The Mordell Conjecture - two (plus one) proofs

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"Solving" polynomial equations

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$$X^5 + 2X + 2 = 0, \quad Y^3 - X^4 + 1 = 0, \quad X^3 - 3XY^2 - Z = 0, \quad etc.$$

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- Solving one variable polynomial equations with coefficients in $\mathbb{Q}$
  $\longleftrightarrow$ Understanding (finite-dimensional) subfields of $\overline{\mathbb{Q}}$
  $\overset{\text{Galois Theory}}{\longleftrightarrow}$ Understanding (finite-index) closed subgroups of

$$G_{\mathbb{Q}} := \{\sigma : \overline{\mathbb{Q}} \tilde{\to} \overline{\mathbb{Q}} \text{ field automorphisms s.t.} \sigma(x) = x,\ x \in \mathbb{Q}\}$$
$\overset{\text{Def.}}{=}$ Absolute Galois group of $\mathbb{Q}$ ("Graal" of number theory)

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So, if values equidistributed, number of zeroes $\sim N^{3-d}$ while $d \geq 3$ iff $g_X \geq 2$...

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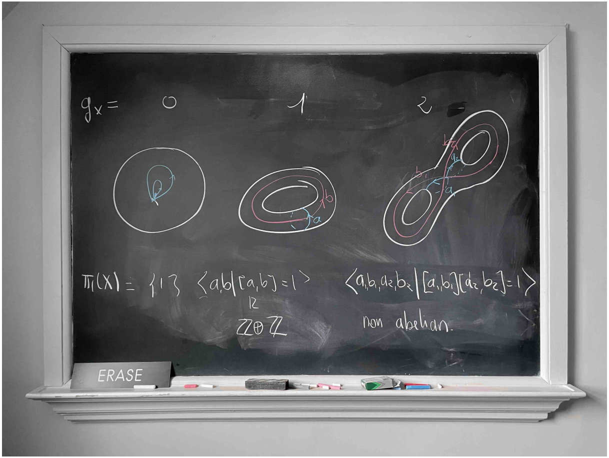

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g_X	$\pi_1^{\text{top}}(X(\mathbb{C}))$	$\pi_1^{\text{top}}(X(\mathbb{C}))^{\text{ab}}$
0	1	0
1	abelian	$\mathbb{Z}^2$
$g \geq 2$	non-abelian	$\mathbb{Z}^{2g}$

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    If $g_X \geq 1$, $X(\mathbb{Q}) \neq \emptyset$ and $P_0 \in X(\mathbb{Q})$, get
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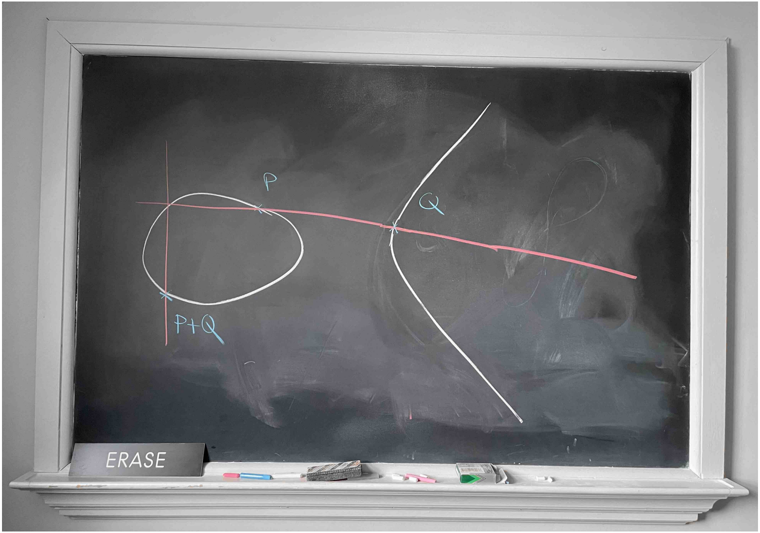
  Thm. (Mordell, 1922 (- Weil, 1929)) X abelian variety $/\mathbb{Q}$, then $X(\mathbb{Q})$ *finitely generated* abelian group *viz*

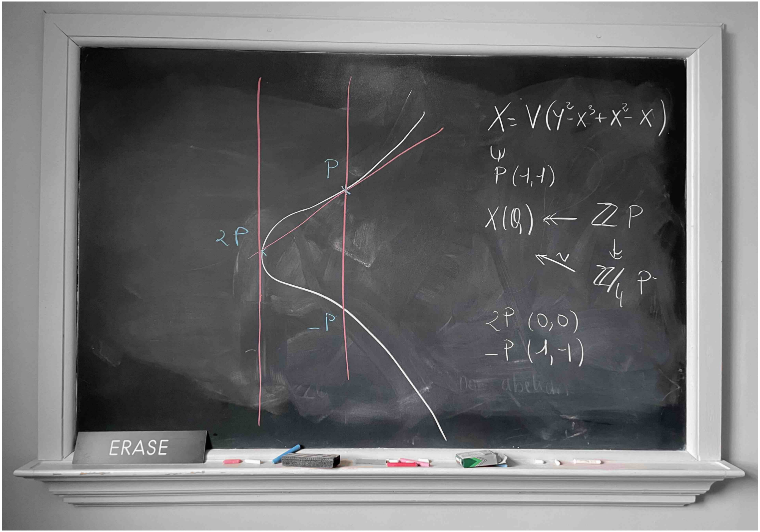
  $$X(\mathbb{Q}) \simeq \mathbb{Z}^{r_X} \oplus T(X)$$

  with $T(X)$ finite abelian group.

$g_X = 1$, Addition law



$$g_X = 1, \quad X(\mathbb{Q}) \simeq \mathbb{Z}/4$$



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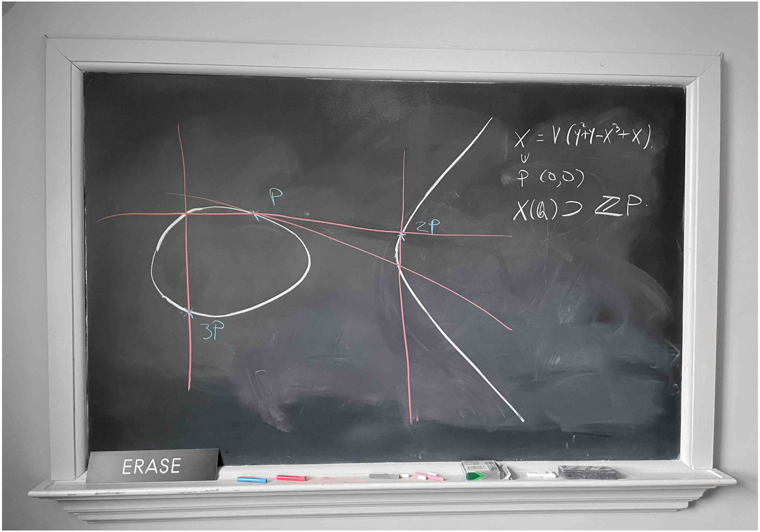

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If X curve	$g_X =$	0	1	≥ 2
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Bombieri-Lang Conjecture $\Rightarrow \#X(\mathbb{Q}) \leq c(g_X)$ (Caporaso-Harris-Mazur, 1997)

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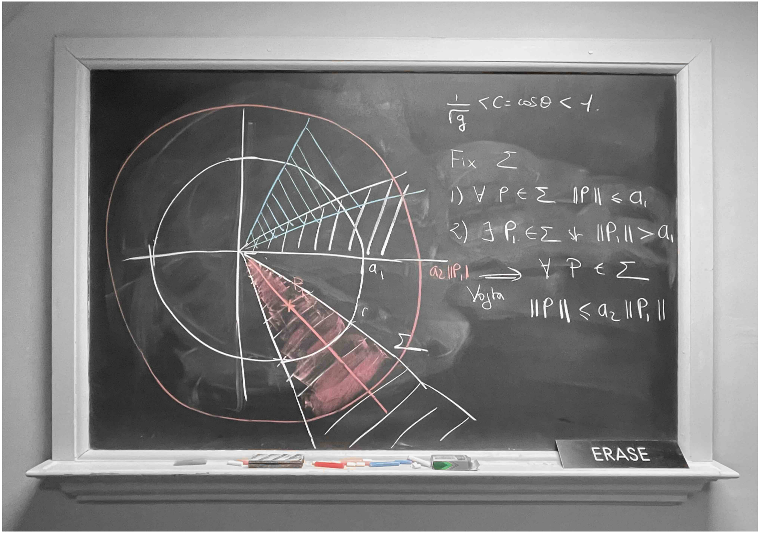
1/√g < C = cos θ < 1.
Fix Σ
1) ∀ P ∈ Σ ||P|| ≤ a₁
2) ∃ P₁ ∈ Σ s.t. ||P₁|| > a₁
a₂||P₁||
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Vojta
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Mordell Conjecture

$$\overset{(*)}{\Longrightarrow} \quad \text{Mordell Conjecture}$$

(*) : Parshin's trick (1968)

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$$\begin{aligned} \text{Tate Conjecture (1963)} \quad &\Rightarrow \quad \text{Shafarevich Conjecture (1963)} \\ &\overset{(*)}{\Rightarrow} \quad \text{Mordell Conjecture} \end{aligned}$$

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Tate Conjecture (1963) $\Rightarrow$ Shafarevich Conjecture (1963)

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- (1) : Finite-to-one. De Franchis theorem (early XXth) : Y, X smooth projective curves of genus ≥ 2, only finitely many finite surjective covers $Y \to X$
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  - (3-2) : **injective** and takes value in subset of **semisimple** representations **(Tate Conjecture for abelian varieties !)**. Additionnally, these are pure of weight 1 with integral characteristic polynomial of Frobenii (Weil Conjecture for abelian varieties - Weil, 1948).

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(1), (3) : Finite-to-one, (2) Injective, image of takes values in subset

$$Rep^{\mathrm{ss,\ int}, \mathbf{1}}_{\mathbb{Q}_\ell, 2\gamma, S}(G_\mathbb{Q}) \subset Rep_{\mathbb{Q}_\ell, 2\gamma, S}(G_\mathbb{Q})$$

of semisimple representations which are pure of weight 2 with integral characteristic polynomial of Frobenii

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- $\mathcal{A}_{\gamma,1}(\mathbb{Z}[S^{-1}])$ is finite (Shafarevich Conjecture)
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In particular,

- $\mathcal{A}_{\gamma,1}(\mathbb{Z}[S^{-1}])$ is finite (Shafarevich Conjecture)
- $X(\mathbb{Q})$ is finite (Mordell Conjecture)

Conjecture (Mordell, 1922) : Assume X smooth, projective, irreducible curve over $\mathbb{Q}$ with genus ≥ 2. Then $X(\mathbb{Q})$ is finite.

Three proofs

- Faltings (1983)
- Vojta (simplified by Bombieri) (1991)
- **Lawrence-Venkatesh** (2017)

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Three proofs

- Faltings (1983)
- Vojta (simplified by Bombieri) (1991)
- **Lawrence-Venkatesh** (2017)

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P & \mapsto & Y_P \qquad Y & \mapsto & Jac(Y) \qquad A & \mapsto & V_\ell(A)
\end{array}
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(1), (3) : Finite-to-one, (2) Injective

Conjecture (Mordell, 1922) : Assume X smooth, projective, irreducible curve over $\mathbb{Q}$ with genus ≥ 2. Then $X(\mathbb{Q})$ is finite.

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Ex : $f : Y \to X$ family of curves, $H^1(Y_{\bar{x}}, \mathbb{Q}_\ell) = V_\ell(Jac(Y_{\bar{x}}))$

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$$\mathcal{V}_\ell := R^i f_* \mathbb{Q}_\ell \longleftrightarrow$$

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Requires the image of $\pi_1^{\mathrm{top}}(X(\mathbb{C}))^\vee$ to be

"huge non abelian"

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Fix a prime p such that $f : Y \to X$ has a nice (smooth, projective *etc.*) model $f : \mathcal{Y} \to \mathcal{X}$ over $\mathbb{Z}_p$.

$$X(\mathbb{Q}) = \mathcal{X}(\mathbb{Z}) \subset \mathcal{X}(\mathbb{Z}_p) = \bigcup_{a \in \mathcal{X}(\mathbb{F}_p)} \Delta_a, \quad \Delta_a := \{x \in \mathcal{X}(\mathbb{Z}_p) \mid x \equiv a\ [p]\}$$

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For each a, has to show : $\Delta_a \cap X(\mathbb{Q})$ not Zariski dense in X (finite)

$f: Y \to X$ smooth, projective, $\quad \begin{array}{ccc} X(\mathbb{Q}) & \xrightarrow{(*)} & Rep^{\text{ss, int},2i}_{\mathbb{Q}_p,d,S}(G_{\mathbb{Q}}) \\ x & \mapsto & H^i(Y_{\bar{x}}, \mathbb{Q}_p) \end{array}$

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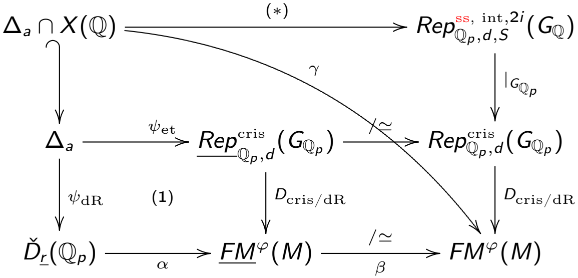

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\Delta_a & \xrightarrow{\psi_{\mathrm{et}}} & \underline{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p,d}(G_{\mathbb{Q}_p}) & \xrightarrow{/\simeq} & Rep^{\mathrm{cris}}_{\mathbb{Q}_p,d}(G_{\mathbb{Q}_p}) \\
\downarrow \psi_{\mathrm{dR}} & (1) & \downarrow D_{\mathrm{cris/dR}} & & \downarrow D_{\mathrm{cris/dR}} \\
\check{D}_{\underline{r}}(\mathbb{Q}_p) & \xrightarrow[\alpha]{} & \underline{FM}^{\varphi}(M) & \xrightarrow[\beta]{/\simeq} & FM^{\varphi}(M)
\end{array}
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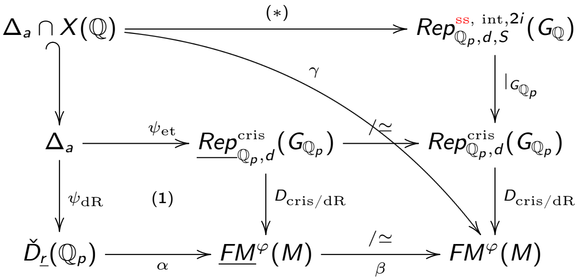
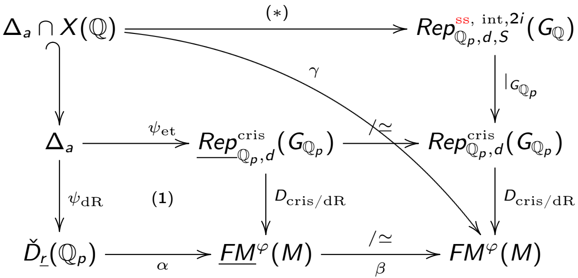


$\psi_{\mathrm{et}} : \Delta_a \ni x \mapsto H^i(Y_{\bar{x}}, \mathbb{Q}_p)$: cristalline representation of $G_{\mathbb{Q}_p} (\subset G_{\mathbb{Q}})$



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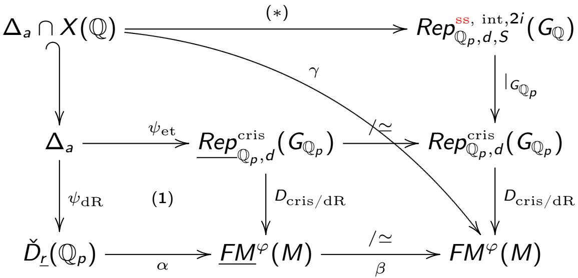
$\psi_{\mathrm{dR}} : \Delta_a \ni x \mapsto H^i_{\mathrm{dR}}(Y_x/\mathbb{Q}_p)$: filtered $\mathbb{Q}_p$-vector space $(M_a, F^{\bullet})$ with fixed numerical data $\underline{r}$



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$\alpha : \check{D}_{\underline{r}}(\mathbb{Q}_p) \ni F^\bullet \mapsto (M_a, \varphi_a, F^\bullet)$: filtered φ-modules $(M, \varphi, F^\bullet)$ with fixed φ-module (M_a, φ_a)

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\Delta_a \cap X(\mathbb{Q}) & \xrightarrow{(*)} & & & Rep^{\mathrm{ss},\,\mathrm{int},2i}_{\mathbb{Q}_p,d,S}(G_{\mathbb{Q}}) \\
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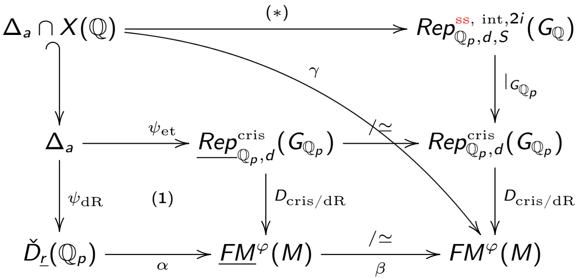
Commutativity of (1) : de Rham / cristalline / p-adic étale comparison thm

$$H^i_{\mathrm{dR}}(Y_x/\mathbb{Q}_p) \simeq H^i_{\mathrm{cris}}(\mathcal{Y}_a/\mathbb{Q}_p) \simeq D_{\mathrm{cris}}(H^i(Y_{\bar{x}}, \mathbb{Q}_p))$$



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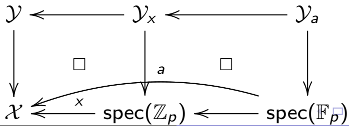

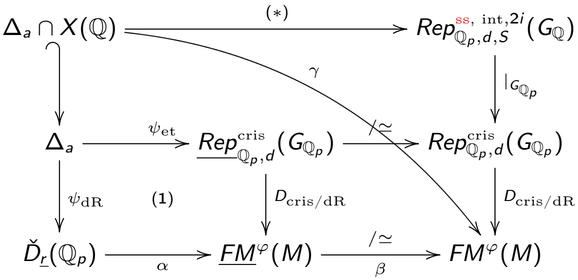

$\psi_{\mathrm{et}} : \Delta_a \ni x \mapsto H^i(Y_{\bar{x}}, \mathbb{Q}_p)$: cristalline representation of $G_{\mathbb{Q}_p} (\subset G_\mathbb{Q})$

$\psi_{\mathrm{dR}} : \Delta_a \ni x \mapsto H^i_{\mathrm{dR}}(Y_x/\mathbb{Q}_p)$: filtered $\mathbb{Q}_p$-vector space $(M_a, F^\bullet)$ with fixed numerical data $\underline{r}$

$\alpha : \check{D}_{\underline{r}}(\mathbb{Q}_p) \ni F^\bullet \mapsto (M_a, \varphi_a, F^\bullet)$: filtered φ-modules $(M, \varphi, F^\bullet)$ with fixed φ-module (M_a, φ_a)

Commutativity of (1) : de Rham / cristalline / p-adic étale comparison thm

$$H^i_{\mathrm{dR}}(Y_x/\mathbb{Q}_p) \simeq H^i_{\mathrm{cris}}(\mathcal{Y}_a/\mathbb{Q}_p) \simeq D_{\mathrm{cris}}(H^i(Y_{\bar{x}}, \mathbb{Q}_p))$$

Fiber of $\beta\alpha$ above $[M_a, \varphi_a, F^\bullet]$: $Z(\varphi_a) \cdot F^\bullet$

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 & \nearrow & \downarrow & & & & \\
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More "conceptual" applications - *E.g.* section conjecture (Betts-Stix), toric points of $\mathbb{Q}_\ell$-local systems on varieties over number fields (Cadoret-Stix)