ON THE TRIVIAL LOCUS OF $\mathbb{Q}_\ell$-LOCAL SYSTEMS

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Abstract. Let k be a number field, let X be a smooth, geometrically connected variety over k and let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X. The unramified Fontaine-Mazur conjecture predicts that the property of being finite is "rigid" in the sense that the following should be equivalent: (i) $\mathcal{V}_\ell$ is finite; (ii) for every $x \in |X|$, $x^*\mathcal{V}_\ell$ is finite; (iii) there exists $x \in |X|$ such that $x^*\mathcal{V}_\ell$ is finite. We prove these equivalences unconditionally when $\mathcal{V}_\ell$ is pure and part of a $\mathbb{Q}$-compatible family. When X is a curve, we also prove these equivalences with condition (iii) replaced by a weaker condition, and under the assumptions that ℓ is large enough compared with the rank of $\mathcal{V}_\ell$ and $\mathcal{V}_\ell$ is Hodge-Tate at at least one finite place of k above ℓ. The proofs use variational p-adic Hodge theory, and, for the second result, a pointwise criterion for $\mathcal{V}_\ell$ to extend over $\mathcal{X}$, and the companion correspondances both of Abe and L. Lafforgue.

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Notation / conventions.

For an algebraic group G, let $G^\circ \subset G$ denote the neutral component of G and, for a closed subgroup $H \subset G$, let $Z_G(H) \subset N_G(H) \subset G$ denote the centralizer and normalizer of H in G respectively; set $Z(G) := Z_G(G)$ for the center of G. For a profinite group Π and a topological field Q, let $\mathrm{Rep}_Q(\Pi)$ denote the category of continuous, finite-dimensional Q-representations of Π.

For a scheme S, write $|S|$ for the set of closed points of S.

For an affine scheme $S = \mathrm{spec}(A)$, we often abbreviate $\pi_1(A) := \pi_1(\mathrm{spec}(A))$ for the étale fundamental group of $\mathrm{spec}(A)$ (and omit fiber functors).

A variety over a field k is a scheme separated and of finite type over k. If S is an integral scheme with generic point η and X is a smooth, geometrically connected variety over $k := k(\eta)$, one says that X admits a **smooth model over** S if it fits into a Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{spec}(k) \arrow[d, "\eta"] \\ \mathcal{X} \arrow[r] & S, \end{tikzcd}$$

with $\mathcal{X}$ integral and $\mathcal{X} \to S$ surjective, smooth, separated and of finite type, and one says that X admits a **smooth model with relative normal crossing compactification (a smooth NCC model for short) over** S if it admits a smooth model $\mathcal{X} \to S$ over S which fits into a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr] & \mathcal{X}^{\mathrm{cpt}} \arrow[d] \\ & S \end{tikzcd}$$

with $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}}$ an open immersion, $\mathcal{X}^{\mathrm{cpt}}$ integral, $\mathcal{X}^{\mathrm{cpt}} \to S$ smooth, proper, and $\mathcal{X}^{\mathrm{cpt}} \setminus \mathcal{X} \to S$ a relative normal crossing divisor.

1. Introduction

Let k be a field and let X be a smooth, geometrically connected variety over k with generic point η.

1.1. **Trivial locus and main conjecture.** Let ℓ be a prime and $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X. For every $x \in X$ and geometric point $\overline{x}$ over x, set $V_\ell := \mathcal{V}_{\ell,\overline{x}}$ and let $\overline{G}_\ell, G_{\ell,x} \subset G_\ell \subset \mathrm{GL}_{V_\ell}$ denote the Zariski closures of the images $\overline{\Pi}_\ell$, $\Pi_{\ell,x}$ and Π_ℓ of the étale fundamental groups $\pi_1(X_{\bar{k}}, \bar{x})$, $\pi_1(x, \bar{x})$ and $\pi_1(X, \bar{x})$ acting on

V_ℓ respectively (so that $\Pi_{\ell,\eta} = \Pi_\ell$, $G_{\ell,\eta} = G_\ell$). Define the **degeneracy locus** or **Tate locus** (restricted to closed points) of $\mathcal{V}_\ell$ as

$$|X|_{\mathcal{V}_\ell} := \{x \in |X| \mid G^\circ_{\ell,x} \subsetneq G^\circ_\ell\}.$$

Informally, $|X|_{\mathcal{V}_\ell}$ is the set of all $x \in |X|$ where $x^*\mathcal{V}_\ell$ degenerates. Under mild assumptions on $\overline{\Pi}_\ell$, one expects that $X(k) \cap |X|_{\mathcal{V}_\ell}$ is not Zariski-dense in X - see [C23]. In this note, we focus on the most degenerate strata of $|X|_{\mathcal{V}_\ell}$, namely the **trivial locus**

$$|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} := \{x \in |X| \mid G^\circ_{\ell,x} = 1\}$$

and the closely related **unipotent locus**

$$|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} := \{x \in |X| \mid G^\circ_{\ell,x} \text{ is unipotent}\},$$

and **centralizing locus**

$$|X|^{\mathrm{cent}}_{\mathcal{V}_\ell} := \{x \in |X| \mid G^\circ_{\ell,x} \subset Z_{G_\ell}(\overline{G}^\circ_\ell)\}.$$

Tautologically $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} \subset |X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$ and if $x^*\mathcal{V}_\ell$ is semisimple for every $x \in |X|$, then $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = |X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$. The following is deeper.

Proposition 1. *Assume k is a number field. Then,*

(1) one has $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} \neq \emptyset \Rightarrow |X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = |X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$;

(2) in general, one always has

$$|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} \subset |X|^{\mathrm{uni}}_{\mathcal{V}_\ell} \subset |X|^{\mathrm{cent}}_{\mathcal{V}_\ell}.$$

Proposition 1 (1) is actually a special case of a more general result - see Corollary 23. The proof of Proposition 1 uses global class field theory and (variational) p-adic Hodge theory, in particular the theorem of Sen and a construction of Beilinson-Petrov. Its proof is carried out in Subsection 5.2.2.2.

The following conjecture predicts that the trivial, unipotent and centralizing loci should all be empty unless $\mathcal{V}_\ell|_{X_{\bar{k}}}$ is finite.

Conjecture A. *Assume k is a number field. For a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X, the following implications hold.*

(T) (i)' $\overline{G}^\circ_\ell = 1$;
$\Leftarrow$ (i) $G^\circ_\ell = 1$;
$\Leftrightarrow$ (ii) $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = |X|$;
$\Leftrightarrow$ (iii) $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} \neq \emptyset$.

(U) (i)' $\overline{G}^\circ_\ell = 1$
$\Leftarrow$ (i) G°_ℓ *is unipotent;*
$\Leftrightarrow$ (ii) $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} = |X|$;
$\Leftrightarrow$ (iii) $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} \neq \emptyset$.

(C) (i)' $\overline{G}^\circ_\ell = 1$;
$\Leftrightarrow$ (i) $\overline{G}^\circ_\ell \subset Z(G^\circ_\ell)$;
$\Leftrightarrow$ (ii) $|X|^{\mathrm{cent}}_{\mathcal{V}_\ell} = |X|$;
$\Leftrightarrow$ (iii) $|X|^{\mathrm{cent}}_{\mathcal{V}_\ell} \neq \emptyset$.

In (T), (U), the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), in (T) the implication (i) $\Rightarrow$ (i)' and, in (C), the implications (i) $\Leftarrow$ (i)' $\Rightarrow$ (ii) $\Rightarrow$ (iii) are tautological. In (C), the implication (i) $\Rightarrow$ (ii) follows from the fact G°_ℓ is generated by $\overline{G}^\circ_\ell$, $G^\circ_{\ell,x}$. The implications (ii) $\Rightarrow$ (i) follow from Hilbert's irreducibility theorem.

Fact 2. (Hilbert's irreducibility - [Ser89, §9.6, 10.6, Thm.]) *Let k be a number field and let X be a smooth, geometrically connected variety over k. For every $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X there exists infinitely many $x \in |X|$ such that $\Pi_{\ell,x} = \Pi_\ell$ - hence such that $G_{\ell,x} = G_\ell$.*

In Subsection 5.2.2, we will attach (see Construction 24) to every $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X and $x \in |X|$ an auxilliary $\mathbb{Q}_\ell$-local system $A_x(\mathcal{V}_\ell)$ with the property that for every $x \in |X|$, $x \in |X|^{\mathrm{cent}}_{\mathcal{V}_\ell}$ if and only if $x \in |X|^{\mathrm{triv}}_{A_x(\mathcal{V}_\ell)}$ and that $\overline{G}^\circ_\ell = 1$ for $A_x(\mathcal{V}_\ell)$ if and only if $\overline{G}^\circ_\ell = 1$ for $\mathcal{V}_\ell$.

In (U), (C) the implication (i) $\Rightarrow$ (i)' follows from geometric class field theory [KL81, Thm. 1]. In (U), this is immediate by reducing to the case where $G^\circ_\ell \simeq \mathbb{G}_{a,\mathbb{Q}_\ell}$. In (C), after possibly replacing S by a connected étale cover, one may assume $\overline{G}_\ell$, G_ℓ are both connected and then, observe that under assumption (i), for every $x \in |X|$, the $\mathbb{Q}_\ell$-local system $A_x(\mathcal{V}_\ell)$ is abelian.

To summarize, one has:

$$\begin{tikzcd} \text{(iii)} \arrow[r, Rightarrow, bend left, "??"] & \text{(ii)} \arrow[l, Rightarrow, bend left] \arrow[r, Leftrightarrow] & \text{(i)} \arrow[r, Rightarrow, bend left] & \text{(i)'} \arrow[l, Rightarrow, bend left, "(3)"] \end{tikzcd}$$

From Proposition 1 (2), one also has

$$[\text{(iii)} \Rightarrow \text{(i) in (C)}] \Longrightarrow [\text{(iii)} \Rightarrow \text{(i) in (U)}] \Longrightarrow [\text{(iii)} \Rightarrow \text{(i) in (T)}]$$

And, using the auxilliary $\mathbb{Q}_\ell$-local systems $A_x(\mathcal{V}_\ell)$, $x \in |X|$,

$$[\text{(iii)} \Rightarrow \text{(i) in (T) for } A_x(\mathcal{V}_\ell) \text{ and some } x \in |X|^{\mathrm{cent}}_{\mathcal{V}_\ell}] \Longrightarrow [\text{(iii)} \Rightarrow \text{(i) in (C) for } \mathcal{V}_\ell]$$

Remark 3. The formulation of Conjecture A may seem cumbersome. However, we adopt this formulation to stress several kind of rigidity phenomena: implications (iii) $\Rightarrow$ (ii) can be thought of as a spreading out property and implication (ii) $\Rightarrow$ (i) as a (pointwise) local to global, or globalization property. These rigidity phenomena will occur - and be a main tool - throughout the paper, where we tried and keep a consistent numbering (i) (global), (ii) (pointwise), (iii) (at one point) for the statements; a statement (x)' will usually indicate a variant or weakening of the corresponding statement (x).

1.1.1. *Reformulation of Conjecture A (T).*

(1) An equivalent formulation of Conjecture A (T) in terms of the degeneracy locus $|X|_{\mathcal{V}_\ell}$ is the following - see Lemma 21. Assume k is a number field and Conjecture A (T) holds (for every $\mathbb{Q}_\ell$-local system on X). Then, for every $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X the following holds. For every $x \in |X|$, $G^\circ_{\ell,x}$ normally generates G°_ℓ. Equivalently,
$$|X|_{\mathcal{V}_\ell} = \{x \in |X| \mid G^\circ_{\ell,x} \subsetneq Nor_{G^\circ_\ell}(G^\circ_{\ell,x})\}.$$

(2) As a compact ℓ-adic Lie group is a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_\ell)$ for some integer $m \geq 0$ [L88, Prop. 4], one has the following diophantine reformulation of Conjecture A (T): Assume k is a number field and let
$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X_0 = X$$
be a projective system of finite étale covers with $X_n \to X$ Galois of group Π_n, $n \geq 0$. Assume $\Pi := \lim \Pi_n$ is a ℓ-adic Lie group of dimension > 0 for some prime ℓ. Then,
$$\lim X_n(k) = \emptyset.$$

1.1.2. *Relation to classical conjectures.*

(1) Assume $k = \mathbb{C}$ and let $\mathcal{V}$ be a polarizable $\mathbb{Z}$-variation of pure Hodge structures on the complex-analytification X^{an} of X. One can define similarly the degeneracy locus or Hodge locus $|X|_{\mathcal{V}}$, the trivial locus $|X|^{\mathrm{triv}}_{\mathcal{V}}$ and the centralizing locus $|X|^{\mathrm{cent}}_{\mathcal{V}}$ of $\mathcal{V}$ using[1] the Mumford-Tate group G_x of $x^*\mathcal{V}$, and the generic Mumford-Tate group G of $\mathcal{V}$ in place of the ℓ-adic algebraic monodromy groups $G_{\ell,x}$, G_ℓ. In that setting, the statements corresponding to Conjecture A (T) and Conjecture A (C) easily follow *e.g.* from the constancy of Hodge numbers[2] and the fact that the neutral component $\overline{G}^\circ$ of the Zariski-closure $\overline{G}$ of the image of $\pi_1(X^{\mathrm{an}})$ acting on $\mathcal{V}_x$ is contained in G [A92, Thm. 1]. In particular, for $\mathbb{Q}_\ell$-local systems arising from motives, Conjecture A (T) and Conjecture A (C) should follow from classical motivic realization conjectures (Hodge [Ho52], Tate [T94]; see also [DLLZ23, Conj. 1.4]).

(2) In whole generality, Conjecture A follows from the unramified Fontaine-Mazur conjecture - see Corollary 28 (1).

Conjecture B. (Unramified Fontaine-Mazur [FoM95, Conj. 5.1a]) *Let k be a number field. Let p be a prime and let $U \subset \mathrm{spec}(\mathcal{O}_k)$ be an open subset containing all finite places of k above p. Then every $\mathbb{Q}_p$-local system on U is finite.*

1.2. Results.

1.2.1. ***$\mathbb{Q}$-compatible families.*** Assume k is a number field with ring of integers $\mathcal{O}_k$. For every finite place v of k with residue characteristic $p := p_v$, let k_v denote the completion of k at v and $\mathcal{O}_v \twoheadrightarrow \kappa_v$ its ring of integers and residue field respectively; let also $\mathbb{Q}_p \subset k_{v,0} \subset k_v$ denote the maximal unramified extension of $\mathbb{Q}_p$ contained in k_v, $m_v := [k_{v,0} : \mathbb{Q}_p]$ its degree and $\sigma : k_{v,0} \tilde{\to} k_{v,0}$ its Frobenius.

For a variety X over k, a closed point $x \in |X|$ with residue field $k(x)$ and a finite place v of $\mathcal{O}_{k(x)}$, write
$$x_v : \mathrm{spec}(k(x)_v) \to \mathrm{spec}(k(x)) \xrightarrow{x} X$$

[1] Recall that these are connected and reductive.

[2] If $|X^{\mathrm{an}}|^{\mathrm{triv}}_{\mathcal{V}} \neq \emptyset$, the only non-zero Hodge number is $h^{0,0}$; equivalently, $G_x = 1$, $x \in X^{\mathrm{an}}$ - hence $G = 1$.

for the resulting $k(x)_v$-point.

1.2.1.1. For a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on $x = X = \mathrm{spec}(k)$, let $U_{\mathcal{V}_\ell} \subset |\mathrm{spec}(\mathcal{O}_k)|$ be the set of all finite places v of k such that, writing $p := p_v$ for the residue characteristic of v, the following holds:

- If $\ell \neq p$, $x_v^*\mathcal{V}_\ell$ is unramified *viz* extends to a $\mathbb{Q}_\ell$-local system over $\mathrm{spec}(\mathcal{O}_v)$;
- If $\ell = p$, $x_v^*\mathcal{V}_p$ is crystalline.

For $v \in U_{\mathcal{V}_\ell}$ and $\ell \neq p$, let $\chi_{x_v,\mathcal{V}_\ell} \in \mathbb{Q}_\ell[T]$ denote the characteristic polynomial of the geometric Frobenius
$$\varphi_{x_v,\ell} : \mathcal{V}_{\ell,\bar{x}} \to \mathcal{V}_{\ell,\bar{x}}$$
and for $\ell = p$, let $\chi_{x_v,\mathcal{V}_p} \in k_{v,0}[T]$ denote the characteristic polynomial of the linearized crystalline Frobenius[3]
$$\varphi_{x_v,\mathrm{cris}} : D_{\mathrm{cris}}(x_v^*\mathcal{V}_p) \to D_{\mathrm{cris}}(x_v^*\mathcal{V}_p).$$
See Subsection 3.1 for a very brief review of basic definitions from p-adic Hodge theory, in particular the one of Fontaine's Riemann-Hilbert functor $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k_v)) \to \mathrm{M}^{\varphi}_{k_{v,0}}$.

One says that $\mathcal{V}_\ell$ is **almost everywhere unramified** (**AEU** for short) if $U_{\mathcal{V}_\ell} \subset |\mathrm{spec}(\mathcal{O}_k)|$ is a non-empty open subset and that it is **$\mathbb{Q}$-rational** (resp. and **pure of weight** $w \in \mathbb{R}$) if there exists a non-empty open subset $U'_{\mathcal{V}_\ell} \subset U_{\mathcal{V}_\ell}$ such that for every $v \in |U'_{\mathcal{V}_\ell}|$ the polynomial $\chi_{x_v} := \chi_{x_v,\mathcal{V}_\ell}$ is in $\mathbb{Q}[T]$ (resp. and χ_{x_v} is pure of weight w, that is for every root α of χ_{x_v} and infinite place $\mathbb{Q}(\alpha) \overset{\infty}{\hookrightarrow} \mathbb{C}$, $|\alpha|_\infty = |\kappa_v|^{\frac{w}{2}}$).

Let $\underline{\mathcal{V}} := (\mathcal{V}_\ell)_\ell$ be a family of $\mathbb{Q}_\ell$-local systems on $x = X = \mathrm{spec}(k)$ (indexed by the set $|\mathrm{spec}(\mathbb{Z})|$ of all rational primes). Write
$$U_{\underline{\mathcal{V}}} := \bigcap_\ell U_{\mathcal{V}_\ell} \subset |\mathrm{spec}(\mathcal{O}_k)|.$$
One says that $\underline{\mathcal{V}}$ is $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{R}$) if $U_{\underline{\mathcal{V}}} \subset |\mathrm{spec}(\mathcal{O}_k)|$ is a non-empty open subset and there exists a non-empty open subset $U'_{\underline{\mathcal{V}}} \subset U_{\underline{\mathcal{V}}}$ such that for every $v \in U'_{\underline{\mathcal{V}}}$ the polynomial $\chi_{x_v} := \chi_{x_v,\mathcal{V}_\ell}$ is in $\mathbb{Q}[T]$ (resp., pure of weight $w \in \mathbb{R}$,) and independent of the prime ℓ.

Let X be a variety over k. One says that a family of $\mathbb{Q}_\ell$-local systems $\underline{\mathcal{V}} := (\mathcal{V}_\ell)_\ell$ on X is $\mathbb{Q}$-compatible (resp. and pure of weight $w \in \mathbb{R}$) if $x^*\underline{\mathcal{V}}$ is, $x \in |X|$. The purity assumption ensures that $\overline{G}_\ell$ is semisimple [D80, Thm. (1.3.8), (1.11), Cor. (3.4.12)].

Classical examples of $\mathbb{Q}$-compatible families $\underline{\mathcal{V}}$ of $\mathbb{Q}_\ell$-local systems on X are those with $\mathcal{V}_\ell = R^i f_*\mathbb{Q}_\ell(j)$ for $f : Y \to X$ a smooth proper morphism and $i \geq 0$, j integers; these are pure of weight $w = i - 2j$ [D80], [KM74].

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1.2.1.2. Our first result is that Conjecture A (T) and Conjecture A (U) hold when $\mathcal{V}_\ell$ is part of a $\mathbb{Q}$-compatible family of pure $\mathbb{Q}_\ell$-local systems on X.

Theorem 4. *Let $\underline{\mathcal{V}}$ be a $\mathbb{Q}$-compatible family of pure $\mathbb{Q}_\ell$-local systems on X. Then,*

(1) $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} := |X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$ is independent of the prime ℓ and $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} = |X|^{\mathrm{triv}}_{\mathcal{V}_\ell}$ for $\ell \gg 0$;

(2) Furthermore, if $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} \neq \emptyset$ then the weight $w = 0$ and G°_ℓ is unipotent (hence $\overline{G}^\circ_\ell = 1$) for every prime ℓ and $G^\circ_\ell = 1$ for $\ell \gg 0$.

In general, Theorem 4 does not imply Conjecture A (C) for $\mathcal{V}_\ell$ part of a $\mathbb{Q}$-compatible family of pure $\mathbb{Q}_\ell$-local systems on X unless one could prove *e.g.* that, for every $x \in |X|$, the family of $\mathbb{Q}_\ell$-local system $A_x(\underline{\mathcal{V}}) := (A_x(\mathcal{V}_\ell))_\ell$ introduced in Subsection 5.2.2 is also $\mathbb{Q}$-compatible. However, Theorem 4 does imply Conjecture A (C) in the following easy *albeit* important cases ("large geometric monodromy").

Corollary 5. *Let $\underline{\mathcal{V}}$ be a $\mathbb{Q}$-compatible family of pure $\mathbb{Q}_\ell$-local systems on X. Assume one of the following conditions hold:*

(1) $\overline{G}^\circ_\ell$ is a Levi subgroup of G°_ℓ;

(2) $\overline{G}^\circ_\ell$ and the homotheties torus $\mathbb{G}_m(V_\ell) \simeq \mathbb{G}_{m,\mathbb{Q}_\ell} \subset \mathrm{GL}(V_\ell)$ generate a Levi subgroup of G°_ℓ.

[3] More precisely, if $\phi_{x_v,\mathrm{cris}} : D_{\mathrm{cris}}(x_v^*\mathcal{V}_p) \to D_{\mathrm{cris}}(x_v^*\mathcal{V}_p)$ denotes the (σ-semilinear) crystalline Frobenius then $\varphi_{x_v,\mathrm{cris}} := \phi^{m_v}_{x_v,\mathrm{cris}}$.

Then, for every prime ℓ, Conjecture A (C) holds for $\mathcal{V}_\ell$, namely $|X|^{\mathrm{cent}}_{\mathcal{V}_\ell} = \emptyset$ unless $\overline{G}^\circ_\ell = 1$ and $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} = \emptyset$ unless G°_ℓ is unipotent.

Proof. As $\overline{G}^\circ_\ell$ is semisimple, the assumptions and conclusions of Conjecture A (C) remain unchanged if one replaces $\mathcal{V}_\ell$ by its semisimplification so that, without loss of generality one may assume that $\mathcal{V}_\ell$ is semisimple, $\ell \in |\mathrm{spec}(\mathbb{Z})|$. In that case, Condition (1) becomes simply $\overline{G}^\circ_\ell = G^\circ_\ell$ and Condition (2) that $\overline{G}^\circ_\ell$ and $\mathbb{G}_m(V_\ell)$ generate G°_ℓ. By construction the family of $\mathbb{Q}_\ell$-local systems $\mathcal{E}_\ell := \mathcal{V}_\ell \otimes \mathcal{V}_\ell^\vee$, $\ell \in |\mathrm{spec}(\mathbb{Z})|$ is pointwise $\mathbb{Q}$-compatible and pure of weight 0. Fix a prime $\ell \in |\mathrm{spec}(\mathbb{Z})|$ and let $x \in |X|^{\mathrm{cent}}_{\mathcal{V}_\ell}$. As $\overline{G}^\circ_\ell$ is semisimple, the assumptions impose that $G^\circ_{\ell,x} \subset \mathbb{G}_m(V_\ell)$ hence $x \in |X|^{\mathrm{triv}}_{\mathcal{E}_\ell}$ and, by Theorem 4, $\overline{G}^{\mathrm{ad}\,\circ}_\ell = 1$ hence $\overline{G}^\circ_\ell = 1$ since $\overline{G}^\circ_\ell$ is semisimple. □

1.2.1.3. Say that $\chi \in \overline{\mathbb{Q}}[T]$ is generalized cyclotomic if all its roots are roots of unity. Theorem 4 (2) easily reduces to proving that the characteristic polynomials of Frobenii χ_{x_v} introduced in Subsection 1.2.1.1 are generalized cyclotomic. Writing $p := p_v$ for the residue characteristic of v, to prove that χ_{x_v} is generalized cyclotomic, it is enough to prove that

$$\text{i) } w = 0; \quad \text{ii) } \chi_{x_v} \in \mathbb{Q}[T]; \quad \text{iii) } \chi_{x_v} \in \mathbb{Z}_\ell[T],\ \ell \neq p; \quad \text{iv) } \chi_{x_v} \in \mathbb{Z}_p[T].$$

The purity assumption plus the fact that $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} \neq \emptyset$ ensure i), the $\mathbb{Q}$-compatibility ensures ii), iii). The proof of iv) relies on a deeper result of variational p-adic Hodge theory (due to Liu-Zhu, Petrov, Shimizu - see Fact 10) ensuring that being potentially unramified is a ("one point to pointwise" - see Remark 3) "rigid" property in the sense that if $x_0^*\mathcal{V}_p$ is potentially unramified for one $x_0 \in |X_{k_v}|$ then $x^*\mathcal{V}_p$ is potentially unramified for every $x \in |X_{k_v}|$. Actually, the proof of iv) is purely local and works for an arbitrary $\mathbb{Q}_p$-local system. The details of the proof of Theorem 4 are carried out in Subsection 5.2.2 and Subsection 5.3.

1.2.2. ***Subquotients of motivic $\mathbb{Q}_\ell$-local systems.*** Let k be a number field. The variational Fontaine-Mazur conjecture of Liu-Zhu [LiZ17, Conj. p.2] predicts that every $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X with $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} \neq \emptyset$ appears as a subquotient of $\mathcal{F}_\ell := R^{2i}f_*\mathbb{Q}_\ell(i)$ for some integer $i \geq 0$ and $f : Y \to X$ a smooth proper morphism, after possibly replacing X by a non-empty open subscheme. So the next case to investigate is the one of arbitrary subquotients $\mathcal{V}_\ell$ of such a $\mathcal{F}_\ell$. As $\underline{\mathcal{F}} := (\mathcal{F}_\ell)_\ell$ is a $\mathbb{Q}$-compatible family of pure $\mathbb{Q}_\ell$-local systems of weight 0 on X, i) and iii) automatically hold for such a $\mathcal{V}_\ell$. As already mentioned, iv) also holds. The main issue to extend the proof of Theorem 4 to $\mathcal{V}_\ell$ is that ii) does not hold in general for arbitrary subquotients of $\mathcal{F}_\ell$.

1.2.2.1. So, to treat more general $\mathbb{Q}_\ell$-local systems, one has to adjust the strategy of the proof of Theorem 4. The idea is to exploit further the fact that, given a finite place $v|\ell$ of k, as soon as $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} \neq \emptyset$ (or as soon as $\mathcal{V}_\ell|_{X_{k_v}}$ is Hodge-Tate and $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} \neq \emptyset$), for every $x \in |X_{k_v}|$, $x^*\mathcal{V}_\ell$ is potentially unramified. More precisely, assume X_{k_v} admits a smooth model $\mathcal{X}_{\mathcal{O}_v}$ over $\mathrm{spec}(\mathcal{O}_v)$, the technical core of our second main result is (a variant of - see Corollary 14) the following basic pointwise criterion (Theorem 12) for $\mathcal{V}_\ell|_{X_{k_v}}$ to extends to $\mathcal{X}_{\mathcal{O}_v}$: assume that $x^*\mathcal{V}_\ell$ is unramified for every x in the image of the map

$$\mathcal{X}_{\mathcal{O}_v}(\overline{\mathcal{O}}_v) \to |X_{k_v}|$$

then $\mathcal{V}_\ell$ extends to a $\mathbb{Q}_\ell$-local system $\widetilde{\mathcal{V}}_\ell$ on $\mathcal{X}_{\mathcal{O}_v}$. This result provides a key step for a general strategy aiming at proving Conjecture A in that it enables to consider the restriction $\widetilde{\mathcal{V}}_\ell|_{\mathcal{X}_v}$ of $\widetilde{\mathcal{V}}_\ell$ to the special fiber $\mathcal{X}_v$ of $\mathcal{X}_{\mathcal{O}_v}$, where one can reformulate the initial problem in terms of overconvergent F-isocrystals and try and exploit the companion correspondances of both Abe [A18] and L. Lafforgue [L02].

1.2.2.2. A first application of this strategy is the following. Let k be a number field. Assume X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to U$ over a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$. For a prime ℓ in the image of $|U| \to |\mathrm{spec}(\mathbb{Z})|$ and a finite place v in U above ℓ, let

$$sp_v : |\mathcal{X}^{\mathrm{cpt}}| \to |\mathcal{X}_v^{\mathrm{cpt}}|$$

denote the specialization map. Let $f : Y \to X$ be a smooth proper morphism and, for some integer $i \geq 0$ and every prime ℓ, set $\mathcal{F}_\ell := R^{2i}f_*\mathbb{Q}_\ell(i)$.

Theorem 6. *Assume X is a curve. For every prime ℓ in the image of $|U| \to |\mathrm{spec}(\mathbb{Z})|$, $\ell \gg 0$ and for every finite place v in U above ℓ, there exists a 0-dimensional Zariski-closed subset $\mathcal{Z}_v \subset \mathcal{X}_v^{\mathrm{cpt}}$ such that for every subquotient $\mathcal{V}_\ell$ of $\mathcal{F}_\ell$ one has*

(i) G°_ℓ is unipotent;
$\Leftrightarrow$ (ii) $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} = |X|$;
$\Leftrightarrow$ (iii)' $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} \not\subset \cap_{v \in U_\ell} sp_v^{-1}(|\mathcal{Z}_v|)$.

In other words, Conjecture A (U) (hence Conjecture A (T)) holds with condition (iii) weakened to condition (iii)'.

Remark 7.

(1) As by assumption $\mathcal{V}_\ell$ is pointwise Hodge-Tate, one actually has $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = |X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$ - see Proposition 20.

(2) Up to shrinking U, one may also assume $f : Y \to X$ admits a smooth proper model $f : \mathcal{Y} \to \mathcal{X}$ over U. Then, for $\ell \gg 0$ and for every finite place v in U above ℓ, the Zariski-closed subset $\mathcal{Z}_v \subset \mathcal{X}_v^{\mathrm{cpt}}$ appearing in the statement of Theorem 6 can be chosen explicitly, namely $\mathcal{X}_v \setminus \mathcal{Z}_v$ is the largest open subset over which the convervent F-isocrystal $R^{2i} f_{\mathrm{cris}*}\mathcal{O}_{\mathcal{Y}_v|\mathcal{X}_v}$ admits a slope filtration (equivalently, has constant Newton polygon). In particular, when $\mathcal{X} = \mathcal{X}^{\mathrm{cpt}}$ and $R^{2i} f_{\mathrm{cris}*}\mathcal{O}_{\mathcal{Y}_v|\mathcal{X}_v}$ has constant Newton polygon at least for one place v above ℓ, then (iii)' reads $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} \neq \emptyset$; in other words, under these assumptions, Conjecture A (U) (hence Conjecture A (T)) holds.

(3) Actually, Theorem 6 is a special case of a more general, and purely local, statement - See Theorem 29. In particular, the condition that $\mathcal{V}_\ell$ be a subquotient of a motivic $\mathbb{Q}_\ell$-local system can be relaxed to get, *e.g.* the following variant:

Theorem 8. *Assume X is a curve. Let ℓ be a prime in the image of $|U| \to |\mathrm{spec}(\mathbb{Z})|$. Then for every $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$ on X with $\ell > rank_{\mathbb{Q}_\ell}(\mathcal{V}_\ell) + 1$ and finite place v in U above ℓ, such that $\mathcal{V}_\ell|_{X_{k_v}}$ is Hodge-Tate, there exists a 0-dimensional Zariski-closed subset $\mathcal{Z}_v \subset \mathcal{X}_v^{\mathrm{cpt}}$ such that one has*

$$\begin{array}{l} (i)\ G_\ell^\circ \text{ is unipotent;} \\ \Leftrightarrow (ii)\ |X|^{\mathrm{uni}}_{\mathcal{V}_\ell} = |X|; \\ \Leftrightarrow (iii)'\ |X|^{\mathrm{uni}}_{\mathcal{V}_\ell} \not\subset sp_v^{-1}(|\mathcal{Z}_v|). \end{array}$$

1.3. **Outline.** After introducing technical level assumptions in Section 2 and reviewing in Section 3 the results from (variational) p-adic Hodge theory used in our proofs, we devote Section 4 to the proof of Theorem 12 or rather the key propositions - Proposition 16 and Proposition 18 - underlying it. The final Section 5 is devoted to the proofs of the global statements - Proposition 1, Theorem 4, Theorem 6 *etc.*

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2. Level

Some of the proofs and statements involve level assumptions that we list here for the convenience of the reader.

Let S be a connected scheme. Fix a prime ℓ and let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on S. Fix a geometric point $\bar{s}$ on S and set $V_\ell := \mathcal{V}_{\ell,\bar{s}}$; let $\Pi_\ell \subset \mathrm{GL}(V_\ell)$ denote the image of $\pi_1(S, \bar{s})$ acting on V_ℓ. Consider the following "level conditions" on $\mathcal{V}_\ell$:

$\mathrm{Lev}_1(\mathcal{V}_\ell)$ There exists a Π_ℓ-stable $\mathbb{Z}_\ell$-lattice $V_\ell^\circ \subset V_\ell$ such that $\Pi_\ell \subset Id + \tilde{\ell}\mathrm{End}_{\mathbb{Z}_\ell}(V_\ell^\circ)$, where $\tilde{\ell} = 4$ if $\ell = 2$ and $\tilde{\ell} = \ell$ otherwise.

$\mathrm{Lev}_2(\mathcal{V}_\ell)$ Π_ℓ is torsion free.

$\mathrm{Lev}_3(\mathcal{V}_\ell)$ Π_ℓ is pro-ℓ.

$\mathrm{Lev}_4(\mathcal{V}_\ell)$ The torsion elements in Π_ℓ are of prime-to-ℓ order.

One easily checks the following implications.

$$\mathrm{Lev}_1(\mathcal{V}_\ell) \Longrightarrow \mathrm{Lev}_2(\mathcal{V}_\ell) \Longleftrightarrow (\mathrm{Lev}_3(\mathcal{V}_\ell) + \mathrm{Lev}_4(\mathcal{V}_\ell)) \Longrightarrow \mathrm{Lev}_4(\mathcal{V}_\ell) \Longleftarrow \ell > \dim(V_\ell) + 1.$$

In practice, $\mathrm{Lev}_1(\mathcal{V}_\ell)$ can always be achieved after replacing S by a connected étale cover.

3. Pointwise versus global properties of $\mathbb{Q}_p$-local systems

Let k be a p-adic field with ring of integers and residue field $k \supset \mathcal{O}_k \twoheadrightarrow \kappa$; let v denote the closed point of $\mathrm{spec}(\mathcal{O}_k)$. Let $\mathbb{Q}_p \subset k_0 \subset k$ be the maximal unramified extension of $\mathbb{Q}_p$ contained in k and $\sigma : k_0 \tilde{\to} k_0$ its arithmetic Frobenius.

3.1. **Brief recollection of classical p-adic Hodge theory.**

3.1.1. Let $B_{\mathrm{cris}} \subset B_{\mathrm{dR}} =: B_{\mathrm{dR}}(k) = B_{\mathrm{dR}}(\overline{k})$ and $B_{\mathrm{HT}} = Gr(B_{\mathrm{dR}})$ denote Fontaine's period rings and the associated "Riemann-Hilbert" $\otimes$-functors

$$D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k)) \to \mathrm{M}^{\varphi}_{k_0},\ V \mapsto (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\pi_1(k)}$$
$$D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k)) \to F\text{-}\mathrm{M}_k,\ V \mapsto (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{\pi_1(k)},$$
$$D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k)) \to \mathrm{Gr}_k,\ V \mapsto (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{\pi_1(k)},$$

Here $\mathrm{M}^{\varphi}_{k_0}$ (resp. $F\text{-}\mathrm{M}_k$, resp. Gr_k) denote the category of k_0-modules of finite rank D equipped with a σ-semilinear endormorphism $\phi : D \to D$ (resp. of k-modules of finite rank D equipped with a descending separated exhaustive filtration $F^\bullet$ by k-submodules, resp. of k-modules of finite rank D equipped with a direct sum decomposition $D^\bullet$ by k-submodules). Let

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\pi_1(k)) \subset \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(k)) \subset \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(\pi_1(k)) \subset \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(k))$$

denote the full subcategories of crystalline (*viz* such that $\mathrm{rank}_{\mathbb{Q}_p}(V) = \mathrm{rank}_{k_0}(D_{\mathrm{cris}}(V))$), de Rham (*viz* such that $\mathrm{rank}_{\mathbb{Q}_p}(V) = \mathrm{rank}_k(D_{\mathrm{dR}}(V))$) and Hodge-Tate (*viz* such that $\mathrm{rank}_{\mathbb{Q}_p}(V) = \mathrm{rank}_k(D_{\mathrm{HT}}(V))$) representations. The functors $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(k)) \to F\text{-}\mathrm{M}_k$ and $D_{\mathrm{HT}} : \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(\pi_1(k)) \to \mathrm{Gr}_k$ are faithful exact $\otimes$-functors

3.1.2. The following implications are classical. Note that being Hodge-Tate with single Hodge-Tate weight 0 is the same thing as being $\mathbb{C}_p$-admissible. In particular, being Hodge-Tate and unipotent - hence a successive extension of the trivial representation $\mathbb{Q}_p$, implies being $\mathbb{C}_p$-admissible.

$$\begin{tikzcd}
\mathbb{C}_p\text{-admissible} \arrow[r, Leftrightarrow, "(1)"] & \text{potentially unramified} \arrow[r, Rightarrow] \arrow[dr, phantom, "\square\ (2)"] & \text{potentially crystalline} \arrow[r, Rightarrow] & \text{de Rham} \arrow[r, Rightarrow] & \text{Hodge-Tate} \\
& \text{unramified} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{crystalline} \arrow[u, Rightarrow] & &
\end{tikzcd}$$

The implication $\overset{(1)}{\Rightarrow}$ is a theorem of Sen [Se80, Cor. to Thm. 11]. For the fact that (2) is "Cartesian", namely that

$$(2) \quad \text{crystalline} + \text{potentially unramified} \Rightarrow \text{unramified},$$

see *e.g.* [Ca19, Prop. 4.3.2]. Let us also make the following observation, the proof of which was explained to us by Benjamin Schraen.

Lemma 9. *Let $V_p \in \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(\mathcal{O}_k))$. Then the elementary divisors of*

- *the image $\varphi_p : V_p \tilde{\to} V_p$ of the geometric Frobenius $\varphi \in \pi_1(\kappa) \simeq \pi_1(\mathcal{O}_k)$;*
- *the linearized crystalline Frobenius $\varphi : D_{\mathrm{cris}}(V_p) \tilde{\to} D_{\mathrm{cris}}(V_p)$,*

coincide. In particular, the characteristic polynomial of $\varphi : D_{\mathrm{cris}}(V_p) \tilde{\to} D_{\mathrm{cris}}(V_p)$ is in $\mathbb{Z}_p[T]$ and its roots are v-adic units.

Proof. Let $I_k := \ker(\pi_1(k) \to \pi_1(\mathcal{O}_k))$ denote the inertia group. Let also $k_0 \subset k_0^{\mathrm{ur}} \subset \overline{k}$ denote the maximal unramified extension of k_0 and $\widehat{k}_0^{\mathrm{ur}}$ its completion. Recall that by definition $\varphi = \phi^m : D_{\mathrm{cris}}(V_p) \to D_{\mathrm{cris}}(V_p)$, where $m := [k_0 : \mathbb{Q}_p]$ and $\phi : D_{\mathrm{cris}}(V_p) \to D_{\mathrm{cris}}(V_p)$ is the crystalline Frobenius. As V_p is crystalline, and using that $(B_{\mathrm{cris}})^{I_k} = \widehat{k}_0^{\mathrm{ur}}$ (*e.g.* [Fo94, Prop. 5.1.2]),

$$D_{\mathrm{cris}}(V_p) = (V_p \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\pi_1(k)} = (V_p \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}})^{I_k})^{\pi_1(\kappa)} = (V_p \otimes_{\mathbb{Q}_p} \widehat{k}_0^{\mathrm{ur}})^{\pi_1(\kappa)} =: D_{\widehat{k}_0^{\mathrm{ur}}}(V_p)$$

has k_0-dimension $\dim_{\mathbb{Q}_p}(V_p)$. In other words, V_p is $\widehat{k}_0^{\mathrm{ur}}$-admissible, hence the canonical $\widehat{k}_0^{\mathrm{ur}}$-linear injective morphism

$$\alpha : D_{\widehat{k}_0^{\mathrm{ur}}}(V_p) \otimes_{k_0} \widehat{k}_0^{\mathrm{ur}} \to V_p \otimes_{\mathbb{Q}_p} \widehat{k}_0^{\mathrm{ur}}$$

is an isomorphism, which is equivariant with the following structures:

- The $\pi_1(k)$-action (with $D_{\widehat{k}_0^{\mathrm{ur}}}(V_p)$ viewed as a trivial $\pi_1(k)$-representation);

- The crystalline Frobenii (with the crystalline Frobenius on V_p being the identity and the one on $\widehat{k}_0^{\mathrm{ur}}$ the lift $\sigma : \widehat{k}_0^{\mathrm{ur}} \tilde{\to} \widehat{k}_0^{\mathrm{ur}}$ of the arithmetic Frobenius on the residue field).

In particular, $\alpha : D_{\widehat{k}_0^{\mathrm{ur}}}(V_p) \otimes_{k_0} \widehat{k}_0^{\mathrm{ur}} \tilde{\to} V_p \otimes_{\mathbb{Q}_p} \widehat{k}_0^{\mathrm{ur}}$ exchanges

$$Id \otimes_{k_0} \sigma^m \longleftrightarrow \varphi_p^{-1} \otimes_{\mathbb{Q}_p} \sigma^m, \quad \phi \otimes_{k_0} \sigma \longleftrightarrow Id \otimes_{\mathbb{Q}_p} \sigma.$$

As a result,

$$\alpha \circ (\phi^m \otimes_{k_0} Id) \circ \alpha^{-1} = \alpha \circ (\phi \otimes_{k_0} \sigma)^m (Id \otimes_{k_0} \sigma^m)^{-1} \circ \alpha^{-1} = (Id \otimes_{\mathbb{Q}_p} \sigma)^m (\varphi_p^{-1} \otimes_{\mathbb{Q}_p} \sigma^m)^{-1} = \varphi_p \otimes_{\mathbb{Q}_p} Id.$$

This shows the two k_0-linear morphisms $\varphi_p \otimes_{\mathbb{Q}_p} Id_{k_0} : V_p \otimes_{\mathbb{Q}_p} k_0 \tilde{\to} V_p \otimes_{\mathbb{Q}_p} k_0$ and $\phi^m : D_{\widehat{k}_0^{\mathrm{ur}}}(V_p) \tilde{\to} D_{\widehat{k}_0^{\mathrm{ur}}}(V_p)$ have the same invariant factors hence, in particular, the same characteristic polynomial. □

3.2. **Pointwise versus global properties.** Let X be a smooth variety over k. Let $\mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on X, write

$$|X|^{\mathrm{ur}}_{\mathcal{V}_p} \subset |X|^{\mathrm{cris}}_{\mathcal{V}_p} \subset |X|^{\mathrm{dR}}_{\mathcal{V}_p} \subset |X|^{\mathrm{HT}}_{\mathcal{V}_p} \subset |X|$$

for the subsets of all $x \in |X|$ such that $x^*\mathcal{V}_p$ is unramified, crystalline, de Rham and Hodge-Tate respectively. Say that $\mathcal{V}_p$ is **pointwise unramified** if $|X|^{\mathrm{ur}} = |X|$; define similarly the notion of being **pointwise crystalline**, **pointwise de Rham** and **pointwise Hodge-Tate**.

3.2.1. *Hodge-Tate, de Rham and crystalline local systems.* There are also global notions of crystalline, de Rham and Hodge-Tate $\mathbb{Q}_p$-local systems on X defined using geometric versions of Fontaine's Riemann-Hilbert functors. More precisely, let $X^{\mathrm{an}} \to X$ denote the rigid-analytification of X. The natural morphism of sites $X^{\mathrm{an}}_{\mathrm{et}} \to X_{\mathrm{et}}$ induces a faithful exact $\otimes$-functor

$$(-)^{\mathrm{an}} : \mathrm{Loc}_{\mathbb{Z}_p}(X_{\mathrm{et}}) \to \mathrm{Loc}_{\mathbb{Z}_p}(X^{\mathrm{an}}_{\mathrm{et}})$$

from the category $\mathrm{Loc}_{\mathbb{Z}_p}(X_{\mathrm{et}})$ of $\mathbb{Z}_p$-local systems on X_{et} to the category $\mathrm{Loc}_{\mathbb{Z}_p}(X^{\mathrm{an}}_{\mathrm{et}})$ of $\mathbb{Z}_p$-local systems on $X^{\mathrm{an}}_{\mathrm{et}}$ hence, passing to the isogeny category, a faithful exact $\otimes$-functor

$$(-)^{\mathrm{an}} : \mathrm{Loc}_{\mathbb{Q}_p}(X_{\mathrm{et}}) \to \mathrm{Loc}_{\mathbb{Q}_p}(X^{\mathrm{an}}_{\mathrm{et}}).$$

- Let $\mathrm{Higgs}(X^{\mathrm{an}})$ denote the category of vector bundles with a nilpotent Higgs field on X^{an}. If

$$D_{\mathrm{HT}} : \mathrm{Loc}_{\mathbb{Q}_p}(X^{\mathrm{an}}_{\mathrm{et}}) \to \mathrm{Higgs}(X^{\mathrm{an}})$$

denotes the natural Hodge-Tate Riemann-Hilbert functor constructed in [LiZ17, §2.1], one says that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on $X^{\mathrm{an}}_{\mathrm{et}}$ is **Hodge-Tate** if

$$\mathrm{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) = \mathrm{rank}(D_{\mathrm{HT}}(\mathcal{V}_p)),$$

and that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on X_{et} is Hodge-Tate if $\mathcal{V}_p^{\mathrm{an}}$ is.

- Let $F\text{-}\mathrm{Vect}^{\nabla}(X^{\mathrm{an}})$ denote the category of filtered vector bundles on X^{an} with a flat connection satisfying Griffith's transversality. If

$$D_{\mathrm{dR}} : \mathrm{Loc}_{\mathbb{Q}_p}(X^{\mathrm{an}}_{\mathrm{et}}) \to F\text{-}\mathrm{Vect}^{\nabla}(X^{\mathrm{an}})$$

denotes the natural de Rham Riemann-Hilbert functor constructed in [LiZ17, §3.2], one says that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on $X^{\mathrm{an}}_{\mathrm{et}}$ is **de Rham** if

$$\mathrm{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) = \mathrm{rank}(D_{\mathrm{dR}}(\mathcal{V}_p)),$$

and that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on X_{et} is de Rham if $\mathcal{V}_p^{\mathrm{an}}$ is.

Assume furthermore $X \to \mathrm{spec}(k)$ admits a model $\mathcal{X} \to \mathrm{spec}(\mathcal{O}_k)$, smooth, separated and of finite type. Write $\widehat{\mathcal{X}}$ for the formal completion of $\mathcal{X}$ along the closed fiber $\mathcal{X}_v$. Let $\widehat{\mathcal{X}}_\eta$ denote the rigid-analytic fiber of $\widehat{\mathcal{X}}$ so that one gets an open immersion $\widehat{\mathcal{X}}_\eta \hookrightarrow X^{\mathrm{an}}$ of rigid analytic spaces.

- Let $F\text{-}\mathrm{wIsoc}(\mathcal{X}_v/\mathcal{O}_{k_0})$ denote the category of weak F-isocrystals on $\mathcal{X}_v/\mathcal{O}_{k_0}$ [GY24, Def. 5.10]. If

$$D^{\mathrm{an}}_{\mathrm{cris}} : \mathrm{Loc}_{\mathbb{Q}_p}(\widehat{\mathcal{X}}_{\eta,\mathrm{et}}) \to F\text{-}\mathrm{wIsoc}(\mathcal{X}_v/\mathcal{O}_{k_0})$$

denotes the natural crystalline Riemann-Hilbert functor constructed in [GY24, Thm. 1.10] and one defines

$$D_{\mathrm{cris}} : \mathrm{Loc}_{\mathbb{Z}_p}(X^{\mathrm{an}}_{\mathrm{et}}) \xrightarrow{|_{\widehat{\mathcal{X}}_\eta}} \mathrm{Loc}_{\mathbb{Q}_p}(\widehat{\mathcal{X}}_{\eta,\mathrm{et}}) \xrightarrow{D^{\mathrm{an}}_{\mathrm{cris}}} F\text{-}\mathrm{wIsoc}(\mathcal{X}_v/\mathcal{O}_{k_0}),$$

one says that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on $X^{\mathrm{an}}_{\mathrm{et}}$ is crystalline if $D_{\mathrm{cris}}(\mathcal{V}_p)$ has constant rank $\mathrm{rank}(D_{\mathrm{cris}}(\mathcal{V}_p))$ and

$$\mathrm{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) = \mathrm{rank}(D_{\mathrm{cris}}(\mathcal{V}_p)).$$

One says that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on X_{et} is **crystalline** (with respect to $\widehat{\mathcal{X}}_\eta$) if $\mathcal{V}_p^{\mathrm{an}}$ is.

The following summarizes the relation between the pointwise and global Hodge-Tate, de Rham and crystalline properties.

Fact 10. *Let X be a smooth, geometrically connected variety over k. Let $\mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on X. Then,*

(1) ([P23, §7]; see also [Shim18]) *One has*

$$\text{(i) } \mathcal{V}_p \text{ is Hodge-Tate} \quad \Leftrightarrow \quad \text{(ii) } |X|^{\mathrm{HT}}_{\mathcal{V}_p} = |X| \quad \Leftrightarrow \quad \text{(iii) } |X|^{\mathrm{HT}}_{\mathcal{V}_p} \neq \emptyset;$$

Furthermore, if $\mathcal{V}_p$ is Hodge-Tate, the multiset $HT(\mathcal{V}_p) := HT(x^*\mathcal{V}_p)$ of Hodge-Tate weights of $x^*\mathcal{V}_p$ is independent of $x \in |X|$.

(2) ([LiZ17, Thm. 1.1, Thm. 1.3]) *One has*

$$\text{(i) } \mathcal{V}_p \text{ is de Rham} \quad \Leftrightarrow \quad \text{(ii) } |X|^{\mathrm{dR}}_{\mathcal{V}_p} = |X| \quad \Leftrightarrow \quad \text{(iii) } |X|^{\mathrm{dR}}_{\mathcal{V}_p} \neq \emptyset.$$

(3) ([GY24, Thm. 7.2]) *Assume furthermore X admits a smooth model $\mathcal{X} \to \mathrm{spec}(\mathcal{O}_k)$. One has*

$$\text{(i) } \mathcal{V}_p \text{ is crystalline (with respect to } \widehat{\mathcal{X}}_\eta) \quad \Leftrightarrow \quad \text{(ii)' } |X|^{\mathrm{cris}}_{\mathcal{V}_p} \supset \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_k) \to |X|).$$

Note that, in particular, the property of being a Hodge-Tate, de Rham or crystalline $\mathbb{Q}_p$-local system is preserved by passing to subquotients.

Corollary 11. *Let X be a smooth, geometrically connected variety over k and let $\mathcal{V}_p$ a $\mathbb{Q}_p$-local system on X. The following properties are equivalent*
(ii) for every $x \in |X|$, $x^*\mathcal{V}_p$ is potentially unramified;
(iii) there exists $x \in |X|$ such that $x^*\mathcal{V}_p$ is potentially unramified.

If these hold, then $|X|^{\mathrm{ur}}_{\mathcal{V}_p} = |X|^{\mathrm{cris}}_{\mathcal{V}_p}$ and for every $x \in |X|^{\mathrm{cris}}_{\mathcal{V}_p}$ the characteristic polynomials of

- *the geometric Frobenius $\varphi_{x,p} : \mathcal{V}_{p,\bar{x}} \tilde{\to} \mathcal{V}_{p,\bar{x}}$;*
- *the linearized crystalline Frobenius $\varphi_{x,\mathrm{cris}} : D_{\mathrm{cris}}(x^*\mathcal{V}_p) \tilde{\to} D_{\mathrm{cris}}(x^*\mathcal{V}_p)$,*

coincide. In particular, the characteristic polynomial of $\varphi_{x,\mathrm{cris}} : D_{\mathrm{cris}}(x^*\mathcal{V}_p) \tilde{\to} D_{\mathrm{cris}}(x^*\mathcal{V}_p)$ is in $\mathbb{Z}_p[T]$ and its roots are v-adic units.

Proof. According to the equivalence (1) of Subsection 3.1.2, the equivalence (ii) $\Leftrightarrow$ (iii) is a special case of Fact 10 (1). The equality $|X|^{\mathrm{ur}}_{\mathcal{V}_p} = |X|^{\mathrm{cris}}_{\mathcal{V}_p}$ follows from the implication (2) in Subsection 3.1.2. The last part of the assertion then follows from Lemma 9. □

3.2.2. *Unramified and tamely ramified local systems.* Assume X admits a smooth model $\mathcal{X} \to \mathrm{spec}(\mathcal{O}_k)$.

3.2.2.1.One says that a $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on X is **unramified with respect to** $\mathcal{X}$ if the corresponding representation of $\pi_1(X)$ on $V_p := \mathcal{V}_{p,\bar{x}}$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X})$ (*viz* $\mathcal{V}_p$ extends to a $\mathbb{Q}_p$-local system on $\mathcal{X}$) and that $\mathcal{V}_p$ is **tamely ramified with respect to** $\mathcal{X}$ if the corresponding representation factors through the tame étale fundamental group $\pi_1(X) \twoheadrightarrow \pi_1^t(\mathcal{X}; \mathcal{X}_v)$ (*viz* $\mathcal{V}_p$ is tamely ramified along $\mathcal{X}_v$). Let

$$|X|^{\mathrm{tr}}_{\mathcal{V}_p} \subset |X|$$

denote the subset of all $x \in |X|$ such that $x^*\mathcal{V}_p$ is tamely ramified.

Theorem 12. *Let $\mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on X. One has*

$$\text{(1) (i) } \mathcal{V}_p \text{ is unramified with respect to } \mathcal{X} \quad \Leftrightarrow \quad \text{(ii)' } |X|^{\mathrm{ur}}_{\mathcal{V}_p} \supset \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_k) \to |X|).$$
$$\text{(2) (i) } \mathcal{V}_p \text{ is tamely ramified with respect to } \mathcal{X} \quad \Leftrightarrow \quad \text{(ii)' } |X|^{\mathrm{tr}}_{\mathcal{V}_p} \supset \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_k) \to |X|).$$

Remark 13.

(1) Guo-Yang proved Theorem 12 (1) in the setting of smooth p-adic formal schemes over $\mathcal{O}_k$ - see [GY24, Thm. 6.31].

(2) Properties (i), (ii)' really depend on the smooth model $\mathcal{X} \to \mathrm{spec}(\mathcal{O}_k)$ of $X \to \mathrm{spec}(k)$. However, if $X \to \mathrm{spec}(k)$ is proper, then Properties (i), (ii)' for a given smooth proper model $\mathcal{X} \to \mathrm{spec}(\mathcal{O}_k)$ are also equivalent to the property

$$\text{(ii) } |X|^{\mathrm{ur}}_{\mathcal{V}_p} = |X|,$$

which is independent of the smooth proper model $\mathcal{X} \to \mathrm{spec}(\mathcal{O}_k)$.

We postpone the proof of Theorem 12 to Section 4.

3.2.2.2. Assume X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$; write $\mathcal{D} := \mathcal{X}^{\mathrm{cpt}} \setminus \mathcal{X}$. Then, by Abhyankar's lemma

$$\pi_1^{\mathrm{t}}(\mathcal{X}; \mathcal{X}_v) \simeq \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}; \mathcal{D}) \times_{\pi_1(v)} \pi_1^{\mathrm{t}}(\mathcal{O}_k; v) \simeq \pi_1^{\mathrm{t}}(\mathcal{X}_v^{\mathrm{cpt}}; \mathcal{D}_v) \times_{\pi_1(v)} \pi_1^{\mathrm{t}}(\mathcal{O}_k; v)$$

while

$$\pi_1(\mathcal{X}) \simeq \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}; \mathcal{D}) \simeq \pi_1^{\mathrm{t}}(\mathcal{X}_v^{\mathrm{cpt}}; \mathcal{D}_v).$$

(1) If $\mathrm{Lev}_3(\mathcal{V}_p)$ holds and $|X|^{\mathrm{tr}}_{\mathcal{V}_p} \supset \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_k) \to |X|)$, then the action of $\pi_1(X)$ on $\mathcal{V}_{p,\bar{x}}$ factors through $\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X}^{\mathrm{cpt}})$.

(2) Say that a connected étale cover $X' \to X$ is **good with respect to** $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$ if the following holds. Let k' be the algebraic closure of k in the function field of X' and $\mathcal{X}' \to \mathcal{X}$, $\mathcal{X}'^{\mathrm{cpt}} \to \mathcal{X}^{\mathrm{cpt}}$ the normalization of $\mathcal{X}$, $\mathcal{X}^{\mathrm{cpt}}$ in $X' \to X \to \mathcal{X}$, $X' \to X \to \mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}}$ respectively. Then the resulting canonical sequence of morphisms $\mathcal{X}' \to \mathcal{X}'^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_{k'})$ is again a smooth NCC over $\mathrm{spec}(\mathcal{O}_{k'})$. Say that an open subgroup $U \subset \pi_1(X)$ is good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$ if the corresponding connected étale cover $X_U \to X$ is. The open subgroups

$$U_1 \times_{\pi_1(v)} U_2 \subset \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}; \mathcal{D}) \times_{\pi_1(v)} \pi_1^{\mathrm{t}}(\mathcal{O}_k; v)$$

with $U_1 \subset \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}; \mathcal{D})$, $U_2 \subset \pi_1^{\mathrm{t}}(\mathcal{O}_k; v)$ open subgroups form a cofinal family of open subgroups of

$$\pi_1^{\mathrm{t}}(\mathcal{X}; \mathcal{X}_v) \simeq \pi_1^{\mathrm{t}}(\mathcal{X}^{\mathrm{cpt}}; \mathcal{D}) \times_{\pi_1(v)} \pi_1^{\mathrm{t}}(\mathcal{O}_k; v)$$

and, if X is a curve, the inverse images of these groups in $\pi_1(X)$ are good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$.

These observations combined with Corollary 11 and Theorem 12 yields the following variant / strengthening of Theorem 12 (1) in the case X is a curve.

Corollary 14. *Let X be a curve. Assume X admits a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$. Let $\mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on X such that $\mathrm{Lev}_4(\mathcal{V}_p)$ holds. Assume there exists $x \in |X|$ such that $x^*\mathcal{V}_p$ is potentially unramified. Then there exists a connected étale cover $X' \to X$, which is good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$ and such that $\mathrm{Lev}_3(\mathcal{V}_p|_{X'})$ holds. In particular, the following properties are equivalent*

(i) the action of $\pi_1(X')$ on $\mathcal{V}_{p,\bar{x}}$ factors through $\pi_1(X') \twoheadrightarrow \pi_1(\mathcal{X}'^{\mathrm{cpt}})$ (viz $\mathcal{V}_p|_{X'}$ extends to a $\mathbb{Q}_p$-local system on $\mathcal{X}'^{\mathrm{cpt}}$);
(ii) for every $x' \in |X'|$, $x'^*\mathcal{V}_p$ is unramified,

where $\mathcal{X}'^{\mathrm{cpt}} \to \mathcal{X}^{\mathrm{cpt}}$ denotes the normalization of $\mathcal{X}^{\mathrm{cpt}}$ in $X' \to X \to \mathcal{X} \to \mathcal{X}^{\mathrm{cpt}}$.

Proof. Assume there exists $x \in |X|$ such that $x^*\mathcal{V}_p$ is potentially unramified. Then, from Corollary 11, for every $x \in |X|$, $x^*\mathcal{V}_p$ is potentially unramified hence, as $\mathrm{Lev}_4(\mathcal{V}_p)$ holds, tamely ramified. From Theorem 12 (2) (ii)' $\Rightarrow$ (i), $\mathcal{V}_p$ is tamely ramified with respect to $\mathcal{X}$ and, from the observation in (2) above, there exists a connected étale cover $X' \to X$ which is good with respect to $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$ and such that $\mathrm{Lev}_1(\mathcal{V}_p|_{X'})$ - hence $\mathrm{Lev}_3(\mathcal{V}_p|_{X'})$ hold. The implication (ii)' $\Rightarrow$ (i) then follows from the observation (1) above (the implication (i) $\Rightarrow$ (ii)' is straightforward). $\square$

4. Pointwise versus global ramification properties

As the results of this section might be of independent interest, we work in a slightly more general setting than the one of p-adic fields.

For a normal scheme X and a normal crossing divisor $D \hookrightarrow X$, let $\pi_1^{\mathrm{t}}(X; D)$ denote the fundamental group classifying finite connected covers $Y \to X$, which are étale over $X \setminus D$ and tamely ramified along D, namely such that for every generic point $\xi \in D$, the corresponding valuation ring $\mathcal{O}_{X,\xi}$ is tamely ramified in the extension of function fields $k(X) \hookrightarrow k(Y)$. This gives rise to an exact diagram of profinite groups:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & I_D^{\mathrm{w}} \arrow[r, equal] \arrow[d] & I_D^{\mathrm{w}} \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & I_D \arrow[r] \arrow[d] & \pi_1(X \setminus D) \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & I_D^{\mathrm{t}} \arrow[r] \arrow[d] & \pi_1^{\mathrm{t}}(X; D) \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

4.1. **Notation and definitions.**

4.1.1. Let $\mathcal{O}$ be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, fraction field k, of characteristic 0 and perfect residue field κ, of characteristic $p > 0$. Set $S := \mathrm{spec}(\mathcal{O}) = \{\eta, s\}$, where η is the generic point and s the closed point of S. Fix a separable($=$ algebraic) closure $k \hookrightarrow \overline{k}$. Considering the normal crossing divisor $s \hookrightarrow S$, we use the more classical notation:

$$I_k := I_s,\ I_k^{\mathrm{w}} := I_s^{\mathrm{w}},\ I_k^{\mathrm{t}} := I_s^{\mathrm{t}}$$

and

$$G_k := \pi_1(\eta) = \pi_1(S \setminus s),\ G_k^{\mathrm{ur}} := \pi_1(S) \xleftarrow{\sim} \pi_1(s) =: G_\kappa,\ G_k^{\mathrm{t}} := \pi_1^{\mathrm{t}}(S; s)$$

Correspondingly, one has the diagram of field extensions

$$\begin{tikzcd}
k \arrow[r, hook] \arrow[r, bend left, dash, "G_k^{\mathrm{ur}}=G_\kappa"] \arrow[rr, bend right, dash, "G_k^{\mathrm{t}}"'] & k^{\mathrm{ur}} \arrow[r, hook] \arrow[r, bend left, dash, "I_k^{\mathrm{t}}"] \arrow[rr, bend left=50, dash, "I_k"] & k^{\mathrm{t}} \arrow[r, hook] \arrow[r, bend left, dash, "I_k^{\mathrm{w}}"] & \overline{k}
\end{tikzcd}$$

For $\# := \overline{}$, t, ur, *etc.* let $\mathcal{O}^\#$ denote the valuation ring of $k^\#$, $\mathfrak{m}^\#$ its maximal ideal and $\kappa^\#$ its residue field; set $S^\# := \mathrm{spec}(\mathcal{O}^\#) = \{\eta^\#, s^\#\}$, where $\eta^\#$ is the generic point and $s^\#$ the closed point of $S^\#$.

4.1.2. Let $\mathcal{X} \to S$ be a morphism, smooth, separated and of finite type. Set $X := \mathcal{X}_\eta$; up to replacing S by its normalization in $\mathcal{X} \to S$, we may and will assume that X is geometrically connected over k. Let $\mathcal{X}_s = \mathcal{X}_{s,1} \sqcup \cdots \sqcup \mathcal{X}_{s,m}$ denote the decomposition of $\mathcal{X}_s$ into irreducible (*viz* connected) components.

4.1.3. Let $\mathcal{X} \to$ be a morphism, smooth, separated and of finite type, let $\psi : Y \to X$ be a Galois (in particular finite, étale and connected) cover and let $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ denote the normalization of $\mathcal{X}$ in $Y \xrightarrow{\psi} X \hookrightarrow \mathcal{X}$. The morphism $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is finite but not smooth in general.

Assume Y is geometrically integral over k. Let $k \hookrightarrow k'$ be a finite field extension and let $S' := \mathrm{spec}(\mathcal{O}') \to S$ denote the normalization of S in $\mathrm{spec}(k') \to \mathrm{spec}(k) \to S$. Let $\mathcal{Y}_1' \to \mathcal{X}$ denote the normalization of $\mathcal{X} \times_S S'$ in $Y \times_k k' \to X \times_k k' \hookrightarrow \mathcal{X} \times_S S'$ and let $\mathcal{Y}_2' \to \mathcal{X} \times_S S'$ denote the normalization of $\mathcal{X} \times_S S'$ in $\mathcal{Y} \times_S S' \to \mathcal{X} \times_S S'$. Note that, by the universal property of normalization the morphism $\mathcal{Y}_i' \to \mathcal{X} \times_S S' \to \mathcal{X}$ factors canonically as $\mathcal{Y}_i' \to \mathcal{Y} \to \mathcal{X}$, $i = 1, 2$. Furthermore the canonical morphism $\mathcal{Y}_1' \to \mathcal{Y}_2'$ is an isomorphism and if $S' \to S$ is étale, $\mathcal{Y} \times_S S' \to \mathcal{Y}_2'$ is an isomorphism [Stacks, Tag 03GV].

For a closed point $x \in |X|$ with residue field $k(x)$, write $\mathcal{O}_x$ for the valuation ring of $k(x)$, $S_x := \mathrm{spec}(\mathcal{O}_x) = \{x, s_x\}$, with ($x$ the generic point and) s_x the closed point of S_x. For a subset $\Sigma \subset |X|$, say that $\psi : Y \to X$ is Σ-pointwise unramified (resp. tame) if for every $x \in \Sigma$ and $y \in Y_x$ the resulting cover $S_y \to S_x$ is étale (resp. tamely ramified along s_x). We will apply this terminology to the following subsets:

- $\Sigma = |X|^{\mathrm{int}} := \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}) \hookrightarrow X(\overline{k}) \to X)$; one can easily check that $|X|^{\mathrm{int}}$ is also the subset of all $x \in |X|$ such that

$$\begin{tikzcd} \mathrm{spec}(k(x)) \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ \mathrm{spec}(\mathcal{O}_x) \arrow[r, dotted, "\exists!\,\tilde{x}"] & \mathcal{X} \end{tikzcd}$$

- For $\# = \mathrm{ur}, \mathrm{t}$, $\Sigma =]u[^{\#}$ for some $u \in |\mathcal{X}_s|$, where $]u[^{\#} := \mathrm{im}(\mathcal{X}(\mathcal{O}^{\#})_u \hookrightarrow X(k^{\#}) \to X) (\subset |X|^{\mathrm{int}})$ and $\mathcal{X}(\mathcal{O}^{\#})_u$ denotes the fiber of $\mathcal{X}(\mathcal{O}^{\#}) \to \mathcal{X}(\overline{\kappa}) \to |\mathcal{X}_s|$ over u; one can easily check that $]u[^{\#}$ is also the subset of all $x \in |X|$ such that

$$\begin{tikzcd} & \mathrm{spec}(k(x)) \arrow[rr, "x"] \arrow[d] & & X \arrow[d] \\ \mathrm{spec}(\mathcal{O}^{\#}) \arrow[r] & \mathrm{spec}(\mathcal{O}_x) \arrow[rr, dotted, "\exists!\,\tilde{x}"] & & \mathcal{X} \\ \mathrm{spec}(\overline{\kappa}) \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & \mathrm{spec}(\kappa(s_x)) \arrow[r] \arrow[u] \arrow[urr, phantom, "\square"] & \mathrm{spec}(\kappa(u)) \arrow[r, "u"] & \mathcal{X}_s \arrow[u] \end{tikzcd}$$

Let $k \hookrightarrow k'$ be a finite field extension with ring of integer $\mathcal{O}'$ and residue field κ'. Write $S' := \mathrm{spec}(\mathcal{O}') \to S$ for the corresponding connected cover, $s' \in S'$ for the closed point of S' and let $\mathcal{X}_{s,i} \times_\kappa \kappa' = \sqcup_{1 \leq j \leq m_i} \mathcal{X}_{s',i,j}$ be the decomposition into irreducible (*viz* connected) components of $\mathcal{X}_{s,i} \times_\kappa \kappa'$, $i = 1, \ldots, m$. For every non-empty open subset $\mathcal{X}^\circ_{s',i,j} \subset \mathcal{X}_{s',i,j}$, the image $\mathcal{X}^o_{s,i} \subset \mathcal{X}_s$ of $\sqcup_{1 \leq j \leq m_i} \mathcal{X}^\circ_{s',i,j}$ *via* $\mathcal{X}_{s'} \to \mathcal{X}_s$ is again a non-empty open subset of $\mathcal{X}_{s,i}$. Further,

Lemma 15. *Assume $k \subset k^{\mathrm{ur}}$ (resp. $k \subset k^{\mathrm{t}}$). Then for every $u \in \mathcal{X}_s$, the square*

$$\begin{tikzcd} \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathrm{ur}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] &]u[^{\mathrm{ur}} \arrow[d, hook] \\ |X \times_k k'| \arrow[r, two heads] & |X| \end{tikzcd} \quad \textit{(resp. the square} \quad \begin{tikzcd} \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathrm{t}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] &]u[^{\mathrm{t}} \arrow[d, hook] \\ |X \times_k k'| \arrow[r, two heads] & |X| \end{tikzcd})$$

is (well-defined and) Cartesian.

Proof. Let $x \in |X|$ and $x' \in (X \times_k k')_x$. We are to prove that $x \in]u[^{\mathrm{ur}}$ if and only if $x' \in \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathrm{ur}}$ (resp. $x \in]u[^{\mathrm{t}}$ if and only if $x' \in \bigcup_{u' \in (\mathcal{X}_{s'})_u}]u'[^{\mathrm{t}}$). The if part of the assertion, which ensures that the upper horizontal arrow is well defined, follows from the definition of $]-[^{\mathrm{ur}}$ (resp. $]-[^{\mathrm{t}}$). Let us prove the only if part. Assume $x \in]u[^{\mathrm{ur}}$ (resp. $x \in]u[^{\mathrm{t}}$). The fact that $x' : \mathrm{spec}(k(x')) \to X \times_k k'$ extends (uniquely) to $\tilde{x}' : S_{x'} \to \mathcal{X} \times_S S'$ follows from the fact that $x : \mathrm{spec}(k(x)) \to X$ extends to $\tilde{x} : S_x \to \mathcal{X}$ and the properness of $\mathcal{X} \times_S S' \to \mathcal{X}$ and the fact that $\mathcal{O}_{x'} \subset \mathcal{O}^{\mathrm{ur}}$ (resp. $\mathcal{O}_{x'} \subset \mathcal{O}^{\mathrm{t}}$) from the fact that $\mathcal{X} \times_S S' \to S$ is étale (resp. at most tamely ramified along $\mathcal{X}_s$). $\square$

4.2. A pointwise criterion for $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ to be étale.

Proposition 16. *Let $\psi : Y \to X$ be a Galois cover. There exists a non-empty open subset $\mathcal{X}^o_{s,i} \subset \mathcal{X}_{s,i}$, $i = 1, \ldots, m$ (depending on $Y \stackrel{\psi}{\to} X \to \mathcal{X}$) such that, for every $u_i \in \mathcal{X}^o_{s,i}$, $i = 1, \ldots, m$, the following conditions are equivalent.*

(U-1) $\ker(\pi_1(X) \twoheadrightarrow \pi_1(\mathcal{X})) \subset \pi_1(Y)$;
(U-2) $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ *is (finite) étale;*
(U-3) $\psi : Y \to X$ *is* $|X|^{\mathrm{int}}$*-pointwise unramified;*
(U-4) $\psi : Y \to X$ *is* $\cup_{1 \leq i \leq m}]u_i[^{\mathrm{ur}}$*-pointwise unramified.*

Proof. The implications (U-1) $\Rightarrow$ (U-2) $\Rightarrow$ (U-3) $\Rightarrow$ (U-4) and (U-2) $\Rightarrow$ (U-1) are (almost) tautological so we are left to prove (U-4) $\Rightarrow$ (U-2). If $\mathcal{X}_s = \emptyset$, there is nothing to prove, so we may and will assume that $\mathcal{X}_s \neq \emptyset$.

Let us first observe that it is enough to prove (U-4) $\Rightarrow$ (U-2) after:

- Replacing k by the algebraic closure k_Y of k in the function field of Y. Let $S_Y \to S$ denote the normalization of S in $\mathrm{spec}(k_Y) \to \mathrm{spec}(k) \to S$. As $\mathcal{X}_s \neq \emptyset$, by condition (U-4) $S_Y \to S$ is dominated by an étale cover of S hence is étale. As $\mathcal{X} \times_S S_Y$ is normal, one has a canonical factorization $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X} \times_S S_Y \to \mathcal{X}$ and $\mathcal{Y} \to \mathcal{X} \times_S S_Y$ is the normalization of Y in $Y \to X \times_k k_Y \to \mathcal{X} \times_S S_Y$. Let $s_Y \in S_Y$ denote the closed point of S_Y and κ_Y its residue field. Let $\mathcal{X}_{s,i} \times_\kappa \kappa_Y = \sqcup_{1\leq j\leq m_i} \mathcal{X}_{s_Y,i,j}$ be the decomposition into irreducible (*viz* connected) components of $\mathcal{X}_{s,i} \times_\kappa \kappa_Y$, $i = 1, \ldots, m$. For every non-empty open subset $\mathcal{X}^\circ_{s_Y,i,j} \subset \mathcal{X}_{s_Y,i,j}$, the image $\mathcal{X}^o_{s,i} \subset \mathcal{X}_s$ of $\sqcup_{1\leq j\leq m_i} \mathcal{X}^\circ_{s_Y,i,j}$ *via* $\mathcal{X}_{s_Y} \to \mathcal{X}_s$ is again a non-empty open subset of $\mathcal{X}_{s,i}$. Now, assume (U-4) $\Rightarrow$ (U-2) holds for $Y \to X \times_k k_Y$ and the open subsets $\mathcal{X}^\circ_{s_Y,i,j} \subset \mathcal{X}_{s_Y,i,j}$, $j = 1, \ldots, m_i$, $i = 1, \ldots, m$. Then (U-4) $\Rightarrow$ (U-2) holds for $Y \to X$ and the open subsets $\mathcal{X}^\circ_{s,i} \subset \mathcal{X}_{s,i}$, $i = 1, \ldots, m$. Indeed, assume for every $u_i \in \mathcal{X}^\circ_{s,i}$, $\psi : Y \to X$ is $]u_i[^{\mathrm{ur}}$-pointwise unramified. Let $x_Y \in]u_{Y,i,j}[^{\mathrm{ur}}$ with image $x \in]u_i[^{\mathrm{ur}}$ (Lemma 15) and for every $y \in Y_{x_Y}$, consider the factorization $S_y \to S_{x_Y} \to S_x$. As $\psi : Y \to X$ is $]u_i[^{\mathrm{ur}}$-pointwise unramified, $S_y \to S_x$ is étale hence $S_y \to S_{x_Y}$ is étale as well. This shows $\psi : Y \to X \times_k k_Y$ is $]u_{Y,i,j}[^{\mathrm{ur}}$-pointwise unramified. By (U-4) $\Rightarrow$ (U-2) for $Y \to X \times_k k_Y$ and the open subsets $\mathcal{X}^\circ_{s_Y,i,j} \subset \mathcal{X}_{s_Y,i,j}$, $j = 1, \ldots, m_i$, $i = 1, \ldots, m$, $\mathcal{Y} \to \mathcal{X} \times_S S_Y$ is étale. As $\mathcal{X} \times_S S_Y \to \mathcal{X}$ is étale, this implies $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is étale.

  So, after possibly replacing k by k_Y, we may and will assume Y is geometrically integral over k.

- Base-change along $\mathrm{spec}(k') \to \mathrm{spec}(k)$ for some finite field extension $k \subset k' \subset k^{\mathrm{ur}}$. Let $S' \to S$ denote the normalization of S in $\mathrm{spec}(k') \to \mathrm{spec}(k) \to S$. By assumption $S' \to S$ is étale hence $\mathcal{Y} \times_S S' \to \mathcal{X} \times_S S'$ is the normalization of $Y \times_k k'$ in $Y \times_k k' \to X \times_k k' \to \mathcal{X} \times_S S'$. Let $s' \in S'$ denote the closed point of S' and κ' its residue field. Let $\mathcal{X}_{s,i} \times_\kappa \kappa' = \sqcup_{1\leq j\leq m_i} \mathcal{X}_{s',i,j}$ be the decomposition into irreducible (*viz* connected) components of $\mathcal{X}_{s',i} \times_\kappa \kappa'$, $i = 1, \ldots, m$. For every non-empty open subset $\mathcal{X}^\circ_{s',i,j} \subset \mathcal{X}_{s',i,j}$, the image $\mathcal{X}^o_{s,i} \subset \mathcal{X}_s$ of $\sqcup_{1\leq j\leq m_i} \mathcal{X}^\circ_{s',i,j}$ *via* $\mathcal{X}_{s'} \to \mathcal{X}_s$ is again a non-empty open subset of $\mathcal{X}_{s,i}$. Now, assume (U-4) $\Rightarrow$ (U-2) holds for $Y \times_k k' \to X \times_k k'$ and the open subsets $\mathcal{X}^\circ_{s',i,j} \subset \mathcal{X}_{s',i,j}$, $j = 1, \ldots, m_i$, $i = 1, \ldots, m$. Then (U-4) $\Rightarrow$ (U-2) holds for $Y \to X$ and the open subsets $\mathcal{X}^\circ_{s,i} \subset \mathcal{X}_{s,i}$, $i = 1, \ldots, m$. Indeed, assume for every $u_i \in \mathcal{X}^\circ_{s,i}$, $\psi : Y \to X$ is $]u_i[^{\mathrm{ur}}$-pointwise unramified. Let $x' \in]u'_{i,j}[^{\mathrm{ur}}$ with image $x \in]u_i[^{\mathrm{ur}}$ (Lemma 15) and for every $y \in Y_{x'}$, consider the factorization $S_y \to S_{x'} \to S_x$. As $\psi : Y \to X$ is $]u_i[^{\mathrm{ur}}$-pointwise unramified, $S_y \to S_x$ is étale hence $S_y \to S_{x'}$ is étale as well. This shows $Y \times_k k' \to X \times_k k_Y$ is $]u'_{i,j}[^{\mathrm{ur}}$-pointwise unramified. By (U-4) $\Rightarrow$ (U-2) for $Y \times_k k' \to X \times_k k'$ and the open subsets $\mathcal{X}^\circ_{s',i,j} \subset \mathcal{X}_{s',i,j}$, $j = 1, \ldots, m_i$, $i = 1, \ldots, m$, $\mathcal{Y} \times_S S' \to \mathcal{X} \times_S S'$ is étale. As $\mathcal{X} \times_S S' \to \mathcal{X}$ is étale, this implies $\mathcal{Y} \times_S S' \to \mathcal{X}$ is étale hence that $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is étale.

  So, after possibly replacing k by a finite $k \subset k' \subset k^{\mathrm{ur}}$, we may and will assume $\mathcal{X}_{s,i}$ is geometrically irreducible (*viz* connected) over κ, $i = 1, \ldots, m$.

Let $\xi_i \in \mathcal{X}_{s,i}$ denote the generic point of $\mathcal{X}_{s,i}$, $i = 1, \ldots m$. As $\mathcal{X}$ is regular, it follows from Zariski-Nagata purity theorem that $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is étale if (and only if) it is étale at ξ_i, $i = 1, \ldots, m$. As $\mathcal{Y}$ is normal, it is regular in codimension 1 hence the non-regular locus $\mathcal{Y}^{\text{n-reg}} \subset \mathcal{Y}$ is a closed subset of codimension ≥ 2 in $\mathcal{Y}$. As $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is finite, $\mathcal{Z} := \psi_{\mathcal{X}}(\mathcal{Y}^{\text{n-reg}}) \subset \mathcal{X}$ is also closed of codimension ≥ 2 in $\mathcal{X}$. On the other hand, for every $i = 1, \ldots, m$, as $\mathcal{X} \to S$ is smooth, $\mathcal{X}_{s,i} \subset \mathcal{X}$ is closed of codimension 1 in $\mathcal{X}$ hence $\mathcal{Z}_{s,i} := \mathcal{Z} \cap \mathcal{X}_{s,i} \subset \mathcal{X}_{s,i}$ is closed of codimension ≥ 1 and $\mathcal{X}^o_{s,i} := \mathcal{X}_{s,i} \setminus \mathcal{Z}_{s,i} \subset \mathcal{X}_{s,i}$ is a non-empty open subscheme; in particular, $\xi_i \in \mathcal{X}^o_{s,i}$. So, setting $\mathcal{X}^o := \mathcal{X} \setminus \mathcal{Z}$, it is enough to prove that $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^o \to \mathcal{X}^o$ is étale at ξ_i, $i = 1, \ldots, m$. So, up to replacing $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^o \to \mathcal{X}^o$, we may and will assume $\mathcal{Y}$ is also regular. Fix $u_i \in \mathcal{X}_{s,i}$, $i = 1, \ldots, m$. Up to replacing further $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ by its base-change along $\mathrm{spec}(\mathcal{O}') \to S$ for some finite $\mathcal{O} \subset \mathcal{O}' \subset \mathcal{O}^{\mathrm{ur}}$ we may and will assume that $\kappa(v) = \kappa$ for every $v \in \mathcal{Y}_{u_i}$ (in particular, $\kappa(u_i) = \kappa$). We argue by contradiction. Assume the subset $\mathcal{Y}^{\text{n-et}} \subset \mathcal{Y}$ of all $y \in \mathcal{Y}$ such that $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is non-étale at y is non-empty. Then, again by Zariski-Nagata purity theorem, $\mathcal{Y}^{\text{n-et}} \subset \mathcal{Y}$ is closed and pure of codimension 1 in $\mathcal{Y}$. Fix a generic point $\xi \in \mathcal{Y}^{\text{n-et}}$. As $\psi : Y \to X = \mathcal{X}_\eta$ is étale, ξ necessarily lies over one of the ξ_i, $i = 1, \ldots, m$ - say ξ_i. But as $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is finite, $\psi_{\mathcal{X}}(\mathcal{Y}^{\text{n-et}}) \subset \mathcal{X}$ is closed in $\mathcal{X}$ hence contains $\mathcal{X}_{s,i}$. For simplicity, write $u := u_i \in \mathcal{X}_{s,i}$ and fix $v \in (\mathcal{Y}^{\text{n-et}})_u$. By Lemma 17 (v) $\Rightarrow$ (ii) applied to $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$, the canonical morphism $\psi^{\#}_{\mathcal{X}} : \mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \to \mathfrak{m}_v/\mathfrak{m}_v^2$ induced by $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ at the level of cotangent spaces is not injective. Recall that, from our preliminary reduction $\kappa = \kappa(u) = \kappa(v)$. Fix $0 \neq a \in \ker(\mathfrak{m}_u/\mathfrak{m}_u^2 \to \mathfrak{m}_v/\mathfrak{m}_v^2)$. The idea is to construct a $\tilde{x} \in \mathcal{X}(\mathcal{O})_u$ such that the resulting morphism of cotangent spaces $\tilde{x}^{\#} : \mathfrak{m}_u/\mathfrak{m}_u^2 \to \mathfrak{m}/\mathfrak{m}^2$ satisfies $\tilde{x}^{\#}(a) \neq 0$. Assume such a $\tilde{x} \in \mathcal{X}(\mathcal{O})_u$ exists and let $x \in]u[^{\mathrm{ur}}$ denote its image in $|X|$. By (U-4), the normalization $\widetilde{Y}_x \to S$ of S in $Y_x \to \mathrm{spec}(k) \to S$ is étale. Actually, $Y_x = \mathrm{spec}(k_{x,1} \times \cdots \times k_{x,t})$ with $k \hookrightarrow k_{x,j}$ a finite unramified extension,

$j = 1, \dots, t$ and $\widetilde{Y}_x = \mathrm{spec}(\mathcal{O}_{x,1} \times \cdots \times \mathcal{O}_{x,t})$, where $\mathcal{O}_{x,j} \subset \mathcal{O}^{\mathrm{ur}}$ is the valuation ring of $k_{x,j}$, $j = 1, \dots, t$. For $j = 1, \dots, t$, write $\widetilde{Y}_{x,j} := \mathrm{spec}(\mathcal{O}_{x,j}) = \{y_j, v_j\}$, where y_j is the generic point and v_j the closed point of $\widetilde{Y}_{x,j}$. As $\widetilde{Y}_x \to S \xrightarrow{\tilde{x}} \mathcal{X}$ also coincides with the normalization of $\mathcal{X}$ in $Y_x \to \mathrm{spec}(k) \to S \xrightarrow{\tilde{x}} \mathcal{X}$, by the universal property of $\widetilde{Y}_x \to S \xrightarrow{\tilde{x}} \mathcal{X}$, one gets a unique factorization

(1)
$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y}_x \arrow[r, dotted] \arrow[d] & \mathcal{Y} \arrow[d, "\psi_{\mathcal{X}}"] \\ \mathrm{spec}(k) \arrow[r] \arrow[rr, bend right, "x"'] & S \arrow[r, "\tilde{x}"] & \mathcal{X} \end{tikzcd}$$

As $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is integral, by the Going down theorem [Stacks, Tag 00H8], there exists $y \in \mathcal{Y}$ specializing to v and mapping to x. By construction $y \in Y_x$ hence coincides with the generic point of one of the irreducible components - say $\widetilde{Y}_{x,j}$ - of $\widetilde{Y}_x$. As $\widetilde{Y}_x \to S$ is finite, v_j maps to v. The commutative square

$$\begin{tikzcd} \widetilde{Y}_{x,j} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\psi_{\mathcal{X}}"] \\ S \arrow[r, "\tilde{x}_i"'] & \mathcal{X} \end{tikzcd}$$

induces a commutative square at the level of cotangent spaces

$$\begin{tikzcd} \mathfrak{m}_{v_j}/\mathfrak{m}_{v_j}^2 & \mathfrak{m}_v/\mathfrak{m}_v^2 \arrow[l] \\ \mathfrak{m}/\mathfrak{m}^2 \arrow[u] & \mathfrak{m}_u/\mathfrak{m}_u^2 \arrow[l, "\tilde{x}^{\#}"] \arrow[u, "\psi_{\mathcal{X}}^{\#}"'] \end{tikzcd}$$

But as $\widetilde{Y}_x \to S$ is étale, the morphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_{v_j}/\mathfrak{m}_{v_j}^2$ is injective (by Lemma 17 (ii) $\Rightarrow$ (v)); this contradicts the fact that $\tilde{x}^{\#}(a) \neq 0$ while $\psi_{\mathcal{X}}^{\#}(a) = 0$.

It remains to construct $\tilde{x} \in \mathcal{X}(\mathcal{O})_u$ such that the resulting morphism of cotangent spaces $\tilde{x}^{\#} : \mathfrak{m}_u/\mathfrak{m}_u^2 \to \mathfrak{m}/\mathfrak{m}^2$ satisfies $\tilde{x}^{\#}(a) \neq 0$. For this, as $\mathcal{X}$ is smooth at u and the residue field of u is κ, $\mathcal{O}[[T_1, \dots, T_n]] \tilde{\to} \widehat{\mathcal{O}}_u$ and, *modulo* this isomorphism, $\widehat{\mathfrak{m}}_u \subset \widehat{\mathcal{O}}_u$ identifies with the ideal $\langle \pi, T_1, \dots, T_n \rangle$, where $\pi \in \mathfrak{m}$ is a uniformizer, $\mathfrak{m}_u/\mathfrak{m}_u^2 \tilde{\to} \kappa\overline{\pi} \oplus \kappa\overline{T}_1 \oplus \cdots \oplus \kappa\overline{T}_n$ and $\mathfrak{m}/\mathfrak{m}^2 \tilde{\to} \kappa\overline{\pi}$. Fix any κ-linear morphism $\overline{f} : \mathfrak{m}_u/\mathfrak{m}_u^2 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ such that $\overline{f}(a) \neq 0$ and $\overline{f}(\overline{\pi}) = \overline{\pi}$ (if $p_i : \mathfrak{m}_u/\mathfrak{m}_u^2 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ denotes the projection onto the $\overline{\pi}$th component for $i = 0$ and the $\overline{T}_i$th component for $i = 1, \dots, n$, and $a = a_0\overline{\pi} + \sum_{1 \leq i \leq n} a_i \overline{T}_i$ then, one can take $\overline{f} = p_0$ if $a_0 \neq 0$ and $\overline{f} = p_0 + p_i$ for some $1 \leq i \leq n$ such that $a_i \neq 0$ if $a_0 = 0$). Let $f(T_1), \dots, f(T_n) \in \mathfrak{m}$ lifting $\overline{f}(\overline{T}_1), \dots, \overline{f}(\overline{T}_n) \in \mathfrak{m}/\mathfrak{m}^2$; these define a unique morphism $f^{\#} : \mathcal{O}[[T_1, \dots, T_n]] \to \mathcal{O}$ of $\mathcal{O}$-algebras, and the resulting $\mathcal{O}$-point

$$\tilde{x} : \mathrm{spec}(\mathcal{O}) \xrightarrow{f} \mathcal{O}[[T_1, \dots, T_n]] \simeq \mathcal{O}_{\mathcal{X},u} \to \mathcal{X}$$

has the expected property.

$\square$

Lemma 17. *Let $\pi : V \to U$ be a finite surjective morphism between integral normal noetherian schemes. Let $v \in V$ and set $u := \pi(v)$. Let $\kappa(u) \twoheadleftarrow \mathcal{O}_u \supset \mathfrak{m}_u$ (resp. $\kappa(v) \twoheadleftarrow \mathcal{O}_v \supset \mathfrak{m}_v$) denote the residue field, local ring and maximal ideal of U at u (resp. of V at v). The following conditions are equivalent*

(i) $\pi : V \to U$ is étale at v;
(ii) $\pi : V \to U$ is unramified at v;
(iii) the canonical morphism $\mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \to \mathfrak{m}_v/\mathfrak{m}_v^2$ is an epimorphism;

Assume furthermore U is regular at u, then these are also equivalent to:

(iv) the canonical morphism $\mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \to \mathfrak{m}_v/\mathfrak{m}_v^2$ is an isomorphism,

(and imply V is regular at v). Assume furthermore V is regular at v, then these are also equivalent to:

(v) the canonical morphism $\mathfrak{m}_u/\mathfrak{m}_u^2 \otimes_{\kappa(u)} \kappa(v) \to \mathfrak{m}_v/\mathfrak{m}_v^2$ is a monomorphism.

(and imply U is regular at u).

Proof. (i) $\Rightarrow$ (ii) is tautological while (ii) $\Rightarrow$ (i) is [G71, Exp. I, Thm. 9.5 (ii)]. (iii) is equivalent to $\mathfrak{m}_u\mathcal{O}_v + \mathfrak{m}_v^2 = \mathfrak{m}_v$ so (ii) $\Rightarrow$ (iii) is tautological while (iii) $\Rightarrow$ (ii) follows from Nakayama's lemma since $\mathfrak{m}_v$ is a finitely generated $\mathcal{O}_v$-module [Stacks, Tag 07RC (4)]. (iv) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are tautological. By Nakayama's lemma and Krull's principal ideal, one always has $\dim_{\kappa(u)}\mathfrak{m}_u/\mathfrak{m}_u^2 \geq \dim\mathcal{O}_u$ with equality if and only if U is regular at ut and similarly for V at v. This shows (iii) $\Rightarrow$ (iv) assuming U is regular at u and (v) $\Rightarrow$ (iv) assuming V is regular at v. $\square$

4.3. **A pointwise criterion for $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ to be tamely ramified along $\mathcal{X}_v$.**

Proposition 18. *Let $\psi : Y \to X$ be a Galois cover. There exists a non-empty open subset $\mathcal{X}_{s,i}^o \subset \mathcal{X}_{s,i}$, $i = 1, \dots, m$ (depending on $Y \stackrel{\psi}{\to} X \to \mathcal{X}$) such that, for every $u_i \in \mathcal{X}_{s,i}^o$, $i = 1, \dots, m$, the following conditions are equivalent.*

(T-1) $\ker(\pi_1(X) \twoheadrightarrow \pi_1^{\mathrm{t}}(\mathcal{X}; \mathcal{X}_s)) \subset \pi_1(Y)$*;*
(T-2) $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ *is (finite) tamely ramified along* $\mathcal{X}_s$*;*
(T-3) $\psi : Y \to X$ *is* $|X|^{\mathrm{int}}$*-pointwise tame;*
(T-4) $\psi : Y \to X$ *is* $\cup_{1\leq i\leq m}]u_i[^{\mathrm{t}}$*-pointwise tame.*

Proof. Again, the implications (T-1) $\Rightarrow$ (T-2) $\Rightarrow$ (T-3) $\Rightarrow$ (T-4) and (T-2) $\Rightarrow$ (T-1) are (almost) tautological so we are left to prove (T-4) $\Rightarrow$ (T-2).

Let $\mathcal{X}^o \subset \mathcal{X}$ be an open subscheme such that $\mathcal{X}^o \cap \mathcal{X}_{s,i} \neq \emptyset$, $i = 1, \dots, m$. By definition of tame ramification, $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is tamely ramified along $\mathcal{X}_s$ if (and only) if $\psi_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{X}^o : \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}^o \to \mathcal{X}^o$ is tamely ramified along $\mathcal{X}_s^o$ so that, in proving (T-4) $\Rightarrow$ (T-2), one may freely replace $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ by its base-change along such an open subscheme $\mathcal{X}^o \hookrightarrow \mathcal{X}$.

For $i = 1, \dots, m$, consider the Cartesian diagram

$$\begin{tikzcd}
 & \mathcal{Y}_s^{\mathrm{red}} \arrow[d, hook] & \mathcal{Y}_{s,i} \arrow[l] \arrow[dd] \\
\mathcal{Y} \arrow[d, "\psi_{\mathcal{X}}"'] \arrow[dr, phantom, "\square"] & \mathcal{Y}_s \arrow[l] \arrow[d, "\psi_{\mathcal{X},s}"] \arrow[r, phantom, "\square"] & \\
\mathcal{X} & \mathcal{X}_s \arrow[l] & \mathcal{X}_{s,i}, \arrow[l]
\end{tikzcd}$$

(where $\mathcal{Y}_s^{\mathrm{red}} \hookrightarrow \mathcal{Y}_s$ denotes the reduced closed subscheme) and write

$$\mathcal{Y}_{s,i} = \bigcup_{1\leq j\leq m_i} \mathcal{Y}_{s,i,j}$$

for the decomposition of $\mathcal{Y}_{s,i}$ into irreducible components. As κ is perfect, the non-regular locus $\mathcal{Y}_{s,i}^{\text{n-reg}} \subsetneq \mathcal{Y}_{s,i}$ is a strict closed subscheme. As $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is finite, $\psi_{\mathcal{X}}(\mathcal{Y}_{s,i}^{\text{n-reg}}) \subsetneq \mathcal{X}_{s,i}$ is again a strict closed subscheme. So that, replacing $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ by its base-change along

$$\mathcal{X}^o := \mathcal{X} \setminus \bigsqcup_{1\leq i\leq m} \psi_{\mathcal{X}}(\mathcal{Y}_{s,i}^{\text{n-reg}}) \hookrightarrow \mathcal{X},$$

we may and will assume that $\mathcal{Y}_{s,i}$ is regular (*viz* smooth over κ as κ is perfect), $i = 1, \dots, m$. Actually, later in the argument we will have to ensure this property holds not only for the normalization $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ of $\mathcal{X}$ in $Y \stackrel{\psi}{\to} X \hookrightarrow \mathcal{X}$ but also for the normalization $\psi'_{\mathcal{X}} : \mathcal{Y}' \to \mathcal{X}$ of $\mathcal{X}$ in $Y' \stackrel{\psi'}{\to} X \hookrightarrow \mathcal{X}$ for some intermediate covers $Y \to Y' \stackrel{\psi'}{\to} X$. But as there are only finitely such intermediate covers, we can do so by shrinking $\mathcal{X}$ further (namely removing not only $\bigsqcup_{1\leq i\leq m} \psi_{\mathcal{X}}(\mathcal{Y}_{s,i}^{\text{n-reg}})$ but the union of all $\bigsqcup_{1\leq i\leq m'} \psi_{\mathcal{X}}(\mathcal{Y}'_{s,i}{}^{\text{n-reg}})$ for $Y \to Y' \stackrel{\psi'}{\to} X$ describing the finitely many intermediate covers of $\psi : Y \to X$).

For a subgroup $H \subset G := \mathrm{Aut}(\psi)$, write $Y_H \to X$ for the corresponding connected étale cover and $\mathcal{Y}_H \to \mathcal{X}$ for the normalization of $\mathcal{X}$ in $Y_H \to X \hookrightarrow \mathcal{X}$. Fix $i = 1, \dots m$, let $\xi := \xi_i \in \mathcal{X}_{s,i}$ denote the generic point of $\mathcal{X}_{s,i}$, $\zeta := \zeta_{i,j} \in \mathcal{Y}_{s,i,j}$ the generic point of $\mathcal{Y}_{s,i,j}$ and write

$$G \supset D := D_{\zeta/\xi} \supset I := I_{\zeta/\xi} \supset I^{\mathrm{w}} := I^{\mathrm{w}}_{\zeta/\xi} \subset 1$$

for the decomposition, inertia and wild inertia groups of ζ/ξ respectively. These yield a commutative diagram

$$\begin{tikzcd} Y \arrow[r] & Y_{I^{\mathrm{w}}} \arrow[r] & Y_I \arrow[r] & Y_D \arrow[r] & X \\ \mathcal{Y} \arrow[r] \arrow[u] & \mathcal{Y}_{I^{\mathrm{w}}} \arrow[r] \arrow[u] & \mathcal{Y}_I \arrow[r] \arrow[u] & \mathcal{Y}_D \arrow[r] \arrow[u] & \mathcal{X}. \arrow[u] \end{tikzcd}$$

Let $\zeta_{I^{\mathrm{w}}}$, ζ_I and ζ_D denote the image of ζ in $\mathcal{Y}_{I^{\mathrm{w}}}$, $\mathcal{Y}_I$ and $\mathcal{Y}_D$ respectively. By construction, $\mathrm{spec}(\mathcal{O}_{\mathcal{Y}_D,\zeta_D}) \to \mathrm{spec}(\mathcal{O}_{\mathcal{X},\eta})$ and $\mathrm{spec}(\mathcal{O}_{\mathcal{Y}_I,\zeta_I}) \to \mathrm{spec}(\mathcal{O}_{\mathcal{Y}_D,\zeta_D})$ are unramified, $\mathrm{spec}(\mathcal{O}_{\mathcal{Y}_{I^{\mathrm{w}}},\zeta_{I^{\mathrm{w}}}}) \to \mathrm{spec}(\mathcal{O}_{\mathcal{Y}_I,\zeta_I})$ is tamely ramified and $\mathrm{spec}(\mathcal{O}_{\mathcal{Y},\zeta}) \to \mathrm{spec}(\mathcal{O}_{\mathcal{Y}_{I^{\mathrm{w}}},\zeta_{I^{\mathrm{w}}}})$ is wildly ramified.

Also, just as in the proof of Proposition 16, it is enough to prove (T-4) $\Rightarrow$ (T-2) after:

- Replacing k by the algebraic closure k_Y of k in the function field of Y. The argument is exactly similar to the one of the proof of Proposition 16, replacing "étale" with "tamely ramified".

  In particular, after possibly replacing k by k_Y, we may and will assume Y is geometrically integral over k.

- Base-change along $\mathrm{spec}(k') \to \mathrm{spec}(k)$ for some finite field extension $k \subset k' \subset k^{\mathrm{t}}$. Again, the argument is exactly similar to the one of the proof of Proposition 16, replacing "étale" with "tamely ramified" and $\mathcal{Y} \times_S S'$ with the normalization of $\mathcal{X} \times_S S'$ in $Y \times_k k' \to X \times_k k' \to \mathcal{X} \times_S S'$.

  So, after possibly replacing k by a finite $k \subset k' \subset k^{\mathrm{t}}$, we may and will assume that $\mathcal{X}_{s,i}$ is geometrically irreducible (*viz* connected) over κ, $i = 1, \ldots, m$ and, by Abhyankar's lemma ([Stacks, Tag 0BRM]), that $\mathrm{spec}(\mathcal{O}_{\mathcal{Y}_{I^{\mathrm{w}}_{i,j}},\zeta_{I^{\mathrm{w}}_{i,j}}}) \to \mathrm{spec}(\mathcal{O}_{\mathcal{X},\xi_i})$ is unramified (in other words, $I_{i,j} = I^{\mathrm{w}}_{i,j}$), $j = 1, \ldots, m_i$, $i = 1, \ldots, m$.

Recall that we may also assume that $\mathcal{Y}_{s,i}$ is smooth over κ, $i = 1, \ldots, m$.

We now define $\mathcal{X}^{\circ}_{s,i} \subset \mathcal{X}_{s,i}$ as in Proposition 16, $i = 1, \ldots, m$.

Fix $1 \leq i \leq m$, $u_i \in |\mathcal{X}^{\circ}_{s,i}|$, $x \in]u_i[^{\mathrm{t}}$ and $y \in Y_x$. By definition, $x : \mathrm{spec}(k(x)) \to X$ extends (uniquely as $\mathcal{X} \to S$ is separated) to $\tilde{x} : S_x \to \mathcal{X}$ and, as $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is finite hence proper, $y : \mathrm{spec}(k(y)) \to Y$ extends uniquely as

$$\begin{tikzcd} \mathrm{spec}(k(y)) \arrow[rr, "y"] \arrow[d] & & \mathcal{Y} \arrow[d, "\psi_{\mathcal{X}}"] \\ S_y \arrow[r] \arrow[urr, dotted, "\exists ! \tilde{y}"] & S_x \arrow[r, "\tilde{x}"'] & \mathcal{X} \end{tikzcd}$$

Then there exists a unique $1 \leq i \leq m$ such that $\tilde{x}_s \in \mathcal{X}_{s,i}$ and a unique (recall that $\mathcal{Y}_s$ is regular) $1 \leq j \leq m_i$ such that $\tilde{y}_s \in \mathcal{Y}_{s,i,j}$. Write $I^{\mathrm{w}} := I^{\mathrm{w}}_{\zeta_{i,j}/\xi_i} (= I_{\zeta_{i,j}/\xi_i})$ and $D := D_{\zeta_{i,j}/\xi_i}$. For $H = D$, I^{w} and 1, let $\mathcal{Y}^o_{H,s} \subset \mathcal{Y}_{H,s}$ denote the irreducible component of ζ_H and set $\mathcal{Y}^o_H := \mathcal{Y}_H \setminus (\mathcal{Y}_{H,s} \setminus \mathcal{Y}^o_{H,s})$. As $\zeta_{D_{i,j}}$ is inert in $\mathcal{Y}_{I^{\mathrm{w}}_{i,j}} \to \mathcal{Y}_{D_{i,j}}$ and $\zeta_{I^{\mathrm{w}}_{i,j}}$ is totally wildly ramified in $\mathcal{Y} \to \mathcal{Y}_{I^{\mathrm{w}}_{i,j}}$, one actually has a partly Cartesian diagram

$$\begin{tikzcd} \mathcal{Y}^o \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}^o_{I^{\mathrm{w}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}^o_D \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, equal] \\ \mathcal{Y} \arrow[r] & \mathcal{Y}_{I^{\mathrm{w}}} \arrow[r] & \mathcal{Y}_D \arrow[r] & \mathcal{X}. \end{tikzcd}$$

In particular, $\mathcal{Y}^o_{I^{\mathrm{w}}} \to \mathcal{Y}^o_D$ is an etale cover and $\mathcal{Y}^o_D \to \mathcal{X}$ is an étale morphism (but *a priori* not finite). Let y^{w} denote the image of y in $Y_{I^{\mathrm{w}}}$ and $\widetilde{y}^{\mathrm{w}} : S_{y^{\mathrm{w}}} = \mathrm{spec}(R_{y^{\mathrm{w}}}) \to \mathcal{Y}^o_{I^{\mathrm{w}}}$ the normalization of $\mathcal{Y}^o_{I^{\mathrm{w}}}$ in $\mathrm{spec}(k(y^{\mathrm{w}})) \xrightarrow{y^{\mathrm{w}}} Y_{I^{\mathrm{w}}} \to \mathcal{Y}^o_{I^{\mathrm{w}}}$ so that one has a commutative diagram

$$\begin{tikzcd} S_y \arrow[r] \arrow[d, "\widetilde{y}"] & S_{y^{\mathrm{w}}} \arrow[rr] \arrow[d, "\widetilde{y}^{\mathrm{w}}"] & & S_x \arrow[d, "\widetilde{x}"] \\ \mathcal{Y}^o \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}^o_{I^{\mathrm{w}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}^o_D \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, equal] \\ \mathcal{Y} \arrow[r] & \mathcal{Y}_{I^{\mathrm{w}}} \arrow[r] & \mathcal{Y}_D \arrow[r] & \mathcal{X}. \end{tikzcd}$$

As $\mathcal{Y}^o_{I^{\mathrm{w}}} \to \mathcal{X}$ is étale, $S_{y^{\mathrm{w}}} \to S_x$ is unramified while as $Y \to Y_{I^{\mathrm{w}}}$ is Galois with group I^{w} of order a power of p, the ramification index of $S_y \to S_{y^{\mathrm{w}}}$ is a power of p. On the other hand, as $x \in]u_i[^{\mathrm{t}}$, by (T-4), $S_y \to S_x$ (hence *a fortiori* $S_y \to S_{y^{\mathrm{w}}}$) is at most tamely ramified. This forces $S_y \to S_x$ to be unramified and proves that $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is $]u_i[^{\mathrm{t}}$-pointwise unramified (hence *a fortiori* $]u_i[^{\mathrm{ur}}$-pointwise unramified).By Proposition 16 (U-4) $\Rightarrow$ (U-2), $\psi_{\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$ is an étale cover. □

4.4. **Proof of Theorem 12.** The only non-obvious implications are b) $\Rightarrow$ a). Consider Assertion (1). By definition, if b) holds then, for every normal open subgroup $U \subset \Pi_p$ the corresponding Galois cover $X_U \to X$ is $|X|^{\mathrm{int}}$-pointwise unramified. By Proposition 16 (U-3) $\Rightarrow$ (U-1), the morphism $\pi_1(X, \bar{x}) \twoheadrightarrow \Pi_p/U$ then factors through $\pi_1(X, \bar{x}) \twoheadrightarrow \pi_1(\mathcal{X}, \bar{x})$. One concludes by passing to the limit on U. This proves (1). The proof of (2) is exactly similar using Proposition 18 (T-3) $\Rightarrow$ (T-1).

5. Applications to Conjecture A (T) and Conjecture A (C)

Let k be a number field and let X be a smooth, geometrically connected variety over k.

5.1. **AEU $\mathbb{Q}_p$-local systems.** Generalizing the definition of an AEU $\mathbb{Q}_p$-local system on $X = x = \mathrm{spec}(k)$ in Paragraph 1.2.1.1, say that a $\mathbb{Q}_\ell$-local $\mathcal{V}_\ell$ on X is AEU if there exists a smooth model $\mathcal{X} \to U$ of X over a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$ such that $\mathcal{V}_\ell$ extends to a $\mathbb{Q}_\ell$-local system on $\mathcal{X}$. Note that, if $\mathcal{X}_i \to U_i$, $i = 1, 2$ are two smooth models of X then there exists a non-empty open subscheme $U \subset U_1 \cap U_2$ such that $\mathcal{X}_1 \times_{U_1} U \tilde{\to} \mathcal{X}_2 \times_{U_2} U$ as U-schemes. In particular, $\mathcal{V}_\ell$ is AEU if and only if for every smooth model $\mathcal{X} \to U$ of X over a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$, there exists a non-empty open subscheme $U' \subset U$ such that $\mathcal{V}_\ell$ extends to a $\mathbb{Q}_\ell$-local system on $\mathcal{X} \times_U U'$.

The property of being pointwise AEU is also rigid.

Fact 19. ([LiZ17, Prop. 4.1], [P23, Prop. 6.1]) *Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X. Consider the following properties (i) $\mathcal{V}_\ell$ is AEU;*
(ii) for every $x \in |X|$, $x^*\mathcal{V}_\ell$ is AEU;
(iii) there exists $x \in |X|$ such that $x^*\mathcal{V}_\ell$ is AEU.
Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) and, if $\mathcal{V}_\ell$ is semisimple, then (iii) $\Rightarrow$ (i).

5.2. **Comparing $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell}$, $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$, $|X|^{\mathrm{cent}}_{\mathcal{V}_\ell}$.**

5.2.1. We begin with the following consequence of local class field theory and Sen's theorem.

Proposition 20. *Let k be a number field. Let ℓ be a prime and $\mathcal{V}_\ell$ a $\mathbb{Q}_\ell$-local system on $x = X = \mathrm{spec}(k)$. Assume $x_v^*\mathcal{V}_\ell$ is Hodge-Tate for every finite place v of k above ℓ. Then $(G_\ell^\circ)^{\mathrm{ab}}$ is reductive.*

Proof. Fix a geometric point $\bar{x}$ over x. Write $V_\ell := \mathcal{V}_{\ell,\bar{x}}$ and let $\rho_\ell : \pi_1(x) = \pi_1(k) \to G_\ell(\mathbb{Q}_\ell) \subset \mathrm{GL}(V_\ell)$ denote the continuous representation corresponding to $\mathcal{V}_\ell$. After possibly replacing k by a finite field extension one may assume $G_\ell = G_\ell^\circ$. Fix a Levi subgroup $L_\ell \subset G_\ell$ and let $N_\ell \subset G_\ell$ denote the smallest normal algebraic subgroup of G_ℓ containing L_ℓ. If G_ℓ^{ab} is not reductive, then $N_\ell \subsetneq G_\ell$. By Lemma 21 below, there exists a G_ℓ-subrepresentation $W_\ell \subset T(V_\ell)$ such that $N_\ell = \ker(G_\ell \to \mathrm{GL}_{W_\ell})$. By construction, the non-trivial unipotent group G_ℓ/N_ℓ acts faithfully on W_ℓ. Let $\mathcal{W}_\ell$ denote the $\mathbb{Q}_\ell$-local system on x corresponding to W_ℓ viewed as a $\pi_1(k)$-representation *via* $\pi_1(k) \to G_\ell(\mathbb{Q}_\ell) \to (G_\ell/N_\ell)(\mathbb{Q}_\ell)$. Then, as $\mathcal{W}_\ell$ lies in the Tannakian category generated by $\mathcal{V}_\ell$, $x_v^*\mathcal{W}_\ell$ is Hodge-Tate for every finite place $v|\ell$ of k. As a result, it is enough to prove that if G_ℓ is unipotent, then it is trivial. If G_ℓ is unipotent non-trivial then there exists a surjective morphism $p : G_\ell \twoheadrightarrow \mathbb{G}_{a,\mathbb{Q}_\ell}$ and a factorization

$$\begin{tikzcd}
\pi_1(x) = \pi_1(k) \arrow[r, "\rho_\ell"] \arrow[d] & G_\ell(\mathbb{Q}_\ell) \arrow[r, "p"] & \mathbb{G}_{a,\mathbb{Q}_\ell}(\mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell \\
\pi_1(k)^{\mathrm{ab}} \arrow[urr, "\rho_\ell^{\mathrm{ab}}"'] & &
\end{tikzcd}$$

such that $\mathrm{im}(\rho_\ell^{\mathrm{ab}}) \simeq \mathbb{Z}_\ell$. As $\mathbb{Z}_\ell$ is torsion-free and as $\mathcal{O}_v^\times \simeq \mathbb{Z}_p^{\oplus r} \times (\mathcal{O}_v^\times)_{\mathrm{tor}}$ for every prime $p \neq \ell$ and finite place v of k above p, it follows from local class field theory that $\rho_\ell^{\mathrm{ab}} : \pi_1(k)^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}_\ell$ factors through $\pi_1(\mathcal{O}_k[\frac{1}{\ell}])^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}_\ell$. On the other hand, for every finite place v of k above ℓ, $x_v^*\mathcal{V}_\ell$ is unipotent, so that it has a single Hodge-Tate weight, which is 0; equivalently (see equivalence (1) in Paragraph 3.1.2), it is potentially unramified. In particular, for every finite place v of k above ℓ, $\rho_\ell^{\mathrm{ab}}|_{\pi_1(k_v)^{\mathrm{ab}}} : \pi_1(k_v)^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}_\ell$ is potentially unramified - hence unramified since $\mathbb{Z}_\ell$ is torsion-free This proves that $\rho_\ell^{ab}:\pi_1(k)^{ab}\twoheadrightarrow\mathbb{Z}_\ell$ actually factors through $\pi_1(\mathcal{O}_k)^{ab}\twoheadrightarrow\mathbb{Z}_\ell$, which contradicts the finiteness of $\pi_1(\mathcal{O}_k)^{ab}$. □

Lemma 21. *Let Q be a field of characteristic 0, V a finite dimensional Q-vector space and $N\subset G\subset \mathrm{GL}_V$ algebraic subgroups with N normal in G. Then there exists a G-subrepresentation*

$$W_N\subset T(V):=\oplus_{m,n}V^{\otimes m}\otimes V^{\vee\otimes n}$$

such that $N=\ker(G\to \mathrm{GL}_{W_N})$.

Proof. By [D82, Prop. 3.1 (a), (b)], there exists a GL_V-subrepresentation $V_1\subset T(V)$ such that N is the stabilizer of a line $L_1\subset V_1$; let $L_1\subset V_2\subset V_1$ denote the smallest G-subrepresentation containing L_1 and let $N_2\subset G_2\subset \mathrm{GL}_{V_2}$ denote the image of N and G acting on V_2 respectively. By construction N_2 is contained in a split torus of GL_{V_2} - hence is reductive. By [D82, Prop. 3.1 (a), (c)], there exists a GL_{V_2}-subrepresentation $V_3\subset T(V_2)(\subset T(V_1)\subset T(V))$ and a finite subset $A\subset V_3$ such that N_2 is the algebraic subgroup of GL_{V_2} fixing the elements in A. Let $A\subset V_4\subset V_3$ denote the smallest G_2-subrepresentation containing A. By construction, $N_2=\ker(G_2\to\mathrm{GL}_{V_4})$ hence one can take $W_N:=V_4$. □

Remark 22. Using Lemma 21, one immediately sees that Conjecture A (T) is also equivalent to the characterization of the degeneracy locus of $\mathcal{V}_\ell$ given in Subsection 1.1.1 (1).

Corollary 23. *Let k be a number field and X a smooth, geometrically connected variety over k. Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X such that $x_v^*\mathcal{V}_\ell$ is Hodge-Tate for some $x\in|X|$ and every finite place v of $k(x)$ above ℓ. Then $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell}=|X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$.*

Proof. From Fact 10 (1), for every $x\in|X|$ and every finite place v of $k(x)$ above ℓ, $x_v^*\mathcal{V}_\ell$ is Hodge-Tate. The assertion thus follows from Proposition 20. □

Corollary 23 applies in particular the case if $\mathcal{V}_{\ell,\overline{\mathbb{Q}}_\ell}$ is simple [P23, Cor. 5.3] (observing that the condition $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell}\neq\emptyset$ forces the character χ appearing in [P23, Cor. 5.3] to be finite).

5.2.2. *Proof of Proposition 1 and Theorem 4 (1).* Let k be a number field and let X be a smooth, geometrically connected variety over k.

5.2.2.1. *A construction.* We begin by recalling the following construction, which is introduced in the proof of [P23, Thm. 8.1], where it is attributed to Beilinson.

Construction 24. Assume $X(k)\neq\emptyset$ and fix $x\in X(k)$, which we regard as a section of the structural morphism $s_X:X\to\mathrm{spec}(k)$. Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X and let

$$A_x(\mathcal{V}_\ell)\subset E_x(\mathcal{V}_\ell):=\mathcal{V}_\ell^\vee\otimes s_X^*(x^*\mathcal{V}_\ell)$$

denote the minimal sub-local system $\mathcal{S}_\ell\subset E_x(\mathcal{V}_\ell)$ such that $\mathcal{S}_{\ell,\bar{x}}$ contains $Id_{\mathcal{V}_{\ell,\bar{x}}}$; explicitly, it corresponds to the $\pi_1(X,\bar{x})$-subrepresentation

$$A_x(\mathcal{V}_\ell)_{\bar{x}}=\mathbb{Q}_\ell[\overline{\Pi}_\ell]\subset E_x(\mathcal{V}_\ell)_{\bar{x}}=\mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{V}_{\ell,\bar{x}}).$$

Note that, by definition, $E_x(\mathcal{V}_\ell)_{\bar{x}}$ is $E_x(\mathcal{V}_\ell)_{\bar{x}}\simeq\mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{V}_{\ell,\bar{x}})$ equipped with the action

$$\pi\cdot f=(xs_X)(\pi)\cdot f\cdot\pi^{-1},\ \ \pi\in\pi_1(X,\bar{x}),\ f\in E_x(\mathcal{V}_\ell)_{\bar{x}}.$$

In particular, one has

- a canonical quotient morphism $A_x(\mathcal{V}_\ell)\otimes s_X^*(x^*\mathcal{V}_\ell)^\vee\twoheadrightarrow\mathcal{V}_\ell^\vee$ (sending $g\otimes\phi$ to the linear form $a\mapsto\phi(g(a))$);
- If $\overline{G}_\ell^\circ=\overline{G}_\ell$, then $x\in|X|^{\mathrm{cent}}_{\mathcal{V}_\ell}$ if and only if $x\in|X|^{\mathrm{triv}}_{A_x(\mathcal{V}_\ell)}$.

Fact 25. ([P23, Prop. 8.2]) *For every finite place v of k above ℓ, the $\mathbb{Q}_\ell$-local system $A_x(\mathcal{V}_\ell)|_{X_{k_v}}$ is de Rham.*

5.2.2.2. *Proof of Proposition 1.* Proposition 1 (1) is a special case of Corollary 23 as every $x\in|X|^{\mathrm{triv}}_{\mathcal{V}_\ell}$ satisfies the assumption of Corollary 23.

For Proposition 1 (2), if $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell}=\emptyset$ there is nothing to prove. Otherwise, one may replace k by a finite field extension hence assume $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell}\cap X(k)\neq\emptyset$ and X by a connected étale cover hence assume $\overline{G}_\ell^\circ=\overline{G}_\ell$. Let $x\in|X|^{\mathrm{uni}}_{\mathcal{V}_\ell}\cap X(k)$. With the notation of Construction 24, $x\in|X|^{\mathrm{uni}}_{E_x(\mathcal{V}_\ell)}\subset|X|^{\mathrm{uni}}_{A_x(\mathcal{V}_\ell)}$. But then, from Fact 25 and Corollary 23,

$$|X|^{\mathrm{triv}}_{A_x(\mathcal{V}_\ell)}=|X|^{\mathrm{uni}}_{A_x(\mathcal{V}_\ell)},$$

so that the conclusion follows from the fact that $x \in |X|^{\mathrm{cent}}_{\mathcal{V}_\ell}$ if and only if $x \in |X|^{\mathrm{triv}}_{A_x(\mathcal{V}_\ell)}$.

5.2.2.3. *Proof of Theorem 4 (1).* The first part of Theorem 4 (1) follows from the fact that one can also describe $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$ as

$$|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} := \{x \in |X| \mid \mathrm{rank}(G^\circ_{\ell,x}) = 0\},$$

and that $\mathrm{rank}(G^\circ_{\ell,x})$ is independent of $\ell \in |\mathrm{spec}(\mathbb{Z})|$ [Ser81, §3]. For the second part of Theorem 4 (1), by definition of $\mathbb{Q}$-compatibility, for every $x \in |X|$ there exists a non-empty open subset $U_x \subset |\mathrm{spec}(\mathcal{O}_{k(x)})|$ such that for every prime ℓ and finite place $v \in U_x$ above ℓ, $x_v^*\mathcal{V}_\ell$ is crystalline - hence Hodge-Tate. Fix $x_0 \in |X|$; up to replacing k with a finite field extension, one may assume $k(x_0) = k$. For primes $\ell \gg 0$, U_{x_0} contains all the finite places of k above ℓ so that, by assumption, for every finite place v of k above ℓ, $x_{0,v}^*\mathcal{V}_\ell$ is Hodge-Tate and the conclusion follows from Corollary 23.

5.2.2.4. Construction 24 can also be used to prove the following.

Corollary 26. *Conjecture A (T) $\Rightarrow$ Conjecture A (C).*

Proof. Let $\mathcal{V}_\ell$ be a $\mathbb{Q}_\ell$-local system on X such that $|X|^{\mathrm{cent}}_{\mathcal{V}_\ell} \neq \emptyset$. One may replace k by a finite field extension hence assume $|X|^{\mathrm{cent}}_{\mathcal{V}_\ell} \cap X(k) \neq \emptyset$ and X by a connected étale cover hence assume $\overline{G}^\circ_\ell = \overline{G}_\ell$. Let $x \in |X|^{\mathrm{cent}}_{\mathcal{V}_\ell} \cap X(k)$. Then $x \in |X|^{\mathrm{triv}}_{A_x(\mathcal{V}_\ell)}$. In particular, $|X|^{\mathrm{triv}}_{A_x(\mathcal{V}_\ell)} \neq \emptyset$ so that by Conjecture A (T) for $A_x(\mathcal{V}_\ell)$, the étale fundamental group $\pi_1(X)$ - hence *a fortiori* $\pi_1(X_{\bar{k}})$, acts on $A_x(\mathcal{V}_\ell)_{\bar{x}} = \mathbb{Q}_\ell[\overline{\Pi}_\ell]$ through a finite quotient. But this means in particular that the orbit $\overline{\Pi}_\ell \simeq \overline{\Pi}_\ell \cdot Id$ is finite. □

5.3. **Proof of Theorem 4 (2).** It is enough to prove that $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} = |X|$. Indeed, from Theorem 4 (1) this implies $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = |X|$ for primes $\ell \gg 0$. By Fact 2, the condition $|X|^{\mathrm{uni}}_{\mathcal{V}_\ell} (= |X|^{\mathrm{uni}}_{\underline{\mathcal{V}}}) = |X|$ implies G°_ℓ is unipotent while the condition $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = |X|$ implies $G^\circ_\ell = 1$.

As the assumptions of Theorem 4 and the property $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} = |X|$ are invariant under base-change, one may freely replace X by a connected étale cover hence assume that $\mathrm{Lev}_1(\mathcal{V}_{\ell_0})$ holds for at least one prime ℓ_0, which implies the following. For every $x \in |X|$, and finite place v of $U'_{x^*\underline{\mathcal{V}}}$ above a prime $\ell \neq \ell_0$, the subgroup $\Xi_{x_v} \subset \overline{\mathbb{Q}}^\times$ generated by the roots of $\chi_{x_v} (= \chi_{x_v,\mathcal{V}_\ell} = \chi_{x_v,\mathcal{V}_{\ell_0}})$ is torsion-free.

For every $x \in |X|$, let $(x^*\mathcal{V}_\ell)^{\mathrm{ss}}$ denote the semisimplification of $x^*\mathcal{V}_\ell$. To prove that $x \in |X|^{\mathrm{uni}}_{\underline{\mathcal{V}}}$ it is enough to prove that $(x^*\mathcal{V}_\ell)^{\mathrm{ss}}$ is trivial. By the Cebotarev density theorem, to prove that $(x^*\mathcal{V}_\ell)^{\mathrm{ss}}$ is trivial it is enough to prove that for all $v \in |U'_{x^*\underline{\mathcal{V}}}|$, χ_{x_v} is a power of $T-1$. From our preliminary reduction, this is equivalent to proving that the roots of χ_{x_v} are all roots of unity, *viz* that

$$\text{i) } w = 0; \quad \text{ii) } \chi_{x_v} \in \mathbb{Q}[T]; \quad \text{iii) } \chi_{x_v} \in \mathbb{Z}_\ell[T],\ \ell \neq p; \quad \text{iv) } \chi_{x_v} \in \mathbb{Z}_p[T].$$

Property i) follows from the assumption that $|X|^{\mathrm{uni}}_{\underline{\mathcal{V}}} \neq \emptyset$, and Properties ii), iii) from the $\mathbb{Q}$-compatibility assumption. It remains to prove Property iv). Fix $x_0 \in |X|^{\mathrm{uni}}_{\mathcal{V}_\ell}$ and let $x \in |X|$ arbitrary. Property iv) is equivalent to saying that the roots $\alpha_1, \ldots, \alpha_r$ of χ_{x_v} in $\overline{\mathbb{Q}}$ are integral over $\mathbb{Z}_p$. As for every integer $n \geq 1$, the ring $\mathbb{Z}_p[\alpha_1, \ldots, \alpha_r]$ is integral over $\mathbb{Z}_p[\alpha_1^n, \ldots, \alpha_r^n]$, one may freely replace k by a finite field extension. In particular, one may assume $k(x) = k = k(x_0)$. For primes $p \gg 0$, $U'_{x_0^*\underline{\mathcal{V}}}$ and $U'_{x^*\underline{\mathcal{V}}}$ both contain all finite places of k above p. Fix such a prime p. Up to replacing further X by a connected étale cover one may assume $\mathrm{Lev}_2(\mathcal{V}_p)$ holds. Let v be a finite place of k above p (so that $v \in |U'_{x_0^*\underline{\mathcal{V}}}| \cap |U'_{x^*\underline{\mathcal{V}}}|$). By assumption $x_{0,v}^*\mathcal{V}_p$ is both crystalline (hence Hodge-Tate) and unipotent. Thus, from Corollary 11, $x_v^*\mathcal{V}_p$ is unramified and $\chi_{x_v} = \chi_{x_v,x^*\mathcal{V}_p}$ is in $\mathbb{Z}_p[T]$. This concludes the proof of Theorem 4.

5.4. **Relation to the unramified Fontaine-Mazur conjecture.** Let k be a number field. Assume X admits a smooth model $\mathcal{X} \to U$ over a non-empty open subscheme $U \subset \mathrm{spec}(\mathcal{O}_k)$. The following is a consequence of Theorem 12 (1).

Corollary 27. *Let $\mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on $\mathcal{X}[\frac{1}{p}]$. Assume that either* $\mathrm{Lev}_2(\mathcal{V}_p)$ *holds or that, for every finite place v of k above p, one has*

$$\mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_v) \to |X_{k_v}|) \subset |X_{k_v}|^{\mathrm{cris}}_{\mathcal{V}_p}.$$

Then , if $|X|^{\mathrm{triv}}_{\mathcal{V}_p} \neq \emptyset$ the $\mathbb{Q}_p$-local system $\mathcal{V}_p$ on $\mathcal{X}[\frac{1}{p}]$ extends to a $\mathbb{Q}_p$-local system on the whole $\mathcal{X}$.

Let $f : Y \to X$ be a smooth proper morphism. Then for $p \gg 0$ depending only on f, every subquotient $\mathcal{V}_p$ of $R^i f_* \mathbb{Q}_p(j)$ satisfies the condition $\mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_v) \to |X_{k_v}|) \subset |X_{k_v}|^{\mathrm{cris}}_{\mathcal{V}_p}$ in Corollary 27.

Proof. Let v be a finite place of k above p. If $|X|^{\mathrm{triv}}_{\mathcal{V}_p} \neq \emptyset$, it follows from Corollary 11 that for every $x \in |X_{k_v}|$, $x_v^* \mathcal{V}_p|_{X_{k_v}}$ is potentially unramified; under our assumptions this implies that for every $x \in \mathrm{im}(\mathcal{X}(\overline{\mathcal{O}}_v) \to |X_{k_v}|)$, $x^* \mathcal{V}_p$ is unramified. From theorem 12 (1), the $\mathbb{Q}_p$-local system $\mathcal{V}_p|_{X_{k_v}}$ extends to a $\mathbb{Q}_p$-local system on $\mathcal{X}_{\mathcal{O}_v}$, namely the corresponding representation of $\pi_1(X_{k_v})$ on $V_p := \mathcal{V}_{p,\bar{x}}$ factors through $\pi_1(X_{k_v}) \twoheadrightarrow \pi_1(\mathcal{X}_{\mathcal{O}_v})$. In particular, the inertia group of the generic point of each connected component of $\mathcal{X}_v$ acts trivially on V_p. By Zariski-Nagata purity, this implies $\mathcal{V}_p$ extends to a $\mathbb{Q}_p$-local system on $\mathcal{X} \times_U (U[\frac{1}{p}] \cup \{v\})$. □

In particular, for $\mathbb{Q}_p$-local system as in Corollary 27, Conjecture A (T) should follow from the following higher-dimensional generalization of Conjecture B.

Conjecture B'. (Variational unramified Fontaine-Mazur) *Let p be a prime such that*[4] *U contains all finite places of k above p. Then every $\mathbb{Q}_p$-local system on $\mathcal{X}$ is finite.*

Corollary 28. *(1) Conjecture B implies Conjecture A (T).*

(2) Let p be a prime such that[4] *U contains all finite places of k above p and let $\mathcal{V}_p$ be a $\mathbb{Q}_p$-local system on $\mathcal{X}$. Then Conjecture B implies that $|X|^{\mathrm{triv}}_{\mathcal{V}_p} \neq \emptyset$.*

In particular, Conjecture B and Conjecture B' are equivalent. But because of its geometric features, one may hope Conjecture B' for X of dimension ≥ 1 to be more tractable.

Proof. (1) From the discussion following the statement of Conjecture A, it it enough to prove that Conjecture B implies i) $\Rightarrow$ ii) in Conjecture A (T). Up to replacing X by a connected étale cover and k by a finite field extension, one may assume $\mathrm{Lev}_2(\mathcal{V}_\ell)$ holds. We retain the notation and assumptions of Conjecture A (T). If $|X|^{\mathrm{triv}}_{\mathcal{V}_\ell} = \emptyset$, there is nothing to prove. Otherwise, from Fact 19 and from Corollary 11, for every $x \in |X|$, $x^* \mathcal{V}_\ell$ is AEU and for every place v of $k(x)$ above ℓ, $x_v^* \mathcal{V}_\ell$ is potentially unramified - hence unramified by $\mathrm{Lev}_2(\mathcal{V}_\ell)$. But then, by Conjecture B, $x \in |X|^{\mathrm{triv}}_{\mathcal{V}_\ell}$.

(2) For every finite place v of k above p the subset $\mathcal{X}(\mathcal{O}_v) \subset X(k_v)$ is a non-empty[4] open subset so that it follows from a classical Corollary of [MB89, Thm. 1.3] - see *e.g.* [Co06, Cor. 1.5], that there exists a finite field extension k'/k, with U_p totally split in k', and $x' \in X(k')$ such that for every finite place v of k above p and finite place v' of k' above v, $x'_{v'} \in \mathcal{X}(\mathcal{O}_v)$. In other words, there exists a non-empty open subscheme $U'_x \subset U' := U \times_{\mathcal{O}_k} \mathcal{O}_{k'}$ containing all the finite places of k' above p such that $x' \in \mathcal{X}(U'_x)$. But then, Conjecture B imposes that $x'^* \mathcal{V}_p$ is finite that it $x' \in |X|^{\mathrm{triv}}_{\mathcal{V}_p}$.

□

5.5. **Proof of Theorem 6.** The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)' are straightforward. To prove (iii)' $\Rightarrow$ (i), it is enough to prove that (iii)' implies $\mathcal{V}_\ell|_{X_{\bar{k}}}$ is finite. By invariance of étale fundamental group under extensions of algebraically closed field in characteristic 0, it is enough to show that for some finite place v of k, (iii)' $\Rightarrow$ $\mathcal{V}_\ell|_{X_{\bar{k}_v}}$ is finite. This follows from the purely local Theorem 29 below.

Let k be a p-adic field with ring of integers and residue field $k \supset \mathcal{O}_k \twoheadrightarrow \kappa$; let v denote the closed point of $\mathrm{spec}(\mathcal{O}_k)$. Let $\mathbb{Q}_p \subset k_0 \subset k$ be the maximal unramified extension of $\mathbb{Q}_p$ contained in k and $\sigma : k_0 \tilde{\to} k_0$ its arithmetic Frobenius. Let X be a smooth, geometrically connected variety over k admitting a smooth NCC model $\mathcal{X} \hookrightarrow \mathcal{X}^{\mathrm{cpt}} \to \mathrm{spec}(\mathcal{O}_k)$ over $\mathrm{spec}(\mathcal{O}_k)$. Let

$$sp_v : |\mathcal{X}^{\mathrm{cpt}}| \to |\mathcal{X}^{\mathrm{cpt}}_v|$$

denote the specialization map.

Theorem 29. *Assume X is a curve. Let $\mathcal{V}_p$ be a $\mathbb{Q}_p$ local system on X. Assume one of the following holds:*
a) *$\mathcal{X} = \mathcal{X}^{\mathrm{cpt}}$ and $\mathcal{V}_p$ is crystalline;*
b) *$\mathcal{V}_p$ is Hodge-Tate and $\mathrm{Lev}_4(\mathcal{V}_p)$ holds (e.g. $p > \mathrm{rank}_{\mathbb{Q}_p}(\mathcal{V}_p) + 1$).*
Then there exists a 0-dimensional Zariski-closed subset $\mathcal{Z}_v \subset \mathcal{X}^{\mathrm{cpt}}_v$ such that

(i) G_p° is unipotent;
$\Leftrightarrow$ (ii) $|X|^{\mathrm{uni}}_{\mathcal{V}_p} = |X|$;
$\Leftrightarrow$ (iii)' $|X|^{\mathrm{uni}}_{\mathcal{V}_p} \not\subset sp_v^{-1}(|\mathcal{Z}_v|)$.

[4] Recall that, by definition of a smooth model, $\mathcal{X} \to U$ is surjective.

Before proving Theorem 29, we recall some facts about F-isocrystals. Let $\mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$, $\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$ denote respectively the categories of convergent and overconvergent F-isocrystals on $\mathcal{X}_v/\mathcal{O}_k$ with scalar extended from k to $\overline{\mathbb{Q}}_p$ [A18, 1.4, 2.14 et seq.]. From [Ke04, Thm. 1.1], there is a fully faithful[5] exact $\otimes$-functor

$$\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p) \to \mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p).$$

Let

$$\mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0 \subset \mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$$

denote the full subcategory of unit-root (*viz* isoclinic of slope 0) convergent F-isocrystals on $\mathcal{X}_v/\mathcal{O}_k$ and

$$\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0 \subset \mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v//\overline{\mathbb{Q}}_p)$$

the full subcategory of unit-root overconvergent ones *viz* of those objects in $\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$ whose image in $\mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$ lies in $\mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0$. From [K73, Prop. 4.1.1], [Cr87, 2.2, Thm.] there is a canonical equivalence of Tannakian categories

$$\mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0 \tilde{\to} \mathrm{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}_v)) := \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(\mathcal{X}_v)) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$$

which restricts to an equivalence of Tannakian categories ([Ts98, Thm. 7.2.3], [Shi11, Prop. 4.2])

$$\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0 \tilde{\to} \mathrm{Rep}^{\dagger}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}_v))$$

onto the full subcategory $\mathrm{Rep}^{\dagger}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}_v)) \subset \mathrm{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}_v))$ of potentially unramified representations. These equivalences preserve characteristic polynomials of Frobenii on both sides.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)' are straightforward. We prove the implication (iii)' $\Rightarrow$ (i).

- Observe first that one may assume $\mathcal{V}_p$ is simple. Indeed, if $\mathcal{V}_p$ is arbitrary, consider a Jordan-Holder filtration
$$\mathcal{V}_{p,0} = 0 \subsetneq \mathcal{V}_{p,1} \subsetneq \cdots \subsetneq \mathcal{V}_{p,r-1} \subsetneq \mathcal{V}_{p,r} = \mathcal{V}_p$$
and set $\mathcal{S}_{p,i} := \mathcal{V}_{p,i-1}/\mathcal{V}_{p,i-1}$, $i = 1, \dots, r$ for its simple graded pieces. If (iii) holds for each of the $\mathcal{S}_{p,i}$, $i = 1, \dots, r$ then (iii) also holds for $\mathcal{V}_p$. Hence it is enough to check that if a) (resp. b), resp. (i)') holds for $\mathcal{V}_p$ then it holds for each of the $\mathcal{S}_{p,i}$, $i = 1, \dots, r$. For (i)', this follows from the tautological inclusions
$$|X|^{\mathrm{uni}}_{\mathcal{V}_p} \subset |X|^{\mathrm{uni}}_{\mathcal{S}_{p,i}}, \quad i = 1, \dots, r.$$
For a) (resp. b)), this follows from the fact that a subquotient of a crystalline (resp. Hodge-Tate, resp. satisfying $\mathrm{Lev}_4(\mathcal{V}_p)$) local system is again crystalline (resp. Hodge-Tate, resp. satisfies $\mathrm{Lev}_4(\mathcal{V}_p)$) (See Fact 10). So, from now on, assume $\mathcal{V}_p$ is simple.
- By assumption, $\mathcal{V}_p$ is Hodge-Tate (with constant Hodge-Tate weights) and $|X|^{\mathrm{uni}}_{\mathcal{V}_p} \neq \emptyset$, hence, by Fact 10 (1) (i) $\Rightarrow$ (ii), for every $x \in |X|$, $x^*\mathcal{V}_p$ is $\mathbb{C}_p$-admissible *viz* pointwise potentially unramified.
  - In case a), for every $x \in \mathcal{X}(\overline{\mathcal{O}}_k)$, $x^*\mathcal{V}_p$ is both potentially unramified and crystalline, which implies that it is unramified by implication (2) in 3.1.2. From Theorem 12 (ii)' $\Rightarrow$ (i), $\mathcal{V}_p$ extends to a $\mathbb{Q}_p$-local system $\widetilde{\mathcal{V}}_p$ on $\mathcal{X}$.
  - In case b), as $\mathrm{Lev}_4(\mathcal{V}_p)$ holds, one can fix a connected étale cover $X' \to X$ as in Corollary 14 so that $\mathcal{V}_p|_{X'}$ extends to a $\mathbb{Q}_p$-local system $\widetilde{\mathcal{V}}'_p$ on $\mathcal{X}'^{\mathrm{cpt}}$. From the canonical chain of morphisms arising from specialization [G71, X]

(2)
$$\begin{tikzcd}
\pi_1(\mathcal{X}'^{\mathrm{cpt}}_v) \arrow[r, "\simeq"] & \pi_1(\mathcal{X}'^{\mathrm{cpt}}) & \pi_1(X'^{\mathrm{cpt}}) \arrow[l, two heads] & \pi_1(X') \arrow[l, two heads] \arrow[r, hook, "\mathrm{open}", "\triangleleft"'] & \pi_1(X) \\
\pi_1(\mathcal{X}'^{\mathrm{cpt}}_{\bar{v}}) \arrow[u, hook] & & \pi_1(X'^{\mathrm{cpt}}_{\bar{k}}) \arrow[ll, two heads] \arrow[u, hook] & \pi_1(X'_{\bar{k}}) \arrow[l, two heads] \arrow[u, hook] \arrow[r, hook, "\mathrm{open}", "\triangleleft"'] & \pi_1(X_{\bar{k}}) \arrow[u, hook]
\end{tikzcd}$$

  one gets that

  * $\mathcal{V}_p$ is unipotent (*viz* finite) if and only if $\widetilde{\mathcal{V}}'_p|_{\mathcal{X}'^{\mathrm{cpt}}_v}$ is unipotent (*viz* finite);

[5]Actually, we will only apply these facts when $\mathcal{X}_v = \mathcal{X}_v^{\mathrm{cpt}}$, in which case $\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p) \to \mathrm{Isoc}^{\varphi}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$ is an equivalence. But, to clarify the structure of the proof, we do not make these assumptions here.

* $\mathcal{V}_p$ is semisimple if and only if $\widetilde{\mathcal{V}}'_p|_{\mathcal{X}'^{\mathrm{cpt}}_v}$ is semisimple.

Also, in Theorem 29, one can freely replace $\mathcal{Z}_v$ by a larger 0-dimensional closed subscheme of $\mathcal{X}_v$. In particular, for every 0-dimensional closed subscheme $\mathcal{Z}'_v \subset \mathcal{X}'^{\mathrm{cpt}}_v$ with image $\mathcal{Z}_v \subset \mathcal{X}^{\mathrm{cpt}}_v$ *via* $\mathcal{X}'^{\mathrm{cpt}}_v \to \mathcal{X}^{\mathrm{cpt}}_v$, up to replacing $\mathcal{Z}'_v \subset \mathcal{X}'^{\mathrm{cpt}}_v$ with the inverse image $\mathcal{Z}''_v(\supset \mathcal{Z}'_v)$ of $\mathcal{Z}_v$ in $\mathcal{X}'^{\mathrm{cpt}}_v$, one has:

* $|X|^{\mathrm{uni}}_{\mathcal{V}_p} \not\subset sp_v^{-1}(|\mathcal{Z}_v|)$ if and only if $|X'|^{\mathrm{uni}}_{\mathcal{V}_p} \not\subset sp_v^{-1}(|\mathcal{Z}'_v|)$..

So, without loss of generality, one may assume $\mathcal{X} = \mathcal{X}^{\mathrm{cpt}}$ and $\mathcal{V}_p$ extends to a $\mathbb{Q}_p$-local system $\widetilde{\mathcal{V}}_p$ on $\mathcal{X}$ whose restriction $\widetilde{\mathcal{V}}_p|_{\mathcal{X}_v}$ is semisimple. Furthermore, one has $|\mathcal{X}_v|^{\mathrm{uni}}_{\widetilde{\mathcal{V}}_p} \neq \emptyset$, which implies that for every simple summand $\mathcal{S}_p$ of the scalar extension $(\widetilde{\mathcal{V}}_p|_{\mathcal{X}_v})_{\overline{\mathbb{Q}}_p}$, one also has $|\mathcal{X}_v|^{\mathrm{uni}}_{\mathcal{S}_p} \neq \emptyset$ hence that $\det(\mathcal{S}_p)$ is finite. We are to show that $\widetilde{\mathcal{V}}_p|_{\mathcal{X}_v}$ is finite.

- Let $\mathfrak{V}_p$ denote the overconvergent F-isocrystal corresponding to $(\widetilde{\mathcal{V}}_p|_{\mathcal{X}_v})_{\overline{\mathbb{Q}}_p}$ *via* the $\otimes$-equivalence

$$\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0 \tilde{\to} \mathrm{Rep}^{\dagger}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}_v)).$$

For every semisimple $\mathfrak{E}_p \in \mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$, prime ℓ (possibly $\ell = p$) and field isomorphism $\tau : \overline{\mathbb{Q}}_p \tilde{\to} \overline{\mathbb{Q}}_\ell$ let ${}^\tau\mathfrak{E}_p$ denote the unique (up to isomorphism) semisimple τ-companion of $\mathfrak{E}_p$ [L02], [A18].

When $\ell = p$, the companion correspondance induces an action with finite orbits of $\mathrm{Aut}(\overline{\mathbb{Q}}_p)$ on the set of isomorphism classes of semisimple objects in $\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)$. Let $\mathfrak{V}_{p,1} := (\mathfrak{V}_p)_{\overline{\mathbb{Q}}_p}, \dots, \mathfrak{V}_{p,s}$ denote the finitely many (up to isomorphism) semisimple companions of $\mathfrak{V}_p$. By construction, the overconvergent F-isocrystal

$$\mathfrak{F}_p := \mathfrak{V}_{p,1} \oplus \cdots \oplus \mathfrak{V}_{p,s}$$

is semisimple, $\mathbb{Q}$-rational, each of its simple summand has finite determinant and one has

$$|\mathcal{X}_v|^{\mathrm{uni}}_{\mathfrak{V}_{p,1}} = \cdots = |\mathcal{X}_v|^{\mathrm{uni}}_{\mathfrak{V}_{p,s}}.$$

In particular, if for every $x \in |X|$, $\chi_x \in \mathbb{Q}[T]$ denotes the characteristic polynomial of Frobenius attached to $x^*\mathfrak{F}_p$, then ii) $\chi_x \in \mathbb{Q}[T]$ and, as every simple summand of $\mathfrak{F}_p$ has finite determinant, i) χ_x is pure of weight 0 [A18].

Let $\mathcal{U}_v \subset \mathcal{X}_v$ denote the largest (non-empty) open subscheme over which $\mathfrak{F}_p$ admits a slope filtration [K79, Thm. 2.3.1, 2.4.2]

$$0 = S_0(\mathfrak{F}_p|_{\mathcal{U}_v}) \subsetneq S_1(\mathfrak{F}_p|_{\mathcal{U}_v}) \subsetneq \cdots \subsetneq S_s(\mathfrak{F}_p|_{\mathcal{U}_v}) = \mathfrak{F}_p|_{\mathcal{U}_v},$$

with

$$Gr^S_i(\mathfrak{F}_p|_{\mathcal{U}_v}) := S_i(\mathfrak{F}_p|_{\mathcal{U}_v})/S_{i-1}(\mathfrak{F}_p|_{\mathcal{U}_v})$$

of slope q_i and $q_1 < \cdots < q_s$. Set $\mathcal{Z}_v := \mathcal{X}^{\mathrm{cpt}}_v \setminus \mathcal{U}_v$. We distinguish two cases:

– At least one of the q_i is $\neq 0$, which forces $|X|^{\mathrm{uni}}_{\mathcal{V}_p} \subset sp_v^{-1}(\mathcal{Z}_v)$;

– $\mathfrak{F}_p|_{\mathcal{U}_p}$ is unitroot. By semicontinuity of the slope filtration [K79, Thm. 2.3.1], this imposes $\mathcal{Z}_v = \emptyset$ and $\mathfrak{F}_p$ is unit-root. In particular, for every $x \in |X|$, iv) $\chi_x \in \overline{\mathbb{Z}}_p[T]$. Eventually, the fact that every simple summand of $\mathfrak{F}_p$ has finite determinant implies that for every prime ℓ and field isomorphism $\tau : \overline{\mathbb{Q}}_p \tilde{\to} \overline{\mathbb{Q}}_\ell$, the unique semisimple τ-companion ${}^\tau\mathfrak{F}_p$ of $\mathfrak{F}_p$ is *étale*; in particular, for every $x \in |X|$, iii) $\chi_x \in \overline{\mathbb{Z}}_\ell[T]$. Let $\mathcal{F}_p$ denote the potentially unramified $\overline{\mathbb{Q}}_p$-local system corresponding to $\mathfrak{F}_p$ *via*

$$\mathrm{Isoc}^{\varphi,\dagger}(\mathcal{X}_v/\overline{\mathbb{Q}}_p)^0 \tilde{\to} \mathrm{Rep}^{\dagger}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}_v)).$$

We have just shown that for every $x \in |\mathcal{X}_v|$, the characteristic polynomial χ_x of the Frobenius $\varphi_{x,p} : \mathcal{F}_{p,\bar{x}} \tilde{\to} \mathcal{F}_{p,\bar{x}}$ satisfies

$$\text{i) } \chi_x \text{ is pure of weight } w = 0; \quad \text{ii) } \chi_x \in \mathbb{Q}[T]; \quad \text{iii) } \chi_x \in \overline{\mathbb{Z}}_\ell[T],\ \ell \neq p; \quad \text{iv) } \chi_x \in \overline{\mathbb{Z}}_p[T],$$

hence is a product of cyclotomic polynomials. In particular, for every connected étale cover $\mathcal{X}'_v \to \mathcal{X}_v$ such that $\mathrm{Lev}_1(\mathcal{F}_p|_{\mathcal{X}'_v})$ holds, for every $x' \in |\mathcal{X}'_v|$, $\chi_{x'} = (T-1)^r$. By Cebotarev, this implies $\mathcal{F}_p$ - hence *a fortiori* $(\widetilde{\mathcal{V}}_p|_{\mathcal{X}_v})_{\overline{\mathbb{Q}}_p}$, is quasi-unipotent - hence finite (since $\mathcal{F}_p$, $(\widetilde{\mathcal{V}}_p|_{\mathcal{X}_v})_{\overline{\mathbb{Q}}_p}$ are semisimple)[6].

[6] For this part of the argument, see also [Ko17, Prop. 1.1].

□

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