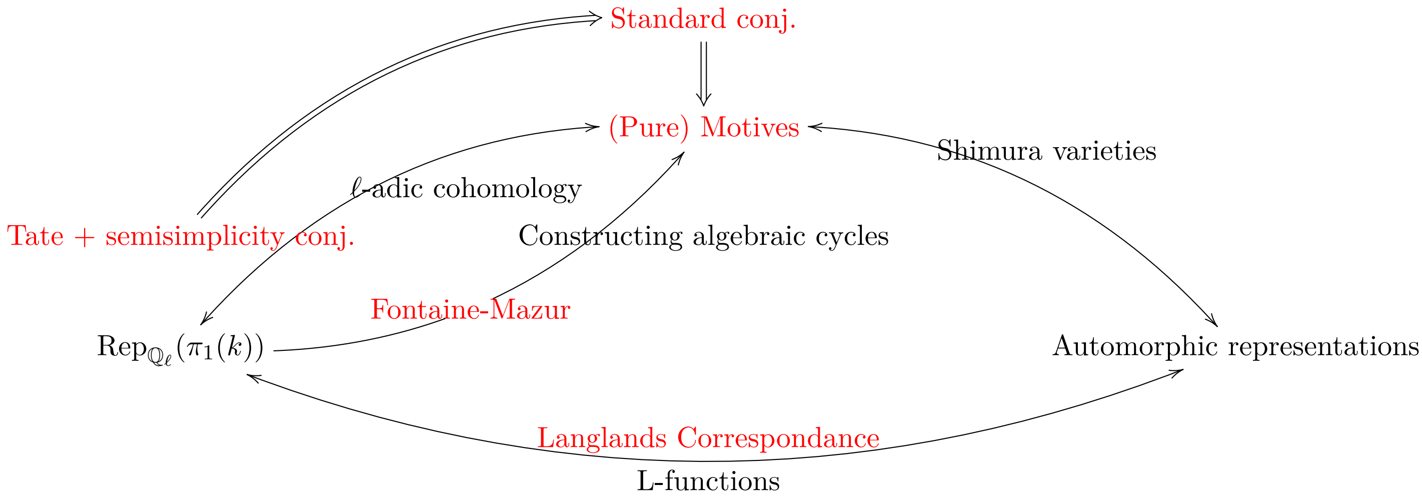
$$\chi(G_\ell) = \chi(\overline{\Phi}_\ell^{\mathrm{zar}}) \subset \overline{\chi(\Phi_\ell)}^{\mathrm{zar}} \subset \mathbb{A}^{r-1}_{\mathbb{Q}_\ell} \times \mathbb{G}_{m,\mathbb{Q}_\ell}.$$

On the other hand, from Lemma 3.3 we also know that $\chi(G_\ell) \subset \mathbb{A}^{r-1}_{\mathbb{Q}_\ell} \times \mathbb{G}_{m,\mathbb{Q}_\ell}$ is Zariski closed, hence $\overline{\chi(\Phi_\ell)}^{\mathrm{zar}} \subset \chi(G_\ell)$. This shows that

$$\overline{\chi(\Phi_\ell)}^{\mathrm{zar}} = \chi(G_\ell).$$

But, by assumption, $\chi(\Phi_\ell) \subset (\mathbb{A}^{r-1} \times \mathbb{G}_m)(\mathbb{Q})$ is indépendant de ℓ; let $\Xi \subset \mathbb{A}^{r-1}_{\mathbb{Q}} \times \mathbb{G}_{m,\mathbb{Q}}$ denotes its Zariski-closure. Then $\Xi_{\mathbb{Q}_\ell} = \overline{\chi(\Phi_\ell)}^{\mathrm{zar}} = \chi(G_\ell)$; in particular, $\dim(\chi(G_\ell)) = \dim(\Xi)$ is independent of ℓ.

3.2. **The motivic philosophy.** $k/\mathbb{Q}$-number field. ℓ-adic representations of $\pi_1(k)$ are part of a coherent network of conjectures which is very roughly encapsulated in the following picture.



[...]

The simplified version of the "horizon conjecture" to keep in mind is the following.

:Horizon

Conjecture 3.6. *$k/\mathbb{Q}$ a number field, X/k smooth, projective variety over k. Let $\Pi_\ell \subset \mathrm{GL}(V_\ell)$ denote the image of the $\pi_1(k)$-representation $V_\ell := \mathrm{H}^i(X_{\bar{k}}, \mathbb{Q}_\ell)$, and $G_\ell := \overline{\Pi}_\ell^{\mathrm{zar}} \subset \mathrm{GL}_{V_\ell}$ its Zariski-closure. There exists a reductive group G over $\mathbb{Q}$ together with a faithfull $\mathbb{Q}$-rational representation $G \hookrightarrow \mathrm{GL}_V$ such that, for every prime ℓ, one has*

$$(G \hookrightarrow \mathrm{GL}_V) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq G_\ell \hookrightarrow \mathrm{GL}_{V_\ell}.$$

Sec:LP

4. The $r = 2$ case of a theorem of Larsen-Pink and a first glance at Frobenius tori

$k/\mathbb{Q}$ number field, $\mathcal{L}$ infinite set of prime numbers. For each $\ell \in \mathcal{L}$, fix $(V_\ell, \rho_\ell) \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$. Write $\Pi_\ell := \rho_\ell(\pi_1(k)) \subset \mathrm{GL}(V_\ell)$ and $G_\ell := \overline{\Pi}_\ell^{\mathrm{zar}} \subset \mathrm{GL}_{V_\ell}$. For $v \in U_{\rho_\ell, \nmid \ell}$, write $\varphi_{v,\ell} = \rho_{v,\ell}(\varphi_{k_v^-}) \in \Pi_\ell$ for a representative of the Π_ℓ-conjugacy class $\Phi_{v,\ell} \subset \Pi_\ell$ of Frobenii at v and $\chi_{v,\ell} := \det(TId - \varphi_{v,\ell}) \in \mathbb{Z}_\ell[T]$ for its characteristic polynomial (which only depends on $\Phi_{v,\ell}$ and not on $\varphi_{v,\ell}$).

The aim of this section is to prove the $r = 2$ case - see Theorem 4.9, of the following theorem.

Thm:LP **Theorem 4.1.** ([LP92, Thm. 9.1, Thm. 9.4])*Assume* $(\underline{V}, \underline{\rho})$ *satisfies* $(\mathrm{RC}/_{\mathbb{Q},\mathrm{et}})$ *and that, for every* $\ell \in \mathcal{L}$, $G_\ell^\circ = G_\ell$ *is reductive connected. Then,*

(1) There exists a finite Galois extension $Q/\mathbb{Q}$ *and a subset* $\mathcal{L}' \subset \mathcal{L}$ *of density* 0 *(containing all places that ramify in* $Q/\mathbb{Q}$*) such that for every* $\ell \in \mathcal{L} \setminus \mathcal{L}'$*, the Weyl group* $W(G_\ell)$ *of* G_ℓ *depends only on the* $Gal(Q|\mathbb{Q})$*-conjugacy class* $\Phi_\ell \subset Gal(Q|\mathbb{Q})$ *of the Frobenii at* ℓ*. In particular, the dimension of* G_ℓ *and the dimension of* $Z(G_\ell)$*.*

(2) Assume furthermore that, for every $\ell \in \mathcal{L}$, (V_ℓ, ρ_ℓ) *is absolutely irreducible. Then the same conclusion holds as in (1) for the root datum of* G_ℓ *(viz. the* $\overline{\mathbb{Q}}_\ell$*-isomorphism class of* $G_{\ell,\overline{\mathbb{Q}}_\ell}$*).*

Recall that one can always reduce to the case where $G_\ell^\circ = G_\ell$, $\ell \in \mathcal{L}$ is reductive connected, after replacing k by a finite Galois extension K/k (Theorem 3.2) and semisimplification.

Theorem 4.1 relies on a subtle analysis of the arithmetico-geometric properties of tori in connected reductive groups. The arguments and techniques involved to treat the general caseof Theorem 4.1 go beyong the ambitions of these lectures but we can still give a good idea of the heuristic by considering the $r = 2$ case. For this, we first review a few results about tori. These play a central part in the structural analysis of algebraic groups. One way to produce tori in G_ℓ is by considering the algebraic enveloppe of the semisimple part of Frobenii elements.

4.1. **Algebraic enveloppe.** Let Q be a field, $char(Q) = 0$ and G/Q an algebraic group over Q.

4.1.1. *Definition.* For every $g \in G(Q)$ define the algebraic enveloppe of g in G is the smallest algebraic subgroup $\mathcal{E}_G(g) \subset G$ containing g *viz* the Zariski-closure of the subgroup $\langle g \rangle \subset G(Q)$ generated by g. We let the following as an exercise.

Lemma 4.2. *If* $g = g^{\mathrm{ss}} g^{\mathrm{u}} = g^{\mathrm{u}} g^{\mathrm{ss}}$ *is the multiplicative Jordan decomposition of* $g \in G$ *then*

$$\mathcal{E}_G(g) \simeq \mathcal{E}_G(g^{\mathrm{ss}}) \times \mathcal{E}_G(g^{\mathrm{u}})$$

with $\mathcal{E}_G(g^{\mathrm{ss}})$ *a group of multiplicative type;*
$\mathcal{E}_G(g^{\mathrm{u}}) \simeq \mathbb{G}_{a,Q}$ *if* $g^{\mathrm{u}} \neq 1$ *and* $\mathcal{E}_G(g^{\mathrm{u}}) \simeq 1$ *if* $g^{\mathrm{u}} = 1$*.*

Write $T_G(g) := \mathcal{E}_G(g^{\mathrm{ss}})^\circ \subset \mathcal{E}_G(g^{\mathrm{ss}})$, which we call the algebraic torus attached to g. The following result requires more work.

Theorem 4.3. *Assume* $G^\circ = G$ *is connected; let* $T \subset G$ *be a torus. Then,*
- ([Bor91, 11.12]) *The centralizer* $\mathrm{Z}_G(T)$ *of* T *in* G *is connected;*
- ([Bor91, 12.1]) *If, furthermore,* $T \subset G$ *is a maximal torus* $\mathrm{Z}_G(T) = T \times R_u(\mathrm{Z}_G(T)) = \mathrm{Nor}_G(\mathrm{Z}_G(T))^\circ$ *and, if* G *is reductive,* $\mathrm{Z}_G(T) = T$*.*

By construction, $T_G(g) \subset \mathcal{E}_G(g) \subset \mathrm{Z}_G(T_G(g))$ hence, if $g = g^{\mathrm{ss}}$ and $T_G(g) \subset G$ is maximal, one gets $T_G(g) = \mathcal{E}_G(g)$ (since the projection of g onto $R_u(\mathrm{Z}_G(T_G(g)))$ is trivial).

Example: For $v \in U_{\rho_\ell, \nmid \ell}$, the $G_\ell(\mathbb{Q}_\ell)$-conjugacy class of $\varphi_{v,\ell,v}$ hence of $\mathcal{E}_{v,\ell} := \mathcal{E}_{G_\ell}(\varphi_{v,\ell}^{\mathrm{ss}})$ and of $T_{v,\ell} := T_{G_\ell}(\varphi_{v,\ell}^{\mathrm{ss}})$ depend only on v and not on the representative $\varphi_{v,\ell}$ in $\Phi_{v,\ell}$ while the $\mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$-conjugacy class of $\mathcal{E}_{v,\ell}$ and $T_{v,\ell}$ only depend on the characteristic polynomial $\chi_{v,\ell} \in \mathbb{Z}_\ell[T]$ of $\varphi_{v,\ell}$. One says that by a slight abuse of language that $T_{v,\ell}$ and $\mathcal{E}_{v,\ell}$ are "the" Frobenius torus and multiplicative group attached to (V_ℓ, ρ_ℓ) at v respectively. If $(\underline{V}, \underline{\rho})$ satisfies $(\mathrm{RC}/_{\mathbb{Q},et})$, $\chi_v := \chi_{v,\ell}$ is in $\mathbb{Q}[T]$ and independent of ℓ hence $\mathcal{E}_{v,\ell}$ and $T_{v,\ell}$ are defined over $\mathbb{Q}$.

4.1.2. *Regular semisimple elements and maximal tori.* If $G = G^\circ$ is connected reductive, one says that $g \in G(Q)$ *regular semisimple* if $g = g^{\mathrm{ss}}$ and $T_G(g) \subset G$ is a maximal torus. Regular semisimple elements are abundant, namely:

hm:SSReg **Theorem 4.4.** ([Bor91, §11-13]) *Assume* $G = G^\circ$ *is connected reductive, there there exists a non-empty (hence Zariski-dense) open subscheme* $U \subset G$*, stable under* $G(\overline{Q})$*-conjugacy and such that every* $g \in U(\overline{Q})$ *is regular semisimple.*

Rem. If one no longer assume G is reductive, it is still true that there there exists a non-empty (hence Zariski-dense) open subscheme $U \subset G$, stable under $G(\overline{Q})$-conjugacy and such that for every $g \in U(\overline{Q})$, $T_G(g) = \mathcal{E}_G(g^{\mathrm{ss}}) \subset G$ is a maximal torus. Indeed, just consider the maximal reductive quotient $p : G \twoheadrightarrow G^{\mathrm{red}} := G/R_u(G)$ and consider a Zariski-dense open subscheme $U \subset G^{\mathrm{red}}$ as in Theorem 4.4. Then $p^{-1}(U) \subset G$ as the requested property.

Combined with the Claim in the proof of Lemma 3.5, one gets

Key1 **Corollary 4.5.** (Frobenius tori versus maximal tori) *Assume $G_\ell^\circ = G_\ell$ is connected. Then the set of all $v \in U_{\rho_\ell,\nmid\ell}$ such that $T_{v,\ell} \subset G_\ell$ is a maximal torus has density* 1.

This provides an alternative proof of Theorem 3.2.

Rem. If one no longer assume that $G_\ell^\circ = G_\ell$ is connected, it is still true that the set of all $v \in U_{\rho_\ell,\nmid\ell}$ such that $T_{v,\ell} \subset G_\ell$ is a maximal torus has density $\geq \frac{1}{|\pi_0(G_\ell)|}(> 0)$.

If (V_ℓ, ρ_ℓ) is part of a family $(\underline{V}, \underline{\rho}))$ satisfying $(\mathrm{RC}/_{\mathbb{Q},\mathrm{et}})$, one has the following technical refinement of Corollary 4.5

Key2 **Theorem 4.6.** ([LP92, §7]) *Assume $G_\ell^\circ = G_\ell$ is connected reductive for every $\ell \in L$. Fix a finite subset $I \subset L$ and for each $\ell \in I$, a maximal torus $T_\ell \subset G_\ell$. Then there exists (infinitely many) $v \in \cap_{\ell \in I} U_{\rho_\ell,\nmid\ell}$ such that, for every $\ell \in I$, $\mathcal{E}_{v,\ell} = T_{v,\ell} \subset G_\ell$ is $G_\ell(\mathbb{Q}_\ell)$-conjugate to the given T_ℓ.*

We refer to [LP92, §7] for the proof which, as the one of Corollary 4.5, combines the general theory of tori in algebraic groups and Cebotarev density theorem but also requires a bit ℓ-adic Lie group theory, in particular the fact that if G_ℓ is a semisimple algebraic group over $\mathbb{Q}_\ell$ and $\Pi_\ell \subset G_\ell(\mathbb{Q}_\ell)$ is a compact subgroup which is Zariski-dense in G_ℓ then Π_ℓ is open in $G_\ell(\mathbb{Q}_\ell)$).

4.1.3. *Description of $\mathcal{E}_G(g)$ in terms of characters.* Assume $g = g^{\mathrm{ss}}$ and $Q = \overline{Q}$. Fix a faithful Q-linear representation $G \hookrightarrow \mathrm{GL}_V \simeq \mathrm{GL}_{r,Q}$ such that $g = (\alpha_1, \ldots, \alpha_r)$ is contained in the diagonal torus $D_{r,Q} \simeq \mathbb{G}_{m,Q}^r \subset \mathrm{GL}_{r,Q}$. One can explicitly describe $\mathcal{E}(g) := \mathcal{E}_G(g) = \mathcal{E}_{\mathrm{GL}_{r,Q}}(g)$ *via* the equivalence of categories $\Theta \mapsto X^*(\Theta)$ from multiplicative groups to finitely generated $\mathbb{Z}$-modules. Using the canonical $\mathbb{Z}$-basis $e_1, \ldots, e_r$ (defined by $e_i(x_1, \ldots, x_r) = x_i$, $i = 1, \ldots, r$), one gets a canonical identification $c : \oplus_{1 \leq i \leq r} \tilde{\to} \mathbb{Z}e_i X^*(D_r)$. With these notation, it follows from the definition of $\mathcal{E}(g)$ and the short exact sequence

$$0 \to X^*(D_r/\mathcal{E}(g)) \to X^*(D_r) \to X^*(\mathcal{E}(g)) \to 0$$

that

$$X^*(D_r/\mathcal{E}(g)) = \{\chi = (a_1, \ldots, a_r) \in \mathbb{Z}^{\oplus r} \mid \alpha_1^{a_1} \cdots \alpha_r^{a_r} = 1\} \subset X^*(D_r).$$

and

$$\mathcal{E}(g) = \bigcap_{\chi \in X^*(D_r/\mathcal{E}(g))} \ker(\chi).$$

Let $\Lambda_V(g) := \{\alpha_1, \ldots, \alpha_r\} \subset Q^\times$ the $\mathbb{Z}$-submodule generated by $\alpha_1, \ldots, \alpha_r$. By the universal property of $\mathbb{Z}^{\oplus r}$, there is a unique (automatically surjective) morphism of $\mathbb{Z}$-modules

$$ev_{\underline{\alpha}} : \mathbb{Z}^{\oplus r} \to \Lambda_V(g),\ \underline{a} \mapsto \alpha_1^{a_1} \cdots \alpha_r^{a_r},$$

which fits into the commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[r] & X^*(D_r/\mathcal{E}(g)) \arrow[r] & X^*(D_r) \arrow[r] & X^*(\mathcal{E}(g)) \arrow[r] & 0 \\
0 \arrow[r] & \ker(ev_{\underline{\alpha}}) \arrow[r] \arrow[u, "\simeq"] & \mathbb{Z}^{\oplus r} \arrow[r, "ev_{\underline{\alpha}}"] \arrow[u, "\simeq", "c"'] & \Lambda_V(g) \arrow[r] \arrow[u, "\simeq"] & 0
\end{tikzcd}$$

In particular, $\Lambda(g) := \Lambda_V(g)$ depends only on g (and not on the faithful linear representation $G \hookrightarrow \mathrm{GL}_V$ and one has

$$\Lambda(g)_{\mathrm{tors}} \simeq X_*(\mathcal{E}(g))_{\mathrm{tors}} \simeq \pi_0(\mathcal{E}(g)),\ \ \mathrm{rank}(T(g)) = \mathrm{rank}(\Lambda(g))\ \ \textit{etc.}$$

Here are two first applications of these observations to ℓ-adic Galois representations.

4.1.3.1. *Homotheties torus.*

Claim: *Assume $Q \subset \mathbb{C}$ and $|\alpha_1| = \cdots = |\alpha_r| =: \alpha \neq 1$. Then $\mathcal{E}(g)$ contains the homotheties torus $Z(\mathrm{GL}_{r,Q}) = H_{r,Q} \simeq \mathbb{G}_{m,Q}$.*

This follows from

$$\begin{aligned} X^*(D_r/\mathcal{E}(g)) &\overset{c}{\simeq} \{\underline{a} \in \mathbb{Z}^{\oplus r} \mid \alpha_1^{a_1} \cdots \alpha_r^{a_r} = 1\} \\ &\subset \{\underline{a} \in \mathbb{Z}^{\oplus r} \mid |\alpha_1|^{a_1} \cdots |\alpha_r|^{a_r} = \alpha^{a_1 + \cdots + a_r} = 1\} \\ &= \{\underline{a} \in \mathbb{Z}^{\oplus r} \mid a_1 + \cdots + a_r = 0\} \overset{c}{\simeq} \ker(X^*(D_r) \twoheadrightarrow X^*(H_{r,Q})) =: K, \end{aligned}$$

whence

$$\begin{tikzcd} 0 \arrow[r] & X^*(D_r/\mathcal{E}(g)) \arrow[r] \arrow[d, hook] & X^*(D_r) \arrow[r] \arrow[d, equal] & X^*(\mathcal{E}(g)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & X^*(D_r) \arrow[r] & X^*(H_{r,Q}) \arrow[r] & 0. \end{tikzcd}$$

Via the equivalence $\Theta \mapsto X^*(\Theta)$, the right square corresponds to a factorization

$$\begin{tikzcd} \mathcal{E}(g) \arrow[r, hook] \arrow[dr, hook] & D_{r,Q} \\ & H_{r,Q} \arrow[u, hook] \end{tikzcd}$$

otheties **Corollary 4.7.** *If (V_ℓ, ρ_ℓ) satisfies $(\mathrm{P}/_w)$ for some weight $w \neq 0$ then $Z(G_\ell)$ contains a 1-dimensional split torus.*

Proof. Just apply the claim to $g := \phi^{\mathrm{ss}}_{v,\ell}$ for some $v \in U_{\rho_\ell, \nmid \ell}$. □

Corollary 4.7 applies for instance to $V_\ell = \mathrm{H}^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$ for X/k a smooth, proper scheme over k and $2i \neq j$.

4.1.3.2. *Connexity and neatness.* From the identifications

$$\Lambda(g)_{\mathrm{tors}} \simeq X_*(\mathcal{E}(g))_{\mathrm{tors}} \simeq \pi_0(\mathcal{E}(g))$$

one sees that $\pi_0(\mathcal{E}(g)) = 1$ if and only if $\Lambda(g)_{\mathrm{tors}} = 1$, in which case one says that $g \in G$ is neat.

onnected **Corollary 4.8.** *If (V_ℓ, ρ_ℓ) satisfies (AEU) and there exists a $\pi_1(k)$-stable $\mathbb{Z}_\ell$-lattice $\Lambda_\ell \subset V_\ell$ such that $\pi_1(k)$ acts trivially on $\Lambda_\ell/\mathfrak{l}$, where $\mathfrak{l} = 4$ if $\ell = 2$ and $\mathfrak{l} = \ell$ otherwise then $G_\ell^\circ = G_\ell$.*

Proof. We argue by contradiction. Assume $\pi_0(G_\ell) \neq 1$ and let $1 \neq a \in \pi_0(G_\ell)$. Then from (classical) Cebotarev applied to $\epsilon_\ell : \pi_1(k) \twoheadrightarrow \pi_0(G_\ell)$, there exists (infinitely many) $v \in U_{\rho_\ell, \nmid \ell}$ such that $\varphi_{v,\ell}$ (equivalently, $\varphi^{\mathrm{ss}}_{v,\ell}$) maps to $a \in \pi_0(G_\ell)$. From the commutative diagram with exact lines

$$\begin{tikzcd} 0 \arrow[r] & T_{G_\ell}(\varphi_{v,\ell}) \arrow[r] \arrow[d, hook] & \mathcal{E}_{G_\ell}(\varphi^{\mathrm{ss}}_{v,\ell}) \arrow[r] \arrow[d, hook] & \pi_0(\mathcal{E}_{G_\ell}(\varphi^{\mathrm{ss}}_{v,\ell})) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G_\ell^\circ \arrow[r] & G_\ell \arrow[r] & \pi_0(G_\ell) \arrow[r] & 0, \end{tikzcd}$$

one sees that $\pi_0(\mathcal{E}_{G_\ell}(\varphi_\ell^{\mathrm{ss}})) \neq 1$ as well. On the other hand, the assumption that $\Pi_\ell \subset Id + \ell \mathrm{End}_{\mathbb{Z}_\ell}(\Lambda_\ell)$ forces $\Lambda(\varphi^{\mathrm{ss}}_{v,\ell})_{\mathrm{tors}} = 1$: a contradiction. □

If $(\underline{V}, \underline{\rho})$ satisfies $\mathrm{RC}/_{\mathbb{Q},\mathrm{et}}$, we know that $K(\underline{V}, \underline{\rho}) := \ker(\epsilon_\ell : \epsilon_\ell : \pi_1(k) \twoheadrightarrow \pi_0(G_\ell))$ is independent of ℓ. From Corollary 4.8, for every prime ℓ and $\pi_1(k)$-stable $\mathbb{Z}_\ell$-lattice $\Lambda_\ell \subset V_\ell$ one has

$$K(\underline{V}, \underline{\rho}) \supset \ker(\pi_1(k) \to \mathrm{GL}(\Lambda_\ell/\mathfrak{l})).$$

4.2. **The $r=2$ of Theorem 4.1.** We retain the notation and assumption of the beginning of Section 4. Assume furthermore that

$$r := \dim_{\mathbb{Q}_\ell}(V_\ell) = 2.$$

Let $U \subset \cap_\ell U_{\rho_\ell} \subset \Sigma_k$ be the cofinite subset of "good places" attached to $(\underline{V}, \underline{\rho})$ in the definition of $(\mathrm{RC}/_{\mathbb{Q},\mathrm{et}})$ and for every $v \in U$, let $\chi_v (= \chi_{v,\ell}) = T^2 + a_v T + b_v \in \mathbb{Q}[T]$ the characteristic polynomial of Frobenii at v. Set $d_v := a_v^2 - 4b_v$ and $E_v := \mathbb{Q}(\sqrt{d_v})$ for the splitting field of χ_v.

Let $\Sigma_k^{max} \subset U$ denote the set of all $v \in U$ such that $\mathcal{E}_{v,\ell} = T_{v,\ell} \subset G_\ell$ is a maximal torus (*viz* $\varphi_{v,\ell}^{ss} \in G_\ell(\mathbb{Q}_\ell)$ is regular semisimple). From Corollaire 4.5, $\Sigma_k^{max} \subset \Sigma_k$ has density 1. Write $\Omega \subset \overline{\mathbb{Q}}$ for the compositum of all E_v, $v \in \Sigma_k^{max}$ (in a given algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$) and E for the intersection of all E_v, $v \in \Sigma_k^{max}$ (in particular, $[E : \mathbb{Q}] \leq 2$).

hm:LPr=2 **Theorem 4.9.** (Larsen-Pink [LP92, Intro]) *One and only one of the following three cases occurs.*
(1) $[\Omega : \mathbb{Q}] = \infty$, $E = \mathbb{Q}$ *and there exists a subset* $\mathcal{L}' \subset \mathcal{L}$ *of density* 0 *such that* $G_\ell = \mathrm{GL}_{2,\mathbb{Q}_\ell}$, $\ell \in \mathcal{L} \setminus \mathcal{L}'$.
(2) $\Omega = E = \mathbb{Q}$ *and* $G_\ell = \mathbb{G}_{m,\mathbb{Q}_\ell}^2$, $\ell \in \mathcal{L}$
(3) $[\Omega : \mathbb{Q}] = 2$, $E = \Omega$ *and if* ℓ *does not ramifies in* $\Omega/\mathbb{Q}_\ell$,
- $G_\ell = \mathrm{Res}_{\mathbb{Q}_\ell^{[2]}|\mathbb{Q}_\ell}(\mathbb{G}_{m,\mathbb{Q}_\ell^{[2]}})$ *if the Frobenii at* ℓ *in* $\mathrm{Gal}(\Omega|\mathbb{Q}) = \mathbb{Z}/2$ *maps to* 1;
- $G_\ell = \mathbb{G}_{m,\mathbb{Q}_\ell}^2$ *if the Frobenii at* ℓ *in* $\mathrm{Gal}(\Omega|\mathbb{Q}) = \mathbb{Z}/2$ *maps to* 0.

Here, $\mathbb{Q}_\ell^{[2]}/\mathbb{Q}_\ell$ *is the unique quadratic unramified extension of* $\mathbb{Q}_\ell$.

4.2.1. *Classification of maximal tori in* $\mathrm{GL}_{2,Q}$. Let Q be a field of characteristic 0 and let $T \subset \mathrm{GL}_{2,Q}$ be a maximal torus. As T is $\mathrm{GL}_2(\overline{Q})$-conjugate to $D_{r,\overline{Q}}$, T contains the homotheties torus $H_{2,Q} = Z(\mathrm{GL}_{2,Q})$. From the short exact sequence of tori over Q

$$1 \to H_{2,Q} \to T \to T/H_{2,Q} \to 1$$

one gets a short exact sequence of discrete $\pi_1(Q)$-modules

$$0 \longrightarrow X^*(T_{\overline{Q}}/H_{2,\overline{Q}}) \longrightarrow X^*(T_{\overline{Q}}) \longrightarrow X^*(H_{2,\overline{Q}}) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow X^*(T_{\overline{Q}}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

The action of $\pi_1(Q)$ on the RHS $\mathbb{Z}$ is trivial and the one on the LHS is given by a quadratic character $\chi_T : \pi_1(Q) \to \{\pm 1\} = \mathbb{Z}^\times$. This imposes that
i) $\chi_T = 1$ and $T \simeq \mathbb{G}_{m,Q}^2$ is split;
ii) or $\chi_T \neq 1$ and if Q_T/Q is the quadratic extension corresponding to $\ker(\chi_T)$, $T_{Q_T} \simeq \mathbb{G}_{m,Q_T}^2$ is split.
Indeed, ii) follows from i) applied to T_{Q_T}. For i), the action of $\pi_1(Q)$ on $X^*(T_{\overline{Q}})$ is given by a morphism of the form

$$\pi_1(Q) \to \mathrm{GL}_2(\mathbb{Z}), \quad \sigma \mapsto \begin{pmatrix} 1 & c(\sigma) \\ 0 & 1 \end{pmatrix}$$

where $c : \pi_1(Q) \to (\mathbb{Z}, +)$ is a continuous discrete morphism hence, in particular, as *finite* image. But as $(\mathbb{Z}, +)$ is torsion-free, this forces $c = 0$.

This reduces the problem to the following. Given a quadratic extension Q'/Q:

i) Classify all rank-2 torus T over Q such that $T_{Q'} \simeq \mathbb{G}_{m,Q'}^2$ is split. *Via* the equivalence of categories X^*, the tori in i) corresponds to $\mathrm{GL}_2(\mathbb{Z})$-conjugacy classes of order ≤ 2 matrices in $\mathrm{GL}_2(\mathbb{Z})$. Up to $\mathrm{GL}_2(\mathbb{Q})$-conjugacy there are only three possibilities:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponding to the invariant factors $X-1|(X-1)^2$, $X+1|(X+1)^2$ and X^2-1. In $\mathrm{GL}_2(\mathbb{Z})$, the third class splits into two $\mathrm{GL}_2(\mathbb{Z})$-conjugacy classes: the one of

$$U^- := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and the one of } U^+ := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, reducing modulo 2 one sees that these two matrices are not $\mathrm{GL}_2(\mathbb{F}_2)$-conjugate hence, *a fortiori* not $\mathrm{GL}_2(\mathbb{Z})$-conjugate. On the other hand, $M \in \mathrm{GL}_2(\mathbb{Z})$ has invariant factor $X^2 - 1$, one can find a rank-1 submodule $L \subset \mathbb{Z}^{\oplus 2}$ such that M acts trivially and $\mathbb{Z}^2/L$ is torsion free *i.e.* M is $\mathrm{GL}_2(\mathbb{Z})$-conjugate to a matrix of the form

$$\begin{pmatrix} 1 & \mathrm{a} \\ 0 & \text{-1} \end{pmatrix}.$$

One can then check that if $2 \not| a$, M is $\mathrm{GL}_2(\mathbb{Z})$-conjugate to U^- and if $2|a$, M is $\mathrm{GL}_2(\mathbb{Z})$-conjugate to U^+. *In fine*, One has 4-isomorphisms classes of d rank-2 tori which are split over Q':

$\pi_1(Q)$-module structure	Geometric description
Id	$\mathbb{G}_{m,Q}^2$
$-Id$	$A_1^{Q'} \times A_1^{Q'}$
U^-	$\mathrm{Res}_{Q'\vert Q}(\mathbb{G}_{m,Q'})$
u^+	$\mathbb{G}_{m,Q} \times A_1^{Q'}$.

Here $A_1^{Q'}$ denotes the unique anisotropic rank-1 torus over Q which splits over Q', namely the torus corresponding to the quadratic character $\pi_1(Q) \twoheadrightarrow \mathrm{Gal}(Q'|Q) \simeq \{\pm 1\} = \mathbb{Z}^\times$.

ii) Check which of the rank-2 tori in i) can be embedded into $\mathrm{GL}_{2,Q}$. As every rank-2 torus $T \subset \mathrm{GL}_{2,Q}$ contains $H_{2,Q}$, one has $\mathbb{Z} \simeq X^*(H_{2,\overline{Q}}) \subset X^*(T_{\overline{Q}})^{\pi_1(Q)}$. In particular, $A_1^{Q'} \times A_1^{Q'}$ cannot be embedded into $\mathrm{GL}_{2,Q}$. Actually, $T \simeq \mathbb{G}_{m,Q} \times A_1^{Q'}$ cannot either be embedded into $\mathrm{GL}_{2,Q}$. Indeed, otherwise, $H_{2,Q} \simeq \mathbb{G}_{m,Q} \subset T$ while $A_1^{Q'} \subset \mathrm{SL}_{2,Q} = \ker(\det : \mathrm{GL}_{2,Q} \to \mathbb{G}_{m,Q})$. But as all maximal tori of $\mathrm{SL}_{2,Q}$ are $\mathrm{SL}_2(\overline{Q})$-conjugate, they all contain $-Id$ hence one woudl have $-Id \in A_1^{Q'} \cap H_{2,Q}$, contradicting the fact that the product $T \simeq H_{2,Q} \times A_1^{Q'}$ should be direct.

Eventually, the only rank-2 connected reductive subgroups of $\mathrm{GL}_{2,Q}$ are $\mathrm{GL}_{2,Q}$ and its maximal tori.

4.2.2. *Proof of Theorem 4.9.*

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