

UNIFORM BOUNDS FOR OBSTRUCTIONS TO THE INTEGRAL TATE CONJECTURE

ANNA CADORET AND ALENA PIRUTKA

ABSTRACT. Assuming natural variational realization conjectures, we give uniform bounds for the obstruction to the integral Tate conjecture in 1-dimensional families of algebraic varieties over an infinite finitely generated field.

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1. INTRODUCTION

For an abelian group A , write $A_{\text{tors}} \subset A$ for its torsion subgroup and $A \twoheadrightarrow A^{\text{free}} := A/A_{\text{tors}}$ for its maximal torsion-free quotient. For an algebraic group G , let $G^\circ \subset G$ denote its neutral component and $G \twoheadrightarrow \pi_0(G) := G/G^\circ$ its group of connected components.

A variety over a field k is a separated scheme of finite type over k .

In this paper k will denote an infinite field of characteristic $p \geq 0$, finitely generated over its prime subfield. We fix a separable closure $k \hookrightarrow \bar{k}$ and write $\pi_1(k) = \text{Gal}(\bar{k}|k)$ for the absolute Galois group.

1.1. Tate conjectures. Let X be a smooth projective variety over k . For every integer $i \geq 0$, let $\text{CH}^i(X)$ denote the group of algebraic cycles of codimension i on X modulo rational equivalence, and for every ring R , set $\text{CH}^i(X)_R := \text{CH}^i(X) \otimes_{\mathbb{Z}} R$. For a prime $\ell \neq p$, set

$$V_{\mathbb{Z}_\ell} := H^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i)).$$

Let $G_\ell \subset \text{GL}(V_{\mathbb{Z}_\ell})$ denote the Zariski-closure of the image of $\pi_1(k)$ acting on $V_{\mathbb{Q}_\ell} := V_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and let

$$\tilde{V}_{\mathbb{Q}_\ell} := (V_{\mathbb{Q}_\ell})^{G_\ell^\circ} \subset V_{\mathbb{Q}_\ell}$$

denote the $\mathbb{Q}_\ell$ -vector space of Tate classes, which can also be described as

$$\tilde{V}_{\mathbb{Q}_\ell} = \text{colim}_U (V_{\mathbb{Q}_\ell})^U,$$

where U varies among all open subgroups of $\pi_1(k)$. The cycle class map $c_\ell : \text{CH}^i(X_{\bar{k}}) \rightarrow V_{\mathbb{Z}_\ell}$ for $\mathbb{Z}_\ell$ -étale cohomology fits into the following canonical Cartesian diagram

$$(1) \quad \begin{array}{ccccccc} & & & \xrightarrow{c_\ell} & & & \\ & & & \text{---} & & & \\ \text{CH}^i(X_{\bar{k}}) & \longrightarrow & \text{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} & \longrightarrow & V_{\mathbb{Z}_\ell}^a & \hookrightarrow & \tilde{V}_{\mathbb{Z}_\ell} & \hookrightarrow & V_{\mathbb{Z}_\ell} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & & & V_{\mathbb{Z}_\ell}^{\text{free}, a} & \hookrightarrow & \tilde{V}_{\mathbb{Z}_\ell}^{\text{free}} & \hookrightarrow & V_{\mathbb{Z}_\ell}^{\text{free}} \\ & & & & \downarrow & & \downarrow & & \downarrow \\ \text{CH}^i(X_{\bar{k}})_{\mathbb{Q}} & \longrightarrow & \text{CH}^i(X_{\bar{k}})_{\mathbb{Q}_\ell} & \longrightarrow & V_{\mathbb{Q}_\ell}^a & \hookrightarrow & \tilde{V}_{\mathbb{Q}_\ell} & \hookrightarrow & V_{\mathbb{Q}_\ell}, \end{array}$$

where $V_{\mathbb{Z}_\ell}^a$ (resp. $V_{\mathbb{Q}_\ell}^a$) is the image of the cycle class map $c_\ell \otimes \mathbb{Z}_\ell : \text{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \rightarrow V_{\mathbb{Z}_\ell}$ (resp. $c_\ell \otimes \mathbb{Q}_\ell$) and where $\tilde{V}_{\mathbb{Z}_\ell}$ and $\tilde{V}_{\mathbb{Q}_\ell}^{\text{free}}$ are defined by the rightmost Cartesian squares of the diagram.

The (classical) rational $\mathbb{Q}_\ell$ -Tate conjecture for codimension i cycles on X [Ta65]

$$\text{Tate}_{\mathbb{Q}_\ell}(X, i) \quad V_{\mathbb{Q}_\ell}^a = \tilde{V}_{\mathbb{Q}_\ell}$$

asserting that Tate classes are $\mathbb{Q}_\ell$ -linear combinations of algebraic classes admits the following integral variants:

$$\begin{aligned} \text{Tate}_{\mathbb{Z}_\ell}^{\text{free}}(X, i) \quad V_{\mathbb{Z}_\ell}^{\text{free}, a} &= \tilde{V}_{\mathbb{Z}_\ell}^{\text{free}} & (\text{Integral Tate conjecture modulo torsion}); \\ \text{Tate}_{\mathbb{Z}_\ell}(X, i) \quad V_{\mathbb{Z}_\ell}^a &= \tilde{V}_{\mathbb{Z}_\ell} & (\text{Integral Tate conjecture}). \end{aligned}$$

While, tautologically,

$$\text{Tate}_{\mathbb{Z}_\ell}(X, i) \Rightarrow \text{Tate}_{\mathbb{Z}_\ell}^{\text{free}}(X, i) \Rightarrow \text{Tate}_{\mathbb{Q}_\ell}(X, i),$$

it is known that, in general, the converse implications fail as soon as $i > 1$ (see e.g. [CTS10, AtH62] for the failure of $\text{Tate}_{\mathbb{Z}_\ell}(X, i)$ and [CTS10, Ko90, To13, P22] for the failure of $\text{Tate}_{\mathbb{Z}_\ell}^{\text{free}}(X, i)$).

The aim of this note is to analyze the obstructions to $\text{Tate}_{\mathbb{Z}_\ell}(X, i)$, $\text{Tate}_{\mathbb{Z}_\ell}^{\text{free}}(X, i)$ when X varies in family. Our arguments provide a new application of the structure theorem of the degeneration locus of ℓ -adic local systems of [CT13] (see Fact A), in the spirit of [CC20, C23].

Before considering the variational setting, we make some elementary remarks. By definition, the obstructions to $\text{Tate}_{\mathbb{Q}_\ell}(X, i)$, $\text{Tate}_{\mathbb{Z}_\ell}^{\text{free}}(X, i)$, $\text{Tate}_{\mathbb{Z}_\ell}(X, i)$ are, respectively:

$$\tilde{C}_{\mathbb{Q}_\ell} := \tilde{V}_{\mathbb{Q}_\ell} / V_{\mathbb{Q}_\ell}^a, \quad \tilde{C}_{\mathbb{Z}_\ell}^{\text{free}} := \tilde{V}_{\mathbb{Z}_\ell}^{\text{free}} / V_{\mathbb{Z}_\ell}^{\text{free}, a}, \quad \tilde{C}_{\mathbb{Z}_\ell} := \tilde{V}_{\mathbb{Z}_\ell} / V_{\mathbb{Z}_\ell}^a.$$

1.1.1. $\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}}$ versus $\tilde{C}_{\mathbb{Z}_\ell}$. The short exact sequence

$$(2) \quad 0 \rightarrow (V_{\mathbb{Z}_\ell})_{\text{tors}} / (V_{\mathbb{Z}_\ell}^a)_{\text{tors}} \rightarrow \tilde{C}_{\mathbb{Z}_\ell} \rightarrow \tilde{C}_{\mathbb{Z}_\ell}^{\text{free}} \rightarrow 0$$

realizes $\tilde{C}_{\mathbb{Z}_\ell}$ as an extension of $\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}}$ by a finite group which is a quotient of $(V_{\mathbb{Z}_\ell})_{\text{tors}}$. As $(V_{\mathbb{Z}_\ell})_{\text{tors}}$ is constant in family, the problems of bounding uniformly $\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}}$ and $\tilde{C}_{\mathbb{Z}_\ell}$ are essentially equivalent.

1.1.2. $\tilde{C}_{\mathbb{Q}_\ell}$ versus $\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}}$. From $\tilde{C}_{\mathbb{Q}_\ell} = \tilde{C}_{\mathbb{Z}_\ell}^{\text{free}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and the short exact sequence (2), one has the following equivalences

$$\text{Tate}_{\mathbb{Q}_\ell}(X, i) \Leftrightarrow (\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}})_{\text{tors}} = \tilde{C}_{\mathbb{Z}_\ell}^{\text{free}} \Leftrightarrow (\tilde{C}_{\mathbb{Z}_\ell})_{\text{tors}} = \tilde{C}_{\mathbb{Z}_\ell}$$

and, in case they hold, (2) reads

$$(3) \quad 0 \rightarrow (V_{\mathbb{Z}_\ell})_{\text{tors}} / (V_{\mathbb{Z}_\ell}^a)_{\text{tors}} \rightarrow (\tilde{C}_{\mathbb{Z}_\ell})_{\text{tors}} \rightarrow (\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}})_{\text{tors}} \rightarrow 0.$$

So that, assuming $\text{Tate}_{\mathbb{Q}_\ell}(X, i)$, the obstructions we are interested in are $(\tilde{C}_{\mathbb{Z}_\ell})_{\text{tors}}$, $(\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}})_{\text{tors}}$. The obstruction $(\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}})_{\text{tors}}$ can be described without involving the $\mathbb{Z}_\ell$ -module $\tilde{V}_{\mathbb{Z}_\ell}^{\text{free}}$ of Tate classes. Indeed, writing

$$C_{\mathbb{Z}_\ell}^{\text{free}} := V_{\mathbb{Z}_\ell}^{\text{free}} / V_{\mathbb{Z}_\ell}^{\text{free}, a},$$

it follows from the short exact sequence

$$0 \rightarrow C_{\mathbb{Z}_\ell}^{\text{free}} \rightarrow \tilde{C}_{\mathbb{Z}_\ell}^{\text{free}} \rightarrow V_{\mathbb{Z}_\ell}^{\text{free}} / \tilde{V}_{\mathbb{Z}_\ell}^{\text{free}} \rightarrow 0$$

and the fact that $V_{\mathbb{Z}_\ell}^{\text{free}} / \tilde{V}_{\mathbb{Z}_\ell}^{\text{free}}$ is torsion-free that

$$(C_{\mathbb{Z}_\ell}^{\text{free}})_{\text{tors}} = (\tilde{C}_{\mathbb{Z}_\ell}^{\text{free}})_{\text{tors}}.$$

1.2. Variational conjectures. Our main results - Theorem A, Theorem B in Subsection 1.3 - involve some classical variational realization conjectures, which we discuss first.

1.2.1. *Characteristic 0.*

- **Singular cohomology:** Fix an embedding $\infty : k \hookrightarrow \mathbb{C}$, let $(-)_\infty$ denote the base-change functor along $\text{Spec}(\mathbb{C}) \xrightarrow{\sim} \text{Spec}(k)$ and $(-)^{\text{an}}$ the analytification functor from varieties over $\mathbb{C}$ to complex analytic spaces. For every $s_\infty \in S_\infty(\mathbb{C})$ the cycle class maps for singular cohomology

$$c : \text{CH}^i(X_\infty)_\mathbb{Q} \rightarrow H^{2i}(X_\infty^{\text{an}}, \mathbb{Q}(i)), \quad c_{s_\infty} : \text{CH}^i(X_{s_\infty})_\mathbb{Q} \rightarrow H^{2i}(X_{s_\infty}^{\text{an}}, \mathbb{Q}(i))$$

fit into a canonical commutative diagram

$$\begin{array}{ccc} \text{CH}^i(X_\infty)_\mathbb{Q} & \xrightarrow{|_{X_\infty, s}} & \text{CH}^i(X_{s_\infty})_\mathbb{Q} \\ \downarrow c & & \downarrow c_{s_\infty} \\ H^{2i}(X_\infty^{\text{an}}, \mathbb{Q}(i)) & \xrightarrow{\epsilon} H^0(S_\infty^{\text{an}}, R^{2i}f_{\infty*}^{\text{an}}\mathbb{Q}(i)) \hookrightarrow & H^{2i}(X_{s_\infty}^{\text{an}}, \mathbb{Q}(i)), \end{array}$$

where $\epsilon : H^{2d}(X_\infty^{\text{an}}, \mathbb{Q}(i)) \twoheadrightarrow E_\infty^{0,i} \hookrightarrow E_2^{0,i} = H^0(S_\infty^{\text{an}}, R^{2i}f_{\infty*}^{\text{an}}\mathbb{Q}(i))$ is the edge morphism from the Leray spectral sequence for $f_\infty^{\text{an}} : X_\infty^{\text{an}} \rightarrow S_\infty^{\text{an}}$.

$\text{VSing}^0(f_\infty, i)$ For every $s_\infty \in S_\infty(\mathbb{C})$ and $\alpha_{s_\infty} \in H^0(S_\infty^{\text{an}}, R^{2i}f_{\infty*}^{\text{an}}\mathbb{Q}(i)) \subset H^{2i}(X_{s_\infty}^{\text{an}}, \mathbb{Q}(i))$ the following properties are equivalent:

- 1) $\alpha_{s_\infty} \in \text{im}[c_{s_\infty, \mathbb{Q}} : \text{CH}^i(X_{s_\infty})_\mathbb{Q} \rightarrow H^{2i}(X_{s_\infty}^{\text{an}}, \mathbb{Q}(i))];$
- 2) there exists $\tilde{\alpha} \in \text{CH}^i(X_\infty)_\mathbb{Q}$ such that $c_{s_\infty}(\tilde{\alpha}|_{X_{s_\infty}}) = \alpha_{s_\infty}$.

Though it does not involve Hodge classes, the statement $\text{VSing}^0(f_\infty, i)$ is often referred to as the variational Hodge conjecture for codimension i cycles because, by the fixed part theorem, it follows from the Hodge conjecture for any smooth compactification of X_∞ - see *e.g.* [CS13, §3.1] for details and an equivalent formulation using de Rham cohomology. *A priori* the statement $\text{VSing}^0(f_\infty, i)$ is not preserved by base-change along finite covers of smooth varieties while the obstructions $\text{Ob}_{\mathbb{Z}_\ell, s}$, $s \in S$ are. So we will rather consider the following "stabilized" variant $\text{VSing}(f_\infty, i)$. For finite covers $S'_\infty \rightarrow S'_\infty \rightarrow S_\infty$ of smooth varieties, consider the notation in the base-change diagram:

$$\begin{array}{ccccc} X''_\infty & \longrightarrow & X'_\infty & \longrightarrow & X_\infty \\ f''_\infty \downarrow & & \square & f'_\infty \downarrow & \square & \downarrow f_\infty \\ S''_\infty & \longrightarrow & S'_\infty & \longrightarrow & S_\infty. \end{array}$$

$\text{VSing}(f_\infty, i)$ There exists a finite cover $S'_\infty \rightarrow S_\infty$ of smooth varieties over $\mathbb{C}$ such that for every finite cover $S''_\infty \rightarrow S'_\infty$ of smooth varieties over $\mathbb{C}$, $\text{VSing}^0(f''_\infty, i)$ holds.

Let us point out that if the Zariski-closure of the image of $\pi_1(S_\infty^{\text{an}})$ acting on $H^{2i}(X_{s_\infty}^{\text{an}}, \mathbb{Q}(i))$ is connected then $\text{VSing}(f_\infty, i)$ and $\text{VSing}^0(f_\infty, i)$ are both equivalent to

For every finite cover $S'_\infty \rightarrow S_\infty$ of smooth varieties over $\mathbb{C}$, $\text{VSing}^0(f'_\infty, i)$ holds.

In particular, if $S'_\infty \rightarrow S_\infty$ is chosen in such a way that the Zariski-closure of the image of $\pi_1(S_\infty^{\text{an}})$ acting on $H^{2i}(X_{s_\infty}^{\text{an}}, \mathbb{Q}(i))$ is connected then $\text{VSing}^0(f'_\infty, i)$ implies $\text{VSing}(f_\infty, i)$.

- **Étale $\mathbb{Q}_\ell$ -cohomology:** The following is the $\mathbb{Q}_\ell$ -étale counterpart of $\text{VSing}^0(f_\infty, i)$:

$\text{Vet}_{\mathbb{Q}_\ell}^0(f, i)$ For every $s \in |S|$ and $\alpha_s \in H^0(S_{\bar{k}}, R^{2i}f_*\mathbb{Q}_\ell(i)) \subset H^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i))$ the following properties are equivalent:

- 1) $\alpha_s \in \text{im}[c_{X_{\bar{s}}, \ell} : \text{CH}^i(X_{\bar{s}})_\mathbb{Q} \rightarrow H^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i))];$
- 2) there exists $\tilde{\alpha} \in \text{CH}^i(X_{\bar{k}})_\mathbb{Q}$ such that $c_{X_{\bar{s}}, \ell}(\tilde{\alpha}|_{X_{\bar{s}}}) = \alpha_s$.

One could also consider the seemingly weaker variant $\text{WVet}_{\mathbb{Q}_\ell}^0(f, i)$ where $\text{CH}^i(X_{\bar{s}})_\mathbb{Q}$, $\text{CH}^i(X_{\bar{k}})_\mathbb{Q}$ are replaced with $\text{CH}^i(X_{\bar{s}})_{\mathbb{Q}_\ell}$, $\text{CH}^i(X_{\bar{k}})_{\mathbb{Q}_\ell}$, and the stabilized variants $\text{WVet}_{\mathbb{Q}_\ell}(f, i)$, $\text{Vet}_{\mathbb{Q}_\ell}(f, i)$. Note that the statements $\text{WVet}_{\mathbb{Q}_\ell}^0(f, i)$, $\text{Vet}_{\mathbb{Q}_\ell}^0(f, i)$ also make sense when $p > 0$.

Proposition 1. *If $p = 0$, one has*

$$\text{WVet}_{\mathbb{Q}_\ell}^0(f, i) \Leftrightarrow \text{Vet}_{\mathbb{Q}_\ell}^0(f, i) \Leftrightarrow \text{VSing}^0(f_\infty, i).$$

In general, one always has $\mathrm{Vet}_{\mathbb{Q}_\ell}^0(f, i) \Rightarrow \mathrm{WVet}_{\mathbb{Q}_\ell}^0(f, i)$ and $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_\eta, i) \Rightarrow \mathrm{WVet}_{\mathbb{Q}_\ell}(f, i)$.

We will give a proof of Proposition 1 in section 3.1.3. In particular, when $p = 0$, $\mathrm{VSing}^0(f_\infty, i)$ is independent of the embedding $\infty : k \hookrightarrow \mathbb{C}$ and $\mathrm{WVet}_{\mathbb{Q}_\ell}^0(f, i)$, $\mathrm{Vet}_{\mathbb{Q}_\ell}^0(f, i)$ are independent of the prime ℓ .

Let us also point out that if the Zariski-closure of the image of $\pi_1(S_{\bar{k}})$ acting on $H^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i))$ is connected then $\mathrm{Vet}_{\mathbb{Q}_\ell}(f, i)$ and $\mathrm{Vet}_{\mathbb{Q}_\ell}^0(f, i)$ are both equivalent to

For every finite cover $S' \rightarrow S$ of smooth varieties, $\mathrm{Vet}_{\mathbb{Q}_\ell}^0(f', i)$ holds.

In particular, if $S' \rightarrow S$ is chosen in such a way that the Zariski-closure of the image of $\pi_1(S'_k)$ acting on $H^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i))$ is connected then $\mathrm{Vet}_{\mathbb{Q}_\ell}^0(f', i)$ implies $\mathrm{Vet}_{\mathbb{Q}_\ell}(f, i)$. The same considerations apply to $\mathrm{WVet}_{\mathbb{Q}_\ell}(f, i)$ and $\mathrm{WVet}_{\mathbb{Q}_\ell}^0(f, i)$.

1.2.2. Characteristic $p > 0$. According to Proposition 1, a first substitute for $\mathrm{VSing}(f_\infty, i)$ when $p > 0$ is $\mathrm{WVet}_{\mathbb{Q}_\ell}(f, i)$. Another natural substitute is the variational realization conjecture in crystalline cohomology $\mathrm{VCrys}(f, i)$. This is more subtle. Indeed, as crystalline cohomology is only well-behaved over a perfect residue field, one has first to spread out all the involved data over a finite base field. Another difficulty is to relate crystalline and étale data; for singular data, this is Artin's comparison isomorphism between étale and singular cohomology. But there is no such a direct functorial comparison isomorphism between crystalline and étale cohomology; to remedy this, one has to invoke a weak form - $\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f, i)$ of the motivic conjecture predicting that homological and numerical equivalence should coincide (combined with a theorem of Ambrosi - see Fact 13).

We now state $\mathrm{VCrys}(f, i)$ and $\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f, i)$. Let F denote the algebraic closure of $\mathbb{F}_p$ in k and let $\mathcal{X}$ be a smooth, separated, geometrically connected scheme over F with generic point $\eta_{\mathcal{X}} : \mathrm{Spec}(k) \rightarrow \mathcal{X}$, let $\mathcal{S} \rightarrow \mathcal{X}$ be a smooth, separated and geometrically connected morphism and $f : \mathcal{X} \rightarrow \mathcal{S}$ a smooth proper morphism fitting in the following Cartesian diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{f} & \mathcal{S} & \longrightarrow & \mathcal{K} \\ \uparrow & \square & \uparrow & \square & \uparrow \eta_{\mathcal{X}} \\ X & \xrightarrow{f} & S & \longrightarrow & k \end{array}$$

Let K denote the fraction field of the ring W of Witt vectors of F . For a F -scheme $\mathcal{Z}$, write $H_{\mathrm{crys}}^i(\mathcal{Z}) := H_{\mathrm{crys}}^i(\mathcal{Z}/W)_K$ for the crystalline cohomology with K -coefficients and

$$c_{\mathrm{crys}} : \mathrm{CH}^i(\mathcal{Z})_{\mathbb{Q}} \rightarrow H_{\mathrm{crys}}^{2i}(\mathcal{Z})$$

for the cycle class map. For every $t \in |\mathcal{S}|$ the cycle class maps

$$c_{\mathrm{crys}} : \mathrm{CH}^i(\mathcal{X}) \rightarrow H_{\mathrm{crys}}^{2i}(\mathcal{X}), \quad c_{\mathrm{crys}, t} : \mathrm{CH}^i(\mathcal{X}_t) \rightarrow H_{\mathrm{crys}}^{2i}(\mathcal{X}_t)$$

fit into a canonical commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}} & \xrightarrow{|\mathcal{X}_t} & \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \\ c_{\mathrm{crys}} \downarrow & & \downarrow c_{\mathrm{crys}, t} \\ H_{\mathrm{crys}}^{2i}(\mathcal{X}) & \xrightarrow{\epsilon} H^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/W})_K \hookrightarrow & H_{\mathrm{crys}}^{2i}(\mathcal{X}_t), \end{array}$$

where $\epsilon : H_{\mathrm{crys}}^{2i}(\mathcal{X}) \rightarrow E_{\infty}^{0,i} \hookrightarrow H^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/W})_K$ is, again, the edge morphism from the Leray spectral sequence for $f : \mathcal{X} \rightarrow \mathcal{S}$ in crystalline cohomology - see [M23, §1] and the references therein for details. The following is the crystalline analogue of $\mathrm{VSing}^0(f_\infty, i)$, $\mathrm{Vet}_{\mathbb{Q}_\ell}^0(f, i)$ [M23, Conj. 0.1].

$\mathrm{VCrys}^0(f, i)$ For every $t \in |\mathcal{S}|$ and $\alpha_t \in H^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/W})_{\mathbb{Q}} \subset H_{\mathrm{crys}}^{2i}(\mathcal{X}_t)$ the following properties are equivalent:

- 1) $\alpha_t \in \mathrm{im}[c_{\mathrm{crys}, t} : \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \rightarrow H_{\mathrm{crys}}^{2i}(\mathcal{X}_t)]$;
- 2) there exists $\tilde{\alpha} \in \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}}$ such that $c_{\mathrm{crys}, t}(\tilde{\alpha}|_{\mathcal{X}_t}) = \alpha_t$.

As before, let $\mathrm{VCrys}(f, i)$ denote its stabilized variant.

Also, consider the following statement

$\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f, i)$ For every $t \in |S|$, the kernel of the cycle class maps

$$c_{\mathrm{crys}, t} : \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \rightarrow H_{\mathrm{crys}}^{2i}(\mathcal{X}_t), \quad c_{\ell, t} : \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \rightarrow H^{2i}(\mathcal{X}_t, \mathbb{Q}_\ell)$$

coincide,

which follows from the standard conjecture predicting that homological and numerical equivalences should coincide, which, in turn, is a consequence of the conjecture predicting that the category of effective motives should be abelian semisimple [J92].

1.3. Statements.

1.3.1. Let now S be a smooth, geometrically connected variety over k , with generic point η , and $f : X \rightarrow S$ a smooth projective morphism. For $s \in S$, denote by a subscript $(-)_s$ the various modules attached to X_s introduced above (e.g. $V_{\mathbb{Z}_\ell, s} := H^{2i}(X_s, \mathbb{Z}_\ell(i))$, $V_{\mathbb{Z}_\ell, s}^a := \mathrm{im}[\mathrm{CH}^i(X_s)_{\mathbb{Z}_\ell} \rightarrow V_{\mathbb{Z}_\ell, s}]$ etc.). One would like to investigate how the obstruction

$$\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell, s} := |(\widetilde{C}_{\mathbb{Z}_\ell, s})_{\mathrm{tors}}|$$

to the integral Tate conjecture for X_s varies with $s \in |S|$. In particular, the vanishing of the obstruction group $(\widetilde{C}_{\mathbb{Z}_\ell, s})_{\mathrm{tors}}$ reads as $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell, s} = 1$.

Assume first $p = 0$. The main conjecture of [C23] predicts that the obstruction to the integral Tate conjecture should be uniformly bounded in fibers over points with residue field of bounded degree. More precisely, for every integer $d \geq 1$, let $|S|^{\leq d} \subset |S|$ denote the set of all closed points $s \in |S|$ with residue degree $[k(s) : k] \leq d$.

Conjecture 2. Assume $p = 0$. For every integer $d \geq 1$, one has

$$\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq d} := \sup\{\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell, s} \mid s \in |S|^{\leq d}\} < +\infty$$

and $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq d} = 1$ for $\ell \gg 0$.

Our first main result is that Conjecture 2 holds when S is a curve *modulo* the variational realization conjectures discussed in Subsection 1.2.1.

Theorem A. Assume S is a curve, $p = 0$ and $\mathrm{VSing}(f_\infty, i)$ holds for one (equivalently every) embedding $\infty : k \hookrightarrow \mathbb{C}$. Then, for every integer $d \geq 1$, one has $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq d} < +\infty$ and $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq d} = 1$ for $\ell \gg 0$ (depending on d).

Assume now $p > 0$. One has a variant of Theorem A for $d = 1$ involving the variational realization conjectures discussed in Subsection 1.2.2 but it is slightly more technical. To state it, one has to make a mild assumption on the $\mathbb{Q}_\ell$ -local system $\mathcal{V}_{\mathbb{Q}_\ell} := R^{2i}f_*\mathbb{Q}_\ell(i)$, namely that it is GLU - see Subsection 2.2.1.2 for the definition.

Theorem B. Assume S is a curve, $p > 0$, $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLU and either (i) $\mathrm{WVet}_{\mathbb{Q}_\ell}(f, i)$ or (ii) $\mathrm{VCrys}(f, i) + \mathrm{CrysEt}_{\mathbb{Q}_\ell}(f, i)$ holds. Then, one has $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq 1} < +\infty$.

Remark 3. We do not know if, under the assumptions of Theorem B, $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq 1} = 0$ for $\ell \gg 0$.

1.3.2. *An unconditional example.* From Proposition 1, one has the following special case of Theorem A and Theorem B.

Corollary 4. Assume S is a curve and $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_\eta, i)$ holds. Then,

- 1) if $p = 0$, for every integer $d \geq 1$, one has $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq d} < +\infty$ and $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq d} = 1$ for $\ell \gg 0$ (depending on d).
- 2) if $p > 0$ and $R^{2i}f_*\mathbb{Q}_\ell(i)$ is GLU, then one has $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell}^{\leq 1} < +\infty$.

To get unconditional examples, one thus only has to check $\text{Tate}_{\mathbb{Q}_\ell}(X_\eta, i)$ holds. This is for instance the case if $U = U(2n, \delta)$ is the moduli space of degree δ -smooth hypersurfaces in $\mathbb{P}^{2n+1}$, $f : X \rightarrow U$ is the universal family and $i = n$. Indeed, in that case, $V_{\mathbb{Q}_\ell}^a$ has dimension ≥ 1 as it contains the class h^n , where $h \in H^2(X_{\bar{\eta}}, \mathbb{Q}_\ell(1))$ is the class of a hyperplane section. On the other hand, if $\bar{G}_\ell$ denotes the Zariski-closure of the geometric étale fundamental group $\pi_1(S_{\bar{k}}, \bar{\eta})$ acting on $V_{\mathbb{Q}_\ell}$, Beauville proves that the $\mathbb{Q}_\ell$ -vector subspace $(V_{\mathbb{Q}_\ell})^{\bar{G}_\ell^\circ} \subset V_{\mathbb{Q}_\ell}$ of $\bar{G}_\ell^\circ$ -invariant vectors is exactly $\mathbb{Q}_\ell h^n$ [B85, Thm. 2]. From the tautological inclusions

$$\tilde{V}_{\mathbb{Q}_\ell} = (V_{\mathbb{Q}_\ell})^{G_\ell^\circ} \subset (V_{\mathbb{Q}_\ell})^{\bar{G}_\ell^\circ} = \mathbb{Q}_\ell h^n \subset V_{\mathbb{Q}_\ell}^a \subset (V_{\mathbb{Q}_\ell})^{G_\ell^\circ} = \tilde{V}_{\mathbb{Q}_\ell}$$

this ensures $\text{Tate}_{\mathbb{Q}_\ell}(X_\eta, n)$. So, if $S \rightarrow U$ is a geometrically connected curve such that $\pi_1(S_{\bar{\mathbb{Q}}}) \rightarrow \pi_1(U_{\bar{\mathbb{Q}}})$ (there exists plenty of such curves by Bertini), the base-changed family $f_S : X_S := X \times_U S \rightarrow S$ satisfies the assumptions of Corollary 4 hence, for the hypersurfaces X_s with $s \in |S|^{\leq d}$, the obstruction to $\text{Tate}_{\mathbb{Z}_\ell}(X_s, n)$ is uniformly bounded with d, ℓ and trivial for $\ell \gg 0$ (depending on d). The same argument applies with complete intersections [B85, Thm. 5].

1.3.3. Unramified cohomology. When $i = 2$, $(\tilde{C}_{\mathbb{Z}_\ell, s})_{\text{tors}}$ can be described in terms of degree 3 unramified cohomology. More precisely, set $C_{\mathbb{Z}_\ell, s} := V_{\mathbb{Z}_\ell} / V_{\mathbb{Z}_\ell, s}^a$. From the short exact sequence

$$0 \rightarrow \tilde{C}_{\mathbb{Z}_\ell, s} \rightarrow C_{\mathbb{Z}_\ell, s} \rightarrow V_{\mathbb{Z}_\ell, s} / \tilde{V}_{\mathbb{Z}_\ell, s} \rightarrow 0$$

and the fact that $V_{\mathbb{Z}_\ell, s} / \tilde{V}_{\mathbb{Z}_\ell, s}$ is torsion-free, one has $(\tilde{C}_{\mathbb{Z}_\ell, s})_{\text{tors}} = (C_{\mathbb{Z}_\ell, s})_{\text{tors}}$. If $i = 2$, [CTK13, Thm. 2.2] states that $(C_{\mathbb{Z}_\ell, s})_{\text{tors}}$ is isomorphic to

$$H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{ndiv}} \stackrel{\text{def}}{=} \text{coker}[H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{div}} \rightarrow H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))].$$

Here for an abelian group A , we let $A_{\text{div}} \subset A$ denote its maximal divisible subgroup.

Hence Theorem A and Theorem B for $i = 2$ imply:

Corollary 5. *Assume S is a curve.*

(1) *Assume $p = 0$ and $\text{VSing}(f_\infty, i)$ for some embedding $\infty : k \hookrightarrow \mathbb{C}$ holds. Then, for every integer $d \geq 1$,*

$$\sup\{|H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{ndiv}}| \mid s \in |S|^{\leq d}\} < +\infty,$$

and $H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{ndiv}} = 0$, $s \in |S|^{\leq d}$ for $\ell \gg 0$ (depending on d).

(2) *Assume $p > 0$, $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLU and either (i) $\text{WVet}_{\mathbb{Q}_\ell}(f, i)$ or (ii) $\text{VCrys}(f, i) + \text{CrysEt}_{\mathbb{Q}_\ell}(f, i)$ holds. Then,*

$$\sup\{|H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{ndiv}}| \mid s \in S(k)\} < +\infty,$$

and $H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{ndiv}} = 0$, $s \in S(k)$ for $\ell \gg 0$.

For integers $a \geq 0$, b, c and $A_\ell = \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{Q}_\ell / \mathbb{Z}_\ell$ etc., Schreieder introduces refined unramified cohomology groups $H_{c, \text{nr}}^a(X_{\bar{s}}, A_\ell(b))$ [S23, §1.2] which, when $c = 0$, coincide with the usual unramified cohomology groups. By [S23, Thm. 1.8], for every integer $i \geq 0$ one has:

$$(\tilde{C}_{\mathbb{Z}_\ell, s})_{\text{tors}} \simeq H_{i-2, \text{nr}}^{2i-1}(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(i))_{\text{ndiv}} \stackrel{\text{def}}{=} \text{coker}[H_{i-2, \text{nr}}^{2i-1}(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(i))_{\text{div}} \rightarrow H_{i-2, \text{nr}}^{2i-1}(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(i))].$$

So, Corollary 5 holds more generally with $H_{\text{nr}}^3(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))_{\text{ndiv}}$ replaced by $H_{i-2, \text{nr}}^{2i-1}(X_{\bar{s}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(i))_{\text{ndiv}}$.

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* * *

In Section 2.1 we review basic properties of cycle class maps for étale $\mathbb{Z}_\ell$ -cohomology in families, introduce the notion of $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points and describe the general strategy for the proof of Theorem A and Theorem B. In Section 3, we inject comparison with singular cohomology - Subsection 3.1, to prove Proposition 1 and conclude the proofs of Theorem A, and with crystalline cohomology - Subsection 3.2, to conclude the proof of Theorem B. In Subsection 3.1.5, we also explain how to derive from Theorem A its variant in the setting of the integral Hodge conjecture.

2. ÉTALE CYCLE CLASS MAPS IN FAMILIES AND GLOBAL STRATEGY

2.1. Étale $\mathbb{Z}_\ell$ -local systems. Let S be a smooth, geometrically connected variety over k . For every $s \in S$, fix a geometric point $\bar{s}$ over it and an étale path $\alpha_{\bar{s}} : (-)_{\bar{s}} \xrightarrow{\sim} (-)_{\bar{\eta}}$. In particular, for every $\mathbb{Z}_\ell$ -local system $\mathcal{V}_{\mathbb{Z}_\ell}$ on S , one identifies $\mathcal{V}_{\mathbb{Z}_\ell, \bar{s}} \xrightarrow{\sim} \mathcal{V}_{\mathbb{Z}_\ell, \bar{\eta}}$ equivariantly with respect to the isomorphism of étale fundamental groups $\pi_1(S, \bar{s}) \xrightarrow{\sim} \pi_1(S, \bar{\eta})$, $\gamma \mapsto \alpha_{\bar{s}} \gamma \alpha_{\bar{s}}^{-1}$. As a result, we will in general omit fiber functors from our notation and simply write

$$V_{\mathbb{Z}_\ell} := \mathcal{V}_{\mathbb{Z}_\ell, \bar{s}} \xrightarrow{\sim} \mathcal{V}_{\mathbb{Z}_\ell, \bar{\eta}}, \quad V_{\mathbb{Q}_\ell} := V_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Let $f : X \rightarrow S$ be a smooth projective morphism.

2.1.1. Notational conventions. Consider the $\mathbb{Z}_\ell$ -étale local system $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i} f_* \mathbb{Z}_\ell(i)$ on S . Let $G_\ell \subset \mathrm{GL}(V_{\mathbb{Q}_\ell})$ denote the Zariski-closure of the image of $\pi_1(S)$ acting on $V_{\mathbb{Q}_\ell}$; let also $\overline{G}_\ell \subset G_\ell$ and, for every $s \in S$, $G_{\ell, s} \subset G_\ell$ denote the Zariski closure of the images of $\pi_1(S_{\bar{k}})$ and $\pi_1(s)$ acting on $V_{\mathbb{Q}_\ell}$ by restriction along the functorial morphisms $\pi_1(S_{\bar{k}}) \rightarrow \pi_1(S)$ and $\pi_1(s) \rightarrow \pi_1(S)$ respectively (in particular $G_{\ell, \eta} = G_\ell$). As S is geometrically connected over k , the functorial sequence

$$1 \rightarrow \pi_1(S_{\bar{k}}) \rightarrow \pi_1(S) \rightarrow \pi_1(k) \rightarrow 1$$

is exact, hence $\overline{G}_\ell \subset G_\ell$ is a normal subgroup, and for every closed point $s \in |S|$, one has $G_\ell^\circ = \overline{G}_\ell^\circ G_{\ell, s}^\circ$.

2.1.2. Specialization and extension of algebraically closed fields. We recall the following two properties of the cycle class map for étale $\mathbb{Z}_\ell$ -cohomology.

2.1.2.1. Compatibility with specialization of algebraic cycles. For every $s \in S$, one has a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^i(X_{\bar{k}}) & \xrightarrow{|X_{\bar{\eta}}|} & \mathrm{CH}^i(X_{\bar{\eta}}) \\ |X_{\bar{s}}| \downarrow & \swarrow \mathrm{sp}_{\eta, s} & \downarrow c_{\ell, \eta} \\ \mathrm{CH}^i(X_{\bar{s}}) & \xrightarrow{c_{\ell, s}} & V_{\mathbb{Z}_\ell} \end{array}$$

(see [F98, § 20.3, Ex. 20.3.1 and 20.3.5]).

2.1.2.2. "Invariance" under extension of algebraically closed field. Let $\Omega \hookrightarrow \Omega'$ be an extension of algebraically closed fields of characteristic $\neq \ell$ and let Y be a smooth proper variety over Ω . Consider the canonical commutative square

$$\begin{array}{ccc} \mathrm{CH}^i(Y) & \xrightarrow{c_\ell} & \mathrm{H}^{2i}(Y, \mathbb{Z}_\ell(i)) \\ |Y_{\Omega'}| \downarrow & & \downarrow \simeq \\ \mathrm{CH}^i(Y_{\Omega'}) & \xrightarrow{c_\ell} & \mathrm{H}^{2i}(Y_{\Omega'}, \mathbb{Z}_\ell(i)). \end{array}$$

Then¹,

$$\mathrm{im}[c_\ell \circ -|Y_{\Omega'}|] : \mathrm{CH}^i(Y) \rightarrow \mathrm{H}^{2i}(Y_{\Omega'}, \mathbb{Z}_\ell(i)) = \mathrm{im}[c_\ell : \mathrm{CH}^i(Y_{\Omega'}) \rightarrow \mathrm{H}^{2i}(Y_{\Omega'}, \mathbb{Z}_\ell(i))].$$

In particular, $V_{\mathbb{Z}_\ell, s}^a, V_{\mathbb{Z}_\ell, s}^{\mathrm{free}, a}$ etc. are independent of the geometric point $\bar{s}$ over s .

2.1.3. The lattice $\Lambda_{\mathbb{Z}_\ell}$. For every $s \in S$, define

$$\Lambda_{\mathbb{Z}_\ell, s} := \mathrm{im}[\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \rightarrow \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Z}_\ell} \xrightarrow{c_{\ell, s}} V_{\mathbb{Z}_\ell}^{\mathrm{free}}] \subset V_{\mathbb{Z}_\ell}^{\mathrm{free}}.$$

By construction and 2.1.2, one has

$$\Lambda_{\mathbb{Z}_\ell, s} \subset V_{\mathbb{Z}_\ell, \eta}^{\mathrm{free}, a} \subset V_{\mathbb{Z}_\ell, s}^{\mathrm{free}, a} \subset V_{\mathbb{Z}_\ell}^{\mathrm{free}}.$$

Lemma 6. *The lattice $\Lambda_{\mathbb{Z}_\ell} := \Lambda_{\mathbb{Z}_\ell, s} \subset V_{\mathbb{Z}_\ell}^{\mathrm{free}}$ is independent of s (modulo the identifications $V_{\mathbb{Z}_\ell} = \mathcal{V}_{\mathbb{Z}_\ell, \bar{s}} \simeq \mathcal{V}_{\mathbb{Z}_\ell, \bar{\eta}}$).*

¹In fact, a cycle $\xi \in \mathrm{CH}^i(Y_{\Omega'})$ is defined over a finitely generated algebraically closed field $\Omega'' \subset \Omega'$. One could then find a smooth and proper model of Y over a small affine scheme U over Ω with generic point Ω'' and use the specialization at a Ω -point of U , as in 2.1.2.1.

Proof. This follows from the fact that the restriction morphism $H^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i)) \rightarrow H^{2i}(X_{\bar{s}}, \mathbb{Z}_\ell(i)) = V_{\mathbb{Z}_\ell}$ factors through the edge morphism $\epsilon : H^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i)) \rightarrow E_\infty^{0,i} \hookrightarrow E_2^{0,i} = H^0(S_\infty, R^{2i}f_*\mathbb{Z}_\ell(i))$ of the Leray spectral sequence for $f : X \rightarrow S$ as

$$\begin{array}{ccc} \mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} & \xrightarrow{|_{X_{\bar{s}}}} & \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Z}_\ell} \\ \downarrow c_\ell & & \downarrow c_{\ell,s} \\ H^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i)) & \xrightarrow{\epsilon} H^0(S_{\bar{k}}, R^{2i}f_*\mathbb{Z}_\ell(i)) \xrightarrow{(-)_{\bar{s}}} & V_{\mathbb{Z}_\ell}^{\mathrm{free}} \end{array}$$

and the fact the embedding

$$V_{\mathbb{Z}_\ell}^{\mathrm{free}} \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} = \mathrm{im}[H^0(S_{\bar{k}}, R^{2i}f_*\mathbb{Z}_\ell(i)) \xrightarrow{(-)_{\bar{s}}} V_{\mathbb{Z}_\ell}^{\mathrm{free}}] \subset V_{\mathbb{Z}_\ell}^{\mathrm{free}}$$

is independent of s (*modulo* the identifications $V_{\mathbb{Z}_\ell} = \mathcal{V}_{\mathbb{Z}_\ell, \bar{s}} \simeq \mathcal{V}_{\mathbb{Z}_\ell, \bar{\eta}}$). $\square$

Remark 7. Assume² there exists a smooth compactification $X \hookrightarrow X^{\mathrm{cpt}}$. Then the surjectivity of the restriction morphism $\mathrm{CH}^i(X_k^{\mathrm{cpt}}) \rightarrow \mathrm{CH}^i(X_{\bar{k}})$ and the functoriality of cycle class maps shows that $\Lambda_{\mathbb{Z}_\ell}$ can also be described as

$$\Lambda_{\mathbb{Z}_\ell} = \mathrm{im}[\mathrm{CH}^i(X_k^{\mathrm{cpt}})_{\mathbb{Z}_\ell} \xrightarrow{c_\ell} H^{2i}(X_k^{\mathrm{cpt}}, \mathbb{Z}_\ell(i)) \rightarrow H^{2i}(X_s^{\mathrm{cpt}}, \mathbb{Z}_\ell(i)) \rightarrow V_{\mathbb{Z}_\ell}^{\mathrm{free}}].$$

In particular, if $\bar{k} \hookrightarrow \Omega$ is an extension of algebraically closed fields and s_Ω a geometric point on S_Ω over $\bar{s}$, then 2.1.2.2 shows that

$$\Lambda_{\mathbb{Z}_\ell} = \mathrm{im}[\mathrm{CH}^i(X_\Omega)_{\mathbb{Z}_\ell} \rightarrow \mathrm{CH}^i(X_{s_\Omega})_{\mathbb{Z}_\ell} \xrightarrow{c_{\ell, s_\Omega}} V_{\mathbb{Z}_\ell}^{\mathrm{free}}].$$

2.2. Strategy for the proof of Theorem A and Theorem B. We retain the notation and conventions of Subsection 1.3 and Subsection 2.1.1. For every $s \in S$, set

$$\mathrm{Ob}_{\mathbb{Z}_\ell, s}^{\mathrm{free}} := |(C_{\mathbb{Z}_\ell, s}^{\mathrm{free}})_{\mathrm{tors}}|.$$

As

$$\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell, s} \leq |(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}| \mathrm{Ob}_{\mathbb{Z}_\ell, s}^{\mathrm{free}}$$

and as $(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}$ is independent of $s \in S$ and, if³ $p = 0$, $(V_{\mathbb{Z}_\ell})_{\mathrm{tors}} = 0$, $\ell \gg 0$, it is enough to prove Theorem A, Theorem B for $\mathrm{Ob}_{\mathbb{Z}_\ell, s}^{\mathrm{free}}$ instead of $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell, s}$.

2.2.1. $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points. The proofs of Theorem A and Theorem B are parallel and follow from the combination of two independent statements involving $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points. Let $\mathcal{V}_{\mathbb{Z}_\ell}$ be a $\mathbb{Z}_\ell$ -local system on S .

2.2.1.1. $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points. Define the sets of closed $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points to be the subset $|S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\mathrm{gen}} \subset |S|$ of all $s \in |S|$ satisfying the following equivalent conditions

$$G_{\ell, s}^\circ = G_\ell^\circ \Leftrightarrow G_{\ell, s}^\circ \supset G_\ell^\circ \Leftrightarrow G_{\ell, s}^\circ \supset \overline{G}_\ell^\circ,$$

and let $|S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\mathrm{ngen}} := |S| \setminus |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\mathrm{gen}} \subset |S|$ be the subset of closed non- $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points. Note that $|S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\mathrm{gen}}$ is contained in the set of all $s \in |S|$ such that $V_{\mathbb{Q}_\ell, s}^a \subset (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell^\circ}$.

2.2.1.2. Sparsity. Under mild assumptions one expects non- $\mathcal{V}_{\mathbb{Q}_\ell}$ -generic points to be sparse - see [C23] for details. When S is a curve, one has the following unconditional results. Let $\overline{\Pi}_\ell$ denote the image of $\pi_1(S_{\bar{k}})$ acting on $V_{\mathbb{Q}_\ell}$ and, if $p > 0$, let $\overline{\Pi}_\ell^+ (\supset \overline{\Pi}_\ell)$ denote the image of $\pi_1(S_{k\overline{\mathbb{F}}_p})$ acting on $V_{\mathbb{Q}_\ell}$; these are ℓ -adic Lie groups. One says that $\mathcal{V}_{\mathbb{Q}_\ell}$ is:

- GLP (geometrically Lie perfect) if $\mathrm{Lie}(\overline{\Pi}_\ell)$ is a perfect Lie algebra *viz* one has $[\mathrm{Lie}(\overline{\Pi}_\ell), \mathrm{Lie}(\overline{\Pi}_\ell)] = 0$;
- and, if $p > 0$, GLU (geometrically Lie unrelated) if $\mathrm{Lie}(\overline{\Pi}_\ell)$ and $\mathrm{Lie}(\overline{\Pi}_\ell^+)$ have no non-trivial common quotient.

Fact A. ([CT13, Thm. 1]). *Assume $p = 0$, S is a curve and $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLP. Then for every integer $d \geq 1$, the set $|S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\mathrm{ngen}} \cap |S|^{\leq d}$ is finite.*

²If $p = 0$, this is always the case - see [Na62], [Na63], [Hi64].

³This follows from Artin's comparison - see Subsection 3.1.2 and the fact that singular cohomology groups are finitely generated. This is also true if $p > 0$ [G83] but we will not resort to this fact.

Fact B. ([T24]; see also the discussion in [A23, 1.7.1]). Assume $p > 0$, S is a curve and $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLU. Then the set $|S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}} \cap S(k)$ is finite.

The $\mathbb{Z}_\ell$ -local system $\mathcal{V}_\ell = R^{2i}f_*\mathbb{Q}_\ell(i)$ is GLP [D71], [D80]. If $p > 0$, it is not necessarily GLU but still, it is *e.g.* if $\bar{\Pi}_\ell$ is open in the derived subgroup of the image of $\pi_1(S_{\bar{k}})$ acting on $V_{\mathbb{Q}_\ell}$ - see [A23, Rem. 1.7.1.4] for details.

2.2.2. *The main Lemmas.* Fact A immediately reduce the proof of Theorem A to the proof of:

Lemma A. Set $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i}f_*\mathbb{Z}_\ell(i)$. Assume $p = 0$ and $\text{VSing}(f_\infty, i)$ holds for some (equivalently every) embedding $\infty : k \hookrightarrow \mathbb{C}$. Then,

$$\text{Ob}_{\mathbb{Z}_\ell}^{\text{free, gen}} := \sup\{\text{Ob}_{\mathbb{Z}_\ell, s}^{\text{free}} \mid s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}\} < +\infty,$$

and $\text{Ob}_{\mathbb{Z}_\ell}^{\text{free, gen}} = 1$ for $\ell \gg 0$.

The proof of Lemma A will be carried out in Section 3.1.4.

Similarly, Fact B immediately reduces the proof of Theorem B to the proof of:

Lemma B. Set $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i}f_*\mathbb{Z}_\ell(i)$. Assume $p > 0$ and either (i) $\text{WVet}_{\mathbb{Q}_\ell}(f, i)$ or (ii) $\text{VCrys}(f, i) + \text{CrysEt}_{\mathbb{Q}_\ell}(f, i)$ holds. Then, $\text{Ob}_{\mathbb{Z}_\ell}^{\text{free, gen}} < +\infty$.

The proof of Lemma B will be carried out in Section 3.2.2.

Note that Lemma A and Lemma B do not involve any restriction on the dimension of S nor on the degree of the residue field $k(s)$ for $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$.

Remark 8. *A priori*, the assumptions in Lemma A, Lemma B do not imply $\text{Tate}_{\mathbb{Q}_\ell}(X_s, i)$, $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$. However, if one assumes $\text{Tate}_{\mathbb{Q}_\ell}(X_{s_0}, i)$ holds for some $s_0 \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$ then these assumptions indeed imply $\text{Tate}_{\mathbb{Q}_\ell}(X_s, i)$, $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$. Indeed, the proofs of Lemma A, Lemma B will show these assumptions imply $\Lambda_{\mathbb{Q}_\ell} = V_{\mathbb{Q}_\ell, s}^a$, $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$, where $\Lambda_{\mathbb{Q}_\ell} = \Lambda_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Assume furthermore $\text{Tate}_{\mathbb{Q}_\ell}(X_{s_0}, i)$ holds - that is $V_{\mathbb{Q}_\ell, s_0}^a = \tilde{V}_{\mathbb{Q}_\ell, s_0}$, for some $s_0 \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$. But then, for every $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$, one has

$$V_{\mathbb{Q}_\ell, s}^a = \Lambda_{\mathbb{Q}_\ell} = V_{\mathbb{Q}_\ell, s_0}^a = \tilde{V}_{\mathbb{Q}_\ell, s_0} \stackrel{(\alpha)}{=} \tilde{V}_{\mathbb{Q}_\ell, s},$$

where (α) follows from $s_0 \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$.

2.2.3. *Reduction to connected monodromy groups.* To bound $\text{Ob}_{\mathbb{Z}_\ell, s}^{\text{free}}$ uniformly for $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$, one can freely replace $f : X \rightarrow S$ by a base change along a finite cover $\pi : S' \rightarrow S$ of connected smooth varieties over k . Indeed, consider the base-change diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & \square & \downarrow f \\ S' & \longrightarrow & S \end{array}$$

and write $\mathcal{V}'_{\mathbb{Z}_\ell} := R^{2i}f'_*\mathbb{Z}_\ell(i)$. For $s \in |S|$ and $s' \in |S'|$ over $s \in |S|$, let $\bar{s}'$ be a geometric point over s' and let $\bar{s} = \pi \circ \bar{s}'$ denote its image on S . Then, $X'_{\bar{s}'} \xrightarrow{\sim} X_{\bar{s}}$ as $\bar{k}$ -schemes hence, *a fortiori*, $\text{CH}^i(X'_{\bar{s}'}) \xrightarrow{\sim} \text{CH}^i(X_{\bar{s}})$. On the other hand, by proper base change, $\mathcal{V}'_{\mathbb{Z}_\ell} = \pi^*\mathcal{V}_{\mathbb{Z}_\ell}$ hence, one gets a canonical commutative square

$$\begin{array}{ccc} \text{CH}^i(X_{\bar{s}}) & \xrightarrow{c_{\ell, s}} & \text{H}^{2i}(X_{\bar{s}}, \mathbb{Z}_\ell(i)) \\ \simeq \uparrow & & \parallel \\ \text{CH}^i(X'_{\bar{s}'}) & \xrightarrow{c_{\ell, s'}} & \text{H}^{2i}(X'_{\bar{s}'}, \mathbb{Z}_\ell(i)) \end{array}$$

where the vertical arrows are isomorphisms and the right vertical one is equivariant with respect to the functorial morphism $\pi_1(S') \hookrightarrow \pi_1(S)$. In particular, as $\pi_1(S') \hookrightarrow \pi_1(S)$ is open, one has $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{ngen}}$ if and only if $s' \in |S'|_{\mathcal{V}'_{\mathbb{Q}_\ell}}^{\text{ngen}}$.

After base change along a finite cover $S' \rightarrow S$ of smooth varieties (which, working componentwise, we may assume to be connected and, replacing k by a finite field extension, geometrically connected over k), one may assume $\text{VSing}^0(f'_\infty, i)$ (resp. $\text{WVet}_{\mathbb{Q}_\ell}^0(f', i)$, resp. $\text{VCrys}^0(f', i)$) holds for every base change along a finite cover $S'_\infty \rightarrow S_\infty$ (resp. $S' \rightarrow S$, resp. $S' \rightarrow S$) of smooth varieties. Then, the assumptions and conclusions of Theorem A and Theorem B become unchanged by base change along finite covers of smooth varieties, so that one may assume:

- a) the algebraic group $\overline{G}_\ell$ is connected⁴;
- b) the algebraic groups $G_{\ell,s}$, $s \in S$ are all connected⁵.

2.2.4. An elementary lemma. Recall that for every $s \in S$, we identify $V_{\mathbb{Z}_\ell} := \mathcal{V}_{\mathbb{Z}_\ell, \bar{s}} \xrightarrow{\sim} \mathcal{V}_{\mathbb{Z}_\ell, \bar{\eta}}$. For a subset $\Sigma \subset S$, set

$$V_{\mathbb{Z}_\ell, \Sigma}^{\text{free}, a} := \bigcap_{s \in \Sigma} V_{\mathbb{Z}_\ell, s}^{\text{free}, a} \subset V_{\mathbb{Z}_\ell, s}^{\text{free}, a} \subset V_{\mathbb{Z}_\ell}^{\text{free}}.$$

Lemma 9. *For every $\mathbb{Z}_\ell$ -submodule $T_{\mathbb{Z}_\ell} \subset V_{\mathbb{Z}_\ell, \Sigma}^{\text{free}, a}$ and for every $s \in \Sigma$, one has the following implications*

$$T_{\mathbb{Q}_\ell} = V_{\mathbb{Q}_\ell, s}^a \iff [V_{\mathbb{Z}_\ell, s}^{\text{free}, a} : T_{\mathbb{Z}_\ell}] < +\infty \implies \text{Ob}_{\mathbb{Z}_\ell, s}^{\text{free}} \leq c(T_{\mathbb{Z}_\ell}) := |(V_{\mathbb{Z}_\ell}^{\text{free}}/T_{\mathbb{Z}_\ell})_{\text{tors}}|.$$

Proof. The first equivalence is straightforward. The second implication follows from the canonical commutative diagram of short exact sequences

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & T_{\mathbb{Z}_\ell} & \longrightarrow & V_{\mathbb{Z}_\ell}^{\text{free}} & \longrightarrow & V_{\mathbb{Z}_\ell}^{\text{free}}/T_{\mathbb{Z}_\ell} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & V_{\mathbb{Z}_\ell, s}^{\text{free}, a} & \longrightarrow & V_{\mathbb{Z}_\ell}^{\text{free}} & \longrightarrow & C_{\mathbb{Z}_\ell, s}^{\text{free}} \longrightarrow 0 \end{array}$$

which, by the snake lemma, identifies

$$K_{\mathbb{Z}_\ell, s} := \text{coker}[T_{\mathbb{Z}_\ell} \hookrightarrow V_{\mathbb{Z}_\ell, s}^{\text{free}, a}] \xrightarrow{\sim} \ker[V_{\mathbb{Z}_\ell}^{\text{free}}/T_{\mathbb{Z}_\ell} \twoheadrightarrow C_{\mathbb{Z}_\ell, s}^{\text{free}}] =: K_{\mathbb{Z}_\ell, s}.$$

But if $K_{\mathbb{Z}_\ell, s}$ is finite, one gets a short exact sequence

$$0 \rightarrow K_{\mathbb{Z}_\ell, s} \rightarrow (V_{\mathbb{Z}_\ell}^{\text{free}}/T_{\mathbb{Z}_\ell})_{\text{tors}} \rightarrow (C_{\mathbb{Z}_\ell, s}^{\text{free}})_{\text{tors}} \rightarrow 0,$$

whence the assertion. $\square$

Lemma 9 reduces the proof of Lemma A and Lemma B to finding a $\mathbb{Z}_\ell$ -submodule $T_{\mathbb{Z}_\ell} \subset V_{\mathbb{Z}_\ell, \Sigma}^{\text{free}, a}$ such that $T_{\mathbb{Q}_\ell} = V_{\mathbb{Q}_\ell, s}^a$, $s \in \Sigma = |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{gen}}$ and, in the setting of of Lemma A, such that $c(T_{\mathbb{Z}_\ell}) = 0$, $\ell \gg 0$. In all cases, we will consider the $\mathbb{Z}_\ell$ -submodule $T_{\mathbb{Z}_\ell} := \Lambda_{\mathbb{Z}_\ell}$ introduced in Subsection 2.1.3, Lemma 6. As a warm-up, we end this Section with the proof of Lemma B (i).

2.2.5. Proof of Lemma B (i). Let $s \in \Sigma = |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{gen}}$. Assuming $\text{WVet}_{\mathbb{Q}_\ell}(f, i)$, we are to prove that the inclusion $\Lambda_{\mathbb{Q}_\ell} \subset V_{\mathbb{Q}_\ell, s}^a$ is an equality. This follows from the inclusions

$$V_{\mathbb{Q}_\ell, s}^a = V_{\mathbb{Q}_\ell, s}^a \cap \tilde{V}_{\mathbb{Q}_\ell, s} \stackrel{(\alpha)}{=} V_{\mathbb{Q}_\ell, s}^a \cap \tilde{V}_{\mathbb{Q}_\ell, \eta} \stackrel{(\beta)}{\subset} V_{\mathbb{Q}_\ell, s}^a \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} \stackrel{(\gamma)}{=} \Lambda_{\mathbb{Q}_\ell} \subset V_{\mathbb{Q}_\ell},$$

where (α) follows from $s \in |S|_{\mathcal{V}_{\mathbb{Q}_\ell}}^{\text{gen}}$, (β) from the reduction 2.2.3 a), and (γ) is $\text{WVet}_{\mathbb{Q}_\ell}(f, i)$.

3. COMPARISON WITH SINGULAR AND CRYSTALLINE COHOMOLOGIES

3.1. Singular cohomology.

⁴ First, after replacing k by a finite field extension, one may assume $S(k) \neq \emptyset$, so that fixing $s \in S(k)$ yields a splitting $s : \pi_1(s) = \pi_1(k) \hookrightarrow \pi_1(S)$ of the canonical short exact sequence

$$1 \rightarrow \pi_1(S_{\bar{k}}) \rightarrow \pi_1(S) \rightarrow \pi_1(k) \rightarrow 1$$

and a well-defined action by conjugacy of $\pi_1(k)$ on $\pi_1(S)$. Then, let $S'_k \rightarrow S_k$ denote the connected étale cover corresponding to $\ker(\pi_1(S_k) \rightarrow \pi_0(\overline{G}_\ell))$. As $\overline{G}_\ell$ is normal in G_ℓ , the $\pi_1(k)$ -action stabilizes $\pi_1(S'_k)$ hence $s(\pi_1(k))\pi_1(S'_k) \subset \pi_1(S)$ is an open subgroup corresponding to a connected étale cover $S' \rightarrow S$ which, by construction, has the requested property.

⁵After base-change along the connected étale cover $S' \rightarrow S$ trivializing $\mathcal{V}_\ell/\tilde{\ell}$ (with $\tilde{\ell} = 4$ if $\ell = 2$ and $\tilde{\ell} = \ell$ if $\ell \neq 2$, this classically follows from the Chebotarev density theorem, using Frobenius tori.

3.1.1. *Singular $\mathbb{Z}$ -local systems.* Let S_∞ be a connected variety smooth over $\mathbb{C}$. For every $s_{0\infty}, s_\infty \in S_\infty(\mathbb{C}) = S_\infty^{\text{an}}$, fix a topological path $s_\infty \rightarrow s_{0\infty}$, inducing an isomorphism of fiber functors $\alpha_{s_\infty} : (-)_{s_\infty} \xrightarrow{\sim} (-)_{s_{0\infty}}$. In particular, for every singular $\mathbb{Z}$ -local system $\mathcal{V}_\mathbb{Z}$ on S_∞^{an} , one identifies $\mathcal{V}_{\infty, \mathbb{Z}, s_\infty} \xrightarrow{\sim} \mathcal{V}_{\infty, \mathbb{Z}, s_{0\infty}}$ equivariantly with respect to the isomorphism of topological fundamental groups $\pi_1^{\text{top}}(S_\infty^{\text{an}}, s_\infty) \xrightarrow{\sim} \pi_1^{\text{top}}(S_\infty^{\text{an}}, s_{0\infty})$, $\gamma \mapsto \alpha_{s_\infty} \gamma \alpha_{s_\infty}^{-1}$. So that we will in general omit fiber functors from our notation and simply write

$$V_\mathbb{Z} := \mathcal{V}_{\mathbb{Z}, s_\infty} \xrightarrow{\sim} \mathcal{V}_{\mathbb{Z}, s_{0\infty}}.$$

Let $f_\infty : X_\infty \rightarrow S_\infty$ be a smooth projective morphism. The singular $\mathbb{Z}$ -local system $\mathcal{V}_\mathbb{Z} := R^{2i} f_\infty^{\text{an}} \mathbb{Z}(i)$ on S_∞^{an} underlies a polarizable $\mathbb{Z}$ -variation of Hodge structure. Let $G \subset \text{GL}(V_\mathbb{Q})$ denote the generic Mumford-Tate group of $\mathcal{V}_\mathbb{Q} := \mathcal{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$, and for every $s_\infty \in S_\infty(\mathbb{C})$, let $G_{s_\infty} \subset G$ denote the Mumford-Tate group of the polarizable $\mathbb{Q}$ -Hodge structure $s_\infty^* \mathcal{V}_\mathbb{Q}$. Let also $\overline{G} \subset \text{GL}(V_\mathbb{Q})$ denote the Zariski-closure of the image of $\pi_1^{\text{top}}(S_\infty^{\text{an}})$ acting on $V_\mathbb{Q}$. By the fixed part theorem, $\overline{G}^\circ$ a normal closed subgroup of G and, for every $s_\infty \in S_\infty(\mathbb{C})$, one has $G = \overline{G}^\circ G_{s_\infty}$.

As in Subsection 2.1.3, for every $s_\infty \in S_\infty(\mathbb{C})$ set

$$\Lambda_{\mathbb{Z}, s_\infty} := \text{im}[\text{CH}^i(X_\infty) \rightarrow \text{CH}^i(X_{s_\infty}) \xrightarrow{c_{s_\infty}} V_{\mathbb{Z}}^{\text{free}}] \subset V_{\mathbb{Z}}^{\text{free}}.$$

The same argument as in the proof of Lemma 6 (using Leray spectral sequence for singular cohomology) shows that $\Lambda_\mathbb{Z} := \Lambda_{\mathbb{Z}, s_\infty}$ is independent of $s_\infty \in S_\infty(\mathbb{C})$.

3.1.2. *Artin's comparison.* Assume $p = 0$ and fix an embedding $\infty : k \hookrightarrow \mathbb{C}$. Recall that $(-)_{\infty}$ denotes the base-change functor along $\text{Spec}(\mathbb{C}) \xrightarrow{\infty} \text{Spec}(k)$ and $(-)_{\text{an}}$ the analytification functor from varieties over $\mathbb{C}$ to complex analytic spaces. Let S be a geometrically connected, smooth variety over k . For every $s_\infty \in S_\infty(\mathbb{C})$ over $s \in S$ let $k(\bar{s}) \subset \mathbb{C}$ denote the algebraic closure of $k(s)$ determined by $k(s) \hookrightarrow \mathbb{C}$ and let $\bar{s}$ denote the corresponding geometric point over s . Let $f : X \rightarrow S$ be a smooth projective morphism. The local systems $\mathcal{V}_\mathbb{Z} := R^{2i} f_\infty^{\text{an}} \mathbb{Z}(i)$ on S_∞^{an} and $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i} f_{\infty, \ell}^{\text{an}} \mathbb{Z}_\ell(i)$ on S are related by Artin's comparison isomorphism [SGA4, XI]

$$(3) \quad \mathcal{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_\ell \xrightarrow{\sim} \mathcal{V}_{\mathbb{Z}_\ell}^{\text{an}},$$

where we write $\mathcal{V}_{\mathbb{Z}_\ell}^{\text{an}}$ for the pull-back of $\mathcal{V}_{\mathbb{Z}_\ell}$ along⁶ the morphisms of sites $(X_\infty^{\text{an}})_{\text{an}} \rightarrow X_{\infty, \text{et}} \rightarrow X_{\text{et}}$. Equivalently, for every $s_\infty \in S_\infty(\mathbb{C})$ over $s \in |S|$, one has a canonical isomorphism of $\mathbb{Z}_\ell$ -modules

$$(6) \quad V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_\ell = \mathcal{V}_{\mathbb{Z}, s_\infty} \otimes_\mathbb{Z} \mathbb{Z}_\ell \xrightarrow{\sim} \mathcal{V}_{\mathbb{Z}_\ell, \bar{s}} = V_{\mathbb{Z}_\ell}, \quad V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}_\ell \xrightarrow{\sim} V_{\mathbb{Q}_\ell},$$

which is equivariant with respect to the profinite completion morphism composed with the GAGA isomorphism and the projection

$$\pi_1^{\text{top}}(S_\infty^{\text{an}}) \rightarrow \pi_1^{\text{top}}(S_\infty^{\text{an}})^\wedge \xrightarrow{\sim} \pi_1(S_\infty) \xrightarrow{\sim} \pi_1(S_{\bar{k}}) \hookrightarrow \pi_1(S).$$

In particular, $\overline{G} \subset \text{GL}(V_\mathbb{Q})$ identifies, *modulo* (6), with the scalar extension $\overline{G}_{\mathbb{Q}_\ell} \subset \text{GL}(V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}_\ell)$ of $\overline{G} \subset \text{GL}(V_\mathbb{Q})$.

Artin's comparison isomorphism is compatible with cycle class maps on both sides. Namely, for every $s_\infty \in S_\infty(\mathbb{C})$ over $s \in S$ one has a canonical commutative diagram

$$\begin{array}{ccccc} \text{CH}^i(X_{\bar{k}}) & \xrightarrow{|X_{\bar{s}}|} & \text{CH}^i(X_{\bar{s}}) & \xrightarrow{c_{\ell, s}} & V_{\mathbb{Z}_\ell}^{\text{free}} \\ |X_\infty| \downarrow & & |X_{s_\infty}| \downarrow & & \uparrow \\ \text{CH}^i(X_\infty) & \xrightarrow{|X_{s_\infty}|} & \text{CH}^i(X_{s_\infty}) & \xrightarrow{c_{s_\infty}} & V_\mathbb{Z}^{\text{free}} \xrightarrow{-\otimes_\mathbb{Z} \mathbb{Z}_\ell} V_{\mathbb{Z}_\ell}^{\text{free}} \otimes_\mathbb{Z} \mathbb{Z}_\ell. \end{array} \quad \begin{array}{c} \swarrow (6) \\ \simeq \end{array}$$

As a result, we will identify subgroups of $V_\mathbb{Z}^{\text{free}}$ (e.g. $\Lambda_\mathbb{Z}$, $V_{\mathbb{Z}, s_\infty}^{\text{free}, a}$ etc.) with their image in $V_{\mathbb{Z}_\ell}^{\text{free}}$. Set

$$\Lambda_{\ell, \mathbb{Z}} := \text{im}[\text{CH}^i(X_{\bar{k}}) \rightarrow \text{CH}^i(X_{\bar{s}}) \xrightarrow{c_{\ell, s}} V_{\mathbb{Z}_\ell}^{\text{free}}] \subset V_{\ell, \mathbb{Z}, s}^{\text{free}, a} := \text{im}[\text{CH}^i(X_{\bar{s}}) \xrightarrow{c_{\ell, s}} V_{\mathbb{Z}_\ell}^{\text{free}}].$$

Then, from 2.1.2.2 and Remark 7 applied to $\bar{k} \hookrightarrow \mathbb{C}$, one has

$$\Lambda_\mathbb{Z} = \Lambda_{\ell, \mathbb{Z}}, \quad V_{\mathbb{Z}, s_\infty}^{\text{free}, a} = V_{\ell, \mathbb{Z}, s}^{\text{free}, a},$$

⁶More precisely, write $\mathcal{V}_{\mathbb{Z}_\ell} = \lim_n \mathcal{V}_{\mathbb{Z}/\ell^n}$ as a limit of $\mathbb{Z}/\ell^n$ -local systems and define the analytification of $\mathcal{V}_{\mathbb{Z}_\ell}$ as $(\mathcal{V}_{\mathbb{Z}_\ell})^{\text{an}} := \lim_n \mathcal{V}_{\mathbb{Z}/\ell^n}|_{(X_\infty^{\text{an}})_{\text{an}}}$.

hence

$$(7) \quad \Lambda_{\ell, \mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \xrightarrow{\sim} \Lambda_{\mathbb{Z}_{\ell}}, \quad V_{\ell, \mathbb{Z}, s}^{\text{free}, a} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \xrightarrow{\sim} V_{\mathbb{Z}_{\ell}, s}^{\text{free}, a}.$$

3.1.3. *Proof of Proposition 1.* For every $s \in S$, write

$$\Lambda_{\ell, \mathbb{Q}} = \text{im}[\text{CH}^i(X_{\bar{k}})_{\mathbb{Q}} \rightarrow \text{CH}^i(X_{\bar{s}})_{\mathbb{Q}} \xrightarrow{c_{\ell, s}^{\ell, s}} V_{\mathbb{Q}_{\ell}}] \subset V_{\ell, \mathbb{Q}, s}^a := \text{im}[\text{CH}^i(X_{\bar{s}})_{\mathbb{Q}} \xrightarrow{c_{\ell, s}^{\ell, s}} V_{\mathbb{Q}_{\ell}}] \subset V_{\mathbb{Q}_{\ell}, s}^a,$$

$$\Lambda_{\mathbb{Q}_{\ell}} = \text{im}[\text{CH}^i(X_{\bar{k}})_{\mathbb{Q}_{\ell}} \rightarrow \text{CH}^i(X_{\bar{s}})_{\mathbb{Q}_{\ell}} \xrightarrow{c_{\ell, s}^{\ell, s}} V_{\mathbb{Q}_{\ell}}].$$

If $p = 0$, fix an embedding $\infty : k \hookrightarrow \mathbb{C}$ and, for every $s_{\infty} \in S_{\infty}(\mathbb{C})$, write

$$\Lambda_{\mathbb{Q}} = \text{im}[\text{CH}^i(X_{\infty})_{\mathbb{Q}} \rightarrow \text{CH}^i(X_{s_{\infty}})_{\mathbb{Q}} \xrightarrow{c_{s_{\infty}}^{\ell, s}} V_{\mathbb{Q}}] \subset V_{\mathbb{Q}, s_{\infty}}^a.$$

Recall from Subsection 3.1.1 and Subsection 2.1.3 that $\Lambda_{\mathbb{Q}}$ is independent of s_{∞} and $\Lambda_{\ell, \mathbb{Q}}$, $\Lambda_{\mathbb{Q}_{\ell}}$ are independent of s (as the notation suggests) and, if $p = 0$, from Subsection 3.1.2, that $\Lambda_{\ell, \mathbb{Q}} = \Lambda_{\mathbb{Q}}$.

With these notation, $\text{VSing}^0(f_{\infty}, i)$, $\text{Vet}_{\mathbb{Q}_{\ell}}^0(f, i)$ and $\text{WVet}_{\mathbb{Q}_{\ell}}^0(f, i)$ can be reformulated as

$$\begin{aligned} \text{VSing}^0(f_{\infty}, i) &= V_{\mathbb{Q}, s_{\infty}}^a \cap (V_{\mathbb{Q}})^{\bar{G}} \subset \Lambda_{\mathbb{Q}}, & s_{\infty} \in S_{\infty}. \\ \text{Vet}_{\mathbb{Q}_{\ell}}^0(f, i) &= V_{\ell, \mathbb{Q}, s}^a \cap (V_{\mathbb{Q}_{\ell}})^{\bar{G}_{\ell}} \subset \Lambda_{\ell, \mathbb{Q}}, & s \in |S|. \\ \text{WVet}_{\mathbb{Q}_{\ell}}^0(f, i) &= V_{\mathbb{Q}_{\ell}, s}^a \cap (V_{\mathbb{Q}_{\ell}})^{\bar{G}_{\ell}} \subset \Lambda_{\mathbb{Q}_{\ell}}, & s \in |S|. \end{aligned}$$

The implication $\text{Vet}_{\mathbb{Q}_{\ell}}^0(f, i) \Rightarrow \text{WVet}_{\mathbb{Q}_{\ell}}^0(f, i)$ immediately follows from the fact that, for every $s \in S$, $V_{\mathbb{Q}_{\ell}, s}^a$ is the $\mathbb{Q}_{\ell}$ -span of $V_{\ell, \mathbb{Q}, s}^a$.

As $\text{Tate}_{\mathbb{Q}_{\ell}}(X_{\eta}, i)$ is invariant under base-change along finite covers $S' \rightarrow S$ of smooth varieties, to prove $\text{Tate}_{\mathbb{Q}_{\ell}}(X_{\eta}, i) \Rightarrow \text{WVet}_{\mathbb{Q}_{\ell}}(f, i)$ one may first perform such a base-change hence assume:

- $V_{\mathbb{Q}_{\ell}, \eta}^a = \text{im}[\text{CH}^i(X_{\eta})_{\mathbb{Q}_{\ell}} \rightarrow \text{CH}^i(X_{\bar{\eta}})_{\mathbb{Q}_{\ell}} \xrightarrow{c_{\ell, \eta}^{\ell, \eta}} V_{\mathbb{Q}_{\ell}}]$, which, from the surjectivity of the restriction map $\text{CH}^i(X) \rightarrow \text{CH}^i(X_{\eta})$, implies $\Lambda_{\mathbb{Q}_{\ell}} = V_{\mathbb{Q}_{\ell}, \eta}^a$;
- $\bar{G}_{\ell}$ is connected - see Footnote 4, which ensures $V_{\mathbb{Q}_{\ell}, s}^a \cap (V_{\mathbb{Q}_{\ell}})^{\bar{G}_{\ell}} \subset \tilde{V}_{\mathbb{Q}_{\ell}, \eta} \stackrel{(\alpha)}{=} V_{\mathbb{Q}_{\ell}, \eta}^a = \Lambda_{\mathbb{Q}_{\ell}}$, where (α) is $\text{Tate}_{\mathbb{Q}_{\ell}}(X_{\eta}, i)$.

If $p = 0$, for every $s_{\infty} \in S_{\infty}(\mathbb{C})$ above $s \in |S|$, Artin's comparison isomorphism yields the following canonical commutative diagram:

$$(8) \quad \begin{array}{ccc} V_{\mathbb{Q}, s_{\infty}}^a \cap (V_{\mathbb{Q}})^{\bar{G}} & \xrightarrow{\simeq} & V_{\ell, \mathbb{Q}, s}^a \cap (V_{\mathbb{Q}_{\ell}})^{\bar{G}_{\ell}} \\ \downarrow & & \downarrow \\ \Lambda_{\mathbb{Q}} & \xrightarrow{\simeq} & \Lambda_{\ell, \mathbb{Q}}, \end{array}$$

which shows $\text{VSing}^0(f_{\infty}, i) \Leftrightarrow \text{Vet}_{\mathbb{Q}_{\ell}}^0(f, i)$, and the isomorphisms

$$(V_{\ell, \mathbb{Q}, s}^a \cap (V_{\mathbb{Q}_{\ell}})^{\bar{G}_{\ell}}) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} = V_{\mathbb{Q}_{\ell}, s}^a \cap (V_{\mathbb{Q}_{\ell}})^{\bar{G}_{\ell}}, \quad \Lambda_{\ell, \mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} = \Lambda_{\mathbb{Q}_{\ell}},$$

(similar to (7)), which, together with (8), show $\text{WVet}_{\mathbb{Q}_{\ell}}^0(f, i) \Rightarrow \text{Vet}_{\mathbb{Q}_{\ell}}^0(f, i)$.

3.1.4. *Proof of Lemma A.* As we already observed that $\text{VSing}(f_{\infty}, i) \Leftrightarrow \text{WVet}_{\mathbb{Q}_{\ell}}(f, i)$ and $\text{WVet}_{\mathbb{Q}_{\ell}}(f, i) \Rightarrow \Lambda_{\mathbb{Q}_{\ell}} = V_{\mathbb{Q}_{\ell}, s}^a$, $s \in |S|_{\mathbb{Q}_{\ell}}^{\text{gen}}$ - see Subsection 2.2.5, it only remains to prove that $c(\Lambda_{\mathbb{Z}_{\ell}}) = 0$ for $\ell \gg 0$. This follows at once from Artin's comparison isomorphism, which yields the identifications

$$(V_{\mathbb{Z}_{\ell}}^{\text{free}} / \Lambda_{\mathbb{Z}_{\ell}})_{\text{tors}} \simeq (V_{\mathbb{Z}}^{\text{free}} / \Lambda_{\mathbb{Z}})_{\text{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}.$$

and the fact that $(V_{\mathbb{Z}}^{\text{free}} / \Lambda_{\mathbb{Z}})_{\text{tors}}$ is a finite group.

3.1.5. *Obstruction to the integral Hodge conjecture.* In this subsection, we deduce from Artin's comparison and Theorem A uniform bounds for the obstruction to the integral Hodge conjecture.

Let X_∞ be a smooth, projective variety over $\mathbb{C}$. The cycle class map

$$c : \mathrm{CH}^i(X_\infty) \rightarrow V_{\mathbb{Z}} := H^{2i}(X_\infty^{\mathrm{an}}, \mathbb{Z}(i))$$

for $\mathbb{Z}$ -singular cohomology fits into a canonical diagram analogue to (1)

$$\begin{array}{ccccccc} & & & & c & & \\ & & & & \curvearrowright & & \\ \mathrm{CH}^i(X_\infty) & \longrightarrow & V_{\mathbb{Z}}^a & \longrightarrow & \tilde{V}_{\mathbb{Z}} & \longrightarrow & V_{\mathbb{Z}} \\ & \downarrow & \downarrow & & \downarrow & \searrow \square & \downarrow \\ & & V_{\mathbb{Z}}^{\mathrm{free},a} & \longrightarrow & \tilde{V}_{\mathbb{Z}}^{\mathrm{free}} & \longrightarrow & V_{\mathbb{Z}}^{\mathrm{free}} \\ & & \downarrow & & \downarrow & \searrow \square & \downarrow \\ \mathrm{CH}^i(X_\infty)_{\mathbb{Q}} & \longrightarrow & V_{\mathbb{Q}}^a & \longrightarrow & \tilde{V}_{\mathbb{Q}} & \longrightarrow & V_{\mathbb{Q}}, \end{array}$$

where, writing $G \subset \mathrm{GL}(V_{\mathbb{Q}})$ for the Mumford-Tate group of the polarizable $\mathbb{Q}$ -Hodge structure $V_{\mathbb{Q}}$ underlies,

$$\tilde{V}_{\mathbb{Q}} := (V_{\mathbb{Q}})^G$$

is the $\mathbb{Q}$ -vector space of Hodge classes. The (classical) rational $\mathbb{Q}$ -Hodge conjecture in codimension i for X [H52]

$$\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i) \quad V_{\mathbb{Q}}^a = \tilde{V}_{\mathbb{Q}}$$

asserting that Hodge classes are algebraic also admits integral variants:

$$\begin{array}{ll} \mathrm{Hodge}_{\mathbb{Z}}^{\mathrm{free}}(X_\infty, i) & V_{\mathbb{Z}_\ell}^{\mathrm{free},a} = \tilde{V}_{\mathbb{Z}}^{\mathrm{free}} \quad (\text{Integral Hodge conjecture modulo torsion}); \\ \mathrm{Hodge}_{\mathbb{Z}}(X_\infty, i) & V_{\mathbb{Z}}^a = \tilde{V}_{\mathbb{Z}} \quad (\text{Integral Hodge conjecture}). \end{array}$$

Again, the implications

$$\mathrm{Hodge}_{\mathbb{Z}}(X_\infty, i) \Rightarrow \mathrm{Hodge}_{\mathbb{Z}}^{\mathrm{free}}(X_\infty, i) \Rightarrow \mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i)$$

are tautological and, in general, the converse implications are known to fail as soon as $i > 1$ (see e.g. [AtH62, Ge19] for examples of the failure of $\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i)$ and [Ko90, K21, P22] for examples of the failure of $\mathrm{Hodge}_{\mathbb{Z}}^{\mathrm{free}}(X_\infty, i)$). By definition, the obstructions to $\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i)$, $\mathrm{Hodge}_{\mathbb{Z}}^{\mathrm{free}}(X_\infty, i)$, $\mathrm{Hodge}_{\mathbb{Z}}(X_\infty, i)$ are, respectively:

$$\tilde{C}_{\mathbb{Q}} := \tilde{V}_{\mathbb{Q}}/V_{\mathbb{Q}}^a, \quad \tilde{C}_{\mathbb{Z}}^{\mathrm{free}} := \tilde{V}_{\mathbb{Z}}^{\mathrm{free}}/V_{\mathbb{Z}}^{\mathrm{free},a}, \quad \tilde{C}_{\mathbb{Z}} := \tilde{V}_{\mathbb{Z}}/V_{\mathbb{Z}}^a,$$

with the properties that one has the short exact sequence

$$(9) \quad 0 \rightarrow (V_{\mathbb{Z}})_{\mathrm{tors}}/(V_{\mathbb{Z}}^a)_{\mathrm{tors}} \rightarrow \tilde{C}_{\mathbb{Z}} \rightarrow \tilde{C}_{\mathbb{Z}}^{\mathrm{free}} \rightarrow 0$$

and that

$$\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i) \Leftrightarrow (\tilde{C}_{\mathbb{Z}}^{\mathrm{free}})_{\mathrm{tors}} = \tilde{C}_{\mathbb{Z}}^{\mathrm{free}} \Leftrightarrow (\tilde{C}_{\mathbb{Z}})_{\mathrm{tors}} = \tilde{C}_{\mathbb{Z}}$$

in which case, (9) reads

$$0 \rightarrow (V_{\mathbb{Z}})_{\mathrm{tors}}/(V_{\mathbb{Z}}^a)_{\mathrm{tors}} \rightarrow (\tilde{C}_{\mathbb{Z}})_{\mathrm{tors}} \rightarrow (\tilde{C}_{\mathbb{Z}}^{\mathrm{free}})_{\mathrm{tors}} \rightarrow 0.$$

Furthermore,

$$(\tilde{C}_{\mathbb{Z}}^{\mathrm{free}})_{\mathrm{tors}} = (C_{\mathbb{Z}}^{\mathrm{free}})_{\mathrm{tors}} := V_{\mathbb{Z}}^{\mathrm{free}}/V_{\mathbb{Z}}^{\mathrm{free},a}.$$

Assume $p = 0$ and fix an embedding $\infty : k \hookrightarrow \mathbb{C}$. Let X be a smooth projective variety over k . From the observations in Subsection 3.1.2 and the flatness of $\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell$, Artin's comparison isomorphism induces the following identifications

$$((V_{\mathbb{Z}})_{\mathrm{tors}}/(V_{\mathbb{Z}}^a)_{\mathrm{tors}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} (V_{\mathbb{Z}_\ell})_{\mathrm{tors}}/(V_{\mathbb{Z}_\ell}^a)_{\mathrm{tors}}, \quad (C_{\mathbb{Z}}^{\mathrm{free}})_{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} (C_{\mathbb{Z}_\ell}^{\mathrm{free}})_{\mathrm{tors}}.$$

As $V_{\mathbb{Z}}$ is a $\mathbb{Z}$ -module of finite type, this shows, in particular,

a) $(\tilde{C}_{\mathbb{Z}_\ell}^{\mathrm{free}})_{\mathrm{tors}} = 0$ - hence $(C_{\mathbb{Z}_\ell}^{\mathrm{free}})_{\mathrm{tors}} = 0$, for $\ell \gg 0$.

- b) The obstruction $(C_{\mathbb{Z}}^{\text{free}})_{\text{tors}}$ to $\text{Hodge}_{\mathbb{Z}}^{\text{free}}(X_{\infty}, i)$ can be recovered from the obstructions $(C_{\mathbb{Z}_{\ell}}^{\text{free}})_{\text{tors}}$ to $\text{Tate}_{\mathbb{Z}_{\ell}}^{\text{free}}(X, i)$, when ℓ varies as

$$(C_{\mathbb{Z}}^{\text{free}})_{\text{tors}} = \bigoplus_{\ell} (C_{\mathbb{Z}_{\ell}}^{\text{free}})_{\text{tors}}.$$

As in Subsection 1.3, let now S be a smooth, geometrically connected variety over k and $f : X \rightarrow S$ a smooth projective morphism. For $s_{\infty} \in S_{\infty}(\mathbb{C})$ above $s \in S$, denote by a subscript $(-)_s$ the various modules attached to $X_{s_{\infty}} = X_{\infty, s_{\infty}}$ introduced above (*e.g.* $V_{\mathbb{Z}, s_{\infty}} := H^{2i}(X_{s_{\infty}}^{\text{an}}, \mathbb{Z}(i))$, $V_{\mathbb{Z}, s_{\infty}}^a := \text{im}[\text{CH}^i(X_{s_{\infty}}) \rightarrow V_{\mathbb{Z}}]$ *etc.*). Again, one may investigate how

$$\widetilde{\text{Ob}}_{\mathbb{Z}, s} := |(\widetilde{C}_{\mathbb{Z}, s_{\infty}})_{\text{tors}}|$$

vary with $s \in |S|$. A direct consequence of Theorem A and the observations a), b) above is the following.

Corollary 10. *Assume S is a curve and $\text{VSing}(f_{\infty}, i)$ holds. Then, for every integer $d \geq 1$, one has*

$$\widetilde{\text{Ob}}_{\mathbb{Z}}^{\leq d} := \sup\{|\widetilde{\text{Ob}}_{\mathbb{Z}, s_{\infty}}| \mid s \in |S|^{\leq d}\} < +\infty.$$

When $i = 2$, $(\widetilde{C}_{\mathbb{Z}, s_{\infty}})_{\text{tors}}$ can again be described in terms of degree 3 unramified cohomology. More precisely, set $C_{\mathbb{Z}, s_{\infty}} := V_{\mathbb{Z}, s_{\infty}}/V_{\mathbb{Z}, s_{\infty}}^a$. From the short exact sequence

$$0 \rightarrow \widetilde{C}_{\mathbb{Z}, s_{\infty}} \rightarrow C_{\mathbb{Z}, s_{\infty}} \rightarrow V_{\mathbb{Z}, s_{\infty}}/\widetilde{V}_{\mathbb{Z}, s_{\infty}} \rightarrow 0$$

and the fact that $V_{\mathbb{Z}, s_{\infty}}/\widetilde{V}_{\mathbb{Z}, s_{\infty}}$ is torsion-free, one has $(\widetilde{C}_{\mathbb{Z}, s_{\infty}})_{\text{tors}} = (C_{\mathbb{Z}, s_{\infty}})_{\text{tors}}$. If $i = 2$, [CTV12, Thm. 3.7] establishes that $(C_{\mathbb{Z}, s_{\infty}})_{\text{tors}}$ is isomorphic to

$$H_{\text{nr}}^3(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ndiv}} \stackrel{\text{def}}{=} \text{coker}[H_{\text{nr}}^3(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(2))_{\text{div}} \rightarrow H_{\text{nr}}^3(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(2))].$$

Hence Corollary 10 implies:

Corollary 11. *Assume S is a curve and $\text{VSing}(f_{\infty}, i)$ holds. Then, for every integer $d \geq 1$,*

$$\sup\{|H_{\text{nr}}^3(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ndiv}}| \mid s \in |S|^{\leq d}\} < +\infty.$$

Remark 12. a) Using [CTV12, Thm. 3.11] and Corollary 10 for cycles of dimension 1, one has an analogue of Corollary 11 with uniform bounds for the groups $H^{n-3}(X_{\infty, s_{\infty}}^{\text{an}}, \mathcal{H}_{X_{\infty, s_{\infty}}^{\text{an}}}^n(\mathbb{Q}/\mathbb{Z}(n-1)))_{\text{ndiv}}$, where n is the relative dimension of $f : Y \rightarrow X$.

- b) More generally, Corollary 11 holds with $H_{\text{nr}}^3(X_{\infty, s}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ndiv}}$ replaced by Schreieder's refined unramified cohomology [S23, §1.2, Thm. 1.6]:

$$H_{i-2, \text{nr}}^{2i-1}(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(i))_{\text{ndiv}} \stackrel{\text{def}}{=} \text{coker}[H_{i-2, \text{nr}}^{2i-1}(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(i))_{\text{div}} \rightarrow H_{i-2, \text{nr}}^{2i-1}(X_{\infty, s_{\infty}}^{\text{an}}, \mathbb{Q}/\mathbb{Z}(i))].$$

- c) For general properties of deformation and specialization of the obstruction groups to the integral Hodge conjecture in families of complex algebraic varieties see [CTV12, Sec. 5.1].

3.2. Crystalline cohomology. We now turn to the setting and retain the notation and conventions of Subsection 1.2.2.

3.2.1. "Comparison" with crystalline cohomology. A delicate issue when $p > 0$ is to find a suitable analogue of Artin's comparison isomorphism. Following the strategy of [A23], this will be achieved by combining Fact 13 below, which relies - *via* a L -function argument - on the Katz-Messing theorem [KM74] and comparison of various categories of isocrystals, with⁷ the conjectural statement $\text{CrysEt}_{\mathbb{Q}_{\ell}}(f, i)$.

Let $\mathcal{S}$ be a smooth, geometrically connected variety over F and consider a Cartesian square

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{S}} & \longrightarrow & \mathcal{X} \\ f_{\mathcal{S}} \downarrow & \square & \downarrow f \\ \mathcal{S} & \longrightarrow & S. \end{array}$$

Fact 13. [A23, Proof of Thm. 1.6.3.1 - esp. (2.1.2.1), Rem. 1.6.3.2] *Assume the canonical restriction morphism in étale $\mathbb{Q}_{\ell}$ -cohomology*

$$H^0(\mathcal{S}_{\bar{F}}, R^{2i}f_*\mathbb{Q}_{\ell}(i)) \xrightarrow{\sim} H^0(\mathcal{S}_{\bar{F}}, R^{2i}f_*\mathbb{Q}_{\ell}(i))$$

⁷Note that [A23] was focussed on divisors, for which the fact that homological and numerical equivalence coincide is known.

is an isomorphism. Then the canonical restriction morphism in crystalline cohomology

$$H^0(\mathcal{S}, R^{2i} f_{\text{crys},*} \mathcal{O}_{\mathcal{X}/K}) \xrightarrow{\sim} H^0(\mathcal{S}, R^{2i} f_{\mathcal{S}, \text{crys},*} \mathcal{O}_{\mathcal{X}_{\mathcal{S}}/K})$$

is an isomorphism.

3.2.2. *Proof of Lemma B (ii).* Let $s \in |S|_{\mathcal{V}_{\ell, \mathbb{Q}_{\ell}}}^{\text{gen}}$. Recall we are to prove $V_{\mathbb{Q}_{\ell}, s}^a = \Lambda_{\mathbb{Q}_{\ell}}$. Replacing k, F by finite field extensions, one may assume there exists a smooth, separated and geometrically connected scheme $\mathcal{S}$ over F with generic point $\eta_{\mathcal{S}} : \text{Spec}(k(s)) \rightarrow \mathcal{S}$ and such that $\mathcal{S}(F) \neq \emptyset$, and a Cartesian diagram

$$(10) \quad \begin{array}{ccccccc} & & & \curvearrowright & & & \\ \mathcal{X}_t & \longrightarrow & \mathcal{X}_{\mathcal{S}} & \longrightarrow & \mathcal{X} & \longleftarrow & X \longleftarrow X_s \\ f_t \downarrow & \square & f_{\mathcal{S}} \downarrow & \square & f \downarrow & \square & f \downarrow \\ F & \xrightarrow{t} & \mathcal{S} & \longrightarrow & S & \longleftarrow & S \longleftarrow k(s) \\ & \searrow & \downarrow & \nearrow \eta_{\mathcal{S}} & \downarrow & \nearrow \eta_{\mathcal{S}} & \downarrow \\ & & F & \longleftarrow & \mathcal{X} & \longleftarrow & k \end{array}$$

Replacing further k, F by finite field extensions, one may assume that

$$(11) \quad V_{\mathbb{Q}_{\ell}, s}^a = \text{im}[\text{CH}^i(X_s) \rightarrow \text{CH}^i(X_{\bar{s}}) \xrightarrow{c_{\ell, s}} V_{\mathbb{Q}_{\ell}}].$$

From (11), it is enough to show that for every $\tilde{\alpha}_s \in \text{CH}^i(X_s)_{\mathbb{Q}}$ with image $\alpha_{\ell, s} := c_{\ell, s}(\tilde{\alpha}_s) \in V_{\mathbb{Q}_{\ell}}$, there exists $\tilde{\alpha} \in \text{CH}^i(X)_{\mathbb{Q}}$ such that $c_{\ell, s}(\tilde{\alpha}|_{X_s}) = \alpha_{\ell, s}$. We retain the notation and conventions in Diagram (10). Up to shrinking $\mathcal{S}$, one may assume there exists $\tilde{\alpha}_{\mathcal{S}} \in \text{CH}^i(\mathcal{X}_{\mathcal{S}})_{\mathbb{Q}}$ such that $\tilde{\alpha}_{\mathcal{S}}|_{X_s} = \tilde{\alpha}_s$; write $\tilde{\alpha}_t := \tilde{\alpha}_{\mathcal{S}}|_{\mathcal{X}_t} \in \text{CH}^i(\mathcal{X}_t)_{\mathbb{Q}}$. Consider now the canonical commutative diagram

$$\begin{array}{ccccc} \text{CH}^i(\mathcal{X})_{\mathbb{Q}} & \xrightarrow{|\mathcal{X}_{\mathcal{S}}|} & \text{CH}^i(\mathcal{X}_{\mathcal{S}})_{\mathbb{Q}} & & \\ c_{\text{crys}} \downarrow & \searrow |\mathcal{X}_t| & \swarrow |\mathcal{X}_t| & \downarrow c_{\text{crys}, \mathcal{S}} & \\ H_{\text{crys}}^{2i}(\mathcal{X}) & & \text{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} & & H_{\text{crys}}^{2i}(\mathcal{X}_{\mathcal{S}}) \\ \epsilon \downarrow & \searrow |\mathcal{X}_t| & \downarrow c_{\text{crys}, t} & \swarrow |\mathcal{X}_t| & \downarrow \epsilon \\ H^0(\mathcal{S}, R^{2i} f_{\text{crys},*} \mathcal{O}_{\mathcal{X}/K}) & \longrightarrow & H_{\text{crys}}^{2i}(\mathcal{X}_t) & \longleftarrow & H^0(\mathcal{S}, R^{2i} f_{\mathcal{S}, \text{crys},*} \mathcal{O}_{\mathcal{X}_{\mathcal{S}}/K}). \end{array}$$

$\xrightarrow{\quad \simeq \quad}$

As $s \in S_{\mathcal{V}_{\ell, \mathbb{Q}_{\ell}}}^{\text{gen}}$, the canonical restriction morphism

$$H^0(\mathcal{S}_{\bar{F}}, R^{2i} f_* \mathbb{Q}_{\ell}(i)) \xrightarrow{\sim} H^0(\mathcal{S}_{\bar{F}}, R^{2i} f_* \mathbb{Q}_{\ell}(i))$$

is an isomorphism - see [A23, §2.2.2]. Here, we implicitly use the reduction 2.2.3 a), b). Hence, by Fact 13, the bottom horizontal arrow is an isomorphism. This implies that $\alpha_t := c_{\text{crys}, t}(\tilde{\alpha}_t)$ lies in $H^0(\mathcal{S}, R^{2i} f_{\text{crys},*} \mathcal{O}_{\mathcal{X}/K})$. But then, by implication 2) $\implies$ 1) in $\text{VCrys}(f, i)$, there exists $\tilde{\alpha}_{\mathcal{X}} \in \text{CH}^i(\mathcal{X})_{\mathbb{Q}}$ such that $c_{\text{crys}, t}(\tilde{\alpha}_{\mathcal{X}}|_{\mathcal{X}_t}) = c_{\text{crys}}(\tilde{\alpha}_{\mathcal{X}})|_{\mathcal{X}_t} = \alpha_t = c_{\text{crys}, t}(\tilde{\alpha}_t)$. By $\text{CrysEt}_{\mathbb{Q}_{\ell}}(f, i)$, this implies $c_{\ell, t}(\tilde{\alpha}_{\mathcal{X}}|_{\mathcal{X}_t}) = c_{\ell, t}(\tilde{\alpha}_t)$. The assertion thus follows, with $\tilde{\alpha} = \tilde{\alpha}_{\mathcal{X}}|_{\mathcal{X}}$, from the canonical commutative specialization diagram of cycle class maps

$$\begin{array}{ccccc} & & \text{CH}^i(\mathcal{X})_{\mathbb{Q}} & & \\ & \swarrow |\mathcal{X}_t| & \downarrow |\mathcal{X}_s| & \searrow |X| & \\ \text{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} & \xleftarrow{sp_{s, t}} & \text{CH}^i(X_s)_{\mathbb{Q}} & \xleftarrow{|X_s|} & \text{CH}^i(X)_{\mathbb{Q}} \\ c_{\ell, t} \downarrow & & \downarrow c_{\ell, s} & & \\ H^{2i}(\mathcal{X}_t, \mathbb{Q}_{\ell}(i)) & = & H^{2i}(X_s, \mathbb{Q}_{\ell}(i)) & & \end{array}$$

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ANNA CADORET
 IMJ-PRG – SORBONNE UNIVERSITÉ, PARIS, FRANCE
 ANNA.CADORET@IMJ-PRG.FR

ALENA PIRUTKA
 CIMS, NEW YORK UNIVERSITY, NEW YORK, U.S.A.
 PIRUTKA@CIMS.NYU.EDU