On the toric locus of ℓ-adic local systems arising from geometry

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For an algebraic group G, let G° denote its neutral component. Let k be a number field. Let X be a smooth, separated, geometrically connected variety over k and let $|X|$ denote the set of closed points of X. For an infinite $\infty : k \hookrightarrow \mathbb{C}$ (resp. finite $v : k \hookrightarrow k_v$) places, let $X \rightsquigarrow X_\infty \rightsquigarrow X_\infty^{\mathrm{an}}$ (resp. $X \rightsquigarrow X_v \rightsquigarrow X_v^{\mathrm{an}}$) denote the base-change and analytification functors. Fix a prime ℓ and a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X *viz* a continuous representation of the étale fundamental group $\pi_1(X, \bar{x})$ on $V_\ell := \mathcal{V}_{\ell,\bar{x}}$. Write $\overline{G}_\ell$, $G_\ell \subset \mathrm{GL}_{V_\ell}$ for the Zariski-closure of the images of $\pi_1(X_{\bar{k}})$ and $\pi_1(X)$ acting on V_ℓ and, for $x \in X$, write $G_{\ell,x} \subset \mathrm{GL}_{V_\ell}$ for "the"[1] Zariski-closure of the image of $\pi_1(x)$ acting on V_ℓ *via* $\pi_1(x) \to \pi_1(X)$. The degeneracy locus of $\mathcal{V}_\ell$ is the set

$$|X|_{\mathcal{V}_\ell} := \{x \in |X| \mid G_{\ell,x}^\circ \subsetneq G_\ell^\circ\}$$

For a smooth projective morphism $f : Y \to X$, the $\mathbb{Q}_\ell$-local systems of the form $\mathcal{V}_\ell = R^i f_* \mathbb{Q}_\ell(j)$ control certain arithmetico-geometric invariants of the Y_x, $x \in |X|$ (*e.g.* ℓ-primary torsion of the Picard variety or of the Brauer group, rank of the Néron-Severi group, of motivated cycles, rank of the the Picard variety *etc.*) and understanding $|X|_{\mathcal{V}_\ell}$ amounts to understanding how those invariants degenerate in the family Y_x, $x \in |X|$ - see [C23, §3] for details.

The leading conjecture about $|X|_{\mathcal{V}_\ell}$ is the following. For every integer $d \geq 1$ let $|X|^{\leq d}$ denote the set of all $x \in |X|$ such that $[k(x) : k] \leq d$. Then

Conj. 1 *Assume $\overline{G}_\ell$ has finite abelianization. Then $|X|_{\mathcal{V}_\ell} \cap |X|^{\leq d}$ is not Zariski-dense in X.*

For X a curve, Conj. 1 is proved in [CT13]. In contrast, if X has dimension ≥ 2, it is widely open. Actually, the strategy of [CT13] provides a heuristic for Conj. 1 when $d = 1$ - see [C23, §4]; this heuristic relies on the diophantine Lang conjecture (that on a number field k the set of k-rational point of a variety of general type is not Zariski-dense), which seems currently out of reach, even for surfaces.

Assume $\mathcal{V}_\ell = R^i f_* \mathbb{Q}_\ell(j)$ for some smooth projective morphism $f : Y \to X$. In this case $\overline{G}_\ell$ is known to be semisimple - hence, in particular, to have finite abelianization; assume furthermore it is not finite. In this work, we investigate a weaker form of Conj. 1, replacing $|X|_{\mathcal{V}_\ell}$ by the toric locus $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}} := \{x \in |X| \mid G_{\ell,x}^\circ \text{ is a torus}\}$, which, informally, corresponds to the most degenerate members of the family of motivated motive $\mathfrak{h}^i(Y_x)(j)$.

Conj. 2 *With the above assumptions, $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}} \cap |X|^{\leq d}$ is not Zariski-dense in X.*

[1] Actually only well defined up to conjugacy.

Conj. 2 follows from Conj. 1 but it is also a consequence of the Mumford-Tate conjecture and the generalized André-Oort conjecture (an unlikely intersection type conjecture); this implication is deep as it involves the average Colmez conjecture. Actually, the Mumford-Tate conjecture predicts that the points $x \in X_\infty(\mathbb{C})$ lifting those of $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}}$ are exactly the CM points of the polarizable $\mathbb{Q}$-VHS $\mathcal{V}_\infty := R^i f_\infty^{\mathrm{an}} \mathbb{Q}(j)$ and that $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}}$ should play a similar part in controlling the geometry of the exceptional locus $|X|_{\mathcal{V}_\ell}$ as the CM points do in controlling the geometry of the Hodge locus. Unfortunately, in general, we do not know how $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}}$ compares with the CM locus; actually, we do not even know if $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}}$ is independent of ℓ. The only things which are known are that

(1) the solvable locus $|X|_{\mathcal{V}_\ell}^{\mathrm{solv}} := \{x \in |X| \mid G_{\ell,x}^\circ \text{ is solvable}\} \subset |X|_{\mathcal{V}_\ell}$, is independent of ℓ; this follows from class field theory [Se68] (and is true more generally for any $\mathbb{Q}_\ell$-local system which is almost pointwise geometric in the sense of Fontaine-Mazur).
(2) For every prime ℓ, $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}}$ contains the set $|X|_{\mathcal{V}_{\mathrm{mot}}}^{\mathrm{tor}}$ of all $x \in |X|$ such that the connected component of the motivated Galois group (in the sense of André) of the motivated motive $\mathfrak{h}^i(Y_x)(j)$ is a torus.

Our main result is the following. We keep the assumptions in Conj. 2. Consider the level condition: (Lev_ℓ) The image of $\pi_1(X_{\bar{k}})$ acting on V_ℓ is a pro-ℓ group.

Thm. *Assume* (Lev_ℓ) *holds for at least two primes* $\ell_1 \neq \ell_2$. *Then there exists a set* $\mathcal{L}$ *of primes of positive Dirichlet density such that for every* $\ell \in \mathcal{L}$ *the set* $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}-} \cap X(k)$ *is not Zariski-dense in* X. *Furthermore, if the complex period map describing* $\mathcal{V}_\infty$ *is finite-to-one, then* $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}-} \cap X(k)$ *is finite.*

Here, $|X|_{\mathcal{V}_\ell}^{\mathrm{tor}-}$ denotes the subset of all $x \in |X|_{\mathcal{V}_\ell}^{\mathrm{tor}}$ such that $x \in |X|_{\mathcal{V}_{\ell_x}}^{\mathrm{tor}}$ for another $\ell_x \neq \ell$. In particular, $|X|_{\mathcal{V}_{\mathrm{mot}}}^{\mathrm{tor}} \cap X(k)$ is not Zariski-dense in X.

Remark. We hope our strategy to prove Thm. can be extended to prove Conj. 2 in general (namely for arbitrary $d \geq 1$ and without level conditions) but still with the restriction that ℓ belongs to a subset $\mathcal{L}$ of primes of positive Dirichlet density. For the time being, treating all primes ℓ seems to require a truely new idea.

Brief sketch of proof.

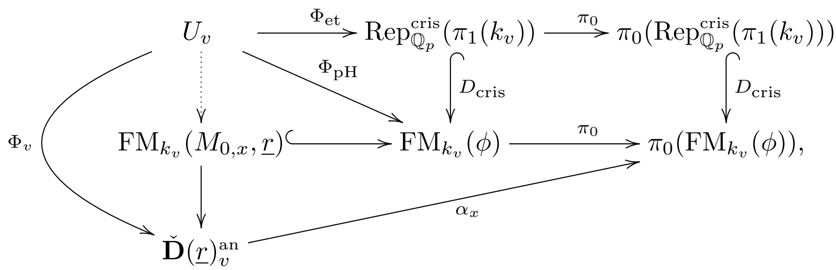
Step 1: "toric points are integral". Fix a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$ and $\mathcal{Y} \xrightarrow{f} \mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to U$ with $\mathcal{X}^{\mathrm{cpt}} \to U$ smooth, projective, $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}}$ an open immersion such that $\mathcal{Z} := \mathcal{X}^{\mathrm{cpt}} \setminus \mathcal{X} \to U$ is a relative normal crossing divisor and $f : \mathcal{Y} \to \mathcal{X}$ a smooth projective morphism with generic fiber $f : Y \to X$. Let $\mathcal{Z}^+ \subset \mathcal{Z}$ denote the union of those irreducible components around which the monodromy of $\mathcal{V}_\infty$ is trivial and set $\mathcal{X}^+ := \mathcal{X} \cup \mathcal{Z}^+$. It is not difficult to check that $\mathcal{V}_\ell$ extends to a $\mathbb{Q}_\ell$-local system on the generic fiber X^+ of $\mathcal{X}^+$. By the nilpotent orbit theorem, $\mathcal{V}_\infty$ also extends to a polarizable $\mathbb{Z}$-VHS on $X_\infty^{+\mathrm{an}}$. The key lemma is the following. Fix $v \in U$, $v|p$ let $\mathcal{O}_v$, k_v denote respectively the completions of $\mathcal{O}_k$, k at v.

Lem. (Good reduction criterion) *Let $\ell \neq p$ be a prime such that the image of $\pi_1(X_{k_v})$ acting on V_ℓ is of prime-to-p order. Then for every $x \in X^+(k_v)$, $x \in \mathcal{X}^+(\mathcal{O}_v)$ iff $x^*\mathcal{V}_\ell$ is unramified at v.*

The proof of Lem. is similar to [PST21, §4]; it relies on the interpretation of Kummer theory in terms of intersection data and the nilpotent orbit theorem. That Lem. applies to toric points follows from the theory of complex multiplication [Se68], Serre-Tate criterion [SeT68], and the level assumption in Thm. It is to apply Lem. that we have to replace $|X|^{\mathrm{tor}}_{\mathcal{V}_\ell}$ with $|X|^{\mathrm{tor}-}_{\mathcal{V}_\ell}$.

Step 2: "Integral toric points are not Zariski-dense". This step follows the strategy of [LawV20] using the enhanced version of the local v-adic period map constructed in [BS22] (building on recent development in variational p-adic Hodge theory - [LiZ17], [Shi20], [DLanLiZ23]). Write $\mathcal{V}_{\mathrm{dR}} := \mathbb{R}^i f_*\Omega_{Y|X}$ for the relative de Rham cohomology; this is a filtered vector bundle with flat connexion. The strategy of [LawV20] relies on the motivic properties of the collection $\mathcal{V}_p$, $\mathcal{V}_{\mathrm{dR}}$, $\mathcal{V}_\infty$ - in particular that $\mathcal{V}_p$ be pointwise pure with characteristic polynomial of Frobenius in $\mathbb{Q}[T]$ and with bounded denominators, that $(\mathcal{V}_\infty, \mathcal{V}^{\mathrm{an}}_{\mathrm{dR},\infty})$ is a polarizable $\mathbb{Z}$-VHS on X^{an}_∞, that $(\mathcal{V}_p, \mathcal{V}^{\mathrm{an}}_{\mathrm{dR},v})$ is a de Rham pair on X^{an}_v *etc.* In step 1, we have extended $\mathcal{V}_p$ from X to X^+ but there is no reason why $f : Y \to X$ should extend to a smooth projective morphism over X^+ (and it does not in general) so, to run the [LawV20] strategy, we have to check that $\mathcal{V}_p$, $\mathcal{V}_{\mathrm{dR}}$, $\mathcal{V}_\infty$ extend from X to X^+ with all the expected properties; this involves the nilpotent orbit theorem (as already mentioned in step 1), the companion conjecture [L02, Thm. VII.6] and the rigidity of the de Rham property [LiZ17, Thm. 3.9 (iv)]. Once this is done, fix $v \in U_p$. Let $\underline{r}$ be the type of the filtration on $\mathcal{V}_{\mathrm{dR}}$ and let $\check{\mathbf{D}}(\underline{r})$ denote the Grassmaniann over k classifying filtration of type $\underline{r}$ on $\mathcal{V}_{\mathrm{dR},x} \simeq k^{\oplus r}$. The crux of the argument is that for every $x_0 \in \Sigma := |X^+|^{\mathrm{tor}}_{\mathcal{V}_p} \cap X(k)$ there exists an admissible open neighbourhood U_v of x_0 in $X^{+\mathrm{an}}_v$ and a v-adic analytic period map $\Phi_v : U_v \to \check{\mathbf{D}}(\underline{r})^{\mathrm{an}}_v$ such that the fibers of $\Phi_v : U_v \to \check{\mathbf{D}}(\underline{r})^{\mathrm{an}}_v$ above points in $\Phi_v(X(k) \cap U_v)$ are not-Zariski dense in X (see [LawV20, §9.2]), and which, for every $x \in U_v$, fits into a commutative diagram:

(*)



where $\Phi_{\mathrm{et}} : U_v \to \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k_v))$ is the map that sends $x \in U_v$ to $x^*\mathcal{V}_v$, which is automatically crystalline - [Shi20], $D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k_v)) \to \mathrm{FM}_{k_v}(\phi)$ is Fontaine's crystalline period functor $V_p \to (V_p \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\pi_1(k_v)}$ (with value in the category $\mathrm{FM}_{k_v}(\phi)$ of filtered ϕ-modules); $\mathrm{FM}_{k_v}(M_{\mathrm{cris},x}, \underline{r})$ denotes the set of filtered ϕ-modules with fixed underlying ϕ-module $(M_{0,x}, \phi_x)$ and filtration of fixed type $\underline{r}$ and $\pi_0(-)$ denotes the set of isomorphism classes (of p-adic $\pi_1(k_v)$-representations and filtered ϕ-modules respectively). From step 1, $\Sigma \subset \mathcal{X}^+(\mathcal{O}_v)$ hence, as $\mathcal{X}^+(\mathcal{O}_v)$ is compact, one can cover Σ by finitely many U_v as above. This reduces the proof of Thm. to showing that (1) $\pi_0 \circ \Phi_{\mathrm{et}}(\Sigma \cap U_v)$ is finite and (2) for every $x \in \Sigma \cap U_v$, the set $\alpha_x^{-1}(\alpha_x \circ \Phi_v(x))$ is finite[2]. Assertion (1) follows from a classical lemma of Faltings [FW84, V.5]. The proof of Assertion (2) is more demanding; the rough idea is as follows. Let $G_{\mathrm{cris},v}$ denote the Galois group of $\Phi_{pH}(x) = (M_{0,x}, \phi_x, F_x^\bullet)$ in $\mathrm{FM}_{k_v}(\phi)$ extended from $\mathbb{Q}_p$ to k_v so that, tautologically its centralizer $Z(G_{\mathrm{cris},v})$ stabilizes $F_x^\bullet$ and is contained in the centralizer $Z(\varphi_x)$ of the linearized crystalline Frobenius φ_x. Assertion (2) amounts to showing that $Z(\varphi_x)$ stabilizes $F_x^\bullet$ so that to conclude, it is enough to show $Z(\varphi_x) = Z(G_{\mathrm{cris},v})$. By the fully faithfullness of $D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k_v)) \to \mathrm{FM}_{k_v}(\phi)$, and the fact that $x \in |X|^{\mathrm{tor}}_{\mathcal{V}_p}$, we already know that $G_{\mathrm{cris},v}$ is a torus so that it is actually enough to show that φ_x has maximal rank in $G_{\mathrm{cris},v}$. This is to ensure the later hold for *every* (*a priori* infinitely many!) $x \in \Sigma$ that we have to restrict to a subset $\mathcal{L}$ of primes of positive Dirichlet density. On top of several ingredients already mentioned, the proof uses Frobenius tori, [KM74] and, as one can guess, the Cebotarev density theorem.

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[2]By construction, the fibers of $\alpha_x : \check{\mathbf{D}}(\underline{r})_v^{\mathrm{an}} \to \pi_0(\mathrm{FM}_{k_v}(\phi))$ are homogeneous spaces under the centralizer $Z(\phi_x)$ of the crystalline Frobenius so, another way to phrase (2) is to say that points in $\alpha_x \circ \phi_v(\Sigma \cap U_v)$ have finite $Z(\phi_x)$-orbits.

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