ℓ-INDEPENDENCE OF TANNAKA GROUPS OF PERVERSE SHEAVES ON ABELIAN VARIETIES

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Abstract. Let k be a finite field of characteristic $p > 0$ and X an abelian variety over k. To every ℓ-adic perverse sheaf $\mathcal{P}$ on X, one can attach a Tannaka group $G(\mathcal{P})$, which is an algebraic group over $\overline{\mathbb{Q}}_\ell$. If $\mathcal{P}_\ell$, $\ell \neq p$ is a compatible family of semisimple perverse sheaves on X, we prove that the neutral component $G(\mathcal{P}_\ell)^\circ$ and the group of connected components $\pi_0(G(\mathcal{P}_\ell))$ of $G(\mathcal{P}_\ell)$ are independent of ℓ. Our arguments combine an explicit description of the group of connected components of $G(\mathcal{P})$, the companions correspondence for perverse sheaves, and a reconstruction theorem of Kazhdan-Larsen-Varshavsky or connected reductive groups.

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1. Introduction

Let k be a finite field of characteristic p. Let X be a normal variety (*viz* a separated scheme of finite type) over k. For a prime $\ell \neq p$, let $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ denote the triangulated category of complexes of étale $\overline{\mathbb{Q}}_\ell$-sheaves with bounded constructible cohomology, and $\mathrm{Loc}(X, \overline{\mathbb{Q}}_\ell) \subset D^b_c(X, \overline{\mathbb{Q}}_\ell)$ the full subcategory of $\overline{\mathbb{Q}}_\ell$-local systems on X. To every $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ and closed point $x \in |X|$ with residue degree $d_x := [k(x) : k]$, one attaches its local L-function at x

$$L_x(\mathcal{K}, T) := \prod_{i\in\mathbb{Z}} \det(1 - t^{d_x}\varphi_x | H^i(\mathcal{K})_{\bar{x}})^{(-1)^{i+1}} \in (1 + \overline{\mathbb{Q}}_\ell[[T]])^\times.$$

By construction and Cebotarev, the resulting map

$$L_\bullet(-, T) : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to \prod_{x\in|X|} (1 + \overline{\mathbb{Q}}_\ell[[T]])^\times, \ \ \mathcal{K} \mapsto (L_x(\mathcal{K}, T))_{x\in|X|}$$

factors through an injective map $L_\bullet(-, T) : K(X, \overline{\mathbb{Q}}_\ell) \hookrightarrow \prod_{x\in|X|}(1 + \overline{\mathbb{Q}}_\ell[[T]])^\times$, where $[-] : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to K(X, \overline{\mathbb{Q}}_\ell)$ denotes the Grothendieck ring of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ [Lau87, Thm. (1.1.2)].

For $\mathcal{K}_1, \mathcal{K}_2 \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are compatible - notation: $\mathcal{K}_1 \sim \mathcal{K}_2$, if the following equivalent conditions hold.

(i) $L_\bullet(\mathcal{K}_1, T) = L_\bullet(\mathcal{K}_2, T)$;

(ii) $[\mathcal{K}_1] = [\mathcal{K}_2]$ in $K(X, \overline{\mathbb{Q}}_\ell)$.

By the trace formula, the Euler-Poincaré characteristic

$$\chi(-) : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to \mathbb{Z}, \ \ \mathcal{K} \mapsto \chi(\mathcal{K}) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\overline{\mathbb{Q}}_\ell}(H^i_c(X_{\bar{k}}, \mathcal{P}))$$

factors through $\chi(-) : K(X, \overline{\mathbb{Q}}_\ell) \to \mathbb{Z}$. Hence, $\mathcal{K}_1 \sim \mathcal{K}_2$ implies $\chi(\mathcal{K}_1) = \chi(\mathcal{K}_2)$.

More generally, let A be a set of pairs $\alpha = (\ell_\alpha, \iota_\alpha)$ consisting of a prime $\ell_\alpha \neq p$ and a field isomorphism $\iota_\alpha : \overline{\mathbb{Q}}_{\ell_\alpha} \tilde{\to} \mathbb{C}$. For every $\alpha, \beta \in A$, $\mathcal{K}_\alpha \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ and $\mathcal{K}_\beta \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\beta})$, say that $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are A-compatible or A-companions - notation: $\mathcal{K}_\alpha \sim_A \mathcal{K}_\beta$ (see [Fu00, §1.2]), if

$$\iota_\alpha L_\bullet(\mathcal{K}_\alpha, T) = \iota_\beta L_\bullet(\mathcal{K}_\beta, T).$$

1.1. **$\overline{\mathbb{Q}}_\ell$-local systems and pure motives.** Fix a prime $\ell \neq p$. One expects that, up to twist, every semisimple object in $\mathrm{Loc}(X, \overline{\mathbb{Q}}_\ell)$ be motivic that is, roughly, be cut out by an algebraic cycle on a $\overline{\mathbb{Q}}_\ell$-local system of the form $Rf_*\overline{\mathbb{Q}}_\ell$ for $f : Y \to X$ a smooth projective morphism. A formal statement reflecting this expectation is provided by the companions conjecture [D80, Conj. 1.2.10]. For a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}$ over X, write $\mathbb{Q} \subset \mathbb{Q}_\mathcal{F} \subset \overline{\mathbb{Q}}_\ell$ for the field subextension generated over $\mathbb{Q}$ by the coefficients of the $\chi_x(\mathcal{F}, T) := L_x(\mathcal{F}, T)^{-1}$, $x \in |X|$.

Conjecture 1. *Let X be a connected normal variety over k. Let $\alpha \in A$ and $\mathcal{F}_\alpha$ a simple $\overline{\mathbb{Q}}_{\ell_\alpha}$-local system on X with finite determinant. Then,*

[Purity] *$\mathcal{F}_\alpha$ is pure of weight 0: for every isomorphism $\iota : \overline{\mathbb{Q}}_{\ell_\alpha} \tilde{\rightarrow} \mathbb{C}$, $x \in |X|$ and root α of $\chi_x(\mathcal{F}_\alpha, T)$, $|\iota\alpha| = 1$;*

[Finiteness] *The field $\mathbb{Q}_{\mathcal{F}_\alpha}$ is a number field.*

[Companions] *For every $\beta \in A$ there exists a (automatically simple and unique) $\overline{\mathbb{Q}}_{\ell_\beta}$-local system $\mathcal{F}_{\alpha\rightsquigarrow\beta}$ on X such that $\mathcal{F}_\alpha \sim_A \mathcal{F}_{\alpha\rightsquigarrow\beta}$.*

Remark 2. The [Companions] part of Conjecture 1 automatically extends to arbitrary $\overline{\mathbb{Q}}_{\ell_\alpha}$-local system $\mathcal{F}_\alpha$ on X. Namely, for every $\beta \in A$ there exists a unique semisimple $\overline{\mathbb{Q}}_{\ell_\beta}$-local system $\mathcal{F}_{\alpha\rightsquigarrow\beta}$ on X such that $\mathcal{F}_\alpha \sim_A \mathcal{F}_{\alpha\rightsquigarrow\beta}$. The unicity of $\mathcal{F}_{\alpha\rightsquigarrow\beta}$ follows from Cebotarev. For the existence, consider the semisimplification $\mathcal{F}_\alpha^{\mathrm{ss}}$ of $\mathcal{F}_\alpha$ in the category of $\overline{\mathbb{Q}}_{\ell_\alpha}$-local systems on X and decompose it as

$$\mathcal{F}_\alpha^{\mathrm{ss}} = \oplus_{i=1}^r \mathcal{S}_{i,\alpha}^{(a_{i,\alpha})}$$

with $\mathcal{S}_{i,\alpha}$ a simple $\overline{\mathbb{Q}}_{\ell_\alpha}$-local system with finite determinant on X and $a_{i,\alpha} \in \overline{\mathbb{Q}}_{\ell_\alpha}$. Then, writing $a_{i,\alpha\rightsquigarrow\beta} := \iota_\beta^{-1}\iota_\alpha(a_{i,\alpha}) \in \overline{\mathbb{Q}}_{\ell_\beta}$, $i = 1, \ldots, r$, the semisimple $\overline{\mathbb{Q}}_{\ell_\beta}$-local system

$$\mathcal{F}_{\alpha\rightsquigarrow\beta} := \oplus_{i=1}^r \mathcal{S}_{i,\alpha\rightsquigarrow\beta}^{(a_{i,\alpha\rightsquigarrow\beta})}.$$

is a A-companion of $\mathcal{F}_\alpha$.

Another expected consequence of the formalism of motives is the following (weakening of) [Ch04, Conj. 1.1], which can be regarded as an upgraded Tannakian version of the [Companions] part in Conjecture 1. For a connected normal variety X over k and a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}$ on X write

$$G(\mathcal{F}) \subset \mathrm{GL}_{V_\mathcal{F}}$$

for the Zariski-closure of the image of $\pi_1(X, \bar{x})$ acting on $V_\mathcal{F} := \mathcal{F}_{\bar{x}}$.

Conjecture 3. *Fix an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let X be a connected normal variety over k. Let $\alpha \in A$ and $\mathcal{F}_\alpha$ a semisimple objects in $\mathrm{Loc}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$. Then, there exists a reductive group G over $\overline{\mathbb{Q}}$ together with a faithfull finite-dimensional $\overline{\mathbb{Q}}$-representation $G \subset \mathrm{GL}_V$ such that for every $\beta \in A$,*

$$(G(\mathcal{F}_{\alpha\rightsquigarrow\beta}) \subset \mathrm{GL}_{V_{\mathcal{F}_{\alpha\rightsquigarrow\beta}}}) \otimes_{\overline{\mathbb{Q}}_{\ell_\beta}, \iota_\beta} \mathbb{C} \simeq (G \subset \mathrm{GL}_V) \otimes_{\overline{\mathbb{Q}}, \iota} \mathbb{C}.$$

When X is a curve, Conjecture 1 was proved by L. Lafforgue [L02], as a consequence of the Langlands correspondence and, somewhat, the motive corresponding to $\mathcal{F}_\alpha$ appears in the moduli stack of Shtukas. For X a smooth variety of higher-dimension, Conjecture 1 was proved by Deligne [De12] and Drinfeld [Dr12], by geometric methods reducing to the case of curves; in particular, they do not provide any construction of the expected motive corresponding to $\mathcal{F}_\alpha$.

1.2. **And beyond...?**

One may ask to what extent the considerations and results of Subsection 1.1 extend to arbitrary objects in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, in particular:

(1) Does the companions correspondence extend to arbitrary $\mathcal{K} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$?

(2) If so, can one formulate upgraded Tannakian variants of the companions correspondence for $\mathcal{K} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ in the spirit of Conjecture 3?

One can answer (1) in a satisfactory way provided one has a good notion of "semisimplification" in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$. Let $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \subset D_c^b(X, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of perverse sheaves on X and let ${}^pH^i(-) : D_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ denote the ith perverse cohomology functor, $i \in \mathbb{Z}$. For $\mathcal{K} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ define the semisimplification of $\mathcal{K}$ in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ as

$$\mathcal{K}^{\mathrm{ss}} := \oplus_{i\in\mathbb{Z}} {}^pH^i(\mathcal{K})^{\mathrm{ss}}[-i],$$

where, here, $(-)^{\mathrm{ss}} : \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ denotes the semisimplification functor in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, and say that $\mathcal{K}$ is semisimple in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ if $\mathcal{K} \simeq \mathcal{K}^{\mathrm{ss}}$ in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$.

Remark 4.

(1) The definition of $\mathcal{K}^{\mathrm{ss}} = \oplus_{i\in\mathbb{Z}} {}^pH^i(\mathcal{K})^{\mathrm{ss}}[-i]$ is tautologically compatible with the usual definition of semisimplification in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ and, if X is smooth over k, with the usual definition of semisimplification in $\mathrm{Loc}(X, \overline{\mathbb{Q}}_\ell)$ (in the sense that, if X is connected, $(\mathcal{F}^{\mathrm{ss}})[d] = (\mathcal{F}[d])^{\mathrm{ss}}$, where $d := \dim(X)$.

(2) If $\mathcal{K} \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is ι-pure for some field isomorphism $\iota : \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{C}$ then, by the decomposition theorem [BBerDG82, Thm. 5.4.5], $\mathcal{K}^{\mathrm{ss}}|_{X_{\bar{k}}} \simeq \mathcal{K}|_{X_{\bar{k}}}$ in $D^b_c(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$.

(3) **Warning:** Let $\mathcal{K}_1, \mathcal{K}_2 \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$.

- If $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, then $\mathcal{K}_1 \sim \mathcal{K}_2$ implies $\mathcal{K}_1^{\mathrm{ss}} \simeq \mathcal{K}_2^{\mathrm{ss}}$ [Lau87, Prop. (1.1.2.1)].
- If $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ are *both* pure of weight w, then $\mathcal{K}_1 \sim \mathcal{K}_2$ implies $\mathcal{K}_1^{\mathrm{ss}} \simeq \mathcal{K}_2^{\mathrm{ss}}$. Indeed, then ${}^pH^i(\mathcal{K}_1), {}^pH^i(\mathcal{K}_2)$ are both pure of weight $w+i$, $i \in \mathbb{Z}$ hence, with the notation of [Z09, §6], $[{}^pH^i(\mathcal{K}_1)] = p_{w+i}[\mathcal{K}_1]$, $[{}^pH^i(\mathcal{K}_2)] = p_{w+i}[\mathcal{K}_2]$ in the Grothendieck group $K(X, \overline{\mathbb{Q}}_\ell)$. But then, by [Z09, Prop. 2.7], ${}^pH^i(\mathcal{K}_1) \sim {}^pH^i(\mathcal{K}_2)$ and the assertion follows from the case of perverse sheaves.

But, in general, $\mathcal{K}_1 \sim \mathcal{K}_2$ does not imply $\mathcal{K}_1^{\mathrm{ss}} \simeq \mathcal{K}_2^{\mathrm{ss}}$ in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. For instance, if X is smooth over k, connected of dimension d and $\mathcal{S} \in \mathrm{Loc}(X, \overline{\mathbb{Q}}_\ell)$ is a simple (non-zero) $\overline{\mathbb{Q}}_\ell$-local system on X, then $\mathcal{S}[d] \sim \mathcal{S}[d-2]$, but $\mathcal{S}[d] \not\simeq \mathcal{S}[d-2]$ in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. So, to get the unicity of the companion $\mathcal{K}_{\alpha \rightsquigarrow \beta}$ one has to strengthen the companion relation $\sim$. For $\alpha, \beta \in A$ and $\mathcal{K}_\alpha \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha})$, $\mathcal{K}_\beta \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\beta})$, say that $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are derived-A-compatible or derived-A-companions - notation: $\mathcal{K}_\alpha \equiv_A \mathcal{K}_\beta$ if ${}^pH^i(\mathcal{K}_\alpha) \sim_A {}^pH^i(\mathcal{K}_\beta)$, $i \in \mathbb{Z}$.

From Remark 4 (3) and the classification of simple objects in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, one deduces from the [Companions] part of Conjecture 1 the following perverse version.

Corollary 5. *Let X be a variety over k. For every $\alpha \in A$ and $\mathcal{P}_\alpha \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ there exists a unique semisimple $\mathcal{P}_{\alpha \rightsquigarrow \beta} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\beta})$ such that $\mathcal{P}_\alpha \sim_A \mathcal{P}_{\alpha \rightsquigarrow \beta}$. More generally, for every $\mathcal{K}_\alpha \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ there exists a unique semisimple $\mathcal{K}_{\alpha \rightsquigarrow \beta} \in D^b_c(X, \overline{\mathbb{Q}}_{\ell_\beta})$ such that $\mathcal{K}_\alpha \equiv_A \mathcal{K}_{\alpha \rightsquigarrow \beta}$.*

One immediately deduces from Corollary 5 that

Proof. The unicity follows from Remark 4 (3) and the second part of the assertion immediately follows from the first part and the definitions. We prove the existence. Without loss of generality, one may assume $\mathcal{P}_\alpha$ is a simple object in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$. From [BBerDG82, Thm. 4.3.1 (ii)], $\mathcal{P}_\alpha = \iota_* j_{!*}(\mathcal{F}_\alpha[d])$ for some closed immersion $\iota : Z \hookrightarrow X$, open immersion $j : U \hookrightarrow Z$ with Zariski-dense image and U smooth over k, irreducible, of dimension d, and $\mathcal{F}_\alpha$ a simple object in $\mathrm{Loc}(U, \overline{\mathbb{Q}}_{\ell_\alpha})$. From the [Companions] part of Conjecture 1, there exists a unique simple object $\mathcal{F}_{\alpha \rightsquigarrow \beta}$ in $\mathrm{Loc}(U, \overline{\mathbb{Q}}_{\ell_\beta})$ such that $\mathcal{F}_\alpha \sim_A \mathcal{F}_{\alpha \rightsquigarrow \beta}$. From the [Purity] part of Conjecture 1, $\mathcal{F}_\alpha$ - hence $\mathcal{F}_{\alpha \rightsquigarrow \beta}$ are both pure so that from [Fu00], $\mathcal{P}_\alpha = \iota_* j_{!*}(\mathcal{F}_\alpha[d]) \sim_A \iota_* j_{!*}(\mathcal{F}_{\alpha \rightsquigarrow \beta}[d])$. So that one can take:

$$\mathcal{P}_{\alpha \rightsquigarrow \beta} := \iota_* j_{!*}(\mathcal{F}_{\alpha \rightsquigarrow \beta}[d]).$$

□

Remark 6. It follows from the explicit construction (and unicity) of $\mathcal{P}_{\alpha \rightsquigarrow \beta}$ that for every semisimple $\mathcal{P}_\alpha \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$,

(1) $\mathcal{P}_\alpha$ is simple in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ if and only if $\mathcal{P}_{\alpha \rightsquigarrow \beta}$ is simple in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\beta})$ and, more generally, that

$$\mathrm{Length}_{\mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})}(\mathcal{P}_\alpha) = \mathrm{Length}_{\mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\beta})}(\mathcal{P}_{\alpha \rightsquigarrow \beta}).$$

(2) $\mathrm{supp}(\mathcal{P}_\alpha) = \mathrm{supp}(\mathcal{P}_{\alpha \rightsquigarrow \beta})$ *etc.*

Formulating a consistent answer to (2) seems more delicate as it is unclear how to upgrade Corollary 5 to a statement for which A-independence would make sense. For instance, and contrary to $\mathrm{Loc}(X, \overline{\mathbb{Q}}_\ell)$, $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ does not carry a natural structure of Tannakian category in general. Still, if X is a connected commutative algebraic group over k, one can attach to objects in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ a Tannakian category and ask for A-independence results in the spirit of Conjecture 3. The main result of this note is the following. Let X be an abelian variety over k. For $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, let $G(\mathcal{P})$ denote the corresponding Tannaka group [FoFrKo25, Chap. 3]; see also Section 2).

Theorem 7. *Let $\alpha \in A$ and $\mathcal{P}_\alpha$ a semisimple object in $\mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$. The group of connected components $\pi_0(G(\mathcal{P}_{\alpha \rightsquigarrow \beta}))$ and the neutral component $G(\mathcal{P}_{\alpha \rightsquigarrow \beta})^\circ$ of $G(\mathcal{P}_{\alpha \rightsquigarrow \beta})$ are independant of $\beta \in A$ in the sense of Conjecture 3.*

Theorem 7 extends the results of [FoFrKo25, §8] (when the base is an abelian variety); it is relies on the combination of the following three ingredients:

- the properties of the companion correspondence for perverse sheaves - Corollary 5 (and its proof);

- the structure of the Tannaka category of perverse sheaves on abelian varieties (See Section 3);
- Kazhdan-Larsen-Varshavsky's reconstruction theorem [KLarV14] for the connected component of a reductive groups from its semiring of representations.

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Notation / conventions: Let Q be a field of characteristic 0.

For an algebraic group G over Q write $G \twoheadrightarrow G^{\mathrm{ab}}$ for its abelianization, $G^{\mathrm{der}} := \ker(G \twoheadrightarrow G^{\mathrm{ab}}) \subset G$ for its derived subgroup, $G \twoheadrightarrow \pi_0(G)$ for its group of connected components, $G^\circ := \ker(G \twoheadrightarrow \pi_0(G)) \subset G$ for its neutral component and $\Xi(G) := \mathrm{Hom}(G, \mathbb{G}_{m,\bar{Q}})$ for its character group; when G is finite, we sometimes write $G^\vee := \Xi(G)$. We use similar notation for proalgebraic groups.

For a neutral Tannaka category $\mathcal{T}$ over Q, a fiber functor $\omega : \mathcal{T} \to \mathrm{Vect}_Q$ and a Tannaka subcategory[1] $\mathcal{T}' \hookrightarrow \mathcal{T}$, write $G_\omega(\mathcal{T}')$ for the corresponding Tannaka group of $\mathcal{T}'$ with respect to the induced fiber functor $\mathcal{T}' \hookrightarrow \mathcal{T} \xrightarrow{\omega} \mathrm{Vect}_Q$. For an object $T \in \mathcal{T}$, write $G_\omega(T) \subset \mathrm{GL}_{\omega(T)}$ for the Tannaka group of T in $\mathcal{T}$ (*viz* of the smallest Tannaka subcategory $\langle T\rangle \hookrightarrow \mathcal{T}$ containing T).

From now on, let X be an abelian variety over k. Let $m : X \times X \to X$ denote the product on X.

2. RECOLLECTION ON TANNAKA CATEGORIES OF PERVERSE SHEAVES ON ABELIAN VARIETIES

See [KrW15], [W16], [FoFrKo25, §3] for more details and proofs. Let X be an abelian variety over k. Fix a prime $\ell \neq p$.

2.1. **The categories** $\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ **and** $\mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$**.** Recall that every $\mathcal{P} \in \mathrm{Perv}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ has positive Euler-Poincaré characteristic $\chi(\mathcal{P}) \geq 0$ [DM25, Prop. 3.7] and say that an object $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ is *negligible* if $\chi(\mathcal{P}) = 0$. The full subcategory $N(X, \overline{\mathbb{Q}}_\ell) \subset \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ of negligible objects is a Serre subcategory and its quotient

$$[-] : \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \twoheadrightarrow P(X, \overline{\mathbb{Q}}_\ell) := \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)/N(X, \overline{\mathbb{Q}}_\ell)$$

is endowed with a natural structure of Tannakian category with monoidal structure given by

$$[\mathcal{P}_1] \otimes [\mathcal{P}_2] = [{}^pH^0(Rm_*(\mathcal{P}_1 \boxtimes^L \mathcal{P}_2))], \quad [\mathcal{P}]^\vee = [D_X(\mathcal{P})], \quad \mathbf{1} = [\delta_0],$$

where $D_X : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X, \overline{\mathbb{Q}}_\ell)$ denotes Verdier duality and $\delta_0 := \iota_{0*}\overline{\mathbb{Q}}_\ell$ is the trivial skyscraper sheaf supported on the origin.

Write

$$(-)_\neg : \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to N(X, \overline{\mathbb{Q}}_\ell), \quad (-)^\neg : \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to N(X, \overline{\mathbb{Q}}_\ell)$$

for the right adjoint ("maximal negligible subobject") and left adjoint ("maximal negligible quotient object") of the inclusion functor

$$N(X, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$$

respectively. Note that, for every $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, $(\mathcal{P}_\neg)|_{X_{\bar{K}}} = (\mathcal{P}|_{X_{\bar{k}}})_\neg \hookrightarrow \mathcal{P}|_{X_{\bar{k}}}$ and $\mathcal{P}|_{X_{\bar{k}}} \twoheadrightarrow (\mathcal{P}^\neg)|_{X_{\bar{k}}} = (\mathcal{P}|_{X_{\bar{k}}})^\neg$. We use the internal description of the Tannaka category $P(X, \overline{\mathbb{Q}}_\ell)$. Namely, consider the full subcategory $\mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \subset \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, whose objects are those $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ with $\mathcal{P}_\neg = \mathcal{P}^\neg = 0$. Then the functor $[-] : \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \twoheadrightarrow P(X, \overline{\mathbb{Q}}_\ell)$ restricts to an equivalence of abelian $\overline{\mathbb{Q}}_\ell$-linear categories $[-] : \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \twoheadrightarrow P(X, \overline{\mathbb{Q}}_\ell)$ with quasi-inverse

$$(-)_{\mathrm{int}} : P(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\simeq} \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$$

given on objects by

$$[\mathcal{P}]_{\mathrm{int}} = (\ker(\mathcal{P} \twoheadrightarrow \mathcal{P}^\neg) + \mathcal{P}_\neg)/\mathcal{P}_\neg.$$

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[1] Recall that a Tannaka subcategory of $\mathcal{T}$ is a strictly full abelian subcategory $\mathcal{T}' \hookrightarrow \mathcal{T}$ stable by $\otimes$, $(-)^\vee$, and such that every subquotient in $\mathcal{T}$ of an object in $\mathcal{T}'$ is again in $\mathcal{T}'$.

Note that, if $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ is semisimple, then the canonical epimorphism $\mathcal{P} \twoheadrightarrow \mathcal{P}^\neg$ restricts to an isomorphism $\mathcal{P}_\neg \tilde{\to} \mathcal{P}^\neg$ so that, in that case, $[\mathcal{P}]_{\mathrm{int}} = \ker(\mathcal{P} \twoheadrightarrow \mathcal{P}^\neg) \simeq \mathcal{P}/\mathcal{P}_\neg$.

The tensor structure on $P(X, \overline{\mathbb{Q}}_\ell)$ induces a tensor structure on $\mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ *via* $(-)_{\mathrm{int}} : P(X, \overline{\mathbb{Q}}_\ell) \overset{\simeq}{\to} \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$, and which is explicitly given by

$$\mathcal{P}_1 \otimes \mathcal{P}_2 = {}^p\mathrm{H}(\mathcal{P}_1 \boxtimes^L \mathcal{P}_2)_{\mathrm{int}}.$$

Note that, as $\mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ is a neutral Tannaka category on $\overline{\mathbb{Q}}_\ell$, the full-subcategory $\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \subset \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ of semisimple objects is again a Tannaka category[2].

We will also need to introduce a geometric variant of $\mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$. The above discussion extends as it is if one replaces X with $X_{\bar{k}}$ except possibly for the assertion that $\mathrm{Perv}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ admits a fiber functor but for our purpose, it will be enough to consider the full subcategory $\mathrm{Perv}^{\mathrm{ss,ar}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Perv}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ of arithmetic objects *viz* of those $\mathcal{P} \in \mathrm{Perv}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ which are a direct factor[3] in $\mathrm{Perv}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ of an object in the essential image of the canonical restriction functor $-|_{X_{\bar{k}}} : \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X_{k'}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ for some finite field extension k'/k. Then $\mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ is a Tannaka category over $\overline{\mathbb{Q}}_\ell$, whose objects are semisimple in $\mathrm{Perv}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$, and the canonical restriction functor $-|_{X_{\bar{k}}} : \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ is an exact $\otimes$-functor.

2.2. **The arithmetic / geometric decomposition.** The exact, fully faithful functor $0_* : \mathrm{Perv}(0, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Perv}(\mathrm{spec}(k), \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ factors through a tensor functor

$$0_* : \mathrm{Perv}(0) \to \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$$

and induces a canonical commutative diagram of Tannaka categories

$$\begin{tikzcd}
\mathrm{Perv}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell) \arrow[r, "0_*"] \arrow[d, hook] & \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \arrow[d, hook] \\
\mathrm{Perv}(0, \overline{\mathbb{Q}}_\ell) \arrow[r, "0_*"] & \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)
\end{tikzcd}$$

Let $\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int},00}(X, \overline{\mathbb{Q}}_\ell) \subset \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ denote the essential image of $0_* : \mathrm{Perv}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$.

Fix a fiber functor $\omega : \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ and, for simplicity, write $\overline{G}_\omega(X)$, $G_\omega(X)$ and $\Gamma_\omega(X)$ for the Tannaka groups of $\mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$, $\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$, and $\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int},00}(X, \overline{\mathbb{Q}}_\ell)$ respectively. The sequence of Tannaka categories

$$\begin{tikzcd}
\mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell) & \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \arrow[l, "(-)|_{X_{\bar{k}}}"'] & \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int},00}(X, \overline{\mathbb{Q}}_\ell) \arrow[l, hook']
\end{tikzcd}$$

then induces a short exact sequence of proreductive groups over $\overline{\mathbb{Q}}_\ell$

$$1 \to \overline{G}_\omega(X) \to G_\omega(X) \to \Gamma_\omega(X) \to 1$$

- see *e.g.* [JKrLeM25, Thm. 4.3]. More generally, for every Tannaka subcategory $\mathcal{T} \hookrightarrow \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$, define $\overline{\mathcal{T}} \hookrightarrow \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ to be the Tannaka subcategory generated by the essential image of $\mathcal{T} \hookrightarrow \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell) \overset{(-)|_{X_{\bar{k}}}}{\to} \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{T}_{00} := \mathcal{T} \cap \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int},00}(X, \overline{\mathbb{Q}}_\ell)$; write $\overline{G}_\omega(\mathcal{T}) := G_\omega(\overline{\mathcal{T}})$ and $\Gamma_\omega(\mathcal{T}) := G_\omega(\mathcal{T}_{00})$. With these notation, the sequence of Tannaka categories

$$\begin{tikzcd}
\overline{\mathcal{T}} & \mathcal{T} \arrow[l, "(-)|_{X_{\bar{k}}}"'] & \mathcal{T}_{00} \arrow[l, hook']
\end{tikzcd}$$

induces a short exact sequence of proreductive groups over $\overline{\mathbb{Q}}_\ell$

$$1 \to \overline{G}_\omega(\mathcal{T}) \to G_\omega(\mathcal{T}) \to \Gamma_\omega(\mathcal{T}) \to 1$$

In particular, when $\mathcal{T} = \langle \mathcal{P} \rangle \subset \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ for some $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$, one has $\overline{\mathcal{T}} = \langle \mathcal{P}|_{X_{\bar{k}}} \rangle \subset \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ and hence, writing $\overline{G}_\omega(\mathcal{P}) := \overline{G}_\omega(\mathcal{T})$, a short exact sequence of algebraic reductive groups over $\overline{\mathbb{Q}}_\ell$

$$1 \to \overline{G}_\omega(\mathcal{P}) \to G_\omega(\mathcal{P}) \to \Gamma_\omega(\mathcal{P}) \to 1 \tag{1}$$

---

[2] The non-obvious fact is that the tensor product of two semisimple objects is again semisimple.

[3] Actually, such a direct factor automatically descends to a finite subextension k' of k - see [BBerDG82, Prop. 5.3.9 (ii)].

* * *

As we are working with Tannaka categories over the algebraically closed field $\overline{\mathbb{Q}}_\ell$, the Tannaka groups $G_\omega(X)$, $G_\omega(\mathcal{P})$ *etc.* appearing depend one the fiber functor ω only up to isomorphism so that, in the following, we will almost always omit the subscript $(-)_\omega$ from the notation and simply write $G(X) := G_\omega(X)$, $G(\mathcal{P}) := G_\omega(\mathcal{P})$ *etc.*

3. STRUCTURE OF $G(X)$ AND $G(\mathcal{P})$

Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$.

3.1. **Structure of $\Gamma(X)$ and $\Gamma(\mathcal{P})$.** The tensor category[4] $\mathrm{Perv}_{\mathrm{int}}(0, \overline{\mathbb{Q}}_\ell) = \mathrm{Perv}(0, \overline{\mathbb{Q}}_\ell)$ is equivalent to the tensor category $\mathrm{Loc}(0, \overline{\mathbb{Q}}_\ell)$ of continuous $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1(0) \simeq \widehat{\mathbb{Z}}$ and the full subcategory $\mathrm{Perv}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Perv}(0, \overline{\mathbb{Q}}_\ell)$ to the full subcategory $\mathrm{Loc}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Loc}(0, \overline{\mathbb{Q}}_\ell)$ of semisimple ones. As $0_* : \mathrm{Perv}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_\ell)$ induces an equivalence of Tannaka categories onto its essential image $\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int},00}(X, \overline{\mathbb{Q}}_\ell)$, one gets that $\Gamma(X)$ fits into a short exact sequence of proalgebraic groups of multiplicative type

$$1 \to \Gamma(X)^\circ \to \Gamma(X) \to \pi_0(\Gamma(X)) \simeq \pi_1(k) \to 1$$

One can 'explicitly' describe the corresponding quotient $\Gamma(X) \twoheadrightarrow \Gamma(\mathcal{P})$ as follows. Let $\mathcal{F} \in \mathrm{Loc}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell)$ be such that $0_*\mathcal{F} \in \langle \mathcal{P} \rangle_{00}$ is a tensor generator of $\langle \mathcal{P} \rangle_{00}$ and let $\phi \in \mathrm{GL}(\mathcal{F}_{\bar{0}})$ denote the image of $1 \in \widehat{\mathbb{Z}}$ so that $\Gamma(\mathcal{P})$ identifies with the Zariski closure $G(\mathcal{F}) \subset \mathrm{GL}_{\mathcal{F}_{\bar{0}}}$ of $\phi^{\mathbb{Z}} \subset \mathrm{GL}(\mathcal{F}_{\bar{0}})$. As $\mathcal{F}$ is semisimple in $\mathrm{Loc}^{\mathrm{ss}}(0, \overline{\mathbb{Q}}_\ell)$, the $\mathrm{GL}(\mathcal{F}_{\bar{0}})$-conjugacy class of ϕ is uniquely determined by the multiset $\Lambda(\mathcal{F}) \subset \overline{\mathbb{Q}}_\ell^\times$ of the eigenvalues of ϕ and $\Xi(\Gamma(\mathcal{P})) = \Xi(G(\mathcal{F}))$ identifies with the subgroup $\langle \Lambda(\mathcal{F}) \rangle \subset \overline{\mathbb{Q}}_\ell^\times$ generated by $\Lambda(\mathcal{F})$ in $\overline{\mathbb{Q}}_\ell^\times$. As $\Gamma(\mathcal{P})$ is of multiplicative type, it is uniquely determined by $\Xi(\Gamma(\mathcal{P}))$.

3.2. **Structure of $\overline{G}(X)$ and $\overline{G}(\mathcal{P})$.** Let $\overline{\Xi}(\mathcal{P}) \subset X(\bar{k})$ denote the subset of all $x \in X(\bar{k})$ such that $\delta_x := x_*\overline{\mathbb{Q}}_\ell \in \langle \mathcal{P}|_{X_{\bar{k}}} \rangle$. As $\delta_{x_1} * \delta_{x_2} = \delta_{x_1+x_2}$, $\overline{\Xi}(\mathcal{P}) \subset X(\bar{k})$ is a subgroup and one gets a canonical commutative square of commutative groups

$$\begin{tikzcd}
\overline{\Xi}(\mathcal{P}) \arrow[r, "\delta_-", "\simeq"'] \arrow[d, hook] & \Xi(\overline{G}(\mathcal{P})) \arrow[d, hook] \\
X(\bar{k}) \arrow[r, "\delta_-", "\simeq"'] & \Xi(\overline{G}(X))
\end{tikzcd}$$

whose horizontal arrows are isomorphisms [W15, Prop. 3]. As $X(\bar{k})$ is a finitely generated (as $\overline{G}(\mathcal{P})$ is algebraic) torsion (as k is finite!) group, it is actually finite; in particular $\pi_0(\overline{G}(\mathcal{P})) \twoheadrightarrow \overline{G}(\mathcal{P})^{\mathrm{ab}}$. But as, on the other hand, $\pi_0(\overline{G}(\mathcal{P}))$ is abelian [W15, End of §2], one eventually gets

$$\pi_0(\overline{G}(\mathcal{P})) \tilde{\to} \overline{G}(\mathcal{P})^{\mathrm{ab}} \tilde{\to} \overline{\Xi}(\mathcal{P})^\vee.$$

This also implies that $\overline{G}(\mathcal{P})$ is semisimple. Indeed, as $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, the finite subgroup $\overline{\Xi}(\mathcal{P}) \subset X(\bar{k})$ is the group of $\bar{k}$-points of a finite étale over k subgroup $\Xi(\mathcal{P}) \subset X$; let $p_{\overline{\Xi}(\mathcal{P})} : X \to X/\Xi(\mathcal{P})$ denote the corresponding étale isogeny. By Tannaka duality, the exact functor

$$p_{\overline{\Xi}(\mathcal{P})*} : \langle \mathcal{P} \rangle \to \langle a_{\mathcal{P}*}\mathcal{P} \rangle$$

corresponds to

$$\begin{tikzcd}
1 \arrow[r] & \overline{G}(\mathcal{P})^\circ = \overline{G}(p_{\overline{\Xi}(\mathcal{P})*}\mathcal{P}) \arrow[r] \arrow[d, hook] & G(p_{\overline{\Xi}(\mathcal{P})*}\mathcal{P}) \arrow[r] \arrow[d, hook] & \Gamma(p_{\overline{\Xi}(\mathcal{P})*}\mathcal{P}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \overline{G}(\mathcal{P}) \arrow[r] & G(\mathcal{P}) \arrow[r] & \Gamma(\mathcal{P}) \arrow[r] & 1
\end{tikzcd}$$

(Observe that $p_{\overline{\Xi}(\mathcal{P})*}\iota_{0*} \simeq \iota_{0*} : \mathrm{Perv}(0, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X_{\overline{\Xi}(\mathcal{P})}, \overline{\mathbb{Q}}_\ell)$). Applying the above considerations to $p_{\overline{\Xi}(\mathcal{P})*}\mathcal{P}$, one gets $\overline{G}(\mathcal{P})^{\circ,\mathrm{ab}} = 0$.

---

[4] Here, we regard 0 as the 0-dimensional abelian vareity over k.

As a result,

$$\overline{G}(\mathcal{P})^\circ = \overline{G}(\mathcal{P})^{\mathrm{der}} = (G(\mathcal{P})^{\mathrm{der}})^\circ = (G(\mathcal{P})^\circ)^{\mathrm{der}}.$$

Passing to the projective limit, one gets, similarly, that $\overline{G}(X)$ is a prosemisimple group with

$$\pi_0(\overline{G}(X))\tilde{\to}\overline{G}(X)^{\mathrm{ab}}\tilde{\to}X(\bar{k})^\vee\tilde{\to}\pi_1(X_{\bar{k}},\bar{0}).$$

Let $\mathrm{Perv}^\circ_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell)\hookrightarrow \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell)$ denote the full Tannaka category generated by the skyscraper sheaves δ_x, $x\in X(\bar{k})$ and set $\langle\mathcal{P}|_{X_{\bar{k}}}\rangle^\circ := \langle\mathcal{P}|_{X_{\bar{k}}}\rangle\cap \mathrm{Perv}^\circ_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell)\hookrightarrow\langle\mathcal{P}|_{X_{\bar{k}}}\rangle$. From the above discussion, the commutative square of Tannaka categories

$$\begin{tikzcd}
\mathrm{Perv}^\circ_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell) \arrow[r, hook] & \mathrm{Perv}^{\mathrm{ar,ss}}_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell) \\
\langle\mathcal{P}|_{X_{\bar{k}}}\rangle^\circ \arrow[u, hook] \arrow[r, hook] & \langle\mathcal{P}|_{X_{\bar{k}}}\rangle \arrow[u, hook]
\end{tikzcd}$$

corresponds to the commutative square of proalgebraic groups

$$\begin{tikzcd}
\pi_0(\overline{G}(X)) \arrow[d, two heads] & \overline{G}(X) \arrow[l, two heads] \arrow[d, two heads] \\
\pi_0(\overline{G}(\mathcal{P})) & \overline{G}(\mathcal{P}). \arrow[l, two heads]
\end{tikzcd}$$

3.3. **Structure of** $\pi_0(G(\mathcal{P}))$**.** Let $\mathrm{Perv}^\circ(0,\overline{\mathbb{Q}}_\ell)\hookrightarrow\mathrm{Perv}^{\mathrm{ss}}(0,\overline{\mathbb{Q}}_\ell)$ denote the full Tannaka subcategory generated by the torsion characters $\pi_1(0)\to\overline{\mathbb{Q}}_\ell^\times$ and let $\mathrm{Perv}^\circ_{\mathrm{int},00}(X,\overline{\mathbb{Q}}_\ell)\hookrightarrow\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$ denote its essential image.

Let $X_0(k)$ denote the set of dimension 0 irreducible algebraic cycles on X or, equivalently, the set $|X|/\pi_1(k)$ of orbits of $\pi_1(k)$ acting on the set of closed points $|X|$ of X; note that for $x_0=\pi_1(k)\cdot x\in X_0(k)$, $\pi_1(x_0)\simeq\pi_1(x)\subset\pi_1(k)$ identifies with the stabilizer of x in $\pi_1(k)$. For $x_0\in X_0(k)$ the $\overline{\mathbb{Q}}_\ell$-linear abelian category $\mathrm{Perv}(x_0,\overline{\mathbb{Q}}_\ell)$ is equivalent to $\mathrm{Loc}(x_0,\overline{\mathbb{Q}}_\ell))$ and the full subcategory $\mathrm{Perv}^{\mathrm{ss}}(x_0,\overline{\mathbb{Q}}_\ell)\hookrightarrow\mathrm{Perv}(x_0,\overline{\mathbb{Q}}_\ell)$ to the full subcategory $\mathrm{Loc}^{\mathrm{ss}}(x_0,\overline{\mathbb{Q}}_\ell)\hookrightarrow\mathrm{Loc}(x_0,\overline{\mathbb{Q}}_\ell)$. Let again $\mathrm{Perv}^\circ(x_0,\overline{\mathbb{Q}}_\ell)\subset\mathrm{Perv}^{\mathrm{ss}}(x_0,\overline{\mathbb{Q}}_\ell)$ denote the full abelian subcategory generated by the torsion characters $\chi:\pi_1(x_0)\to\overline{\mathbb{Q}}_\ell^\times$. The closed immersion corresponding to x_0 induces a fully faithful exact $\overline{\mathbb{Q}}_\ell$-linear functor $x_{0*}:\mathrm{Perv}^{\mathrm{ss}}(x_0,\overline{\mathbb{Q}}_\ell)\to\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$. Let $\mathrm{Perv}^\circ_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)\hookrightarrow\mathrm{Perv}_{\mathrm{int},0}(X,\overline{\mathbb{Q}}_\ell)\hookrightarrow\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$ denote the full Tannaka subcategories generated by the essential images of

$$x_{0*}:\mathrm{Perv}^\circ(x_0,\overline{\mathbb{Q}}_\ell)\to\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell),\ \ x_0\in X_0(k),\ \ \text{and}\ \ x_{0*}:\mathrm{Perv}^{\mathrm{ss}}(x_0,\overline{\mathbb{Q}}_\ell)\to\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell),\ \ x_0\in X_0(k)$$

respectively.

Lemma 8. *The sequence of Tannaka catgeories*

$$\begin{tikzcd}
\mathrm{Perv}^\circ_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell) & \mathrm{Perv}^\circ_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell) \arrow[l, "(-)|_{X_{\bar{k}}}"'] & \mathrm{Perv}^\circ_{\mathrm{int},00}(X,\overline{\mathbb{Q}}_\ell) \arrow[l, hook']
\end{tikzcd} \tag{2}$$

corresponds to the canonical split short exact sequence of proalgebraic groups

$$\begin{tikzcd}
1 \arrow[r] & \pi_0(\overline{G}(X)) \arrow[r] \arrow[d, "\simeq"] & \pi_0(G(X)) \arrow[r] \arrow[d, "\simeq"] & \pi_0(\Gamma(X)) \arrow[r] \arrow[d, "\simeq"] & 1 \\
1 \arrow[r] & X(\bar{k})^\vee \arrow[r] & X(\bar{k})^\vee\rtimes\pi_1(k) \arrow[r] & \pi_1(k) \arrow[r] & 1,
\end{tikzcd}$$

where the semidirect product $X(\bar{k})^\vee\rtimes\pi_1(k)$ *is the one given by the natural action of* $\pi_1(k)$ *on* $X(\bar{k})^\vee$ *viz*

$$X(\bar{k})^\vee\rtimes\pi_1(k)\tilde{\to}\pi_1(X,\bar{0})=\pi_1(X_{\bar{k}},\bar{0})\rtimes\pi_1(0,\bar{0}).$$

Proof. As every simple object in $\mathrm{Perv}^\circ_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell)$ is of the form δ_x for some $x\in X(\bar{k})$ hence satisfies $\mathrm{H}^i(X_{\bar{k}},\delta_x)=0$, $i\neq 0$, one can apply the observation of [JKrLeM25, §4.3], namely that the functor

$$\omega:=\mathrm{H}^0(X_{\bar{k}},-):\mathrm{Perv}^\circ_{\mathrm{int}}(X_{\bar{k}},\overline{\mathbb{Q}}_\ell)\to\mathrm{Vect}_{\overline{\mathbb{Q}}_\ell} \tag{3}$$

is a fiber functor and that it factors through the following commutative diagram of Tannaka categories

$$\begin{tikzcd}
\mathrm{Perv}^{\circ}_{\mathrm{int}}(X_{\bar k},\overline{\mathbb{Q}}_\ell) \arrow[d, "\simeq"] & \mathrm{Perv}^{\circ}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell) \arrow[l, "(-)|_{X_{\bar k}}"'] \arrow[d] & \mathrm{Perv}^{\circ}_{\mathrm{int},00}(X,\overline{\mathbb{Q}}_\ell) \arrow[l, hook'] \arrow[d, "\simeq"] \\
\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(X(\bar k)^\vee) \arrow[dr] & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(X(\bar k)^\vee \rtimes \pi_1(k)) \arrow[l] \arrow[d] & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(k)) \arrow[l, hook'] \arrow[dl] \\
 & \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}. &
\end{tikzcd}$$

By Tannaka duality, it is thus enough to prove that the sequence (2) corresponds to the exact sequence

$$\pi_0(\overline{G}(X)) \to \pi_0(G(X)) \to \pi_0(\Gamma(X)) \to 1.$$

As $\mathrm{Perv}^{\circ}_{\mathrm{int}}(X_{\bar k},\overline{\mathbb{Q}}_\ell) = \overline{\mathrm{Perv}^{\circ}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)}$ and $\mathrm{Perv}^{\circ}_{\mathrm{int},00}(X,\overline{\mathbb{Q}}_\ell) = \mathrm{Perv}^{\circ}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)_{00}$, in view of the discussion in Subsection 2.2 and Subsections 3.1, 3.2, the only thing to check is that, for a simple object $\mathcal{S} \in \mathrm{Perv}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$, $\mathcal{S} \in \mathrm{Perv}^{\circ}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$ if and only if it generates a submonoid of finite rank in the Grothendieck semiring $K^+[\mathrm{Perv}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)]$. But, if $\mathcal{S}$ generates a $\mathbb{Z}_{\geq 0}$-submonoid of finite rank, by Tannaka duality and the short exact sequence (1), $\mathcal{S}|_{X_{\bar k}} \in K^+[\pi_0(\overline{G}(\mathcal{P}))]$ hence is of the form

$$\mathcal{S}|_{X_{\bar k}} = \sum_{x \in \overline{\Xi}(\mathcal{S})} m_x \delta_x$$

for some integers $m_x \geq 0$, $x \in \overline{\Xi}(\mathcal{S})$. In particular, the support of $\mathcal{S}$ is 0-dimensional hence $\mathcal{S} = \delta_{x_0,\chi} := x_{0*}\chi$ for some $x_0 \in X_0(k)$ and torsion character $\chi : \pi_1(x_0) \to \overline{\mathbb{Q}}_\ell^\times$. The converse inclusion is clear. □

Remark 9.

(1) Let X_1, X_2 be two abelian varieties over k. Assume there exists a tensor-equivalence of Tannaka categories $\mathit{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X_1,\overline{\mathbb{Q}}_\ell) \simeq \mathit{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X_2,\overline{\mathbb{Q}}_\ell)$. Then, by Tannaka duality, $G(X_1) \simeq G(X_2)$ as proalgebraic groupsp and, in particular, $\pi_1(X_1,\bar 0) \simeq \pi_1(X_2,\bar 0)$ as profinite groups. This already implies that X_1 and X_2 are isogenous. Indeed, as $\pi_1(X_{i\bar k},\bar 0) \subset \pi_1(X_i,\bar 0)$ is the maximal normal abelian closed subgroup $\Pi \subset \pi_1(X_i,\bar 0)$ such that $\pi_1(X_i,\bar 0)/\Pi$ is infinite procyclic, $i = 1, 2$, every isomorphism of profinite groups $\alpha : \pi_1(X_1,\bar 0) \xrightarrow{\sim} \pi_1(X_2,\bar 0)$ induces an isomorphism $\alpha : \pi_1(X_{1\bar k},\bar 0) \xrightarrow{\sim} \pi_1(X_{2\bar k},\bar 0)$, and hence an isomorphism $\bar\alpha : \pi_1(0,\bar 0) \xrightarrow{\sim} \pi_1(0,\bar 0)$. Fix a prime ℓ and let r_ℓ denote the rank of the pro-ℓ completions $\pi_1(X_i,\bar 0)^{(\ell)}$ and fix an isomorphism $\pi_1(X_i,\bar 0)^{(\ell)} \simeq \mathbb{Z}_\ell^{\oplus r_\ell}$ so that the semidirect product structure $\pi_1(X_{i\bar k},\bar 0)^{(\ell)} \rtimes \pi_1(0,\bar 0) \simeq \mathbb{Z}_\ell^{\oplus r_\ell} \rtimes \pi_1(k)$ is given by the image $M_{\ell,i}(\varphi) \in \mathrm{GL}_{r_\ell}(\mathbb{Z}_\ell)$ of the geometric Frobenius $\varphi \in \pi_1(0,\bar 0)$ acting on $\pi_1(X_i,\bar 0)^{(\ell)} \simeq \mathbb{Z}_\ell^{\oplus r_\ell}$, $i = 1, 2$. Then $\alpha : \pi_1(X_{1\bar k},\bar 0)^{(\ell)} \rtimes \pi_1(0,\bar 0) \xrightarrow{\sim} \pi_1(X_{2\bar k},\bar 0)^{(\ell)} \rtimes \pi_1(0,\bar 0)$ is given by a couple $(N_\ell, u_\ell) \in \mathrm{GL}_{2g}(\mathbb{Z}_\ell) \times \mathbb{Z}_\ell^\times$ such that $N_\ell M_{\ell,1}(\varphi) N_\ell^{-1} = M_{\ell,2}(\varphi)^{u_\ell}$. In particular, the characteristic polynomial of $M_{\ell,1}(\varphi), M_{\ell,2}(\varphi)^{u_\ell} \in \mathrm{GL}_{r_\ell}(\mathbb{Q}_\ell)$ should coincide. But as those of $M_{\ell,1}(\varphi), M_{\ell,2}(\varphi)$ both have roots which are $|k|$-Weil numbers of weight 1, this forces $u = \pm 1$. From Tate's theorem [Ta66] X_1 is thus k-isogenous to X_2 (the case $u_\ell = -1$ corresponds to an isogeny from X_1 to the dual abelian variety X_2^* of X_2 but X_2 and X_2^* are isogenous anyway).

(2) The proof of Lemma 8 extends as it is if k is a field of characteristic 0 provided one replaces $X(\bar k)^\vee$ with $X(\bar k)^\vee_{\mathrm{tors}}$ - see [W15] so that, again, one recovers $\pi_1(X,\bar 0)$ as $\pi_1(X,\bar 0) \xrightarrow{\sim} \pi_0(G(X))$. If one assumes k is finitely generated over $\mathbb{Q}$ and if X_1, X_2 are two abelian varieties over k such that there exists a tensor-equivalence of Tannaka categories $\mathit{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X_1,\overline{\mathbb{Q}}_\ell) \simeq \mathit{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X_2,\overline{\mathbb{Q}}_\ell)$, the resulting isomorphism of profinite groups $\pi_1(X_1,\bar 0) \simeq \pi_1(X_2,\bar 0)$ ensures that there exists a field automorphism $\sigma : k \xrightarrow{\sim} k$ such that X_1 and $X_2^\sigma := X_2 \times_{k,\sigma} k$ are k-isogenous. Indeed, as $\pi_1(X_{i\bar k},\bar 0) \subset \pi_1(X_i,\bar 0)$ is the maximal normal closed subgroup $\Pi \subset \pi_1(X_i,\bar 0)$ which is topologically finitely generated [FriJ08, Prop. 16.11.6], every isomorphism of profinite groups $\alpha : \pi_1(X_1,\bar 0) \xrightarrow{\sim} \pi_1(X_2,\bar 0)$ induces an isomorphism $\alpha : \pi_1(X_{1\bar k},\bar 0) \xrightarrow{\sim} \pi_1(X_{2\bar k},\bar 0)$, and hence an isomorphism $\bar\alpha : \pi_1(0,\bar 0) \xrightarrow{\sim} \pi_1(0,\bar 0)$. By the Neukirch-Uchida-Pop theorem [N69], [U77], [P94], [P95] $\bar\alpha : \pi_1(0,\bar 0) \xrightarrow{\sim} \pi_1(0,\bar 0)$ arises from a field isomorphism $\sigma : \bar k \xrightarrow{\sim} \bar k$ such that $\sigma(k) = k$ and hence, as in Remark 9 (1), the assertion follows from Faltings' theorem [F83], [FW92, IV, VI].

(3) In view of Remarks 9 (1), (2), and if k is finite (resp. finitely generated over $\mathbb{Q}$), one may ask whether X (resp. X up to Galois twists) can be reconstructed from the whole $G(X)$ up to k-isomorphism (and

not only up to k-isogeny) or, in the opposite direction, whether if X_1 and X_2 (resp. X_1 and X_2^σ for some field automorphism $\sigma : k\tilde{\to}k$) are k-isogenous then $G(X_1)\simeq G(X_2)$ as proreductive groups over $\overline{\mathbb{Q}}_\ell$.

Write also

$$\langle\mathcal{P}\rangle^\circ := \langle\mathcal{P}\rangle\cap \mathrm{Perv}^\circ_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)\hookrightarrow\langle\mathcal{P}\rangle,\ \ \langle\mathcal{P}\rangle^\circ_{00} := \langle\mathcal{P}\rangle\cap \mathrm{Perv}^\circ_{\mathrm{int},00}(X,\overline{\mathbb{Q}}_\ell)\hookrightarrow\langle\mathcal{P}\rangle.$$

The sequence of Tannaka categories

$$\begin{tikzcd}\langle\mathcal{P}|_{X_{\bar k}}\rangle^\circ & \langle\mathcal{P}\rangle^\circ \arrow[l, "(-)|_{X_{\bar k}}"'] & \langle\mathcal{P}\rangle^\circ_{00} \arrow[l, hook']\end{tikzcd}$$

induces an exact sequence

$$\pi_0(\overline{G}(\mathcal{P}))\to\pi_0(G(\mathcal{P}))\to\pi_0(\Gamma(\mathcal{P}))\to 1$$

which is not left-exact in general but, by Tannaka duality, the kernel

$$\overline{\pi}_0(G(\mathcal{P})) := \ker(\pi_0(G(\mathcal{P}))\to\pi_0(\Gamma(\mathcal{P})))$$

corresponds to the Tannaka subcategory $\overline{\langle\mathcal{P}\rangle^\circ}\hookrightarrow\langle\mathcal{P}|_{X_{\bar k}}\rangle^\circ$. Explicitly, let $\Xi(\mathcal{P})\subset\overline{\Xi}(\mathcal{P})$ denote the subset of all $x\in\overline{\Xi}(\mathcal{P})$ such that δ_x is a direct factor of an object in the essential image of $(-)|_{X_{\bar k}}:\langle\mathcal{P}\rangle^\circ\to\langle\mathcal{P}|_{X_{\bar k}}\rangle$. Then, again, $\Xi(\mathcal{P})\subset\overline{\Xi}(\mathcal{P})$ is a subgroup and the canonical morphism

$$\delta_- : \Xi(\mathcal{P})\to\Xi(\overline{\pi}_0(G(\mathcal{P})))$$

is an isomorphism.

3.3.1. The equivalence $\omega : \mathrm{Perv}^\circ_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)\tilde{\to}\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(X(\bar k)^\vee\rtimes\pi_1(k))$ provided by the fiber functor (3) can be described explicitly as follows on simple objects in $\mathrm{Perv}^\circ_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$. Let $x_0\in X_0(k)$ and $\chi:\pi_1(x_0)\to\overline{\mathbb{Q}}_\ell^\times$ a torsion character; fix $x\in|x_0|$, giving rise to an identification

$$\pi_1(k)/\pi_1(x)\tilde{\to}|x_0|,\ \ \sigma\mapsto\sigma\cdot x.$$

Note that as $\Xi(\delta_{x_0,\chi})\subset X(\bar k)$ is the subgroup generated by $|x_0|=\pi_1(k)\cdot x$, the kernel of $X(\bar k)^\vee\twoheadrightarrow\Xi(\delta_{x_0,\chi})$ is normalized by $\pi_1(x_0)$ and, as $\pi_1(x_0)$ is the stabilizer of x in $\pi_1(k)$, the resulting semidirect product

$$X(\bar k)^\vee\rtimes\pi_1(x_0)\twoheadrightarrow\Xi(\delta_{x_0,\chi})^\vee\rtimes\pi_1(x_0)=\overline{\Xi}(\delta_{x_0,\chi})^\vee\times\pi_1(x_0)$$

is actually a direct product. Then

$$\omega(\delta_{x_0,\chi})=\mathrm{H}^0(X_{\bar k},\delta_{x_0,\chi})\simeq\mathrm{Ind}_{X(\bar k)^\vee\rtimes\pi_1(x_0)}^{X(\bar k)^\vee\rtimes\pi_1(k)}(\overline{\mathbb{Q}}_\ell(\widetilde{\chi})\otimes\widetilde{\delta}_x)$$

as a $X(\bar k)^\vee\rtimes\rtimes\pi_1(k)$-representation. Here, we write

$$\widetilde{\chi}:X(\bar k)^\vee\rtimes\pi_1(x_0)\twoheadrightarrow\pi_1(x_0)\xrightarrow{\chi}\overline{\mathbb{Q}}_\ell^\times,$$

and

$$\widetilde{\delta}_x:X(\bar k)^\vee\rtimes\pi_1(x_0)\twoheadrightarrow\Xi(\delta_{x_0,\chi})^\vee\times\pi_1(x_0)\twoheadrightarrow\Xi(\delta_{x_0,\chi})^\vee=\Xi(\delta_{x_0})^\vee\xrightarrow{\delta_x}\overline{\mathbb{Q}}_\ell^\times,$$

where $\delta_x:\Xi(\delta_{x_0})^\vee\to\overline{\mathbb{Q}}_\ell^\times$ is the character induced by duality

$$\Xi(\delta_{x_0})^\vee\times\Xi(\delta_{x_0})\to\overline{\mathbb{Q}}_\ell^\times,\ \ (a,x)\mapsto a(x)=:\delta_x(a)$$

3.3.2. As $\mathcal{P}\in\mathrm{Perv}^\circ_{\mathrm{int}}(X,\overline{\mathbb{Q}}_\ell)$ the subgroup $\Xi(\mathcal{P})\subset X(\bar k)$ is $\pi_1(k)$-invariant hence the kernel of $X(\bar k)^\vee\twoheadrightarrow\Xi(\mathcal{P})^\vee$ is normalized by $\pi_1(k)$ and the short exact sequence

$$1\to\overline{\pi}_0(G(\mathcal{P}))\to\pi_0(G(\mathcal{P}))\to\pi_0(\Gamma(\mathcal{P}))\to 1 \tag{4}$$

fits into a commutative exact diagram of proalgebraic groups

$$\begin{tikzcd}1 \arrow[r] & \Xi(\mathcal{P})^\vee \arrow[r] \arrow[d, "\simeq"] & \Xi(\mathcal{P})^\vee\rtimes\pi_1(k) \arrow[r] \arrow[d, "(5-1)"] & \pi_1(k) \arrow[r] \arrow[d, "(5-2)"] & 1\\ 1 \arrow[r] & \overline{\pi}_0(G(\mathcal{P})) \arrow[r] & \pi_0(G(\mathcal{P})) \arrow[r] & \pi_0(\Gamma(\mathcal{P})) \arrow[r] & 1\end{tikzcd} \tag{5}$$

In particular, the canonical projection $\Xi(\mathcal{P})^\vee\rtimes\pi_1(k)\twoheadrightarrow\pi_1(k)$ induces an isomorphism

$$\mathrm{K}(\mathcal{P}):=\ker((5\text{-}1))\tilde{\to}\ker((5\text{-}2)).$$

Let $\varphi\in\pi_1(k)$ denote the Frobenius so that $\ker((5\text{-}2))=\langle\varphi^N\rangle$ for some $N\geq 1$. Then $\ker((5\text{-}1))=\langle(a,\varphi^N)\rangle$ for some (a unique) $a\in\Xi(\mathcal{P})^\vee$. But the condition that $\ker((5\text{-}1))$ be normal in $\overline{\Xi}(\mathcal{P})^\vee\rtimes\pi_1(k)$ imposes that:

- $(0,\varphi)(a,\varphi^N)(0,\varphi^{-1})=(\varphi\cdot a,\varphi^N)\in\langle(a,\varphi^N)\rangle$ - *viz* $\varphi\cdot a=a$ hence $a\in(\Xi(\mathcal{P})^\vee)^{\pi_1(k)}$;
- $(b,1)(a,\varphi^N)(-b,1)=(b+a-\varphi^N b,\varphi^N)\in\langle(a,\varphi^N)\rangle$ - *viz* $\varphi^N\cdot b=b$, $b\in\Xi(\mathcal{P})^\vee$ hence φ^N acts trivially on $\Xi(\mathcal{P})^\vee$.

Let $k^\circ_{\mathcal{P}}/k$ denote the finite extension defined by $\pi_1(k^\circ_{\mathcal{P}})=\ker((5\text{-}2))$. The short exact sequence (4) does not split in general ; it does split if and only if there exists $b\in\Xi(\mathcal{P})$ such that

$$\sum_{0\leq i\leq N-1}\varphi^i b=a.$$

But in general, $\mathrm{H}^2(\mathrm{Gal}(k^\circ_{\mathcal{P}}|k),\overline{\Xi}(\mathcal{P}))\neq 0$ (even though $\mathrm{H}^2(k,\ \overline{\Xi}(\mathcal{P}))=0$). Still, one can compute explicitly $\mathrm{K}(\mathcal{P})$ as follows, which is enough for our purpose. Let $\delta_{x_{0,1},\chi_1},\dots,\delta_{x_{0,s},\chi_s}$ denote a set of representatives of the finitely many isomorphism classes of simple objects in $\langle\mathcal{P}\rangle^\circ$. For each $i=1,\dots,s$, the action of $X(\bar{k})^\vee\rtimes\pi_1(k)$ on $\omega(\delta_{x_{0,i},\chi_i})$ (see Subsection 3.3.1) factors through $X(\bar{k})^\vee\rtimes\pi_1(k)\twoheadrightarrow\Xi(\mathcal{P})\rtimes\pi_1(k)$; write $\mathrm{K}(x_{0,i},\chi_i)\subset X(\bar{k})^\vee\rtimes\pi_1(k)$ for the kernel of $\Xi(\mathcal{P})^\vee\rtimes\pi_1(k)$ acting on $\omega(\delta_{x_{0,i},\chi_i})$. Then

$$\mathrm{K}(\mathcal{P})=\cap_{1\leq i\leq s}\mathrm{K}(x_{0,i},\chi_i)\subset\Xi(\mathcal{P})^\vee\rtimes\pi_1(k).$$

Furthermore, from the above analysis, setting again $N:=[k^\circ_{\mathcal{P}}:k]$, one also has

$$\mathrm{K}(\mathcal{P})=\langle(a,\varphi^N)\rangle\subset(\Xi(\mathcal{P})^\vee)^{\pi_1(k)}\times\pi_1(k),$$

with $a\in(\Xi(\mathcal{P})^\vee)^{\pi_1(k)}$ and $N\geq 1$ characterized by the following properties:

(1) $a(x_i)=\chi_i(\varphi^N)^{-1}$, $i=1,\dots,s$;

(2) Every $(a',\varphi^{N'})\in(\Xi(\mathcal{P})^\vee)^{\pi_1(k)}\times\pi_1(k)$ satisfying (1) is a multiple of (a,φ^N).

Note that, setting $n_i:=|\mathrm{im}(\chi_i)|$, $i=i,\dots,s$, Property (1) and the fact that $x_1,\dots,x_s$ generates $\Xi(\mathcal{P})$ impose that the order of a divides $gcm(n_1,\dots,n_s)$.

4. The companion correspondence in terms of semirings

4.1. **Semirings.** Let Q be a field of characteristic 0 and $\mathcal{T}$ a Tannaka category neutral over Q. Write $K^+[\mathcal{T}]$ for the semiring of $\mathcal{T}$ that is the set of isomorphism classes $[V]$ of semisimple objects V in $\mathcal{T}$ endowed with the laws $[V_1]+[V_2]=[V_1\oplus V_2]$, with neutral element $[0]$, and $[V_1]\cdot[V_2]=[V_1\otimes V_2]$, with neutral element $[Q]$. An element $[V]\in K^+[\mathcal{T}]$ is called simple if the corresponding object V is simple in $\mathcal{T}$. The Tannaka dimension $\dim:[V]\mapsto\dim_Q(V)$ induces a morphism of semirings $\dim:K^+[\mathcal{T}]\to\mathbb{Z}_{\geq 0}$.

If G is an algebraic group over Q, write $K^+[G]$ for the semiring of the Tannaka category $\mathrm{Rep}_Q(G)$ of finite dimensional Q-rational representations of G. Every fiber functor $\omega:\mathcal{T}\to\mathrm{Vect}_Q$ induces an equivalence of Tannaka category $\omega:\mathcal{T}\overset{\simeq}{\to}\mathrm{Rep}_Q(G(\mathcal{T},\omega))$ and, correspondingly, a canonical isomorphism of semirings over $\mathbb{Z}_{\geq 0}$

$$\begin{tikzcd}
K^+[\mathcal{T}] \arrow[rr, "\simeq", "\omega"'] \arrow[dr, "\dim"'] & & K^+[G(\mathcal{T},\omega)] \arrow[dl, "\dim"] \\
 & \mathbb{Z}_{\geq 0} &
\end{tikzcd}$$

Remark 10. One can immediately reconstruct the semirings $K^+[G(\mathcal{T},\omega)^\circ]\hookrightarrow K^+[G(\mathcal{T},\omega)]$ and $K^+[G(\mathcal{T},\omega)^{\mathrm{ab}}]\hookrightarrow K^+[G(\mathcal{T},\omega)]$ from $\dim:K^+[\mathcal{T}]\to\mathbb{Z}_{\geq 0}$ as the largest semiring with underlying $\mathbb{Z}_{\geq 0}$-monoid of finite rank and the semiring generated by $\dim^{-1}(1)\subset K^+[\mathcal{T}]$ respectively.

4.2. **Output of the companion correspondence.** Fix $\alpha,\beta\in A$ and a semisimple $\mathcal{P}_\alpha\in\mathrm{Perv}(X,\overline{\mathbb{Q}}_{\ell_\alpha})$. As $\chi(\mathcal{P}_\alpha)=\chi(\mathcal{P}_{\alpha\rightsquigarrow\beta})$, one gets

$$(\mathcal{P}_\alpha)_\neg\sim(\mathcal{P}_{\alpha\rightsquigarrow\beta})_\neg$$

so that $(\mathcal{P}_\alpha)_{\mathrm{int}}\sim(\mathcal{P}_{\alpha\rightsquigarrow\beta})_{\mathrm{int}}$, and the companions correspondence induces an isomorphism on the set of isomorphisms classes of semisimple objects

$$(-)_{\alpha\rightsquigarrow\beta}:Ob(\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_{\ell_\alpha}))/\simeq\ \tilde{\to}Ob(\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_{\ell_\beta}))/\simeq$$

Furthermore, from [Fu00] and [Z09, Prop. 2.1], [BBerDG82, Thm. 5.4.1], for every $\mathcal{P}_\alpha,\mathcal{Q}_\alpha\in\mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X,\overline{\mathbb{Q}}_{\ell_\alpha})$,

$${}^p\mathrm{H}^0(\mathcal{P}_\alpha\boxtimes\mathcal{Q}_\alpha)\sim_A{}^p\mathrm{H}^0(\mathcal{P}_{\alpha\rightsquigarrow\beta}\boxtimes\mathcal{Q}_{\alpha\rightsquigarrow\beta}),$$

hence

$$\mathcal{P}_\alpha \otimes \mathcal{Q}_\alpha \sim_A \mathcal{P}_{\alpha\rightsquigarrow\beta} \otimes \mathcal{Q}_{\alpha\rightsquigarrow\beta},$$

and

$$\mathcal{P}_\alpha^\vee = D_X(\mathcal{P}_\alpha) \sim_A D_X(\mathcal{P}_{\alpha\rightsquigarrow\beta}) = \mathcal{P}_{\alpha\rightsquigarrow\beta}^\vee.$$

For every $\alpha \in A$, fix a fiber functor

$$\omega_\alpha : \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_{\ell_\alpha}) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_{\ell_\alpha}}$$

and for $\mathcal{P}_\alpha \in \mathrm{Perv}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$, set $G(\mathcal{P}_\alpha) := G_{\omega_\alpha}(\mathcal{P}_\alpha) \subset \mathrm{GL}_{\omega_\alpha(\mathcal{P}_\alpha)}$. The above discussion implies, in particular, the following.

Corollary 11. *For every $\alpha, \beta \in A$ and semisimple $\mathcal{P}_\alpha \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ the companion correspondence induces an isomorphism of semirings*

$$\begin{tikzcd}
K^+[G(\mathcal{P}_\alpha)] \arrow[rr, "\simeq", "\phi_{\alpha\rightsquigarrow\beta}"'] \arrow[dr, "\dim"'] & & K^+[G(\mathcal{P}_{\alpha\rightsquigarrow\beta})]. \arrow[dl, "\dim"] \\
& \mathbb{Z}_{\geq 0} &
\end{tikzcd}$$

5. Proof of Theorem 7

Fix $\alpha, \beta \in A$, and $\mathcal{P}_\alpha \in \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$.

5.1. **A-independence of $\Gamma(\mathcal{P}_\alpha)$.** By construction of and unicity in the companion correspondence, if $0_*\mathcal{F}_\alpha \in \langle \mathcal{P}_\alpha \rangle_{00}$ is a tensor generator of $\langle \mathcal{P}_\alpha \rangle_{00}$ then $0_*\mathcal{F}_{\alpha\rightsquigarrow\beta} \in \langle \mathcal{P}_{\alpha\rightsquigarrow\beta} \rangle_{00}$ is a tensor generator of $\langle \mathcal{P}_{\alpha\rightsquigarrow\beta} \rangle_{00}$. But as $\mathcal{F}_\alpha \sim_A \mathcal{F}_{\alpha\rightsquigarrow\beta}$, one has, by definition of the companions correspondence on local systems, $\iota_\alpha(\Lambda(\mathcal{F}_\alpha)) = \iota_\beta(\Lambda(\mathcal{F}_{\alpha\rightsquigarrow\beta}))$ hence

$$\iota_\alpha \Gamma(\mathcal{P}_\alpha) \tilde{\to} \iota_\beta \Gamma(\mathcal{P}_{\alpha\rightsquigarrow\beta}).$$

5.2. **A-independence of $\pi_0(\overline{G}(\mathcal{P}_\alpha))$, $\pi_0(G(\mathcal{P}_\alpha))$.** For every simple object $\mathcal{S}_\alpha := \delta_{x_0, \chi_\alpha}$ in $\langle \mathcal{P}_\alpha \rangle_0$, $\mathcal{S}_{\alpha\rightsquigarrow\beta} = \delta_{x_0, \chi_{\alpha\rightsquigarrow\beta}}$. In particular

$$\overline{\Xi} := \overline{\Xi}(\mathcal{P}_\alpha) = \overline{\Xi}(\mathcal{P}_{\alpha\rightsquigarrow\beta}), \ \ \Xi := \Xi(\mathcal{P}_\alpha) = \Xi(\mathcal{P}_{\alpha\rightsquigarrow\beta}),$$

and hence

$$\pi_0(\iota_\alpha \overline{G}(\mathcal{P}_\alpha)) = \pi_0(\iota_\beta \overline{G}(\mathcal{P}_{\alpha\rightsquigarrow\beta})), \ \ \overline{\pi}_0(\iota_\alpha \overline{G}(\mathcal{P}_\alpha)) = \overline{\pi}_0(\iota_\beta \overline{G}(\mathcal{P}_{\alpha\rightsquigarrow\beta})).$$

Also, if $\delta_{x_{0,1}, \chi_{1,\alpha}}, \ldots, \delta_{x_{0,s}, \chi_{s,\alpha}}$ is a set of representatives of the finitely many isomorphism classes of simple objects in $\langle \mathcal{P}_\alpha \rangle^\circ$ then $\delta_{x_{0,1}, \chi_{1,\alpha\rightsquigarrow\beta}}, \ldots, \delta_{x_{0,s}, \chi_{s,\alpha\rightsquigarrow\beta}}$ are non-isomorphic simple objects in $\langle \mathcal{P}_{\alpha\rightsquigarrow\beta} \rangle^\circ$ hence, by symmetry of $\sim_A$, is a set of representatives of the finitely many isomorphism classes of simple objects in $\langle \mathcal{P}_{\alpha\rightsquigarrow\beta} \rangle^\circ$ as well. Let $N_i := |x_{0,i}|$ and $n_i := |\mathrm{im}(\chi_i)|$ so that $\chi_{i,\alpha}(\varphi^{N_i}) = \zeta_{n_i}$ for some primitive n_ith root of unity $\zeta_{n_i} \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{\ell_\alpha}$, $i = 1, \ldots, s$. Let $n := gcm(n_1, \ldots, n_s)$ and fix a primitive nth root of unity $\zeta_n \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{\ell_\alpha}$. Then $\iota_\beta^{-1}\iota_\alpha : \overline{\mathbb{Q}}_{\ell_\alpha} \tilde{\to} \overline{\mathbb{Q}}_{\ell_\beta}$ maps ζ_n to $\zeta_n^u \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{\ell_\beta}$ for some $u \in (\mathbb{Z}/n)^\times$ and $\chi_{i,\alpha,\rightsquigarrow\beta} : \pi_1(x_{0,i}) = \pi_1(x_i) \to \overline{\mathbb{Q}}_{\ell_\beta}^\times$ is characterized by $\chi_{i,\alpha,\rightsquigarrow\beta}(\varphi^{N_i}) = \zeta_{n_i}^u$, $i = 1, \ldots, s$. As in Subsection 3.2, write

$$\mathrm{K}(\mathcal{P}_\alpha) = \cap_{1 \leq i \leq s} \mathrm{K}(x_{0,i}, \chi_{i,\alpha}) = \langle (a_\alpha, \varphi^{N_\alpha}) \rangle$$

and

$$\mathrm{K}(\mathcal{P}_{\alpha\rightsquigarrow\beta}) = \cap_{1 \leq i \leq s} \mathrm{K}(x_{0,i}, \chi_{i,\alpha\rightsquigarrow\beta}) = \langle (a_{\alpha\rightsquigarrow\beta}, \varphi^{N_{\alpha\rightsquigarrow\beta}}) \rangle$$

Then the characterization of $\mathrm{K}(\mathcal{P}_\alpha)$, $\mathrm{K}(\mathcal{P}_{\alpha\rightsquigarrow\beta})$ given in *loc. cit* shows that $N := N_\alpha = N_{\alpha\rightsquigarrow\beta}$ and $a_{\alpha\rightsquigarrow\beta} = u a_\alpha$. As the order of a divides n and u is prime to n, the multiplication-by-u map on $\langle a \rangle$ extends to a $\pi_1(k)$-equivariant automorphism $U : \Xi^\vee \tilde{\to} \Xi^\vee$ as follows. For every prime q, let $\Xi^\vee[q^\infty] \subset \Xi^\vee$ denote the q-Sylow of $\Xi^\vee$. Let $I(\Xi^\vee)$ denote the set of primes dividing $|\Xi^\vee|$ and $I(a) \subset I(\Xi^\vee)$ the subset of those dividing the order of a. Then

$$\Xi^\vee = \bigoplus_{q \in I(\Xi^\vee)} \Xi^\vee[q^\infty],$$

and define $U : \Xi^\vee \tilde{\to} \Xi^\vee$ to be the multiplication-by-u automorphism on $\bigoplus_{q \in I(a)} \Xi^\vee[q^\infty]$ and the identity on $\bigoplus_{q \notin I(a)} \Xi^\vee[q^\infty]$. By construction, one gets the following canonical commutative diagram.

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{K}(\mathcal{P}_\alpha) \arrow[r] \arrow[d, "U"', "\simeq"] & \Xi^\vee \rtimes \pi_1(k) \arrow[r] \arrow[d, "U\rtimes Id"', "\simeq"] & \pi_0(G(\mathcal{P}_\alpha)) \arrow[r] \arrow[d, "\simeq"] & 1 \\
1 \arrow[r] & \mathrm{K}(\mathcal{P}_{\alpha\rightsquigarrow\beta}) \arrow[r] & \Xi^\vee \rtimes \pi_1(k) \arrow[r] & \pi_0(G(\mathcal{P}_{\alpha\rightsquigarrow\beta})) \arrow[r] & 1
\end{tikzcd}$$

5.3. A-independence of $G(\mathcal{P}_\alpha)^\circ$.

5.3.1. *Passing to the neutral component.* Let $\mathcal{P} \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$. Retaining the notation of Subsection 3.1 for $p_{\overline{\Xi}(\mathcal{P})} : X \to X/\overline{\Xi}(\mathcal{P})$ and of Subsection 3.2 for $k^\circ_{\mathcal{P}}$,

$$G(p_{\overline{\Xi}(\mathcal{P}),*}\mathcal{P})|_{(X/\overline{\Xi}(\mathcal{P}))_{k^\circ_{\mathcal{P}}}}) = G(\mathcal{P})^\circ.$$

5.3.2. *A-independence of the neutral component.* From Subsection 5.2, one has

$$\overline{\Xi} := \overline{\Xi}(\mathcal{P}_\alpha) = \overline{\Xi}(\mathcal{P}_{\alpha\rightsquigarrow\beta})$$

and

$$k^\circ := k^\circ_{\mathcal{P}_\alpha} = k^\circ_{\mathcal{P}_{\alpha\rightsquigarrow\beta}}.$$

As both $a_{\overline{\Xi}*} : \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X/\overline{\Xi})$ and $(-)|_{(X/\overline{\Xi})_{k^\circ}} : \mathrm{Perv}(X/\Xi, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}((X/\overline{\Xi})_{k^\circ})$ preserve compatibility (and semisimplicity), one has

$$(p_{\overline{\Xi}*}\mathcal{P}_\alpha)|_{(X/\overline{\Xi})_{k^\circ}} \sim_A (p_{\overline{\Xi}*}\mathcal{P}_{\alpha\rightsquigarrow\beta})|_{(X/\overline{\Xi})_{k^\circ}}.$$

Whence, as

$$G(p_{\overline{\Xi}*}\mathcal{P}_\alpha)|_{(X/\overline{\Xi})_{k^\circ}}) = G(\mathcal{P}_\alpha)^\circ,\ \ G(p_{\overline{\Xi}*}\mathcal{P}_{\alpha\rightsquigarrow\beta})|_{(X/\overline{\Xi})_{k^\circ}}) = G(\mathcal{P}_{\alpha\rightsquigarrow\beta})^\circ,$$

and from Corollary 11, an isomorphism of semirings

$$\begin{tikzcd}
K^+[G(\mathcal{P}_\alpha)^\circ] \arrow[rr, "\simeq", "\phi_{\alpha\rightsquigarrow\beta}"'] \arrow[dr, "\mathrm{dim}"'] & & K^+[G(\mathcal{P}_{\alpha\rightsquigarrow\beta})^\circ]^+. \arrow[dl, "\mathrm{dim}"] \\
& \mathbb{Z}_{\geq 0} &
\end{tikzcd}$$

From [KLarV14, Thm. 1.2], $\phi_{\alpha\rightsquigarrow\beta} : K^+[G(\mathcal{P}_\alpha)^\circ]\tilde{\rightarrow}K^+[G(\mathcal{P}_{\alpha\rightsquigarrow\beta})^\circ]$ arises from a unique isomorphism of algebraic groups

$$\iota_\beta G(\mathcal{P}_{\alpha\rightsquigarrow\beta})^\circ\tilde{\rightarrow}\iota_\alpha G(\mathcal{P}_\alpha)^\circ.$$

6. FINAL REMARKS

(1) We already observed that $\pi_0(\iota_\beta\overline{G}(\mathcal{P}_{\alpha\rightsquigarrow\beta}))\tilde{\rightarrow}\pi_0(\iota_\alpha\overline{G}(\mathcal{P}_\alpha))$. From the fact that $\overline{G}(\mathcal{P})^\circ = G(\mathcal{P})^{\circ,\mathrm{der}}$, one immediately gets $\iota_\beta\overline{G}(\mathcal{P}_{\alpha\rightsquigarrow\beta})^\circ\tilde{\rightarrow}\iota_\alpha\overline{G}(\mathcal{P}_\alpha)^\circ$ as well.

(2) From Remark 10, the isomorphism of semirings

$$\phi_{\alpha\rightsquigarrow\beta} : K^+[G(\mathcal{P}_\alpha)]\tilde{\rightarrow}K^+[G(\mathcal{P}_{\alpha\rightsquigarrow\beta})]$$

restricts to an isomorphism of semirings

$$\phi_{\alpha\rightsquigarrow\beta} : K^+[G(\mathcal{P}_\alpha)^{\mathrm{ab}}]\tilde{\rightarrow}K^+[G(\mathcal{P}_{\alpha\rightsquigarrow\beta})^{\mathrm{ab}}],$$

and hence $\iota_\alpha G(\mathcal{P}_\alpha)^{\mathrm{ab}}$ is also independent of $\alpha \in A$ (as an algebraic group of multiplicative type can be reconstructed from its semiring of representations).

(3) From Remark 10, the isomorphism of semirings

$$\phi_{\alpha\rightsquigarrow\beta} : K^+[G(\mathcal{P}_\alpha)]\tilde{\rightarrow}K^+[G(\mathcal{P}_{\alpha\rightsquigarrow\beta})]$$

also restricts to an isomorphism of semirings

$$\phi_{\alpha\rightsquigarrow\beta} : K^+[\pi_0(\iota_\alpha G(\mathcal{P}_\alpha))]\tilde{\rightarrow}K^+[\pi_0(\iota_\beta G(\mathcal{P}_{\alpha\rightsquigarrow\beta}))].$$

Still, in general, for a finite group G, $\mathrm{dim} : K^+[G] \to \mathbb{Z}_{\geq 0}$ does not determine the isomorphism class of G - for instance the diedral group D_8 of order 8 and the quaternion group $\mathbb{H}_8$ have the same character table hence *a fortiori*, the same semiring $\mathrm{dim} : K^+[G] \to \mathbb{Z}_{\geq 0}$. So, we really need to carry out the non-formal analysis of Subsection 3.3.

(4) If instead of a single $\mathcal{P}_\alpha \in \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ one considers the whole Tannaka category $\mathcal{T}_\alpha := \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$, one gets a canonical isomorphisms (of pro-algebraic groups)

$$\begin{tikzcd}
1 \arrow[r] & \pi_0(G(\overline{\mathcal{T}}_\alpha)) \arrow[r] \arrow[d, "\simeq"] & \pi_0(G(\mathcal{T}_\alpha)) \arrow[r] \arrow[d, "\simeq"] & \pi_0(G(\mathcal{T}_{\alpha,00})) \arrow[r] \arrow[d, "\simeq"] & 1 \\
1 \arrow[r] & X(\bar{k})^\vee \arrow[r] & X(\bar{k})^\vee \rtimes \pi_1(k) \arrow[r] & \pi_1(k) \arrow[r] & 1,
\end{tikzcd}$$

as well as isomorphisms

$$\iota_\alpha G^\circ(\overline{\mathcal{T}}_\alpha)(= \lim \iota_\alpha \overline{G}^\circ(\mathcal{P}_\alpha)) \simeq (\lim \iota_\beta \overline{G}^\circ(\mathcal{P}_{\alpha\rightsquigarrow\beta}) =)\iota_\beta G^\circ(\overline{\mathcal{T}}_\beta).$$

$$\iota_\alpha G^\circ(\mathcal{T}_\alpha)(= \lim \iota_\alpha G^\circ(\mathcal{P}_\alpha)) \simeq (\lim \iota_\beta G^\circ(\mathcal{P}_{\alpha\rightsquigarrow\beta}) =)\iota_\beta G^\circ(\mathcal{T}_\beta).$$

(5) For every $\mathcal{P}_\alpha \in \mathrm{Perv}^{\mathrm{ss}}_{\mathrm{int}}(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ there exists a finite field extension $Q$of $\mathbb{Q}$ such that $\mathcal{P}_{\alpha\rightsquigarrow\beta}$, $\beta \in A$ is Q-compatible (in the sense of [Fu00]). One may then ask whether, up to replacing Q by a finite field extension, the common $\overline{\mathbb{Q}}$-form G of the $G(\mathcal{P}_{\alpha\rightsquigarrow\beta})$, $\beta \in A$ descends to Q (compare with [Ch04, Thm. 1.4]).

(6) For the Tannaka category of $\overline{\mathbb{Q}}_\ell$-local systems on smooth varieties X, stronger ℓ-independence results hold - see [Dr18]. One may ask whether the enhanced reconstruction result à la Kazhan-Larsen-Varshavsky [Dr12, Thm. 4.3.7] could be adapted to the setting of perverse $\overline{\mathbb{Q}}_\ell$-sheaves on abelian varieties to show the A-independence of $\iota_\alpha G(\mathcal{P}_\alpha)$.

(7) One may also ask whether similar A-independence results hold for the Tannaka categories of perverse sheaves on arbitrary commutative algebraic groups X over k.

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