VARIATION OF TANNAKA GROUPS OF PERVERSE SHEAVES IN FAMILY

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Abstract. Let k be a field of characteristic 0, let S be a smooth, geometrically connected variety over k, with generic point η, and $f : \mathcal{X} \to S$ a morphism separated and of finite type. Fix a prime ℓ. Let $\mathcal{P}$ be an f-universally locally acyclic relative perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}/S$. We prove that if for some (equivalently, every) geometric point $\bar{\eta}$ over η the restriction $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple as a perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}_{\bar{\eta}}$, then there is a non-empty open subscheme $U \subset S$ such that, for every geometric point $\bar{s}$ on U, the restriction $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple as a perverse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}_{\bar{s}}$. When $f : \mathcal{X} \to S$ is an abelian scheme, we give applications of this result to the variation with $s \in S$ of the Tannaka group of $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$.

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1. Introduction

Let k be a field of characteristic 0, let S be a smooth, geometrically connected variety over k, with generic point η, and let $g : \mathcal{Y} \to S$ be a smooth projective S-scheme of relative dimension d. A general and central question in algebraic geometry is to understand how the fibers $\mathcal{Y}_s$ vary with $s \in S$. For instance, one may ask when some power $\mathcal{Y}_{\bar{s}}^n$ of $\mathcal{Y}_{\bar{s}}$ carries exceptional algebraic cycles. If $k \subset \mathbb{C}$, the Hodge conjecture predicts that this is the same as asking when the Mumford Tate group $G(\mathcal{V})_s$ of the polarizable $\mathbb{Q}$-Hodge structure $\mathrm{H}^\bullet(\mathcal{Y}_s^{\mathrm{an}}, \mathbb{Q}) \simeq s^* R^\bullet g_*^{\mathrm{an}} \mathbb{Q}$ becomes smaller than the generic Mumford-Tate group $G(\mathcal{V})$ of the polarizable $\mathbb{Q}$-variation of Hodge structures $\mathcal{V} := R^\bullet g_*^{\mathrm{an}} \mathbb{Q}$ on the analytification S^{an} of $S \times_k \mathbb{C}$. Here "becomes smaller" makes sense because the Tannaka categories of polarizable $\mathbb{Q}$-variations of Hodge structures are functorial with respect to pullbacks along morphism of complex analytic spaces $S' \to S$ so that one can view naturally $G(\mathcal{V})_s$ as a subgroup of $G(\mathcal{V})$. This leads to introduce and study the Hodge locus

$$S_{\mathcal{V}} := \{s \in S \mid G(\mathcal{V})_s \subsetneq G(\mathcal{V})\}$$

of a polarizable $\mathbb{Q}$-variation of Hodge structures $\mathcal{V}$ on S^{an}. If k is finitely generated over $\mathbb{Q}$, similar considerations apply with ℓ-adic étale local systems on S yielding the introduction of the Tate locus $S_{\mathcal{V}_\ell} \subset S$ of such a $\mathbb{Q}_\ell$-local system $\mathcal{V}_\ell$. Under mild assumptions, general heuristics predict that these exceptional loci $S_{\mathcal{V}}$, $S_{\mathcal{V}_\ell}$ are sparse in some precise sense - *e.g.* that the atypical part of the Hodge locus $S_{\mathcal{V}}$ is not Zariski-dense in S [Kl23] or that the set of k-rational points in the Tate locus $S_{\mathcal{V}_\ell}$ is not Zariski-dense in S [C23]. Proving such sparsity results is notoriously challenging as it requires constructing bridges between the Zariski topology of S and the analytic natures of the coefficients $\mathcal{V}$, $\mathcal{V}_\ell$. The results of this article are also partly motivated by the problem of understanding how the fibers $\mathcal{Y}_s$ vary with $s \in S$ and inspired by the above Tannaka approaches, but in a more restricted setting and with a rather different category of coefficients, which makes the sparsity of the exceptional loci more accessible. Namely, if one assumes $g : \mathcal{Y} \to S$ factors as

$$g : \mathcal{Y} \overset{\iota}{\hookrightarrow} \mathcal{X} \xrightarrow{f} S$$

with $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed immersion and $f : \mathcal{X} \to S$ an abelian scheme, one can consider the f-universally locally acyclic (f-ULA or simply ULA for short) relative perverse sheaf $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_\ell[d]$ on $f : \mathcal{X} \to S$. As the quotient of the category of f-ULA relative perverse sheaves by negligible ones is Tannaka and functorial with respect to pullback along morphism of schemes $S' \to S$, one can attach to each $s \in S$ a Tannaka group $G(\mathcal{P})_s$ which detects some of the symmetric features of $\mathcal{Y}_s$ regarded as a closed subvariety of $\mathcal{X}_s$ and ask for the structure of the corresponding degeneracy locus $S_{\mathcal{P}} \subset S$.

More formally, let $f : \mathcal{X} \to S$ be an abelian scheme. Fix a prime ℓ. Let $D^b_c(\mathcal{X})$ denote the triangulated category of étale $\overline{\mathbb{Q}}_\ell$-sheaves with bounded constructible cohomology on $\mathcal{X}$ and let $D^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^b_c(\mathcal{X})$ denote the full subcategory[1] of those complexes which are f-universally locally acyclic; this is a triangulated subcategory. The convolution product built out from the multiplication on $\mathcal{X}$ endows $D^{\mathrm{ULA}}(\mathcal{X}/S)$ with a structure of $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category. This monoidal structure, in turn, induces a structure of $\overline{\mathbb{Q}}_\ell$-Tannakian category on the quotient $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \twoheadrightarrow P^{\mathrm{ULA}}(\mathcal{X}/S)$ of the full subcategory $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ of f-ULA relative perverse $\overline{\mathbb{Q}}_\ell$-sheaf by the Serre subcategory of negligible objects. When S is a point, one recovers the usual construction $\mathrm{Perv}(X) \twoheadrightarrow P(X)$ of the $\overline{\mathbb{Q}}_\ell$-Tannakian category of perverse sheaves on an abelian variety X. Further, in the relative setting, for every $s \in S$ and geometric point $\bar{s}$ over s, the canonical restriction functors

$$\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \overset{|_{\mathcal{X}_s}}{\to} \mathrm{Perv}(\mathcal{X}_s) \overset{|_{\mathcal{X}_{\bar{s}}}}{\to} \mathrm{Perv}(\mathcal{X}_{\bar{s}})$$

induce exact tensor functors

$$P^{\mathrm{ULA}}(\mathcal{X}/S) \overset{|_{\mathcal{X}_s}}{\to} P(\mathcal{X}_s) \overset{|_{\mathcal{X}_{\bar{s}}}}{\to} P(\mathcal{X}_{\bar{s}}).$$

In particular, fixing a fiber functor $\omega_{\bar{s}} : P(\mathcal{X}_{\bar{s}}) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$, one may ask, for $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, how the corresponding Tannaka groups

$$G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \subset G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \subset G(\mathcal{P}, \omega_{\bar{s}})$$

vary[2] with $s \in S$. Further, as S is smooth over k, the canonical functor

$$-|_{\mathcal{X}_\eta} : \langle \mathcal{P} \rangle \to \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle$$

is an equivalence of categories and, for every $s \in S$ one gets a natural (up to inner automorphisms) cospecialization diagram (see Section 3):

(1)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{\bar{\eta}}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{\bar{\eta}}) \arrow[r] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] & 1,
\end{tikzcd}$$

where, for $t \in S$, the category $\langle \mathcal{P}|_{\mathcal{X}_t} \rangle_0 \subset \langle \mathcal{P}|_{\mathcal{X}_t} \rangle$ denotes the full subcategory whose objects are of the form $0_{t*}\mathcal{L}$ for $\mathcal{L} \in \mathrm{Perv}(\mathrm{spec}(k(t)))$ and $0_t : \mathrm{spec}(k(t)) \to \mathcal{X}_t$ the zero-section. As the existence of this cospecialization diagram does not depend on the choice of the fiber functors, we omit

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[1] The assumption f-ULA is not very restrictive as, for every $\mathcal{K} \in D^b_c(\mathcal{X})$ there exists a non-empty open subscheme $U \subset S$ such that $\mathcal{K}|_{\mathcal{X} \times_S U} \in D^{\mathrm{ULA}}(\mathcal{X} \times_S U/U)$; see [SGA4 1/2, Thm. 2.13, p. 242] and [B24, Lemma 3.10].

[2] Recall that if k is algebraically closed and K/k is an extension of algebraically closed fields then the canonical restriction functor $P(X) \to P(X_K)$, is a fully faithful tensor functor with image stable under subquotients, so that the category $\langle \mathcal{X}_{\bar{s}} \rangle$ and the group $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ do not depend on the choice of the geometric point $\bar{s}$ over $s \in S$ but only on s itself.

them from the notation from now on. In other words, one would like to understand the arithmetico-geometric structure of the following degeneracy loci:

$$S_{\mathcal{P}}^{?} := \{s \in S \mid G(\mathcal{P}|_{\mathcal{X}_s})^{?} \subsetneq G(\mathcal{P}|_{\mathcal{X}_\eta})^{?}\}$$
$$S_{\mathcal{P}}^{\mathrm{geo},?} := \{s \in S \mid G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{?} \subsetneq G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{?}\},$$

where *e.g.*

?=	no decoration	G	;
	$\circ$	G°	:= neutral component of G;
	der	G^{der}	:= derived subgroup of G;
	$\circ$, der	$G^{\circ,\mathrm{der}}$.	

For $S_{\mathcal{P}}^{\mathrm{geo}}$, this question has been tackled in Krämer's dissertation thesis [Kr13, 3.7]; in particular Krämer observes that one cannot expect, in general, that $S_{\mathcal{P}}^{\mathrm{geo}}$ be a strict, Zariski-closed subset of S unless the determinant $\det(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ of $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is torsion and uniformly bounded with s [Kr13, Ex. 3.17 a)]. In the converse direction, Krämer proves the following.

Fact 1.1. ([Kr13, Prop. 3.20)], [KW15, Prop. 7.4]) *Let* $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. *Assume that for every geometric point* $\bar{s}$ *over* $s \in S$, *the restriction* $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ *is simple in* $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ *with torsion determinant*[3] *and that the order of* $\det(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ *is uniformly bounded with* $s \in S$. *Then* $S_{\mathcal{P}}^{\mathrm{geo}}$ *is not Zariski-dense in* S.

Our main result is about the simplicity assumption.

Theorem 1.2. *Let* $f : \mathcal{X} \to S$ *be a morphism, separated and of finite type. For every* $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, *after possibly replacing* S *by a non-empty open subscheme (depending on* $\mathcal{P}$*) the following holds. For every* $s \in S$,

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}).$$

In particular, if $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ *is simple (resp. semisimple) in* $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$ *then* $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ *is simple (resp. semisimple) in* $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$.

Corollary 1.3. *Assume furthermore* $f : \mathcal{X} \to S$ *is an abelian scheme. Then for every* $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, *after possibly replacing* S *by a non-empty open subscheme (depending on* $\mathcal{P}$*) the following holds. For every* $s \in S$,

$$\mathrm{length}_{P(\mathcal{X}_{\bar{\eta}})}(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) = \mathrm{length}_{P(\mathcal{X}_{\bar{s}})}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}).$$

In particular, if $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ *is simple (resp. semisimple) in* $P(\mathcal{X}_{\bar{\eta}})$ *then* $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ *is simple (resp. semisimple) in* $P(\mathcal{X}_{\bar{s}})$.

Theorem 1.2 yields the following generalization of Fact 1.1 to arbitrary semisimple perverse sheaves.

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[3] As a connected reductive group G over an algebraically closed field Q of characteristic 0 admits an irreducible faithful representation if and only if its center is $\mathbb{G}_{m,Q}$ or finite cyclic, the condition that $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is simple with torsion determinant imposes that $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ}$ is semisimple with finite cyclic center.

Corollary 1.4. *Let $f : \mathcal{X} \to S$ be an abelian scheme and let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$.*

(1) Assume $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ for every $s \in S$. Then $S_{\mathcal{P}}^{\mathrm{geo}}$ is a countable union of strict, Zariski-closed subvarieties of S.

(2) Assume $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple in $P(\mathcal{X}_{\bar{\eta}})$. Then $S_{\mathcal{P}}^{\mathrm{geo}}$ is contained in a countable union of strict, Zariski-closed subvarieties of S.

If k is countable, we do not know if $S_{\mathcal{P}}^{\mathrm{geo}} \subsetneq S$ in general though we suspect it is true. Still, combined with Fact 1.1 and some tannakian formalism Theorem 1.2 yields the following. For an algebraic group G over a field Q, let $R(G) \subset G$ denote its solvable radical (*viz* its largest connected normal solvable subgroup) and $G \twoheadrightarrow G^{\mathrm{ss}} := G/R(G)$ its maximal semisimple quotient.

Corollary 1.5. *Let $f : \mathcal{X} \to S$ be an abelian scheme and let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$.*

(1) Assume $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple in $P(\mathcal{X}_{\bar{\eta}})$ with torsion determinant. Then $S_{\mathcal{P}}^{\mathrm{geo}}$ is not Zariski-dense in S.

(2) Up to replacing S by a non-empty open subscheme, one may assume that for all $s \in S$ the canonical morphism induced by cospecialization

$$G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ} \hookrightarrow G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ} \twoheadrightarrow G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{ss}}$$

factors through an isogeny

$$\begin{tikzcd}
G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ} \arrow[r, hook] \arrow[d, two heads] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ} \arrow[d, two heads] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ,\mathrm{ss}} \arrow[r, dotted] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{ss}}.
\end{tikzcd}$$

In particular,

(a) if $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is semisimple (e.g. $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple with torsion determinant in $P(\mathcal{X}_{\bar{\eta}})$) then $S_{\mathcal{P}}^{\mathrm{geo},\circ}$ is not Zariski-dense in S.

(b) if $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is reductive (viz $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is semisimple in $P(\mathcal{X}_{\bar{\eta}})$) then $S_{\mathcal{P}}^{\mathrm{geo},\circ,\mathrm{der}}$ is not Zariski-dense in S.

Here is a sample of geometric application of Corollary 1.5 (see also Remark 5.3).

Corollary 1.6. (Corollary 5.2) *Let $\mathcal{X} \to S$ be an abelian scheme of relative dimension $g \geq 3$ and $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth and geometrically connected over S. Assume $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ has ample normal bundle and trivial stabilizer. Then the set of all $s \in S$ such that $\mathcal{Y}_{\bar{s}}$ is a product is Zariski-dense in S (if and) only if $\mathcal{Y}_{\bar{\eta}}$ is itself a product.*

We refer to Section 5 for more details.

As for $S_{\mathcal{P}}$, at least if k is arithmetically rich enough, the non-Zariski density of $S_{\mathcal{P}}^{\mathrm{geo}}$ in S automatically implies that $S_{\mathcal{P}}$ is sparse in the following sense. For an integer $d \geq 1$, write

$$|S|^{\leq d} := \{s \in |S| \mid [k(s) : k] \leq d\}.$$

Proposition 1.7. *Let $f : \mathcal{X} \to S$ be an abelian scheme and let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume $S_{\mathcal{P}}^{\mathrm{geo}}$ is not Zariski-dense in S. Assume furthermore that S has dimension > 0 and that k is Hilbertian (e.g. finitely generated over $\mathbb{Q}$). Then there exists an integer $d \geq 1$ such that $|S|^{\leq d} \setminus (S_{\mathcal{P}} \cap |S|^{\leq d})$ is infinite.*

When S is a curve, k is a number field and $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$ is semisimple, the conclusion of Proposition 1.7 can be strengthened to: for every integer $d \geq 1$ the set $S_{\mathcal{P}} \cap |S|^{\leq d}$ is finite. This applies, for instance, to the intersection complex $\iota_* \overline{\mathbb{Q}}_\ell[d]$ for $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed immersion such that $\mathcal{Y} \to S$ is smooth, geometrically connected of relative dimension d and symmetric in the sense that $[-1]^*\mathcal{Y} = \mathcal{Y}$.

Organization of the paper. In Section 2, we briefly review the Tannakian formalism of perverse sheaves on abelian schemes, both in the absolute and relative setting. In Section 3, we elucidate the existence of the specialization diagram (1), giving two constructions. The proofs of Theorem 1.2, its corollaries and Proposition 1.7 are performed in Section 4. The final Section 5 is devoted to a sample of geometric applications of Corollary 1.5.

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Notation and conventions

For an additive functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ between abelian categories, we write $F : \mathcal{A}_1 \overset{\approx}{\overset{\sim}{\to}} \mathcal{A}_2$ if it is fully faithful with image stable under subquotients and $F : \mathcal{A}_1 \overset{\simeq}{\to} \mathcal{A}_2$ if it is an equivalence.

For a rigid symmetric monoidal category $(\mathcal{T}, \otimes)$ with unit $\mathbb{I}$ and an object X in $\mathcal{T}$, let $X^\vee$ denote its dual and, for every integers $m, n \geq 0$, set $T^{m,n}(X) := X^{\otimes m} \otimes X^{\vee \otimes n}$; write $T(X) := \oplus_{m,n \geq 0} T^{m,n}(X)$. If $(\mathcal{T}, \otimes)$ is

Tannakian, for every integers $m, n \geq 0$, let also $I^{m,n}(X) \subset T^{m,n}(X)$ denote the sum of all subobjects of $T^{m,n}(X)$ which are isomorphic to $\mathbb{I}$ in $\mathcal{T}$ (so that $\mathrm{Hom}_{\mathcal{T}}(\mathbb{I}, I^{m,n}(X)) \tilde{\rightarrow} \mathrm{Hom}_{\mathcal{T}}(\mathbb{I}, T^{m,n}(X)))$. If $\mathcal{T}$ is Tannakian with fiber functor $\omega : \mathcal{T} \rightarrow \mathrm{Vect}_Q$, let $G(\mathcal{T}, \omega)$ denote its Tannaka group; recall that $G(\mathcal{T}, \omega)$ may depend on ω but that if Q is algebraically closed then $G(\mathcal{T}, \omega)$ is uniquely determined up to non-canonical isomorphism. For an object X in $\mathcal{T}$ let $\langle X \rangle \subset \mathcal{T}$ denote the smallest Tannakian category containing X and, given a fiber functor $\omega : \langle X \rangle \rightarrow \mathrm{Vect}_Q$ set $G(X, \omega) := G(\langle X \rangle, \omega)$.

In the whole paper, we fix a prime ℓ. For a scheme S, let $\mathrm{Loc}(S)$ denote the category of étale $\overline{\mathbb{Q}}_\ell$-local systems on S and $D^b_c(S)$ the triangulated category of étale $\overline{\mathbb{Q}}_\ell$-sheaves with bounded constructible cohomology on S.

A variety over a field K is a scheme separated and of finite type over K.

When S is a variety, let $\mathrm{Perv}(S) \subset D^b_c(S)$ denote the full subcategory of perverse sheave and, for a morphism $f : X \rightarrow S$ of varieties, write

$$D_{X/S}(-) := \mathrm{RHom}(-, Rf^!\overline{\mathbb{Q}}_\ell) : D^b_c(X)^{\mathrm{op}} \rightarrow D^b_c(X)$$

for the relative Verdier duality functor. When S is a point, we simply set $D_X(-) := D_{X/S}(-)$.

For morphisms of varieties $S_1 \rightarrow S \leftarrow S_2$, one writes

$$\boxtimes^L_S : D^b_c(S_1) \times D^b_c(S_2) \rightarrow D^b_c(S_1 \times_S S_2), \ \ (\mathcal{K}_1, \mathcal{K}_2) \mapsto p_1^*\mathcal{K}_1 \otimes^L p_2^*\mathcal{K}_2,$$

for the outer tensor product, where $p_i : S_1 \times_S S_2 \rightarrow S_i$ denotes the ith projection, $i = 1, 2$. When $S = \mathrm{spec}(k)$, one simply writes $\boxtimes^L := \boxtimes^L_S$.

2. Tannakian category of relative perverse sheaves

2.1. **Absolute setting.** Let K be a field of characteristic 0 and let X be an abelian variety over K with group law $m : X \times_K X \rightarrow X$.

2.1.1. *Construction.* See [Kr13], [KrW15] for details, and [JKrLM25, §3.1] for a shorter overview. The convolution product

$$* : D^b_c(X) \times D^b_c(X) \rightarrow D^b_c(X), \ \ (\mathcal{K}_1, \mathcal{K}_2) \mapsto \mathcal{K}_1 * \mathcal{K}_2 := Rm_*(\mathcal{K}_1 \boxtimes^L \mathcal{K}_2)$$

endows $D^b_c(X)$ with the structure of a $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category with duality functor

$$(-)^\vee : D^b_c(X) \rightarrow D^b_c(X), \ \ \mathcal{K} \mapsto \mathcal{K}^\vee := [-1]^* D_X(\mathcal{K})$$

and unit the rank one skyscraper sheaf $\delta_0 := \iota_{0*}\overline{\mathbb{Q}}_\ell \in D^b_c(X)$ supported on 0.

The full subcategory $\mathrm{Perv}(X) \subset D^b_c(X)$ is abelian and stable under Verdier duality, but not under convolution. To remedy this, one can mod out by negligible objects. Recall that every $\mathcal{P} \in \mathrm{Perv}(X)$ has non negative Euler-Poincaré characteristic:

$$\chi(X, \mathcal{P}) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\overline{\mathbb{Q}}_\ell}(\mathrm{H}^i(X_{\bar{K}}, \mathcal{P})) \geq 0$$

Let ${}^p\mathrm{H}^n(-) : D^b_c(X) \to \mathrm{Perv}(X)$, $n \in \mathbb{Z}$ denote the perverse cohomology functors and let $N(X) \subset D^b_c(X)$ denote the full subcategory of all $\mathcal{K} \in D^b_c(X)$ such that $\chi(X, {}^p\mathrm{H}^n(\mathcal{K})) = 0$ for all $n \in \mathbb{Z}$; this is a null system such that the convolution bifunctor $* : D^b_c(X) \times D^b_c(X) \to D^b_c(X)$, the dualization functor $(-)^\vee : D^b_c(X)^{\mathrm{op}} \to D^b_c(X)$ and the perverse cohomology ${}^pH^0(-) : D^b_c(X) \to \mathrm{Perv}(X)$ restrict to

$$* : N(X) \times D^b_c(X) \to N(X), \quad * : D^b_c(X) \times N(X) \to N(X)$$
$$(-)^\vee : N(X)^{\mathrm{op}} \to N(X)$$

and

$${}^pH^0(-) : N(X) \to N(X) \cap \mathrm{Perv}(X).$$

Consider the quotient functor

$$\mathrm{Perv}(X) \to P(X) := \mathrm{Perv}(X)/(N(X) \cap \mathrm{Perv}(X))$$

so that one gets

$$\begin{tikzcd}
\mathrm{Perv}(X) \times \mathrm{Perv}(X) \arrow[r, "*"] \arrow[d] & D^b_c(X) \arrow[r, "{}^pH^0(-)"] & \mathrm{Perv}(X) \arrow[d] \\
P(X) \times P(X) \arrow[rr, dotted, "*"'] & & P(X)
\end{tikzcd}$$

The abelian category $P(X)$ endowed with

$$* : P(X) \times P(X) \to P(X)$$

is Tannakian with duality functor induced by

$$\begin{tikzcd}
\mathrm{Perv}(X)^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d] & \mathrm{Perv}(X) \arrow[d] \\
P(X)^{\mathrm{op}} \arrow[r, dotted, "(-)^\vee"'] & P(X)
\end{tikzcd}$$

and unit the image of δ_0 in $P(X)$.

2.1.2. *Extension of the base field.* See [JKrLM25, Sec. 4] for details. Let L/K be a field extension and let $K^L \subset L$ denote the algebraic closure of K in L; assume K^L/K is Galois. Let $\mathcal{T} \subset P(X)$ be a full abelian $\otimes$-subcategory and let $\mathcal{T}_L \subset P(X_L)$ denote the full abelian $\otimes$-subcategory generated by the essential image of $\mathcal{T} \hookrightarrow P(X) \overset{-|_{X_L}}{\to} P(X_L)$, namely the full subcategory of all $\mathcal{Q} \in P(X_L)$ such that there exists $\mathcal{P} \in \mathcal{T}$ with $\mathcal{Q}$ a subquotient of $\mathcal{P}|_{X_L}$. For instance, for every $\mathcal{P} \in P(X)$, $\langle \mathcal{P} \rangle_L = \langle \mathcal{P}|_{X_L} \rangle$. The structure of $\mathcal{T}$ is closely related to the structure of $\mathcal{T}_L$ and the structure of the category

$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(K^L/K))$ of finite dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations of the Galois group $\mathrm{Gal}(K^L/K)$ of K^L/K. More precisely,

- The canonical functor

$$-|_{X_L} : \mathrm{Perv}(X) \overset{|_{X_L}}{\to} \mathrm{Perv}(X_L) \to P(X_L)$$

is an exact functor of $\overline{\mathbb{Q}}_\ell$-linear categories which induces a faithful functor of Tannakian categories

$$\begin{tikzcd} \mathrm{Perv}(X) \arrow[r, "|_{X_L}"] \arrow[d] & \mathrm{Perv}(X_L) \arrow[d] \\ P(X) \arrow[r, dotted, "|_{X_L}"] & P(X_L) \end{tikzcd} .$$

- For simplicity, write $\mathrm{Perv}(K) := \mathrm{Perv}(\mathrm{spec}(K))$. The canonical functor

$$0_* : \mathrm{Perv}(K) \overset{0_*}{\to} \mathrm{Perv}(X) \to P(X)$$

is an exact fully faithful functor of Tannakian categories with essential image $P_0(X) \subset P(X)$ stable under subquotients. Precomposing $0_* : \mathrm{Perv}(K) \to P_0(X) \hookrightarrow P(X)$ with the fully faithful exact $\otimes$-tensor functor $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(K^L/K)) \hookrightarrow \mathrm{Perv}(K)$, one gets an exact fully faithful functor of Tannakian categories

$$0_*^L : \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(K^L/K)) \to P(X);$$

let $P_0^L(X) \subset P_0(X)$ denote its essential image.

Consider the full Tannakian subcategory $\mathcal{T}_0^L := \mathcal{T} \cap P_0^L(X) \subset \mathcal{T}$. Then, for every fiber functor $\omega : \mathcal{T}_L \to \mathrm{Vect}_{\mathbb{Q}_\ell}$, the sequence of Tannakian categories

$$\mathcal{T}_0^L \to \mathcal{T} \overset{|_{X_L}}{\to} \mathcal{T}_L$$

induces a short exact sequence of proalgebraic groups

$$1 \longrightarrow G(\mathcal{T}_L, \omega) \longrightarrow G(\mathcal{T}, \omega) \longrightarrow G(\mathcal{T}_0^L, \omega) \longrightarrow 1,$$

from which one immediately deduces that

- (2.1.2-1) For every $\mathcal{P} \in P(X)$, the sequence of Tannakian categories

$$\langle \mathcal{P} \rangle_0^L \to \langle \mathcal{P} \rangle \overset{|_{X_L}}{\to} \langle \mathcal{P}|_{X_L} \rangle$$

induces a short exact sequence of algebraic groups

$$1 \longrightarrow G(\mathcal{P}|_{X_L}, \omega) \longrightarrow G(\mathcal{P}, \omega) \longrightarrow G(\langle \mathcal{P} \rangle_0^L, \omega) \longrightarrow 1.$$

- (2.1.2-2) If K is algebraically closed, the restriction functor $-|_L : \mathcal{T} \overset{\simeq}{\to} \mathcal{T}_L$ is an equivalence of Tannakian categories. In particular, for every $\mathcal{P} \in P(X)$, $G(\mathcal{P}|_{X_L}, \omega) \tilde{\to} G(\mathcal{P}, \omega)$.

We drop the superscript $(-)^L$ when $L = \overline{K}$.

2.2. **Relative setting.**

Let k be a field of characteristic 0, S a smooth, geometrically connected variety over k with generic point η. Let $f : \mathcal{X} \to S$ be an abelian scheme.

2.2.1. *Construction.* See [HS23] for details. Let $D^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^b_c(\mathcal{X})$ denote the full subcategory of f-universally locally acyclic (f-ULA or just ULA for short) complexes on $\mathcal{X}/S$; this is a triangulated subcategory. As in the absolute setting, the convolution product

$$* : D^{\mathrm{ULA}}(\mathcal{X}/S) \times D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(\mathcal{X}/S), \quad (\mathcal{K}_1, \mathcal{K}_2) \mapsto \mathcal{K}_1 * \mathcal{K}_2 := Rm_*(\mathcal{K}_1 \boxtimes^L_S \mathcal{K}_2)$$

endows $D^{\mathrm{ULA}}(\mathcal{X}/S)$ with the structure of a $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category with duality functor

$$(-)^\vee : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(\mathcal{X}/S), \quad K \mapsto \mathcal{K}^\vee := [-1]^* D_{\mathcal{X}/S}(\mathcal{K})$$

and unit $\delta_{S,0} := 0_* \overline{\mathbb{Q}}_\ell \in D^b_c(\mathcal{X})$, where $0 : S \hookrightarrow \mathcal{X}$ is the 0-section. By construction and proper base change, for every $s \in S$, the pull-back functor $-|_{\mathcal{X}_s} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_s)$ is a tensor functor.

Let $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ denote the full subcategory of ULA relative perverse sheaves on $\mathcal{X}$. This is an abelian category, stable by relative Verdier duality $D_{\mathcal{X}/S}(-) : D^b_c(\mathcal{X})^{\mathrm{op}} \to D^b_c(\mathcal{X})$ and such that for every $s \in S$, the pull-back functor $-|_{\mathcal{X}_s} : D^b_c(\mathcal{X}) \to D^b_c(\mathcal{X}_s)$ restricts to an exact functor $-|_{\mathcal{X}_s} : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_s)$ which, when $s = \eta$, is fully faithful with essential image stable under subquotients. Actually, $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ is the heart of a t-structure $D^{\mathrm{ULA},\leq 0}(\mathcal{X}/S), D^{\mathrm{ULA},\geq 0}(\mathcal{X}/S) \subset D^{\mathrm{ULA}}(\mathcal{X}/S)$ - the relative perverse t-structure with associated truncation functors ${}^{p/S}\tau^{\leq 0} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA},\leq 0}(\mathcal{X}/S)$, ${}^{p/S}\tau^{\geq 0} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA},\geq 0}(\mathcal{X}/S)$ and perverse cohomology functors

$${}^{p/S}\mathrm{H}^n : D^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S), \quad n \in \mathbb{Z}.$$

As $f : \mathcal{X} \to S$ is proper, $Rf_* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(S)$ factors as $Rf_* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(S/S) = D^b_{\mathrm{liss}}(S)$ [B24, Lem. (ii), p.20, Lem. (i), p.21]. Combined with the fact that, for every $t \in S$, the following diagrams

$$\begin{tikzcd} D^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "-|_{\mathcal{X}_{\bar{t}}}"] \arrow[d, "{}^{p/S}\mathrm{H}^n(-)"'] & D^b_c(\mathcal{X}_{\bar{t}}) \arrow[d, "{}^{p}\mathrm{H}^n(-)"] \\ \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "-|_{\mathcal{X}_{\bar{t}}}"'] & \mathrm{Perv}(\mathcal{X}_{\bar{t}}) \end{tikzcd} \quad , \quad n \in \mathbb{Z}$$

commute, one gets that, for $\mathcal{K} \in D^{\mathrm{ULA}}(\mathcal{X}/S)$, the following properties are equivalent:

(i) $\mathcal{K}|_{\mathcal{X}_\eta} \in N(\mathcal{X}_\eta)$;
(ii) For every $s \in S$, $\mathcal{K}|_{\mathcal{X}_s} \in N(\mathcal{X}_s)$;
(iii) There exists $s \in S$ such that $\mathcal{K}|_{\mathcal{X}_s} \in N(\mathcal{X}_s)$.

In other words, for every $s \in S$, the following null systems of $D^{\mathrm{ULA}}(\mathcal{X}/S)$

$$\begin{aligned} N^{\mathrm{ULA}}(\mathcal{X}/S) \quad &:= \ker(D^{\mathrm{ULA}}(\mathcal{X}/S) \overset{-|_{\mathcal{X}_\eta}}{\to} D^b_c(\mathcal{X}_\eta) \to D^b_c(\mathcal{X}_\eta)/N(\mathcal{X}_\eta)) \\ &= \ker(D^{\mathrm{ULA}}(\mathcal{X}/S) \overset{-|_{\mathcal{X}_s}}{\to} D^b_c(\mathcal{X}_s) \to D^b_c(\mathcal{X}_s)/N(\mathcal{X}_s)). \end{aligned} \tag{2}$$

coincide.

By construction the functors $* : D^{\mathrm{ULA}}(\mathcal{X}/S) \times D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^{\mathrm{ULA}}(\mathcal{X}/S)$, $(-)^\vee : D^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \to D^{\mathrm{ULA}}(\mathcal{X}/S)$, ${}^{p/S}H^0(-) : D^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ and $-|_{\mathcal{X}_s} : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_s)$, $s \in S$ restrict to

$$* : N^{\mathrm{ULA}}(\mathcal{X}/S) \times D^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S), \;\; * : D^{\mathrm{ULA}}(\mathcal{X}/S) \times N^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S)$$

$$(-)^\vee : N^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \to N^{\mathrm{ULA}}(\mathcal{X}/S)$$

$${}^{p/S}H^0(-) : N^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S) \cap \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$$

and

$$-|_{\mathcal{X}_s} : N^{\mathrm{ULA}}(\mathcal{X}/S) \to N(\mathcal{X}_s), \;\; s \in S.$$

Consider the quotient functor

$$\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to P^{\mathrm{ULA}}(\mathcal{X}/S) := \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)/(\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \cap N^{\mathrm{ULA}}(\mathcal{X}/S))$$

so that one gets

$$\begin{tikzcd} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \times \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "*"] \arrow[d] & D^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "{}^{p/S}H^0(-)"] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[d] \\ P^{\mathrm{ULA}}(\mathcal{X}/S) \times P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[rr, dotted, "*"'] & & P^{\mathrm{ULA}}(\mathcal{X}/S) \end{tikzcd}$$

The abelian category $P^{\mathrm{ULA}}(\mathcal{X}/S)$ endowed with

$$* : P^{\mathrm{ULA}}(\mathcal{X}/S) \times P^{\mathrm{ULA}}(\mathcal{X}/S) \to P^{\mathrm{ULA}}(\mathcal{X}/S)$$

is a $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal category with duality functor induced by

$$\begin{tikzcd} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \arrow[r, "(-)^\vee"] \arrow[d] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[d] \\ P^{\mathrm{ULA}}(\mathcal{X}/S)^{\mathrm{op}} \arrow[r, dotted, "(-)^\vee"'] & P^{\mathrm{ULA}}(\mathcal{X}/S) \end{tikzcd}$$

and unit the image of $\delta_{S,0}$ in $P^{\mathrm{ULA}}(\mathcal{X}/S)$. For every $s \in S$, the exact pull-back functor $-|_{X_s} : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_s)$ induces an exact faithful functor of $\overline{\mathbb{Q}}_\ell$-linear rigid symmetric monoidal categories

$$\begin{tikzcd} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, "-|_{\mathcal{X}_s}"] \arrow[d] & \mathrm{Perv}(\mathcal{X}_s) \arrow[d] \\ P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[r, dotted, "-|_{\mathcal{X}_s}"'] & P(\mathcal{X}_s), \end{tikzcd} \tag{3}$$

which, when $s = \eta$, is fully faithful with essential image stable under subquotients. In particular, $-|_{\mathcal{X}_\eta} : P^{\mathrm{ULA}}(\mathcal{X}/S) \to P(\mathcal{X}_\eta)$ identifies $P^{\mathrm{ULA}}(\mathcal{X}/S)$ with a full Tannakian subcategory of $P(\mathcal{X}_\eta)$ and for every $\mathcal{P} \in P^{\mathrm{ULA}}(\mathcal{X}/S)$, induces an equivalence of Tannakian categories

$$-|_{\mathcal{X}_\eta} : \langle \mathcal{P} \rangle \overset{\simeq}{\to} \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle.$$

3. Specialization

In the following, to simplify notation, given an exact tensor functor of Tannakian categories $F : \mathcal{T}' \to \mathcal{T}$ and a fiber functor ω on $\mathcal{T}$, we will again write $\omega := \omega \circ F$ for the resulting fiber functor on $\mathcal{T}'$; as the functor $F : \mathcal{T}' \to \mathcal{T}$ should always be clear from the context, this should not give rise to confusion.

Let k be a field of characteristic 0, let S a smooth, geometrically connected variety over k with generic point η. Let $f : \mathcal{X} \to S$ be an abelian scheme. From (2), the canonical diagram of $\overline{\mathbb{Q}}_\ell$-linear abelian categories (4-1) induces the canonical diagram of Tannakian categories (4-2)

(4)

(4-1)

$$\begin{tikzcd}
\mathrm{Perv}(\mathcal{X}_s)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{s}}) \\
 & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & \\
\mathrm{Perv}(\mathcal{X}_\eta)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

(4-2)

$$\begin{tikzcd}
P(\mathcal{X}_s)_0 \arrow[r] & P(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & P(\mathcal{X}_{\bar{s}}) \\
 & P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & \\
P(\mathcal{X}_\eta)_0 \arrow[r] & P(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & P(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

and for every $\mathcal{P} \in P^{\mathrm{ULA}}(\mathcal{X}/S)$,

(5)

$$\begin{tikzcd}
\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_s} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}} \rangle \\
 & \langle \mathcal{P} \rangle \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & \\
\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}} \rangle .
\end{tikzcd}$$

For every $t \in S$, fix a fiber functor $\omega_{\bar{t}} : \langle \mathcal{P}|_{\mathcal{X}_{\bar{t}}} \rangle \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$. We claim that for every $s \in S$ and choice of an isomorphism of fiber functors

$$\omega_{\bar{s}} \circ -|_{\mathcal{X}_{\bar{s}}} \tilde{\to} \omega_{\bar{\eta}} \circ -|_{\mathcal{X}_{\bar{\eta}}} : P^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell},$$

the diagram (5) induces a diagram of algebraic groups with exact lines, which can be completed as indicated by the dotted arrows

(6)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \arrow[r] \arrow[ddd, hook, dotted, "csp_{\bar{\eta},\bar{s}}"] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[ddd, dotted, "(csp_{\bar{\eta},\bar{s}})_0"] & 1 \\
 & & G(\mathcal{P}, \omega_{\bar{s}}) \arrow[d, "\simeq"] & & \\
 & & G(\mathcal{P}, \omega_{\bar{\eta}}) & & \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{\bar{\eta}}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{\bar{\eta}}) \arrow[r] \arrow[u, "\simeq"'] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] & 1
\end{tikzcd}$$

The exactness of the lines is (2.1.2-1). The fact that $G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \hookrightarrow G(\mathcal{P}, \omega_{\bar{s}})$ is a closed immersion is formal[4].

Note that the existence of the dotted arrows is independent of the choice of the isomorphism $\omega_{\bar{s}} \circ -|_{\mathcal{X}_{\bar{s}}} \tilde{\to} \omega_{\bar{\eta}} \circ -|_{\mathcal{X}_{\bar{\eta}}}$ hence of the fiber functors $\omega_{\bar{\eta}}$, $\omega_{\bar{s}}$. So we will be free to choose $\omega_{\bar{\eta}}$, $\omega_{\bar{s}}$. Note also that the existence of the arrow $csp_{\bar{\eta},\bar{s}}$ is equivalent to the one of the arrow $(csp_{\bar{\eta},\bar{s}})_0$. We provide two constructions of (6), one *via* a construction of $csp_{\bar{\eta},\bar{s}}$ and one *via* a construction of $(csp_{\bar{\eta},\bar{s}})_0$. In both cases, we actually complete (4-1), (4-2) (5) by introducing some intermediate categories (to be defined) - $(*)_{\bar{\eta},\bar{s}}$ to construct $csp_{\bar{\eta},\bar{s}}$ and $((*)_{\bar{\eta},\bar{s}})_0$ to construct $(csp_{\bar{\eta},\bar{s}})_0$ as indicated in (7-1), (7-2) and (8) below.

(7)

(7-1)
$$\begin{tikzcd}
\mathrm{Perv}(\mathcal{X}_s)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{s}}) \\
((*)_{\bar{\eta},\bar{s}})_0 \arrow[u, dotted] \arrow[r, dotted] \arrow[d, dotted, "\simeq"] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[r, dotted] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & (*)_{\bar{\eta},\bar{s}} \arrow[u, dotted] \arrow[d, dotted, "\simeq"] \\
\mathrm{Perv}(\mathcal{X}_\eta)_0 \arrow[r] & \mathrm{Perv}(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

(7-2)
$$\begin{tikzcd}
P(\mathcal{X}_s)_0 \arrow[r] & P(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & P(\mathcal{X}_{\bar{s}}) \\
((*)_{\bar{\eta},\bar{s}})_0 \arrow[u, dotted] \arrow[r, dotted] \arrow[d, dotted, "\simeq"] & P^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[r, dotted] \arrow[d, "|_{\mathcal{X}_\eta}"', "\approx"] & (*)_{\bar{\eta},\bar{s}} \arrow[u, dotted] \arrow[d, dotted, "\simeq"] \\
P(\mathcal{X}_\eta)_0 \arrow[r] & P(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & P(\mathcal{X}_{\bar{\eta}})
\end{tikzcd}$$

---

[4] Indeed, by definition of $\langle \mathcal{P}|_{\mathcal{X}_s} \rangle$, every object in $\langle \mathcal{P}|_{\mathcal{X}_s} \rangle$ is a subquotient of some $T^{m,n}(\mathcal{P}|_{\mathcal{X}_s}) \simeq T^{m,n}(\mathcal{P})|_{\mathcal{X}_s}$ for some integers $m, n \geq 0$.

$$\begin{tikzcd}
\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_s}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}}\rangle \\
((*)_{\bar{\eta},\bar{s}})_0 \arrow[u, dotted] \arrow[r, dotted] \arrow[d, dotted, "\simeq"] & \langle \mathcal{P}\rangle \arrow[r, dotted] \arrow[u, "|_{\mathcal{X}_s}"'] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & (*)_{\bar{\eta},\bar{s}} \arrow[u, dotted] \arrow[d, dotted, "\simeq"] \\
\langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_\eta}\rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}\rangle
\end{tikzcd}\tag{8}$$

Remark 3.1. Actually, the constructions of (7-1) do not use that $f : \mathcal{X} \to S$ is an abelian scheme; they only require that $f : \mathcal{X} \to S$ be separated and of finite type (with a section).

3.1. A construction of $(csp_{\bar{\eta},\bar{s}})_0$.

We begin with the following observation.

Lemma 3.2. *Let $f : \mathcal{X} \to S$ be a separated morphism of finite type with a section $\iota : S \to \mathcal{X}$. The following commutative diagram*

$$\begin{tikzcd}
\mathrm{Loc}(S) = \mathrm{Perv}^{\mathrm{ULA}}(S/S) \arrow[r, "\iota_*"] \arrow[d, "\eta^*"'] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[d, "|_{\mathcal{X}_\eta}"] \\
\mathrm{Perv}(\eta) \arrow[r, "\iota_{\eta *}"] & \mathrm{Perv}(\mathcal{X}_\eta)
\end{tikzcd}$$

is cartesian. Namely for every $\mathcal{L}_{[\eta]} \in \mathrm{Perv}(\eta)$, if there exists $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ such that $\iota_{\eta *}\mathcal{L}_{[\eta]} \simeq \mathcal{P}|_{\mathcal{X}_\eta}$ then there exists $\mathcal{L} \in \mathrm{Loc}(S)$ such that $\eta^*\mathcal{L} \simeq \mathcal{L}_{[\eta]}$ and $\iota_*\mathcal{L} \simeq \mathcal{P}$.

Proof. As $f : \mathcal{X} \to S$ is separated, $\iota : S \hookrightarrow \mathcal{X}$ is a closed immersion; let $j : U := \mathcal{X} \setminus \iota(S) \hookrightarrow \mathcal{X}$ denote the complementary open immersion. Then $(j^*\mathcal{P})|_{U_\eta} \simeq j_\eta^*(\mathcal{P}|_{\mathcal{X}_\eta}) \simeq j_\eta^*\iota_{\eta *}\mathcal{L}_{[\eta]} \simeq 0$. As $j^*\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(U/S)$ and $-|_{U_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(U/S) \to \mathrm{Perv}(U_\eta)$ is fully faithful, this forces $j^*\mathcal{P} = 0$. From the distinguished triangle

$$j_!j^*\mathcal{P} \to \mathcal{P} \to \iota_*\iota^*\mathcal{P} \overset{+1}{\to}$$

in $D_c^b(\mathcal{X})$, $\mathcal{P} \tilde{\to} \iota_*\iota^*\mathcal{P}$ hence, by [B24, Lem. 3.6 (iv)], $\iota^*\mathcal{P} \in D^{\mathrm{ULA}}(S/S)$. From [B24, Lem. 3.7 (ii)], $\iota^*\mathcal{P} \in D^b_{\mathrm{liss}}(S)$. But $\eta^*\iota^*\mathcal{P} \simeq \iota_\eta^*(\mathcal{P}|_{\mathcal{X}_\eta}) \simeq \iota_\eta^*\iota_{\eta *}\mathcal{L}_{[\eta]} \simeq \mathcal{L}_{[\eta]}$, so that $\mathcal{L} := \iota^*\mathcal{P}$ lies in $\mathrm{Loc}(S)$ and has the requested property. □

We return to the case where $f : \mathcal{X} \to S$ is an abelian scheme and $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. From Lemma 3.2, one can complete (5) as

(9)
$$\begin{tikzcd}
\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_s} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}} \rangle \\
\langle \mathcal{P} \rangle_0 \arrow[r] \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & \langle \mathcal{P} \rangle \arrow[u, "|_{\mathcal{X}_s}"] \arrow[d, "|_{\mathcal{X}_\eta}"', "\simeq"] & \\
\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0 \arrow[r] & \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}} \rangle,
\end{tikzcd}$$

which, as claimed, formally yields a commutative diagram of algebraic groups:

(10)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{\bar{s}}) \arrow[r] \arrow[ddd, hook, dotted, "csp_{\bar{\eta},\bar{s}}"] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{\bar{s}}) \arrow[r] \arrow[d, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[d] \arrow[ddd, bend left, "(csp_{\bar{\eta},\bar{s}})_0"] & 1 \\
 & & G(\mathcal{P}, \omega_{\bar{s}}) \arrow[r] \arrow[d, "\simeq"] & G(\langle \mathcal{P} \rangle_0, \omega_{\bar{s}}) \arrow[r] \arrow[d, "\simeq"] & 1 \\
 & & G(\mathcal{P}, \omega_{\bar{\eta}}) \arrow[r] & G(\langle \mathcal{P} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] & 1 \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{\bar{\eta}}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{\bar{\eta}}) \arrow[r] \arrow[u, "\simeq"] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{\bar{\eta}}) \arrow[r] \arrow[u, "\simeq"] & 1.
\end{tikzcd}$$

3.2. A construction of $csp_{\bar{\eta},\bar{s}}$.

3.2.1. *Absolutely integrally closed valuation rings and nearby cycles.* Recall that a valuation ring V is said to be absolutely integrally closed (AIC for short) if it satisfies the following equivalent conditions

(AIC-1) The fraction field of V is algebraically closed;

(AIC-2) Every monic polynomial of degree ≥ 1 in $V[T]$ has a root in V.

In particular, such a valuation ring V is strictly henselian.

Fact 3.3. ([HS23, Thm. 1.7, Thm. 6.1 (ii), Cor. 4.2]) *Let $S = \mathrm{spec}(V)$ be the spectrum of an AIC valuation ring with generic point η and closed point s. Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite presentation and write $\mathcal{X}_\eta \overset{\beta}{\hookrightarrow} \mathcal{X} \overset{\alpha}{\hookleftarrow} \mathcal{X}_s$ for the inclusions of the generic and closed fibers respectively. Then,*

(1) The functor $\beta^* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_\eta)$ is an equivalence of categories with quasi-inverse $R\beta_* : D^b_c(\mathcal{X}_\eta) \to D^{\mathrm{ULA}}(\mathcal{X}/S)$. In particular, $\beta^* : D^{\mathrm{ULA}}(\mathcal{X}/S) \to D^b_c(\mathcal{X}_\eta)$ restricts to an equivalence of categories

$\beta^* : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_\eta)$ *with quasi-inverse* $R\beta_* : \mathrm{Perv}(\mathcal{X}_\eta) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$.

(2) *The nearby cycle functor* $R\psi_f = \alpha^* R\beta_* : D(\mathcal{X}_\eta) \to D(\mathcal{X}_s)$ *restricts to a functor* $R\psi_f : D^b_c(\mathcal{X}_\eta) \to D^b_c(\mathcal{X}_s)$ *which is t-exact with respect to the perverse t-structures hence induces an exact functor* $R\psi_f : \mathrm{Perv}(\mathcal{X}_\eta) \to \mathrm{Perv}(\mathcal{X}_s)$.

(3) *Assume furthermore that* $f : \mathcal{X} \to S$ *is an abelian scheme. Then* $R\psi_f : D^b_c(\mathcal{X}_\eta) \to D^b_c(\mathcal{X}_s)$ *is a tensor functor and* $N(\mathcal{X}_\eta) = \ker(D^b_c(\mathcal{X}_\eta) \overset{R\psi_f}{\to} D^b_c(\mathcal{X}_s) \to D^b_c(\mathcal{X}_s)/N(\mathcal{X}_s)$*; in particular,* $R\psi_f : \mathrm{Perv}(\mathcal{X}_\eta) \to \mathrm{Perv}(\mathcal{X}_s)$ *induces a faithful exact tensor functor* $R\psi_f : P(\mathcal{X}_\eta) \to P(\mathcal{X}_s)$.

From [BhM21, Lem. 3.28], for a quasi-compact, quasi-separated scheme T, a specialization $t_1 \rightsquigarrow t$ of points on T, one can always find a morphism $S \to T$ with source the spectrum $S = \mathrm{Spec}(V)$ of an AIC valuation ring V, mapping the generic point η (resp. the closed point s) of S to t_1 (resp. t). We will call such a morphism - usually written as $(S, \eta, s) \to (T, t_1, t)$, a *witness* for $t_1 \rightsquigarrow t$ in T. The proof of [BhM21, Lem. 3.28] shows that, if furthermore one fixes a geometric point $\bar{t}_1$ over t_1, one can choose S in such a way that $\eta \to t_1$ factors as $\eta \to \bar{t}_1 \to t_1$; if we want to specify a geometric point over which $\eta \to t_1$ factors, we will rather write $(S, \eta, s) \to (T, \bar{t}_1, t)$.

3.2.2. We return to the case where S is a smooth, geometrically connected variety over k and $f : \mathcal{X} \to S$ is an abelian scheme. For every specialization $\eta \rightsquigarrow s$ of points on S, fix a witness $(S', \eta', s') \to (S, \eta, s)$ and geometric points $\eta' \to \bar{\eta} \to \eta$, $s' \to \bar{s} \to s$. Set $f' : \mathcal{X}' := \mathcal{X} \times_S S' \to S'$. From Fact 3.3, one gets a canonical diagram of $\overline{\mathbb{Q}}_\ell$-linear abelian categories

$$\begin{tikzcd}
 & \mathrm{Perv}(\mathcal{X}_s) \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \mathrm{Perv}(\mathcal{X}_{\bar{s}}) \arrow[r, "|_{\mathcal{X}'_{s'}}"] & \mathrm{Perv}(\mathcal{X}'_{s'}) \\
\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \arrow[ur, "|_{\mathcal{X}_s}"] \arrow[r, "|_{\mathcal{X}'}"] \arrow[dr, "\approx", "|_{\mathcal{X}_\eta}"'] & \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S') \arrow[urr, bend right=10, "|_{\mathcal{X}'_{s'}}"] \arrow[drr, bend left=10, "\simeq", "|_{\mathcal{X}'_{\eta'}}"'] & & \\
 & \mathrm{Perv}(\mathcal{X}_\eta) \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"'] & \mathrm{Perv}(\mathcal{X}_{\bar{\eta}}) \arrow[r, "|_{\mathcal{X}'_{\eta'}}"'] & \mathrm{Perv}(\mathcal{X}'_{\eta'}) \arrow[uu, dotted, bend right, "R\psi_{f'}"']
\end{tikzcd}$$

which induces, for every $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$, a canonical diagram of Tannakian categories

(11)
$$\begin{tikzcd}
 & \langle \mathcal{P}|_{\mathcal{X}_s} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}} \rangle \arrow[r, "|_{\mathcal{X}'_{s'}}", "\simeq"'] & \langle \mathcal{P}|_{\mathcal{X}'_{s'}} \rangle \\
\langle \mathcal{P} \rangle \arrow[ur, "|_{\mathcal{X}_s}"] \arrow[r, "|_{\mathcal{X}'}"] \arrow[dr, "\simeq", "|_{\mathcal{X}_\eta}"'] & \langle \mathcal{P}|_{\mathcal{X}'} \rangle \arrow[urr, bend right, "|_{\mathcal{X}'_{s'}}"] \arrow[drr, bend left, "\simeq", "|_{\mathcal{X}'_{\eta'}}"'] & & \\
 & \langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{\eta}}}"'] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}} \rangle \arrow[r, "\simeq", "|_{\mathcal{X}'_{\eta'}}"'] & \langle \mathcal{P}|_{\mathcal{X}'_{\eta'}} \rangle \arrow[uu, dotted, bend right, "R\psi_{f'}"']
\end{tikzcd}$$
,

which in turn, as claimed, formally yields a commutative diagram of algebraic groups

(12)
$$\begin{tikzcd}
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}, \omega_{s'}) \arrow[r] \arrow[dddd, dotted, bend right, "csp_{\bar{\eta},\bar{s}}"'] & G(\mathcal{P}|_{\mathcal{X}_s}, \omega_{s'}) \arrow[r] \arrow[dd, hook] & G(\langle \mathcal{P}|_{\mathcal{X}_s} \rangle_0, \omega_{s'}) \arrow[r] \arrow[dd] \arrow[dddd, dotted, bend left, "(csp_{\bar{\eta},\bar{s}})_0"] & 1 \\
 & G(\mathcal{P}|_{\mathcal{X}'_{s'}}, \omega_{\bar{s}}) \arrow[u, "\simeq"'] \arrow[d] \arrow[dd, bend right, "R\psi_{f'}"'] & & & \\
 & G(\mathcal{P}|_{\mathcal{X}'}, \omega_{s'}) \arrow[r, hook] & G(\mathcal{P}, \omega_{s'}) \arrow[r] & G(\langle \mathcal{P} \rangle_0, \omega_{s'}) \arrow[r] & 1 \\
 & G(\mathcal{P}|_{\mathcal{X}'_{\eta'}}, \omega_{s'}) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & & & \\
1 \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}, \omega_{s'}) \arrow[r] & G(\mathcal{P}|_{\mathcal{X}_\eta}, \omega_{s'}) \arrow[r] \arrow[uu, "\simeq"'] & G(\langle \mathcal{P}|_{\mathcal{X}_\eta} \rangle_0, \omega_{s'}) \arrow[r] \arrow[uu, "\simeq"'] & 1
\end{tikzcd}$$

For simplicity, we now omit fiber functors from the notation.

4. Proofs

Unless otherwise stated, in this Section k denotes a field of characteristic 0, S a smooth geometrically connected variety over k with generic point η and $f : \mathcal{X} \to S$ a morphism, separated and of finite type.

4.1. **Proof of Theorem 1.2 and Corollary 1.3.**

4.1.1. *Recollection on artinian and noetherian abelian categories.*

4.1.1.1. Let $\mathcal{A}$ be a an artinian and noetherian abelian category. Then,

(1) For every $A \in \mathcal{A}$ and $\phi \in \mathrm{End}_{\mathcal{A}}(A)$, one has a ϕ-stable direct sum decomposition (Fitting lemma): $A \simeq A_{\phi,0} \oplus A_{\phi,\infty}$, with the property that the induced morphism $\phi : A_{\phi,0} \to A_{\phi,0}$ is nilpotent and $\phi : A_{\phi,\infty} \to A_{\phi,\infty}$ is an automorphism; explicitly $A_{\phi,0} = \ker(\phi^n)$, for $n \gg 0$, $A_{\phi,\infty} = \mathrm{im}(\phi^n)$, for $n \gg 0$. In particular, for every $A \in \mathcal{A}$, A is indecomposable in $\mathcal{A}$ if and only if $\mathrm{End}_{\mathcal{A}}(A)$ is a local ring and every $A \in \mathcal{A}$ admits a Krull-Schmidt decomposition: A decomposes into a direct sum $A = \oplus_{1 \leq i \leq r} A_i$ with $A_1, \ldots, A_r \in \mathcal{A}$ indecomposable and the indecomposable objects $A_1, \ldots, A_r$ (counted with multiplicity) are unique up to isomorphism and called the Krull-Schmidt or indecomposable factors of A. In particular, A is semisimple if and only if its indecomposable factors are simple.

(2) Every $A \in \mathcal{A}$ admits a composition series that is a filtration

$$0 = A_{r+1} \subsetneq A_r \subsetneq \cdots \subsetneq A_2 \subsetneq A_1 := A$$

in $\mathcal{A}$ with $S_i := A_i/A_{i+1}$ a simple object in $\mathcal{A}$, $i = 1, \ldots, r$. Furthermore, the simple objects $S_1, \ldots, S_r$ (counted with multiplicity) are unique up to isomorphism and called the Jordan-Hölder or simple factors of A. In particular the length $\mathrm{length}_{\mathcal{A}}(A) := r$ of A is a well-defined integer.

(3) Let $\mathcal{N} \subset \mathcal{A}$ be a Serre subcategory and let $p : \mathcal{A} \to \overline{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ denote the resulting quotient functor, which is exact and essentially surjective. Then, for every simple object S in $\mathcal{A}$ not lying in $\mathcal{N}$, $p(S)$ is again a simple object in $\overline{\mathcal{A}}$. This follows from the definition of morphisms in $\overline{\mathcal{A}}$. Indeed, consider a diagram

$$X \overset{f}{\to} Y \overset{s}{\hookleftarrow} S$$

in $\mathcal{A}$ with $N := \mathrm{coker}(s) \in \mathcal{N}$ such that the resulting morphism

$$p(X) \overset{p(f)}{\to} p(Y) \overset{p(s)^{-1}}{\overset{\simeq}{\to}} p(S)$$

is injective. In particular, the morphism $p(S) \overset{p(s)}{\overset{\simeq}{\to}} p(Y) \to p(Y/X)$ is surjective. So either $p(Y/X) = 0$ and $p(X) \overset{p(f)}{\to} p(Y)$ is an isomorphism in $\overline{\mathcal{A}}$ or the morphism $S \overset{s}{\to} Y \to Y/X$ is non-zero hence injective. But then the morphism $p(S) \overset{p(s)}{\overset{\simeq}{\to}} p(Y) \to p(Y/X)$ is an isomorphism in $\overline{\mathcal{A}}$ hence so is $p(Y) \to p(Y/X)$, which imposes $p(X) = 0$. In particular, for every $A \in \mathcal{A}$, if one defines $\mathrm{length}_{\mathcal{A},\mathcal{N}}(A) \leq \mathrm{length}_{\mathcal{A}}(A)$ to be the number of Jordan-Hölder factors of A which lies in $\mathcal{N}$, one has

$$\mathrm{length}_{\overline{\mathcal{A}}}(p(A)) + \mathrm{length}_{\mathcal{A},\mathcal{N}}(A) = \mathrm{length}_{\mathcal{A}}(A).$$

4.1.1.2. Let now $\mathcal{A}_1$, $\mathcal{A}_2$ be artinian and noetherian abelian categories and let $F : \mathcal{A}_1 \to \mathcal{A}_2$ be an additive functor.

(1) Assume $F : \mathcal{A}_1 \to \mathcal{A}_2$ is fully faithful. Then for every $A_1 \in \mathcal{A}_1$, A_1 is indecomposable in $\mathcal{A}_1$ if and only if $F(A_1)$ is indecomposable in $\mathcal{A}_2$.

(2) Consider the following conditions

($\mathrm{S}_{F,[??]}$) For every $A_1 \in \mathcal{A}_1$, A_1 is simple in $\mathcal{A}_1$ [??] $F(A_1)$ is simple in $\mathcal{A}_2$;
($\mathrm{SS}_{F,[??]}$) For every $A_1 \in \mathcal{A}_1$, A_1 is semisimple in $\mathcal{A}_1$ [??] $F(A_1)$ is semisimple in $\mathcal{A}_2$.

with [??] one of $\Leftarrow$, $\Rightarrow$, $\Leftrightarrow$.

(a) Then ($\mathrm{S}_{F,\Rightarrow}$) always implies ($\mathrm{SS}_{F,\Rightarrow}$) and, if furthermore $F : \mathcal{A}_1 \to \mathcal{A}_2$ is exact, then for every $A_1 \in \mathcal{A}_1$,

$$\mathrm{length}_{\mathcal{A}_1}(A_1) = \mathrm{length}_{\mathcal{A}_2}(F(A_1)).$$

(b) If $F : \mathcal{A}_1 \to \mathcal{A}_2$ is fully faithful then ($\mathrm{S}_{F,\Leftarrow}$) implies ($\mathrm{SS}_{F,\Leftarrow}$). Indeed, let $A_1 \in \mathcal{A}_1$ and consider the Krull-Schmidt decomposition $A_1 = \oplus_{1\leq i\leq r} A_{1,i}$ of A_1 in $\mathcal{A}_1$. Then $F(A_1) = \oplus_{1\leq i\leq r} F(A_{1,i})$ is the Krull-Schmidt decomposition of $F(A_1)$ in $\mathcal{A}_2$. So, one has

$$\begin{aligned} F(A_1) \text{ is semisimple in } \mathcal{A}_2 &\iff F(A_{1,i}) \text{ is simple in } \mathcal{A}_2,\ i = 1,\dots r \\ &\overset{(\mathrm{S}_{F,\Leftarrow})}{\Longrightarrow} A_{1,i} \text{ is simple in } \mathcal{A}_1,\ i = 1,\dots r \\ &\iff A_1 \text{ is semisimple in } \mathcal{A}_1. \end{aligned}$$

(3) Assume $F : \mathcal{A}_1 \to \mathcal{A}_2$ is exact. Let $\mathcal{N}_2 \subset \mathcal{A}_2$ be a Serre subcategory and let $p_2 : \mathcal{A}_2 \to \overline{\mathcal{A}}_2 := \mathcal{A}_2/\mathcal{N}_2$ denote the resulting quotient functor. The full subcategory

$$\mathcal{N}_1 := \ker(\mathcal{A}_1 \overset{F}{\to} \mathcal{A}_2 \overset{p_2}{\to} \overline{\mathcal{A}}_2) \subset \mathcal{A}_1$$

is also a Serre subcategory and the resulting quotient functor $p_1 : \mathcal{A}_1 \to \overline{\mathcal{A}}_1 := \mathcal{A}_1/\mathcal{N}_1$ fits into a canonical commutative diagram of exact functors

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "p_1"] \arrow[d, "F"'] & \overline{\mathcal{A}}_1 \arrow[d, "\overline{F}"] \\ \mathcal{A}_2 \arrow[r, "p_2"'] & \overline{\mathcal{A}}_2. \end{tikzcd}$$

As $\overline{F} : \overline{\mathcal{A}}_1 \to \overline{\mathcal{A}}_2$ is faithful exact, ($\mathrm{S}_{\overline{F},\Leftarrow}$) always holds.

(a) Also, ($\mathrm{S}_{F,\Rightarrow}$) always implies ($\mathrm{S}_{\overline{F},\Rightarrow}$). Indeed, for every $A_1 \in \mathcal{A}_1$, consider a Jordan-Hölder filtration

$$\underline{A}_1 : 0 = A_{1,r+1} \subsetneq A_{1,r} \subsetneq \cdots \subsetneq A_{1,2} \subsetneq A_{1,1} := A_1$$

of A_1 in $\mathcal{A}_1$. Assume ($\mathrm{S}_{F,\Rightarrow}$) holds. Then

$$F(\underline{A}_1) : 0 = F(A_{1,r+1}) \subsetneq F(A_{1,r}) \subsetneq \cdots \subsetneq F(A_{1,2}) \subsetneq F(A_{1,1}) := F(A_1)$$

is again a Jordan-Hölder filtration of $F(A_1)$ in $\mathcal{A}_2$. If $p_1(A_1)$ is simple in $\overline{\mathcal{A}}_1$, then all but one of the $A_{1,i}/A_{1,i+1}$ lie in $\mathcal{N}_1$ which, again by definition of $\mathcal{N}_1$, ensures that all but one of the $F(A_{1,i})/F(A_{1,i+1})$ lie in $\mathcal{N}_2$, hence that $\overline{F}(p_1(A_1))$ is simple in $\overline{\mathcal{A}}_2$.

(b) The argument in (3) (a) shows more precisely that for every $A_1 \in \mathcal{A}_1$ with a Jordan-Hölder filtration

$$\underline{A}_1 : 0 = A_{1,r+1} \subsetneq A_{1,r} \subsetneq \cdots \subsetneq A_{1,2} \subsetneq A_{1,1} := A_1$$

in $\mathcal{A}_1$, if $F(A_{1,i})/F(A_{1,i+1})$ is a simple object in $\mathcal{A}_2$, $i = 1, \ldots, r$, then one has

$$\mathrm{length}_{\mathcal{A}_1}(A_1) = \mathrm{length}_{\mathcal{A}_2}(F(A_1)), \;\; \mathrm{length}_{\overline{\mathcal{A}}_1}(p_1(A_1)) = \mathrm{length}_{\overline{\mathcal{A}}_2}(\overline{F}(p_1(A_1)))$$

and

$$A_1 \text{ is semisimple in } \mathcal{A}_1 \Rightarrow F(A_1) \text{ is semisimple in } \mathcal{A}_2$$
$$p_1(A_1) \text{ is semisimple in } \overline{\mathcal{A}}_1 \Rightarrow \overline{F}(p_1(A_1)) \text{ is semisimple in } \overline{\mathcal{A}}_2.$$

4.1.2. *Preliminary reductions.*

4.1.2.1. *Independence of the geometric point.* The following observation will enable us to choose geometric points freely. Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Let $t \in S$ and let $\bar{t}_1, \bar{t}_2$ be two geometric points over t. Then

(1) $\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_1})$ if and only if $\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}$ is simple (resp. semisimple) in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_2})$ and one has

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{t}_1})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}) = \mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar{t}_2})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}).$$

(2) Assume furthermore that $f : \mathcal{X} \to S$ is an abelian scheme. Then $\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}$ is simple (resp. semisimple) in $P(\mathcal{X}_{\bar{t}_1})$ if and only if $\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}$ is simple (resp. semisimple) in $P(\mathcal{X}_{\bar{t}_2})$ and one has

$$\mathrm{length}_{P(\mathcal{X}_{\bar{t}_1})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_1}}) = \mathrm{length}_{P(\mathcal{X}_{\bar{t}_2})}(\mathcal{P}|_{\mathcal{X}_{\bar{t}_2}}).$$

Indeed, by considering a geometric point $\bar{t}$ over both $\bar{t}_1$ and $\bar{t}_2$ one immediately reduces to the case where, say, $\bar{t}_2$ is over $\bar{t}_1$. As the restrictions functors

$$-|_{\mathcal{X}_{\bar{t}_2}} : \mathrm{Perv}(\mathcal{X}_{\bar{t}_1}) \to \mathrm{Perv}(\mathcal{X}_{\bar{t}_2}), \;\; -|_{\mathcal{X}_{\bar{t}_2}} : P(\mathcal{X}_{\bar{t}_1}) \to P(\mathcal{X}_{\bar{t}_2})$$

are exact, fully faithful (*e.g.* [JKrLM25, Lem. A.1]), the observations in Paragraph 4.1.1.2 (2) reduce the proof of (1) and (2) to showing respectively that for every $\mathcal{P}_1 \in \mathrm{Perv}(\mathcal{X}_{\bar{t}_1})$,

(1)' $\mathcal{P}_1$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_1})$ if and only if $\mathcal{P}_1|_{\mathcal{X}_{\bar{t}_2}}$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_2})$;

(2)' (if $f : \mathcal{X} \to S$ is an abelian scheme) $\mathcal{P}_1$ is simple in $P(\mathcal{X}_{\bar{t}_1})$ if and only if $\mathcal{P}_1|_{\mathcal{X}_{\bar{t}_2}}$ is simple in $P(\mathcal{X}_{\bar{t}_2})$

while the observation of Paragraph 4.1.1.2 (3) (a) shows that Assertion (2)' follows from Assertion (1)'. Let us prove Assertion (1)'. The if part of follows from the fact that $-|_{\mathcal{X}_{\bar{t}_2}} : \mathrm{Perv}(\mathcal{X}_{\bar{t}_1}) \to \mathrm{Perv}(\mathcal{X}_{\bar{t}_2})$ is exact and fully faithful. For the only if part, from [BeBerDG82, Thm. 4.3.1 (ii)] every simple object $\mathcal{S}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{t}_1})$ is of the form $\mathcal{S} = \iota_{1*}j_{1!*}\mathcal{F}_1[d]$ for some irreducible closed subscheme $\iota_1 : Z_1 \hookrightarrow \mathcal{X}_{\bar{t}_1}$, non-empty open subscheme $j_1 : U_1 \hookrightarrow Z_1$, smooth over $k(\bar{t}_1)$ and pure of dimension d, and simple $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}_1$ on U_1. Consider the base-change diagram

$$\begin{tikzcd}
U_2 \arrow[r, hook, "j_2"] \arrow[d] \arrow[dr, phantom, "\square"] & Z_2 \arrow[r, hook, "\iota_2"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{\bar{t}_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{spec}(k(\bar{t}_2)) \arrow[d] \\
U_1 \arrow[r, hook, "j_1"] & Z_1 \arrow[r, hook, "\iota_1"] & \mathcal{X}_{\bar{t}_1} \arrow[r] & \mathrm{spec}(k(\bar{t}_1))
\end{tikzcd}$$

and set $\mathcal{F}_2 := \mathcal{F}_1|_{U_2}$. Then

$$\mathcal{S}|_{\mathcal{X}_{\bar{t}_2}} = (\iota_{1*}j_{1!*}\mathcal{F}_1[d])|_{\mathcal{X}_{\bar{t}_2}} \simeq \iota_{2*}j_{2!*}\mathcal{F}_2[d]$$

and the assertion follows from the fact that the restriction functor

$$-|_{U_2} : \mathrm{Loc}(U_1) \to \mathrm{Loc}(U_2)$$

maps simple objects to simple objects since the canonical morphism $\pi_1(U_2) \to \pi_1(U_1)$ is surjective (*e.g.* [St25, Tag 0387]).

4.1.2.2. Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. Every witness $(S', \eta', s') \to (S, \eta, s)$ induces a canonical exact functor

$$R\psi_{f'} : \mathrm{Perv}(\mathcal{X}_{\eta'}) \overset{\simeq}{\to} \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S') \to \mathrm{Perv}(\mathcal{X}_{s'}),$$

where the notation are as follows

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

Assume furthermore $f : \mathcal{X} \to S$ is an abelian scheme. Then

$$N(\mathcal{X}_{\eta'}) \cap \mathrm{Perv}(\mathcal{X}_{\eta'}) = \ker(\mathrm{Perv}(\mathcal{X}_{\eta'}) \overset{R\psi_{f'}}{\to} \mathrm{Perv}(\mathcal{X}_{s'}) \to P(\mathcal{X}_{s'})).$$

So, Paragraph 4.1.2.1 and the observations in Paragraph 4.1.1.2 (2) (b) applied to

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, "p"] \arrow[d, "F"'] & \overline{\mathcal{A}}_1 \arrow[d, "\overline{F}"] \\
\mathcal{A}_2 \arrow[r, "p'"'] & \overline{\mathcal{A}}_2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{Perv}(\mathcal{X}_{\eta'}) \arrow[r] \arrow[d, "R\psi_{f'}"'] & P(\mathcal{X}_{\eta'}) \arrow[d] \\
\mathrm{Perv}(\mathcal{X}_{s'}) \arrow[r] & P(\mathcal{X}_{s'}).
\end{tikzcd}$$

reduce the proof of Theorem 1.2 and Corollary 1.3 to the following statement.

Theorem 4.1. *Let $f : \mathcal{X} \to S$ be a morphism, separated and of finite type. Let $\mathcal{P}_i \in \mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$, $i = 1, \dots, r$ be finitely many simple objects in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$. After possibly replacing S by a non-empty open subscheme S the following holds. For every $s \in S$, there exists a witness $(S', \eta', s') \to (S, \overline{\eta}, s)$ such that $R\psi_{f'}(\mathcal{P}_i|_{\mathcal{X}_{\eta'}})$ is simple in $\mathrm{Perv}(\mathcal{X}_{s'})$, $i = 1, \dots, r$.*

4.1.3. *Proof of Theorem 4.1.*

4.1.3.1. *Intermediate extensions and the ULA property.* Let ℓ be a prime. Let $\mathcal{Z} \to S$ be a separated morphism of finite presentation of quasi-compact quasi-separated schemes over $\mathbb{Z}[1/\ell]$. Assume that S has only finitely many irreducible components, so that by [HS23, Theorem 6.7] the relative perverse t-structure exists on $D^{\mathrm{ULA}}(\mathcal{Z}/S)$. Let $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ be an open immersion of finite presentation. Given $\mathcal{K} \in D^{\mathrm{ULA}}(\mathcal{U}/S)$ with $j_!\mathcal{K}$, $j_! D_{\mathcal{U}/S}\mathcal{K} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$, write $j_{!*/S}\mathcal{K}$ for the image of the natural morphism

$$ {}^{p/S}H^0(j_!\mathcal{K}) \to {}^{p/S}H^0(D_{\mathcal{Z}/S} j_! D_{\mathcal{U}/S}\mathcal{K}) $$

in the abelian category $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{Z}/S)$. Observe the following

(1) When S is the spectrum of a field and $\mathcal{K} \in \mathrm{Perv}(\mathcal{U})$, $j_{!*/S}\mathcal{K}$ is the usual middle extension of $\mathcal{K}$ to $\mathcal{Z}$.

(2) As, for ULA objects, both the formation of relative Verdier duality and $j_!$ commute with base change $S' \to S$ ([HS23, Proposition 3.4 (ii)]), the formation of $j_{!*/S}\mathcal{K}$ also commutes with base-changes $S' \to S$. In particular, for every geometric point $\bar{s}$ on S, if $j_{\bar{s}} : \mathcal{U}_{\bar{s}} \hookrightarrow \mathcal{Z}_{\bar{s}}$ denotes the base change of $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ along $\bar{s} \to S$, one has $(j_{!*/S}\mathcal{K})|_{\mathcal{Z}_{\bar{s}}} = j_{\bar{s},!*}(\mathcal{K}|_{\mathcal{U}_{\bar{s}}})$.

Lemma 4.2. *Consider a diagram*

(13)
$$\begin{tikzcd} \widetilde{\mathcal{U}} \arrow[r, hook, "\widetilde{j}"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Z}} \arrow[d, "g"] & \\ \mathcal{U} \arrow[r, hook, "j"] & \mathcal{Z} \arrow[r, "f"] & S \end{tikzcd}$$

of quasi-compact quasi-separated schemes. Assume that

- *$f : \mathcal{Z} \to S$ is separated of finite presentation and $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ is an open immersion with $\mathcal{U} \to S$ smooth, ;*
- *$g : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is proper, and $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is finite étale;*
- *$\widetilde{\mathcal{Z}} \to S$ is smooth and $\widetilde{\mathcal{D}} := \widetilde{\mathcal{Z}} \setminus \widetilde{\mathcal{U}}$ is a divisor on $\widetilde{\mathcal{Z}}$ with strict normal crossings relative to $\widetilde{\mathcal{Z}} \to S$.*

Let ℓ be a prime invertible on S. Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_\ell$-local system on $\mathcal{U}$. Assume that $h^*\mathcal{F}$ is tamely ramified along $\widetilde{\mathcal{D}}$. Then $j_!\mathcal{F}$, $j_!(\mathcal{F}^\vee)$ are constructible sheaves that are ULA relative to $f : \mathcal{Z} \to S$. In particular, if S has only finitely many irreducible components and d denotes the relative dimension *of* $\mathcal{U} \to S$*, then* $j_{!*/S}\mathcal{F}[d]$ *is a well defined object in* $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{Z}/S)$*, whose formation commutes with base-changes.*

Proof. By [St25, Tag 0818], $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ is of finite presentation. From [L81, Proposition 1.4.4], and as $\mathcal{U} \to S$ is smooth, $\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{U}/S)$. So, since $D_{\mathcal{U}/S}(\mathcal{F}[d]) = \mathcal{F}^\vee[d]$, the second part of Lemma 4.2 follows from the first part.

The sheaves $j_!\mathcal{F}$ and $\widetilde{j}_!h^*\mathcal{F}$ are constructible. From [S17, Lemma 3.14], and as $h^*\mathcal{F}$ is tamely ramified along $\widetilde{\mathcal{D}}$, $\widetilde{j}_!h^*\mathcal{F}$ is ULA relative to $\widetilde{\mathcal{Z}} \to S$. From [HS23, p.643], and as $g : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is proper, $Rg_*\widetilde{j}_!h^*\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$. Hence $j_!h_*h^*\mathcal{F} \simeq Rg_*\widetilde{j}_!h^*\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$.

Since $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is finite étale, the natural morphism $\overline{\mathbb{Q}}_{\ell,\mathcal{U}} \to h_*\overline{\mathbb{Q}}_{\ell,\widetilde{\mathcal{U}}}$ of lisse sheaves is the inclusion of a direct summand. By the projection formula [St25, Tag 0F0G], as $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is proper, one has $h_*(h^*\mathcal{F}) = \mathcal{F} \otimes_{\overline{\mathbb{Q}}_\ell} h_*\overline{\mathbb{Q}}_{\ell,\widetilde{\mathcal{U}}}$. Thus, $\mathcal{F} \hookrightarrow h_*h^*\mathcal{F}$ and hence $j_!\mathcal{F} \hookrightarrow j_!h_*h^*\mathcal{F}$ are direct summands. Since universal local acyclicity is preserved under passing to direct summands, one has $j_!\mathcal{F} \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$. As $h^*\mathcal{F}^\vee$ is also tamely ramified along $\widetilde{\mathcal{D}}$, one has $j_!\mathcal{F}^\vee \in D^{\mathrm{ULA}}(\mathcal{Z}/S)$.

□

4.1.3.2. Proof of Theorem 4.1. From [BeBerDG82, Thm. 4.3.1 (ii)], for every $i = 1, \ldots, r$, there exists an integral closed subscheme $\iota_i : Z_i \hookrightarrow \mathcal{X}_{\bar\eta}$ and a non-empty open subscheme $j_i : U_i \hookrightarrow Z_i$, smooth over $k(\bar\eta)$ and pure of dimension d_i, and a simple object $\mathcal{F}_i$ in $\mathrm{Loc}(U_i)$ such that $\mathcal{P}_i = \iota_{i,*}j_{i,!*}\mathcal{F}_i[d_i]$. Fix also a $k(\bar\eta)$-point $u_i \in U_i$ and a smooth normal crossing compactification $U_i \hookrightarrow U_i^{\mathrm{cpt}}$. There exists a finite field extension K_0 of $k(\eta)$ such that, for every $i = 1, \ldots, r$,

$$\begin{tikzcd}
U_i^{\mathrm{cpt}} & U_i \arrow[l, hook'] \arrow[r, hook, "j_i"] & Z_i \arrow[r, hook, "\iota_i"] & \mathcal{X}_{\bar\eta} \arrow[r] & \mathrm{spec}(k(\bar\eta)) \arrow[llll, bend right, "u_i"']
\end{tikzcd}$$

is defined over K_0 and spread out as

$$\begin{tikzcd}
U_i^{\mathrm{cpt}} \arrow[d] \arrow[rd, phantom, "\square"] & U_i \arrow[l, hook'] \arrow[r, hook, "j_i"] \arrow[d] \arrow[rd, phantom, "\square"] & Z_i \arrow[r, hook, "\iota_i"] \arrow[d] \arrow[rd, phantom, "\square"] & \mathcal{X}_{\eta_0} \arrow[r] \arrow[d] & \mathrm{spec}(K_0) \arrow[d, "\eta_0"] \arrow[llll, bend right, "u_i"'] \\
\mathcal{U}_i^{\mathrm{cpt}} & \mathcal{U}_i \arrow[l, hook'] \arrow[r, hook, "j_i"] & \mathcal{Z}_i \arrow[r, hook, "\iota_i"] & \mathcal{X} \times_S S_0 \arrow[r] & S_0 \arrow[llll, bend left, "u_i"]
\end{tikzcd}$$

with

- $\iota_i : \mathcal{Z}_i \hookrightarrow \mathcal{X} \times_S S_0$ a closed immersion and $\mathcal{Z}_i \to S_0$ geometrically irreducible;

- $j_i : \mathcal{U}_i \hookrightarrow \mathcal{Z}_i$ an open immersion and $\mathcal{U}_i \to S_0$ is smooth, pure of relative dimension d_i;
- $\mathcal{U}_i \hookrightarrow \mathcal{U}_i^{\mathrm{cpt}}$ is a relative smooth normal crossing compactification over S_0,

where $S_0 \subset \widetilde{S}_0$ is a non-empty open in the normalization $\widetilde{S}_0 \to S$ of S in $\mathrm{spec}(K_0) \to \mathrm{spec}(k(\eta)) \xrightarrow{\eta} S$.

By Lemma 4.3, as $\mathrm{char}(k) = 0$, shrinking S_0 if necessary, one may furthermore assume that there is a diagram

$$\begin{tikzcd}
\widetilde{\mathcal{U}}_i \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Z}}_i \arrow[d] & \\
\mathcal{U}_i \arrow[r, hook, "j_i"] & \mathcal{Z}_i \arrow[r] & S_0
\end{tikzcd}$$

satisfying the conditions of Lemma 4.2. Then the diagram base changed along a witness $S' \to S_0$ also satisfies the conditions of Lemma 4.2. From $\mathrm{char}(k) = 0$, the tame ramification condition holds.

As the image of every non-empty open subscheme of S_0 contains a non-empty open subscheme of S and as, for every $s_0 \in S_0$ with image $s \in S$ any witness $(S'_0, \eta'_0, s'_0) \to (S_0, \eta_0, s_0)$ induces a witness $(S'_0, \eta'_0, s'_0) \to (S_0, \eta_0, s_0) \to (S, \eta, s)$, one may freely replace S with S_0 so that we remove the subscripts $(-)_0$ from the notation. Let now $s \in S$ and fix a witness $(S', \eta', s') \to (S, \overline{\eta}, s)$. By invariance of étale fundamental group under extensions of algebraically closed fields in characteristic 0, one has $\pi_1(\mathcal{U}_{i,\eta'}) \tilde{\to} \pi_1(U_i)$ hence $\mathcal{F}_i|_{\mathcal{U}_{i,\eta'}}$ is again irreducible. As S' is strictly henselian, $\pi_1(S') = 1$ and as $\mathcal{U}'_i := \mathcal{U}_i \times_S S' \to S'$ has a section, the canonical morphisms

$$\pi_1(\mathcal{U}'_{i,\eta'}) \tilde{\to} \pi_1(\mathcal{U}'_i) \tilde{\leftarrow} \pi_1(\mathcal{U}'_{i,s'})$$

are both isomorphisms [SGA1, XIII, 4.3, 4.4]. In particular, $\mathcal{F}_i|_{\mathcal{U}_{i,\eta'}}$ extends uniquely to an object $\mathcal{F}'_i$ in $\mathrm{Loc}(\mathcal{U}'_i)$, and $\mathcal{F}'_i|_{\mathcal{U}'_{i,s'}}$ is simple in $\mathrm{Loc}(\mathcal{U}'_{i,s'})$. From [BeBerDG82, Thm. 4.3.1 (ii)], it is thus enough to show that

$$R\psi_{f'}(\mathcal{P}_i|_{\mathcal{X}_{\eta'}}) \simeq \iota_{i,s'*} j_{i,s'!*}(\mathcal{F}'_i|_{\mathcal{U}'_{i,s'}}[d_i]). \tag{14}$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{U}'_i \arrow[r, hook, "j'_i"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z}'_i \arrow[r, hook, "\iota'_i"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\
\mathcal{U}_i \arrow[r, hook, "j_i"] & \mathcal{Z}_i \arrow[r, hook, "\iota_i"] & \mathcal{X} \arrow[r, "f"] & S
\end{tikzcd}$$

of schemes with cartesian squares. By Lemma 4.2, $j'_{i,!*/S'}\mathcal{F}'_i[d_i] \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{Z}'_i/S')$. Therefore, $\mathcal{K} := \iota'_{i*} j'_{i,!*/S'}\mathcal{F}'_i[d_i]$ is in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}'/S')$. By Observation (2) in Paragraph 4.1.3.1 and the proper base change theorem, as $\iota'_i : \mathcal{Z}'_i \hookrightarrow$

$\mathcal{X}'$ is proper, $\mathcal{K}|_{\mathcal{X}_{\eta'}}$ is the pullback of $\mathcal{P}_i$ along $\mathcal{X}_{\eta'} \to \mathcal{X}_{\bar{\eta}}$, and $\mathcal{K}|_{\mathcal{X}_{s'}}$ is $\iota_{i,s'*} j_{i,s'!*}(\mathcal{F}'|_{\mathcal{U}'_{i,s'}}[d_i])$, which proves (14).

Lemma 4.3. *Let S be an irreducible scheme with generic point η. Assume that $k(\eta)$ is of characteristic 0. Let $f : \mathcal{Z} \to S$ be a morphism separated of finite presentation, with $\mathcal{Z}_\eta$ integral. Let $U \hookrightarrow \mathcal{Z}_\eta$ be an open subset smooth over $k(\eta)$. Then up to shrinking S to an affine open subset, there is a diagram (13) satisfying the conditions of Lemma 4.2, such that $\mathcal{U}_\eta = U$ and $h : \widetilde{\mathcal{U}} \to \mathcal{U}$ is an isomorphism.*

Proof. By Hironaka's resolution of singularities (see, e.g., [SGA5, I, 3.1.5 b) a)]), as $k(\eta)$ is of characteristic 0, $\mathcal{Z}_\eta$ is strongly desingularizable. As $\mathcal{Z}_\eta$ is integral, there is a proper morphism $\widetilde{Z} \to \mathcal{Z}_\eta$ with $\widetilde{Z}$ smooth over $k(\eta)$, such that the pullback $\widetilde{U} := U \times_{\mathcal{Z}_\eta} \widetilde{Z} \to U$ is an isomorphism, and that $\widetilde{Z} \setminus \widetilde{U}$ is a strict normal crossing divisor. The result follows by spreading out. □

Remark 4.4. The nearby cycles functor may not preserve simplicity of perverse sheaves nor commute with middle extension. Let S be the spectrum of a strictly Henselian discrete valuation ring with generic point η and closed point s. Let $f : \mathcal{X} \to S$ be a proper semi-stable morphism with geometrically integral fibers of dimension d. Assume that the special fiber $\mathcal{X}_s \hookrightarrow \mathcal{X}$ is a strict normal crossing divisor on $\mathcal{X}$. Then there is an open subset $j : \mathcal{U} \hookrightarrow \mathcal{X}$ smooth over S, such that $\mathcal{U}_\eta = \mathcal{X}_\eta$ and that $\mathcal{U}_s$ is Zariski-dense in $\mathcal{X}_s$. Let $R\psi_f : D^b_c(\mathcal{X}_{\bar{\eta}}) \to D^b_c(\mathcal{X}_s)$ be the nearby cycles functor. By [I94, Théorème 3.2 (c) (i)], $H^0 R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar{\eta}}}) \simeq \overline{\mathbb{Q}}_{\ell,\mathcal{X}_s}$. Let $j_s : \mathcal{U}_s \hookrightarrow \mathcal{X}_s$ be the pullback of $j : \mathcal{U} \hookrightarrow \mathcal{X}$ along $s \to S$. Let $\mathrm{IC}_{\mathcal{X}_s} := j_{s,!*}\overline{\mathbb{Q}}_{\ell,\mathcal{U}_s}[d]$ be the intersection cohomology complex on $\mathcal{X}_s$. In general, $H^{-d}\mathrm{IC}_{\mathcal{X}_s}$ is not constant, in which case the perverse sheaf $R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar{\eta}}}[d])$ is not isomorphic to $\mathrm{IC}_{\mathcal{X}_s}$. Also, from [SGA4-III, XV, Thm 2.1], one has

$$\left(R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar{\eta}}})\right)|_{\mathcal{U}_s} = R\psi_{f\circ j}(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar{\eta}}}) = \overline{\mathbb{Q}}_{\ell,\mathcal{U}_s}.$$

Then by [BeBerDG82, Théorème 4.3.1 (ii)], $R\psi_f(\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar{\eta}}}[d])$ is not simple in $\mathrm{Perv}(\mathcal{X}_s)$ (otherwise, it would be isomorphic to the simple object $\mathrm{IC}_{\mathcal{X}_s}$) while $\overline{\mathbb{Q}}_{\ell,\mathcal{X}_{\bar{\eta}}}[d]$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar{\eta}})$.

4.2. **Lifting semisimplicity.**

4.2.1. Let $\mathcal{A}$ be an artinian and noetherian abelian category, let $\iota : \mathcal{N} \hookrightarrow \mathcal{A}$ be a Serre subcategory and let $p : \mathcal{A} \to \overline{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ denote the resulting quotient functor. The inclusion functor $\iota : \mathcal{N} \hookrightarrow \mathcal{A}$ admits both a right adjoint $(-)_\neg : \mathcal{N} \to \mathcal{A}$ ("maximal subobject in $\mathcal{N}$") and a left adjoint $(-)^\neg : \mathcal{N} \to \mathcal{A}$ ("maximal quotient object in $\mathcal{N}$"). Explicitly, for $A \in \mathcal{A}$, $A_\neg = \sum_{N \in S_\neg(A)} N \hookrightarrow A$, where $S_\neg(A)$ denotes the subset of all subobjects of A in $\mathcal{N}$ and $A \twoheadrightarrow A^\neg = A/\cap_{S \in S^\neg(A)} S$, where $S^\neg(A)$ denotes the subobjects S of A such that $A/S \in \mathcal{N}$. Then for every $A \in \mathcal{A}$,

$$A^* := \ker(A/A_\neg \to (A/A_\neg)^\neg)$$

is a subquotient of A in $\mathcal{A}$ satisfying $(A^*)^\neg = (A^*)_\neg = 0$ and $p(A^*) \simeq p(A)$ in $\overline{\mathcal{A}}$. Observing that for every $A_1, A_2 \in \mathcal{A}$ with $A_1^\neg = 0$ and $A_{2,\neg} = 0$ the canonical morphism

$$\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\overline{\mathcal{A}}}(p(A_1), p(A_2))$$

is an isomorphism, one gets that for every $A_1, A_2 \in \mathcal{A}$,

$$p(A_1) \simeq p(A_2) \text{ in } \overline{\mathcal{A}} \text{ if and only if } A_1^* \simeq A_2^* \text{ in } \mathcal{A}.$$

Lemma 4.5. *Let $A \in \mathcal{A}$ such that $p(A)$ is semisimple in $\overline{\mathcal{A}}$. Then A^* is semisimple in $\mathcal{A}$ and*

$$\mathrm{length}_{\mathcal{A}}(A^*) = \mathrm{length}_{\overline{\mathcal{A}}}(p(A)).$$

Proof. Assume first $p(A)$ is simple in $\overline{\mathcal{A}}$. As $p(A^*) \simeq p(A)$ in $\overline{A}$, A^* has a single Jordan-Hölder factor in $\mathcal{A}$ which is not in $\mathcal{N}$. But as $(A^*)_\neg = (A^*)^\neg = 0$, this forces A^* to be simple in $\mathcal{A}$. In general, let $\overline{S}_1, \dots, \overline{S}_r$ denote the simple factors (counted with multiplicities) of $p(A)$ in $\overline{\mathcal{A}}$ and let $S_1, \dots, S_r \in \mathcal{A}$ with $p(S_i) \simeq \overline{S}_i$ in $\overline{\mathcal{A}}$, $i = 1, \dots, r$. Set

$$S := S_1^* \oplus \cdots \oplus S_r^*.$$

Then, S is semisimple in $\mathcal{A}$, $S = S^*$, and $p(S) \simeq p(A)$ in $\overline{\mathcal{A}}$. This shows $A^* \simeq (S^* \simeq) S$ is semisimple in $\mathcal{A}$ and

$$\mathrm{length}_{\mathcal{A}}(A^*) = \mathrm{length}_{\mathcal{A}}(S) = r = \mathrm{length}_{\overline{\mathcal{A}}}(p(A)).$$

□

4.2.2. If X is an abelian variety over a field K of characteristic 0, write

$$(-)_\neg : \mathrm{Perv}(X) \to N(X) \cap \mathrm{Perv}(X), \quad (-)^\neg : \mathrm{Perv}(X) \to N(X) \cap \mathrm{Perv}(X)$$

for the right adjoint ("maximal negligible subobject") and left adjoint ("maximal negligible quotient object") of the inclusion functor

$$N(X) \cap \mathrm{Perv}(X) \hookrightarrow \mathrm{Perv}(X)$$

respectively. By Galois descent [Ri14, Lem. A.6], for every $\mathcal{P} \in \mathrm{Perv}(X)$, $(\mathcal{P}_\neg)|_{X_{\bar{K}}} = (\mathcal{P}|_{X_{\bar{K}}})_\neg \hookrightarrow \mathcal{P}|_{X_{\bar{K}}}$ and $\mathcal{P}|_{X_{\bar{K}}} \twoheadrightarrow (\mathcal{P}^\neg)|_{X_{\bar{K}}} = (\mathcal{P}|_{X_{\bar{K}}})^\neg$; in particular

$$(-)^* \circ -|_{X_{\bar{K}}} \simeq -|_{X_{\bar{K}}} \circ (-)^* : \mathrm{Perv}(X) \to \mathrm{Perv}(X_{\bar{K}}). \tag{15}$$

If $f : \mathcal{X} \to S$ is an abelian scheme, write again

$$(-)_\neg : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S) \cap \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S),$$
$$(-)^\neg : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to N^{\mathrm{ULA}}(\mathcal{X}/S) \cap \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$$

for the right adjoint and left adjoint of the inclusion functor

$$N^{\mathrm{ULA}}(\mathcal{X}/S) \cap \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \hookrightarrow \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$$

respectively. Furthermore, as for every $s \in S$,

$$N^{\mathrm{ULA}}(\mathcal{X}/S) \cap \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) = \ker(\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to P(\mathcal{X}_{\bar{s}})),$$

and, for $s = \eta$, $-|_{\mathcal{X}_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_\eta)$ is fully faithful with essential image stable under subquotients, one has

$$(-)^* \circ -|_{\mathcal{X}_\eta} \simeq -|_{\mathcal{X}_\eta} \circ (-)^* : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_\eta). \tag{16}$$

Combining (15), (16) one gets

$$(-)^* \circ -|_{\mathcal{X}_{\bar\eta}} \simeq -|_{\mathcal{X}_{\bar\eta}} \circ (-)^* : \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_{\bar\eta}).$$

This observation together with Lemma 4.5 yields the following result.

Corollary 4.6. *Let $f : \mathcal{X} \to S$ an abelian scheme. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume that $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $P(\mathcal{X}_{\bar\eta})$. Then $\mathcal{P}^*|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar\eta})$ with*

$$\mathrm{length}_{\mathrm{Perv}(\mathcal{X}_{\bar\eta})}(\mathcal{P}^*|_{\mathcal{X}_{\bar\eta}}) = \mathrm{length}_{P(\mathcal{X}_{\bar\eta})}(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}).$$

and, for every $s \in S$, $\mathcal{P}^*|_{\mathcal{X}_{\bar s}} \simeq \mathcal{P}|_{\mathcal{X}_{\bar s}}$ in $P(\mathcal{X}_{\bar s})$.

4.3. **Proof of Corollary 1.4.** Let $f : \mathcal{X} \to S$ an abelian scheme. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ such that $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $P(\mathcal{X}_{\bar\eta})$. From Corollary 4.6, up to replacing $\mathcal{P}$ with $\mathcal{P}^*$, one may assume $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar\eta})$ and, from Theorem 1.2, up to replacing S by a non-empty open subscheme, one may assume that for every $s \in S$, $\mathcal{P}|_{\mathcal{X}_{\bar s}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar s})$. This reduces the proof of Corollary 1.4 (2) to the one of Corollary 1.4 (1). Under the assumptions of Corollary 1.4 (1), $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is a reductive group and, for every $s \in S$, $G(\mathcal{P}|_{\mathcal{X}_{\bar s}}) \subset G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is a closed reductive subgroup. Recall that [D82, Prop. 3.1 (c)] for a reductive group G over a field Q of characteristic 0, a finite-dimensional Q-rational faithful representation V of G and a closed reductive subgroup $H \subset G$, one has

$$H = \mathrm{Fix}_G(u_H) \subset G,$$

for some integers $m, n \geq 0$ and $0 \neq u_H \in T^{m,n}(V)$. In particular, $H \subsetneq G$ if and only if

$$\dim_Q(I^{m,n}(V)) < \dim_Q(I^{m,n}(V|_H)),$$

for some integers $m, n \geq 0$.

This reduces the proof of Corollary 1.4 (1) to the following.

Corollary 4.7. *Let $f : \mathcal{X} \to S$ an abelian scheme. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ such that $\mathcal{P}|_{\mathcal{X}_{\bar s}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar s})$ for every $s \in S$. Then for every integers $m, n \geq 0$, there exists a strict closed subscheme $S_{m,n} \hookrightarrow S$ such that for every $s \in S$,*

$$\dim_{\overline{\mathbb{Q}}_\ell}(I^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) < \dim_{\overline{\mathbb{Q}}_\ell}(I^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar s}}))$$

if and only if $s \in S_{m,n}$.

Proof. Write

$$\mathcal{P}_{m,n} := {}^{p/S}\mathrm{H}^0(T^{m,n}(\mathcal{P})),$$

which is again in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with the properties that, for every $s \in S$,

$${}^p\mathrm{H}^0(T^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})) \simeq \mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$$

and, by Lemma 4.8, $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ is semisimple in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$. For $s \in S$, decompose $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ as

$$\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}} \simeq (\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}})_{\neg} \oplus \mathcal{S},$$

where $\mathcal{S}$ is the sum of all simple non-negligible subobjects of $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$. As for every $\mathcal{N} \in N(\mathcal{X}_{\bar{s}}) \cap \mathrm{Perv}(\mathcal{X}_{\bar{s}})$ one has

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{N}, \delta_0) = 0,$$

the canonical morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{S}, \delta_0) \to \mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}, \delta_0)$$

is an isomorphism. On the other hand, as for every non-negligible simple objects $\mathcal{S}_1, \mathcal{S}_2$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ the canonical morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_{P(\mathcal{X}_{\bar{s}})}(\mathcal{S}_1, \mathcal{S}_2)$$

is an isomorphism, the canonical morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{S}, \delta_0) \to \mathrm{Hom}_{P(\mathcal{X}_{\bar{s}})}(\mathcal{S}, \delta_0)$$

is also an isomorphism. This proves that

$$\dim_{\overline{\mathbb{Q}}_\ell}(I^{m,n}(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})) = \dim_{\overline{\mathbb{Q}}_\ell}(I^{1,0}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}})) = \dim_{\overline{\mathbb{Q}}_\ell}(\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}, \delta_0)).$$

By Lemma 4.9, there is a quotient $\mathcal{P}_{m,n} \twoheadrightarrow \mathcal{P}_{m,n,\{0\}}$ in $\mathrm{Perv}(\mathcal{X}/S)$ such that for every geometric point $\bar{s}$ on S, $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}} \twoheadrightarrow \mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}$ is the maximal quotient of $\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ with support in $\{0\}$. In particular, the canonical injective morphism

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}, \delta_0) \to \mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{s}}}, \delta_0)$$

is an isomorphism. But as the full subcategory $\mathrm{Perv}_0(\mathcal{X}_{\bar{s}}) \subset \mathrm{Perv}(\mathcal{X}_{\bar{s}})$ of all objects with support in $\{0\}$ identifies with $\mathrm{Perv}(\bar{0}) \simeq \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ *via*

$$0_* : \mathrm{Perv}(\bar{0}) \tilde{\to} \mathrm{Perv}_0(\mathcal{X}_{\bar{s}}),$$

one has $\mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}} \simeq \delta_0^{\oplus \mu_s}$ with

$$\mu_s := \dim_{\overline{\mathbb{Q}}_\ell}(\mathrm{Hom}_{\mathrm{Perv}(\mathcal{X}_{\bar{s}})}(\mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}, \delta_0) = \chi(\mathcal{X}_{\bar{s}}, \mathcal{P}_{m,n,\{0\}}|_{\mathcal{X}_{\bar{s}}}).$$

This eventually reduces Corollary 4.7 to proving that for every $b \geq 0$ the subset

$$U_{\leq b} := \{s \in S \mid \mu_s \leq b\} \subset S$$

is open. As the map $\mu : S \to \mathbb{Z}_{\geq 0}$ is constructible, it is enough to prove that $U_{\leq b}$ is stable under generization. This essentially follows from the existence of the cospecialization morphism since, for every specialization $t_1 \rightsquigarrow t_0$ of points in S, $csp_{\bar{t}_1, \bar{t}_0}$ identifies $G(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_0}})$ with a subgroup

$$G(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_0}}) \subset G(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_1}}) \subset \mathrm{GL}(\omega_{\bar{t}_1}(\mathcal{P}_{m,n}|_{\mathcal{X}_{\bar{t}_1}})),$$

so that $\mu_{t_0} \geq \mu_{t_1}$. □

Lemma 4.8. *Let k be an algebraically closed field of characteristic 0, let X be an abelian variety over k and let $\mathcal{P} \in \mathrm{Perv}(X)$. Assume $\mathcal{P}$ is semisimple in $\mathrm{Perv}(X)$. Then for every integers $m, n \geq 0$, ${}^p\mathrm{H}^0(T^{m,n}(\mathcal{P}))$ is again semisimple in $\mathrm{Perv}(X)$.*

Proof. (Sketch of) This is mentioned as [KrW15, Ex. 5.1]. The fact that $Rm_* : D^b_c(X \times X) \to D^b_c(X)$ preserves direct sums of shifts of simple perverse sheaves follows from Kashiwara's conjecture (Kashiwara's conjecture is reduced to a conjecture of de Jong in [Dr01], and de Jong's conjecture is proved in [BoK06], [G07]) while the fact that the exterior tensor product $\mathcal{P}_1 \boxtimes^L \mathcal{P}_2$ of two simple objects $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Perv}(X)$ is a direct sums of shifts of simple perverse sheaves follows from the structure of simple perverse sheaves and the fact that for every immersion $\iota_i : U_i \hookrightarrow X$ and $\mathcal{L}_i \in \mathrm{Loc}(U_i)$, $i = 1, 2$ one has

$$(\iota_1 \times \iota_2)_{!*}(\mathcal{L}_1 \boxtimes^L \mathcal{L}_2) \simeq \iota_{1,!*}(\mathcal{L}_1) \boxtimes^L \iota_{2,!*}(\mathcal{L}_2).$$

See *e.g.* [MS22, Ex. 10.2.31]. □

Lemma 4.9. *Let $f : \mathcal{X} \to S$ be a separated morphism of finite type and let $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed immersion. For every $\mathcal{P} \in \mathrm{Perv}(\mathcal{X}/S)$, there is a quotient $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}$ in $\mathrm{Perv}(\mathcal{X}/S)$ such that for every geometric point $\bar{s}$ on S, $\mathcal{P}|_{\mathcal{X}_{\bar{s}}} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}|_{\mathcal{X}_{\bar{s}}}$ is the maximal quotient of $\mathcal{P}|_{\mathcal{X}_{\bar{s}}}$ in $\mathrm{Perv}(\mathcal{X}_{\bar{s}})$ with support in $\mathcal{Z}_{\bar{s}}$.*

Proof. Define $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}$ as the image of the composite

$$\mathcal{P} \overset{\mathrm{ad}_\iota}{\to} \iota_*\iota^*\mathcal{P} \to {}^{p/S}\tau^{\geq 0}(\iota_*\iota^*\mathcal{P})$$

of the adjunction morphism for $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ and the relative perverse truncation with respect to $f : \mathcal{X} \to S$. As $\iota_* : D^b_c(\mathcal{Z}) \to D^b_c(\mathcal{X})$ is t-exact and $\iota^* : D^b_c(\mathcal{X}) \to D^b_c(\mathcal{Z})$ is right t-exact with respect to the relative perverse t-structure on $f : \mathcal{X} \to S$, one has

$${}^{p/S}\tau^{\geq 0}(\iota_*\iota^* -) \simeq \iota_* {}^{p/S}\tau^{\geq 0}(\iota^* -) \simeq \iota_* {}^{p/S}\mathrm{H}^0(\iota^* -) : \mathrm{Perv}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}/S).$$

Furthermore, for every geometric point $\bar{s}$ on S, by proper base-change,

$$-|_{\mathcal{X}_{\bar{s}}} \circ \iota_* \simeq \iota_{\bar{s}*} \circ -|_{\mathcal{Z}_{\bar{s}}} : \mathrm{Perv}(\mathcal{Z}/S) \to \mathrm{Perv}(\mathcal{X}_{\bar{s}}),$$

while, by definition of the relative perverse t-structure,

$$-|_{\mathcal{Z}_{\bar{s}}} \circ {}^{p/S}\mathrm{H}^0(-) \simeq {}^p\mathrm{H}^0(-|_{\mathcal{Z}_{\bar{s}}}) : \mathrm{Perv}(\mathcal{Z}/S) \to \mathrm{Perv}(\mathcal{Z}_{\bar{s}}).$$

This proves that for every geometric point $\bar{s}$, the formation of $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathcal{Z}}$ commutes with $-|_{\mathcal{X}_{\bar{s}}} : \mathrm{Perv}(\mathcal{X}/S) \to \mathrm{Perv}(\mathcal{X}_{\bar{s}})$ and reduces the proof of Lemma 4.9 to the case where $S = \mathrm{spec}(\bar{k})$ is the spectrum of an algebraically closed field. By construction $\mathcal{P}_{\mathcal{Z}}$ has support in $\mathcal{Z}$. Let $j : \mathcal{X} \setminus \mathcal{Z} \hookrightarrow \mathcal{X}$ denote the complementary open immersion. Conversely, for every quotient $\mathcal{P} \twoheadrightarrow \mathcal{Q}$ in $\mathrm{Perv}(\mathcal{X})$ with support in $\mathcal{Z}$, the distinguished triangle

$$j_! j^* \mathcal{Q} \to \mathcal{Q} \to \iota_*\iota^*\mathcal{Q} \xrightarrow{+1}$$

ensures that $\mathcal{Q}\tilde{\rightarrow}\iota_*\iota^*\mathcal{Q}$ so that, by adjunction, $\mathcal{P}\twoheadrightarrow\mathcal{Q}\tilde{\rightarrow}\iota_*\iota^*\mathcal{Q}$ factors as

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, two heads] \arrow[dr, "\mathrm{ad}_\iota"'] & & \iota_*\iota^*\mathcal{Q}\ , \\
& \iota_*\iota^*\mathcal{P} \arrow[ur, dotted] &
\end{tikzcd}\tag{17}$$

and, as $\mathcal{Q}\in\mathrm{Perv}(\mathcal{X})$, (17) factors further as

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, two heads] \arrow[dr, "\mathrm{ad}_\iota"'] & & \iota_*\iota^*\mathcal{Q} \\
& \iota_*\iota^*\mathcal{P} \arrow[ur] \arrow[r] & {}^{p/S}\tau^{\geq 0}(\iota_*\iota^*\mathcal{P}), \arrow[u, dotted]
\end{tikzcd}$$

which concludes the proof of Lemma 4.9. □

4.4. **Proof of Corollary 1.5.** Let $f:\mathcal{X}\to S$ be an abelian scheme. We begin with the following observation.

Lemma 4.10. *Let $\mathcal{P}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assume $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ has torsion determinant of order N. Then for every $s\in S$, $\mathcal{P}|_{\mathcal{X}_{\bar s}}$ also has torsion determinant of order dividing N.*

Proof. Let n denote the dimension of $\mathcal{P}$ in $P^{\mathrm{ULA}}(\mathcal{X}/S)$ and $\det(\mathcal{P}):=\wedge^n\mathcal{P}$ its determinant. It follows from the general formalism of Tannakian categories that for every $t\in S$, $\mathcal{P}|_{\mathcal{X}_t}$, $\mathcal{P}|_{\mathcal{X}_{\bar t}}$ again have dimension n and that $(\wedge^n\mathcal{P})|_{\mathcal{X}_t}\simeq\wedge^n(\mathcal{P}|_{\mathcal{X}_t})$, $(\wedge^n\mathcal{P})|_{\mathcal{X}_{\bar t}}\simeq\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar t}})$. In particular, as for every $s\in S$, one has

$$G(\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar s}}))\subset G(\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})),$$

hence $\wedge^n(\mathcal{P}|_{\mathcal{X}_{\bar s}})$ is also torsion, with order dividing N. □

Note that if $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is semisimple then any object in $\langle\mathcal{P}|_{\mathcal{X}_{\bar\eta}}\rangle$ has torsion determinant of order dividing $|\pi_0(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}))|$.

We now turn to the proof of Corollary 1.5 itself. Let $\mathcal{P}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. After possibly replacing $\mathcal{P}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $\mathcal{P}|_{\mathcal{X}_{\bar k}}\in\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}_{\bar k}/S_{\bar k})$, one may assume $k=\bar k$ is algebraically closed.

- Proof of Corollary 1.5 (1). Up to replacing $\mathcal{P}$ with $\mathcal{P}^*$ - see Corollary 4.6, one may assume $\mathcal{P}|_{\mathcal{X}_{\bar\eta}}$ is simple in $\mathrm{Perv}(\mathcal{X}_{\bar\eta})$, and not only in $P(\mathcal{X}_{\bar\eta})$. Then Corollary 1.5 (1) immediately follows from Fact 1.1, Theorem 1.2 and Lemma 4.10.
- Proof of Corollary 1.5 (2). Let us first observe that for a connected reductive group G and a closed subgroup $H\subset G$, the following are equivalent
  (1) $H\subset G\twoheadrightarrow G^{\mathrm{ss}}$ factors through an isogeny $H^{\mathrm{ss}}\twoheadrightarrow G^{\mathrm{ss}}$;
  (2) $\dim(R(G)\cap H)=\dim(R(H))$ and $\dim(H)-\dim(R(H))=\dim(G)-\dim(R(G))$;

(3) $\dim(R(G)\cap H^\circ) = \dim(R(H^\circ))$ and $\dim(H^\circ) - \dim(R(H^\circ)) = \dim(G) - \dim(R(G))$;

(4) $H^\circ \subset G \twoheadrightarrow G^{\mathrm{ss}}$ factors through an isogeny $H^{\circ,\mathrm{ss}} \twoheadrightarrow G^{\mathrm{ss}}$.

In particular, to prove the first part of Corollary 1.5 (2), one can replace $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^\circ$ with $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^\circ$ (which will simplify a bit the notation).

Replacing $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $[N]_*\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ for some integer $N \geq 1$, one may assume $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is connected (see [W15, §2]). From the short exact sequence

$$1 \to G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) \to G(\mathcal{P}) \to G(\langle \mathcal{P} \rangle_0) \to 1$$

and the description of $G(\langle \mathcal{P} \rangle_0)$ in terms of representation of $\pi_1(S)$ (see Subsection 4.5 below), replacing $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $\mathcal{P}|_{\mathcal{X}_{S'}} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}_{S'}/S')$ for some connected étale cover $S' \to S$, one may also assume $G(\mathcal{P})$ is connected.

Let

$$G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}}$$

denote the maximal *adjoint* quotient[5] of $G(\mathcal{P})$. The morphism $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}}$ factors as $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ss}} \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}}$. On the other hand, as $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is normal in $G(\mathcal{P})$, $R(G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})) = (R(G(\mathcal{P})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}))^\circ$ so that the morphism $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}) \hookrightarrow G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{ss}}$ factors through a morphism $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\mathrm{ss}} \to G(\mathcal{P})^{\mathrm{ss}}$ inducing an isogeny onto its image, which is a closed normal subgroup of $G(\mathcal{P})^{\mathrm{ss}}$. By the structure theory of connected semisimple groups, there is a (unique) connected (automatically adjoint) quotient $G(\mathcal{P})^{\mathrm{ad}} \twoheadrightarrow \widetilde{G}$ such that the resulting canonical morphism

$$G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\mathrm{ss}} \to G(\mathcal{P})^{\mathrm{ss}} \twoheadrightarrow G(\mathcal{P})^{\mathrm{ad}} \twoheadrightarrow \widetilde{G}$$

is an isogeny. As $\widetilde{G}$ is adjoint, it admits an irreducible faithful representation corresponding to a simple object $\mathcal{Q} \in \langle \mathcal{P} \rangle$; in particular, $\widetilde{G} = G(\mathcal{Q})$. The commutative diagram of exact tensor functors

$$\begin{tikzcd}
\langle \mathcal{Q}|_{\mathcal{X}_{\bar{\eta}}} \rangle \arrow[d, hook] \arrow[rrr, bend left, "csp_{\bar{\eta},\bar{s}}"] & \langle \mathcal{Q} \rangle \arrow[l, "|_{\mathcal{X}_{\bar{\eta}}}"'] \arrow[r, "|_{\mathcal{X}_s}"] \arrow[d, hook] & \langle \mathcal{Q}|_{\mathcal{X}_s} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] \arrow[d, hook] & \langle \mathcal{Q}|_{\mathcal{X}_{\bar{s}}} \rangle \arrow[d, hook] \\
\langle \mathcal{P}|_{\mathcal{X}_{\bar{\eta}}} \rangle \arrow[rrr, bend right, "csp_{\bar{\eta},\bar{s}}"'] & \langle \mathcal{P} \rangle \arrow[l, "|_{\mathcal{X}_{\bar{\eta}}}"'] \arrow[r, "|_{\mathcal{X}_s}"] & \langle \mathcal{P}|_{\mathcal{X}_s} \rangle \arrow[r, "|_{\mathcal{X}_{\bar{s}}}"] & \langle \mathcal{P}|_{\mathcal{X}_{\bar{s}}} \rangle
\end{tikzcd} \tag{18}$$

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[5] Namely, $G(\mathcal{P})^{\mathrm{ad}} = G(\mathcal{P})^{\mathrm{red}}/Z(G(\mathcal{P})^{\mathrm{red}})$, where $G(\mathcal{P}) \twoheadrightarrow G(\mathcal{P})^{\mathrm{red}} := G(\mathcal{P})/R_u(G(\mathcal{P}))$ is the maximal reductive quotient of $G(\mathcal{P})$.

induces a commutative diagram of algebraic groups

$$\begin{tikzcd}
G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}) \arrow[r, hook] & G(\mathcal{Q}) & G(\mathcal{Q}|_{\mathcal{X}_s}) \arrow[l, hook'] & G(\mathcal{Q}|_{\mathcal{X}_{\bar s}}) \arrow[l, hook'] \arrow[lll, bend right, "csp_{\bar\eta,\bar s}"'] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}} \arrow[u, two heads] \arrow[r] & G(\mathcal{P})^{\mathrm{ss}} \arrow[u, two heads] & & \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}) \arrow[u, two heads] \arrow[r, hook] & G(\mathcal{P}) \arrow[u, two heads] & G(\mathcal{P}|_{\mathcal{X}_s}) \arrow[l, hook'] \arrow[uu] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}}) \arrow[l, hook'] \arrow[uu] \arrow[lll, bend left, "csp_{\bar\eta,\bar s}"]
\end{tikzcd} \tag{19}$$

As $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}} \twoheadrightarrow G(\mathcal{Q})$ is an isogeny, $G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}) \to G(\mathcal{Q})$ is an isomorphism. In particular, $\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}$ is a simple object in $P(\mathcal{X}_{\bar\eta})$ and every object in $\langle \mathcal{Q}|_{\mathcal{X}_{\bar\eta}} \rangle$ has trivial determinant. By Corollary 1.5 (1) applied to $\mathcal{Q}$ up to replacing S by a non-empty open subscheme, one may assume that for every $s \in S$ the cospecialization morphism $csp_{\bar\eta,\bar s} : G(\mathcal{Q}|_{\mathcal{X}_{\bar s}}) \hookrightarrow G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}})$ is an isomorphism so that one has a canonical commutative diagram

$$\begin{tikzcd}
G(\mathcal{Q}|_{\mathcal{X}_{\bar\eta}}) & & G(\mathcal{Q}|_{\mathcal{X}_{\bar s}}) \arrow[ll, "csp_{\bar\eta,\bar s}", "\simeq"'] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}} \arrow[u, two heads] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}})) \arrow[l, hook'] & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}} \arrow[u, two heads] \\
 & G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))^{\circ} \arrow[u, two heads] \arrow[ur, two heads] & \\
G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}}) \arrow[uu, two heads] & & G(\mathcal{P}|_{\mathcal{X}_{\bar s}}). \arrow[ll, hook', "csp_{\bar\eta,\bar s}"'] \arrow[ul, two heads] \arrow[uu, two heads]
\end{tikzcd} \tag{20}$$

In particular,

$$\begin{aligned}
\dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}}) \geq \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}}) \quad & \geq \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))) \\
& = \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))^{\circ}) \\
& \geq \dim(G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}})
\end{aligned}$$

which, as $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ - hence $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}}$ - are connected, imposes that the morphisms

$$G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}})) \overset{\simeq}{\twoheadrightarrow} G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\mathrm{ss}}$$

and

$$G(\mathcal{P}|_{\mathcal{X}_{\bar s}})/(R(G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})) \cap G(\mathcal{P}|_{\mathcal{X}_{\bar s}}))^{\circ} \overset{\simeq}{\twoheadrightarrow} G(\mathcal{P}|_{\mathcal{X}_{\bar s}})^{\mathrm{ss}},$$

are isomorphisms. This concludes the proof of the first part of Corollary 1.5 (2). The second part when $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})$ is semisimple tautologically follows from the first part as, then, $G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ} = G(\mathcal{P}|_{\mathcal{X}_{\bar\eta}})^{\circ,\mathrm{ss}}$ while the second part when $G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is reductive follows from the first part and the fact that, for every $s \in S$ such that $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^\circ \subset G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^\circ$ factors through an isogeny $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ,\mathrm{ss}} \twoheadrightarrow G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{ss}}$, the arrows $(\ast\text{-}\bar{s})$, $(\ast\text{-}\bar{\eta})$ and the right vertical arrow in the canonical commutative diagram

$$\begin{tikzcd}
G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ,\mathrm{der}} \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "(\ast\text{-}\bar{s})"] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^\circ \arrow[r, two heads] \arrow[d, hook] & G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})^{\circ,\mathrm{ss}} \arrow[d] \\
G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{der}} \arrow[r, hook] \arrow[rr, bend right, "(\ast\text{-}\bar{\eta})"'] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^\circ \arrow[r, two heads] & G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})^{\circ,\mathrm{ss}}.
\end{tikzcd}$$

are isogenies.

4.5. **Proof of Proposition 1.7.** Let k be a field of characteristic 0, S a smooth geometrically connected variety over k with generic point η and $f : \mathcal{X} \to S$ an abelian scheme. Let $P^{\mathrm{ULA}}(\mathcal{X}/S)_0 \subset P^{\mathrm{ULA}}(\mathcal{X}/S)$ denote the essential image of

$$0_* : \mathrm{Loc}(S) = \mathrm{Perv}^{\mathrm{ULA}}(S/S) \to \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S) \to P^{\mathrm{ULA}}(\mathcal{X}/S)$$

and for every $\mathcal{P} \in P^{\mathrm{ULA}}(\mathcal{X}/S)$, consider the full subcategory $\langle\mathcal{P}\rangle_0 := \langle\mathcal{P}\rangle \cap P^{\mathrm{ULA}}(\mathcal{X}/S)_0 \subset \langle\mathcal{P}\rangle$. From Lemma 3.2, the equivalence of Tannakian categories $-|_{\mathcal{X}_\eta} : \langle\mathcal{P}\rangle \tilde{\to} \langle\mathcal{P}|_{\mathcal{X}_\eta}\rangle$ restricts to an equivalence

$$-|_{\mathcal{X}_\eta} : \langle\mathcal{P}\rangle_0 \tilde{\to} \langle\mathcal{P}|_{\mathcal{X}_\eta}\rangle_0.$$

This yields an explicit categorical description of the morphism $G(\langle\mathcal{P}|_{\mathcal{X}_s}\rangle_0) \to G(\langle\mathcal{P}|_{\mathcal{X}_\eta}\rangle_0)$ as the composite $G(\langle\mathcal{P}|_{\mathcal{X}_s}\rangle_0) \to G(\langle\mathcal{P}\rangle_0) \tilde{\leftarrow} G(\langle\mathcal{P}|_{\mathcal{X}_\eta}\rangle_0)$ arising from the diagram of Tannakian categories

$$\langle\mathcal{P}|_{\mathcal{X}_s}\rangle_0 \overset{|_{\mathcal{X}_s}}{\leftarrow} \langle\mathcal{P}\rangle_0 \overset{|_{\mathcal{X}_\eta}}{\rightarrow} \langle\mathcal{P}|_{\mathcal{X}_\eta}\rangle_0$$

Assume furthermore $S_{\mathcal{P}}^{\mathrm{geo}} = \emptyset$ that is, for every $s \in S$, the cospecialization morphism $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) \to G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is an isomorphism so that the morphism $G(\langle\mathcal{P}|_{\mathcal{X}_s}\rangle_0) \hookrightarrow G(\langle\mathcal{P}\rangle_0)$ is a closed immersion. Then every $\otimes$-generator $0_*\mathcal{L}$ of $\langle\mathcal{P}\rangle_0$ yields a $\otimes$-generator $(0_*\mathcal{L})|_{\mathcal{X}_s} \simeq 0_{s*}s^*\mathcal{L}$ of $\langle\mathcal{P}|_{\mathcal{X}_s}\rangle_0$ and from the canonical diagram of Tannakian categories

$$\begin{tikzcd}
\langle\mathcal{L}\rangle \arrow[r, hook] \arrow[d, "0_*", "\simeq"'] \arrow[rrr, bend left] & \mathrm{Perv}^{\mathrm{ULA}}(S/S) \arrow[r, "s^*"] \arrow[d, "0_*", "\simeq"'] & \mathrm{Perv}(s) \arrow[d, "0_{s*}", "\simeq"'] & \langle s^*\mathcal{L}\rangle \arrow[l, hook'] \arrow[d, "0_{s*}", "\simeq"'] \\
\langle 0_*\mathcal{L}\rangle \arrow[r, hook] \arrow[rrr, bend right] & P^{\mathrm{ULA}}(\mathcal{X}/S)_0 \arrow[r, "|_{\mathcal{X}_s}"] & P(\mathcal{X}_s)_0 & \langle 0_{s*}s^*\mathcal{L}\rangle \simeq \langle(0_*\mathcal{L})|_{\mathcal{X}_s}\rangle \arrow[l, hook']
\end{tikzcd}$$

the morphism $G(\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0) \hookrightarrow G(\langle \mathcal{P}\rangle_0)$ also describes the functor of Tannakian categories $s^* : \langle \mathcal{L}\rangle \to \langle s^*\mathcal{L}\rangle$ hence corresponds to the embedding

$$G(\mathcal{L})_s \hookrightarrow G(\mathcal{L}) \hookrightarrow \mathrm{GL}(\mathcal{L}_{\bar{s}})$$

of the Zariski-closures of the images

$$\Pi(\mathcal{L})_s \subset \Pi(\mathcal{L}) \subset \mathrm{GL}(\mathcal{L}_{\bar{s}})$$

of $\pi_1(s, \bar{s}) \to \pi_1(S, \bar{s})$ acting on $\mathcal{L}_{\bar{s}}$ respectively.

This observation yields the following.

Lemma 4.11. *Assume S has dimension > 0. Let $\mathcal{P} \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $S^{\mathrm{geo}}_{\mathcal{P}} = \emptyset$. Then,*

(1) if k is Hilbertian, there exists an integer $d \geq 1$ such that $|S|^{\leq d} \setminus S_{\mathcal{P}} \cap |S|^{\leq d}$ is infinite.

(2) if S is a curve, k is finitely generated over $\mathbb{Q}$ and $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$ is semisimple, for every integer $d \geq 1$, $S^{\circ}_{\mathcal{P}} \cap |S|^{\leq d}$ is finite.

Proof. From the exact specialization diagram (6) and the fact that, by our assumptions, for every $s \in |S|$ the morphisms $G(\mathcal{P}|_{\mathcal{X}_{\bar{s}}}) \to G(\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}})$ is an isomorphism and the morphism $G(\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0) \to G(\langle \mathcal{P}\rangle_0)$ is a closed immersion, it is enough to prove that, under the assumptions

- in (1): there exists an integer $d \geq 1$ such that for infinitely many $s \in |S|^{\leq d}$ the closed immersion $G(\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0) \hookrightarrow G(\langle \mathcal{P}\rangle_0)$ is an isomorphism.

  This follows from the defining property of Hilbertian fields and a Frattini argument [Se89, §10.6], which ensures that there exists an integer $d \geq 1$ such that for infinitely many $s \in |S|^{\leq d}$ one has $\Pi(\mathcal{L})_s = \Pi(\mathcal{L})$.

- in (2): for every integer $d \geq 1$ and for all but finitely many $s \in |S|^{\leq d}$ the closed immersion $G(\langle \mathcal{P}|_{\mathcal{X}_s}\rangle_0)^{\circ} \hookrightarrow G(\langle \mathcal{P}\rangle_0)^{\circ}$ is an isomorphism.

  This follows from [CT13, Thm. 1], which asserts that if $\rho : \pi_1(S) \to \mathrm{GL}_N(\mathbb{Z}_\ell)$ is a continuous GLP representation then, for every integer $d \geq 1$ and all but finitely many $s \in |S|^{\leq d}$, $\rho(\pi_1(s)) \subset \rho(\pi_1(S))$ is open. The GLP condition means that every open subgroup of $\Pi := \rho(\pi_1(S_{\bar{k}}))$ has finite abelianization or, equivalently, that the Lie algebra $\mathrm{Lie}(\Pi)$ of Π (as an ℓ-adic Lie group) is perfect. This is for instance the case if (*) one can realize Π as a closed subgroup $\Pi \subset H_0(\mathbb{Q}_\ell)$ of the group of $\mathbb{Q}_\ell$-points of an algebraic group H_0 over $\mathbb{Q}_\ell$ such that the Zariski-closure of Π in H_0 is semisimple. The assumption that $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$ is semisimple ensures that one can reduce to this situation. Indeed, as $G(\mathcal{L}|_{S_{\bar{k}}})$ is a quotient of $G(\mathcal{P}|_{\mathcal{X}_{\bar{k}}})$, $G(\mathcal{L}|_{S_{\bar{k}}})$ is semisimple as well and, as there exists a finite Galois extension Q_ℓ of $\mathbb{Q}_\ell$ such that $\mathcal{L}$ arises from a Q_ℓ-local system on S, one may assume $\Pi(\mathcal{L}) \subset \mathrm{GL}(\mathcal{L}_{\bar{s}}) \simeq \mathrm{GL}_r(Q_\ell)$. But as $\mathrm{GL}_r(Q_\ell)$ has a natural structure of Lie group over $\mathbb{Q}_\ell$, so has $\Pi(\mathcal{L})$ [Se65, L.G., Chap. V, §9] hence, as $\Pi(\mathcal{L})$ is also compact being the continuous image of a profinite group, it admits a faithful embedding into $\mathrm{GL}_N(\mathbb{Z}_\ell)$ for some $N \geq 1$ [Lu88, Prop. 4]. To apply [CT13, Thm. 1] to the resulting ℓ-adic representation $\pi_1(S) \twoheadrightarrow \Pi(\mathcal{L}) \subset \mathrm{GL}_N(\mathbb{Z}_\ell)$, it is thus enough to show that $\Pi := \Pi(\mathcal{L}|_{S_{\bar{k}}})$ satisfies the criterion (*). This follows from the claim below, applied with $K/k = Q_\ell/\mathbb{Q}_\ell$, $\Pi := \Pi(\mathcal{L}|_{S_{\bar{k}}})$ and $G_0 := \mathrm{GL}_{r,\mathbb{Q}_\ell}$.

Claim. *Let K/k be a finite Galois extension and write $R := \mathrm{Res}_{K|k} : Sch/_K \to Sch/_k$ for the Weil restriction functor. Let G_0 be an algebraic group over k and set $G := G_{0,K}$. Let $\Pi \subset G(K) = (RG)(k)$ be a subgroup. Let $\iota : H \hookrightarrow G$ denote the Zariski closure of Π in G and $\iota_0 : H_0 \hookrightarrow RG$ the Zariski-closure of Π in RG. Write $ad : G_0 \hookrightarrow RG$ for the adjunction morphism. Then the morphism $c : H \overset{\iota}{\hookrightarrow} G \overset{ad_K}{\hookrightarrow} (RG)_K$ factors through an isomorphism*

$$\begin{tikzcd}
H \arrow[r, hook, "\iota"] \arrow[drr, dotted, "\simeq", "c"'] & G \arrow[r, hook, "ad_K"] & (RG)_K \\
 & & H_{0,K}. \arrow[u, hook, "\iota_{0,K}"']
\end{tikzcd}$$

Proof of the claim. At the level of K-points, the diagram

$$\begin{tikzcd}
H \arrow[r, hook, "\iota"] & G \arrow[r, hook, "ad_K"] & (RG)_K & H_{0,K} \arrow[l, hook', "\iota_{0,K}"']
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
H(K) \arrow[r, hook] & G(K) \arrow[r, hook] \arrow[dr, equal] & (RG)_K(K) & H_{0,K}(K) \arrow[l, hook'] \\
 & \Pi \arrow[ul, hook] \arrow[rr, hook, bend right] & RG(k) \arrow[u, hook] & H_0(k). \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

As Π is Zariski-dense in H, this already shows the existence of the factorization $c : H \hookrightarrow H_{0,K}$. On the other hand, at the level of k-points the diagram

$$\begin{tikzcd}
RH \arrow[r, hook, "R\iota"] & RG & H_0 \arrow[l, hook', "\iota_0"']
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
RH(k) = H(K) \arrow[r, hook] & (RG)(k) = G(K) \\
\Pi \arrow[u, hook] \arrow[r, hook] & H_0(k) \arrow[u, hook]
\end{tikzcd}$$

As Π is Zariski-dense in H_0, this shows that $H_0 \overset{\iota_0}{\hookrightarrow} RG$ factors as $\iota_0 : H_0 \overset{d_0}{\hookrightarrow} RH \overset{R\iota}{\hookrightarrow} RG$. One thus gets

$$\begin{tikzcd}
RH \arrow[r, hook, "R\iota"] \arrow[rrd, bend right, hook, "Rc"'] & RG \arrow[r, hook, "R(ad_K)"] & R((RG)_K) \\
& H_0 \arrow[ul, hook, "d_0"] \arrow[u, hook, "\iota_0"'] \arrow[r, hook, "ad"] & R(H_{0,K}) \arrow[u, hook, "R(\iota_{0,K})"']
\end{tikzcd}$$

Let $d : H_{0,K} \to H$ denote the morphism corresponding by functoriality, to $d_0 : H_0 \hookrightarrow RH$. Then, by construction, $c \circ d = Id : H_{0,K} \tilde{\to} H_{0,K}$ and $d \circ c = Id : H \tilde{\to} H$. $\square$

Proposition 1.7 (and its strengthening when S is a curve and k is finitely generated over $\mathbb{Q}$) follows from Lemma 4.11 applied to the restriction of $\mathcal{P}$ to $\mathcal{X} \times_S U$, where $U \subset S$ denotes the complement of the Zariski-closure of $S_{\mathcal{P}}^{\mathrm{geo}}$ in S.

5. Geometric applications

Let $\mathcal{X} \to S$ be an abelian scheme and let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed subscheme, smooth and geometrically connected over S.

5.1. **Preliminaries.** As S is smooth, $\mathcal{X} \to S$ is projective [R70, Thm. XI.1.4] hence $\mathcal{X}$ carries a line bundle $\mathcal{O}_{\mathcal{X}}(1)$ which is very ample with respect to $\mathcal{X} \to S$. Let $P \in \mathbb{Q}[T]$ denote the Hilbert polynomial of $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ with respect to $\mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{X}_{\bar{\eta}}}$ and let $\mathfrak{H}^P_{\mathcal{X}/S} \to S$ be the Hilbert scheme classifying closed subschemes of $\mathcal{X} \times_S T$ which are flat over T and with constant Hilbert polynomial P [Gro61, Thm. 3.2]. By construction, $\mathcal{X}$ acts by translation on $\mathfrak{H}^P_{\mathcal{X}/S}$ over S. Let $[\mathcal{Y}] \in \mathfrak{H}^P_{\mathcal{X}/S}(S)$ be the S-point corresponding to $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ and consider the corresponding morphism of S-schemes

$$\phi_{[\mathcal{Y}]} : \mathcal{X} \to \mathfrak{H}^P_{\mathcal{X}/S} \times_S \mathfrak{H}^P_{\mathcal{X}/S}, \ \ x \mapsto ([\mathcal{Y}], [\mathcal{Y} + x]).$$

Let also

$$\Delta : \mathfrak{H}^P_{\mathcal{X}/S} \hookrightarrow \mathfrak{H}^P_{\mathcal{X}/S} \times_S \mathfrak{H}^P_{\mathcal{X}/S}$$

denote the diagonal embedding, which is a closed immersion as $\mathfrak{H}^P_{\mathcal{X}/S} \to S$ is projective. Define the stabilizer $\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y})$ of $\mathcal{Y}$ in $\mathcal{X}$ as the fiber product

$$\begin{tikzcd}
\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\phi_{[\mathcal{Y}]}"] \\
\mathfrak{H}^P_{\mathcal{X}/S} \arrow[r, hook, "\Delta"'] & \mathfrak{H}^P_{\mathcal{X}/S} \times_S \mathfrak{H}^P_{\mathcal{X}/S}.
\end{tikzcd}$$

By construction $\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y}) \hookrightarrow \mathcal{X}$ is a closed subgroup scheme of $\mathcal{X} \to S$, whose formation commutes with arbitrary Noetherian base change $T \to S$. In particular, for every $t \in S$ one has

$$\mathrm{Stab}_{\mathcal{X}/S}(\mathcal{Y})_{\bar t} = \mathrm{Stab}_{\mathcal{X}_{\bar t}}(\mathcal{Y}_{\bar t}). \tag{21}$$

Lemma 5.1. *Let $\mathcal{X} \to S$ an abelian scheme and $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth, geometrically connected, and of relative dimension d over S. Then,*

(1) the relative perverse sheaf $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_{\ell,Y}[d]$ lies in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$;

(2) Assume furthermore $\mathcal{Y}_{\bar\eta} \hookrightarrow \mathcal{X}_{\bar\eta}$ has

  i) ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar\eta}/\mathcal{X}_{\bar\eta}}$ then, after possibly replacing S by a non-empty open subset, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar s} \hookrightarrow \mathcal{X}_{\bar s}$ also has ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar s}/\mathcal{X}_{\bar s}}$

  ii) trivial stabilizer $\mathrm{Stab}_{\mathcal{X}_{\bar\eta}}(\mathcal{Y}_{\bar\eta})$ then, after possibly replacing S by a non-empty open subset, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar s} \hookrightarrow \mathcal{X}_{\bar s}$ also has trivial stabilizer.

Proof. For (1), as $\mathcal{Y} \to S$ is smooth, $\overline{\mathbb{Q}}_{\ell,\mathcal{Y}} \in D^{\mathrm{ULA}}(\mathcal{Y}/S)$ [B24, 3.6, Lemma (i)] and as $\mathcal{Y} \hookrightarrow \mathcal{X}$ is proper, $\iota_* \overline{\mathbb{Q}}_{\ell,\mathcal{Y}} \in D^{\mathrm{ULA}}(\mathcal{X}/S)$ [B24, 3.6, Lemma (ii)] hence $\mathcal{P} := \iota_* \overline{\mathbb{Q}}_{\ell,\mathcal{Y}}[d] \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$. Assertion (2) ii) follows from (21) and [EGAIV$_3$, Thm. 8.10.5. (i)]. For assertion (2) i), under our assumptions for every $t \in S$ and with the notation in the base-change diagram

$$\begin{tikzcd}
\mathcal{Y}_{\bar t} \arrow[r, hook, "\iota_{\bar t}"] \arrow[d] \arrow[dd, bend right=60, "\iota_{\mathcal{Y}_{\bar t}}"'] \arrow[rd, phantom, "\square"] & \mathcal{X}_{\bar t} \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & \mathrm{spec}(k(\bar t))\ , \arrow[d] \\
\mathcal{Y}_t \arrow[r, hook, "\iota_t"] \arrow[d, hook, "\iota_{\mathcal{Y}_t}"'] \arrow[rd, phantom, "\square"] & \mathcal{X}_t \arrow[r] \arrow[d, hook, "\iota_{\mathcal{X}_t}"] \arrow[rd, phantom, "\square"] & \mathrm{spec}(k(t)) \arrow[d] \\
\mathcal{Y} \arrow[r, hook, "\iota"] & \mathcal{X} \arrow[r] & S
\end{tikzcd}$$

the canonical morphisms

$$\iota_{\mathcal{Y}_t}^* \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \to \mathcal{N}_{\mathcal{Y}_t/\mathcal{X}_t}, \quad \iota_{\mathcal{Y}_{\bar t}}^* \mathcal{N}_{\mathcal{Y}/\mathcal{X}} \to \mathcal{N}_{\mathcal{Y}_{\bar t}/\mathcal{X}_{\bar t}} \tag{22}$$

are isomorphisms. Indeed, applying $\iota_{\mathcal{Y}_t}^*$ to the short exact sequence of locally free $\mathcal{O}_{\mathcal{Y}}$-modules [Li02, §6.3, Prop. 3.13]

$$0 \to \mathcal{C}_{\mathcal{Y}/\mathcal{X}} \to \iota^* \Omega^1_{\mathcal{X}|S} \to \Omega^1_{\mathcal{Y}|S} \to 0$$

and using the canonical identifications

$$\iota_{\mathcal{Y}_t}^* \Omega^1_{\mathcal{Y}|S} \simeq \Omega^1_{\mathcal{Y}_t|k(t)}, \quad \iota_{\mathcal{Y}_t}^* \iota^* \Omega^1_{\mathcal{X}|S} \simeq \iota_t^* \iota_{\mathcal{X}_t}^* \Omega^1_{\mathcal{X}|S} \simeq \iota_t^* \Omega^1_{\mathcal{X}_t|k(t)},$$

one gets the short exact sequence of locally free $\mathcal{O}_{\mathcal{Y}_t}$-modules

$$0 \to \iota_{\mathcal{Y}_t}^* \mathcal{C}_{\mathcal{Y}/\mathcal{X}} \to \iota_t^* \Omega^1_{\mathcal{X}_t|k(t)} \to \Omega^1_{\mathcal{Y}_t|k(t)} \to 0,$$

which yields $\iota_{\mathcal{Y}_t}^*\mathcal{C}_{\mathcal{Y}/\mathcal{X}} \simeq \mathcal{C}_{\mathcal{Y}_t/\mathcal{X}_t}$, whence the assertion, by dualizing. On the other hand, as $\mathcal{N}_{\mathcal{Y}_{\bar{\eta}}/\mathcal{X}_{\bar{\eta}}}(\simeq (\mathcal{N}_{\mathcal{Y}/\mathcal{X}})|_{\mathcal{Y}_{\bar{\eta}}})$ is ample, by fpqc descent of ampleness, $(\mathcal{N}_{\mathcal{Y}/\mathcal{X}})|_{\mathcal{Y}_\eta}$ is ample [St25, Tag 0D2P]; the assertion thus follows from [Ha66, Prop. 4.4]. □

5.2. **Sample of rigidity phenomena.** We give here two examples of rigidity phenomena, building on the classification results of [JKrLM25] and Corollary 1.5 (2).

For an abelian variety X over a field K of characteristic 0 and a closed subvariety $Y \subset X$, smooth, geometrically connected and of dimension $d \geq 2$ over K, one says that Y is:

- a product if there exist closed subvarieties $Y_1, Y_2 \subset X$, smooth over K and of dimension > 0, such that the sum map $+ : Y_1 \times Y_2 \to X$ induces an isomorphism $+ : Y_1 \times Y_2 \tilde{\to} Y$;
- a symmetric power of a curve if there is a closed smooth irreducible curve $C \hookrightarrow X$ such that the sum morphism $\mathrm{Sym}^d C \to X$ is a closed embedding with image Y.

Note that if K is algebraically closed and L/K is a field extension, then Y is a product (resp. a symmetric power of a curve) if and only if $Y \times_K L$ is. The only if assertion is straightforward and the if one follows from spreading out and specialization, using Hilbert Nullstellensatz.

Corollary 5.2. *Let $\mathcal{X} \to S$ an abelian scheme of relative dimension $g \geq 3$ and $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth and geometrically connected over S. Assume $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ has ample normal bundle and trivial stabilizer. Then the set of all $s \in S$ such that $\mathcal{Y}_{\bar{s}}$ is a product is Zariski-dense in S (if and) only if $\mathcal{Y}_{\bar{\eta}}$ is itself a product.*

Proof. From Lemma 5.1 (1), $\mathcal{P} := \iota_*\overline{\mathbb{Q}}_{\ell,\mathcal{Y}}[d] \in \mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ and from Lemma 5.1 (2), up to replacing S by a non-empty open subscheme, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar{s}} \hookrightarrow \mathcal{X}_{\bar{s}}$ has ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar{s}}/\mathcal{X}_{\bar{s}}}$ and trivial stabilizer. The if assertion is by spreading out. To prove the only if assertion, observe first that, as $\mathcal{Y}_{\bar{\eta}}$ is smooth and irreducible $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ is simple - hence semisimple [BeBerDG82, Thm. 4.3.1 (ii)] so that Corollary 1.5 (2) (b) applies. The assertion thus follows from [JKrLM25, Thm. 6.1], which asserts that, for any $t \in S$, $\mathcal{Y}_{\bar{t}}$ is a product if and only if $G(\mathcal{P}|_{\mathcal{Y}_{\bar{t}}})^{\circ\,\mathrm{der}}$ is not simple. □

Remark 5.3. For $\mathcal{Y} = \mathcal{X} \to S$, in general, it is not true that the set of all $s \in S$ such that $\mathcal{X}_{\bar{s}}$ is a product is Zariski-dense in S (if and) only if $\mathcal{X}_{\bar{\eta}}$ is itself a product. For instance, let $k = \mathbb{C}$, $S = M_2$ the moduli space of genus 2 smooth projective curves (with suitable level structures) and $\mathcal{X} := \mathrm{Jac}(\mathcal{C}|S) \to S$ the Jacobian of the universal genus 2 curve $\mathcal{C} \to S$. Then the set of all points $s \in S$ such that $\mathcal{X}_{\bar{s}}$ is a product of two elliptic curves[6] is supported on infinitely many irreducible curves $C_d \hookrightarrow S$. Let $S^{\mathrm{prod}} \subset S$ denote the Zariski-closure of the union of all C_d. Then the geometric generic fiber $\mathcal{X}_{\bar{\xi}}$ over the generic point ξ of an irreducible component of S^{prod} of dimension ≥ 2 is not a product of two elliptic curves. See [K16] and the references therein for details.

Corollary 5.4. *Let $\mathcal{X} \to S$ an abelian scheme of relative dimension g and $\iota : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed subscheme, smooth, geometrically connected and of relative dimension $d < \frac{g-1}{2}$ over S. Assume $\mathcal{Y}_{\bar{\eta}} \hookrightarrow \mathcal{X}_{\bar{\eta}}$ has ample normal bundle and trivial stabilizer. Then the set of all $s \in S$ such that $\mathcal{Y}_{\bar{s}}$ is a symmetric power of a curve is Zariski-dense in S (if and) only if $\mathcal{Y}_{\bar{\eta}}$ is itself a symmetric power of a curve.*

Proof. The argument is similar to the one for Corollary 5.2. If $d = 1$ there is nothing to prove so that we may assume $d \geq 2$. Again, $\mathcal{P} := \iota_*\overline{\mathbb{Q}}_{\ell,\mathcal{Y}}[d]$ lies in $\mathrm{Perv}^{\mathrm{ULA}}(\mathcal{X}/S)$ with $\mathcal{P}|_{\mathcal{X}_{\bar{\eta}}}$ simple - hence semisimple, and, up to replacing S by a non-empty open subscheme, one may assume that for all $s \in S$, $\mathcal{Y}_{\bar{s}} \hookrightarrow \mathcal{X}_{\bar{s}}$ has ample normal bundle $\mathcal{N}_{\mathcal{Y}_{\bar{s}}/\mathcal{X}_{\bar{s}}}$ and trivial stabilizer. The if assertion is by spreading out. To prove the only if assertion, let $r := \chi(\mathcal{X}_{\bar{t}}, \mathcal{P}) = (-1)^d\chi(\mathcal{Y}_{\bar{t}})$ denote the Euler-Poincaré characteristic of $\mathcal{P}|_{\mathcal{X}_{\bar{t}}}$ for one (equivalently every) $t \in S$. Assume that for some $s \in S$, $\mathcal{Y}_{\bar{s}} \simeq \mathrm{Sym}^d(C)$ is a symmetric power of a curve. Then from [JKrLM25, Lem. 7.2], $G(\mathcal{P}|_{\mathcal{Y}_{\bar{s}}})^{\circ\,\mathrm{der}}$ acting on $\omega(\mathcal{P}|_{\mathcal{X}_{\bar{s}}})$ identifies with the image of $\mathrm{SL}_{n\overline{\mathbb{Q}}_\ell}$ acting on a wedge power $\wedge^d\mathrm{Std}_n$ of the standard representation of $\mathrm{SL}_{n\overline{\mathbb{Q}}_\ell}$ with $n = -\chi(C, \overline{\mathbb{Q}}_\ell) = 2g_C - 2$ (where g_C denotes the genus of C). Note that, as $r = \binom{n}{d}$, $2g_C - 2 = n =: n(r,d) =$ is uniquely determined by r and d hence is independent of $s \in S$. Furthermore, as $W_d(C) \subset \mathrm{Alb}(C)$ is then automatically smooth, it follows from Riemann's singularity theorem (*e.g.* [GrH78, p. 344]) that C has gonality $\geq d+1$ hence genus $g_C \geq 2d-1$. As $d \geq 2$, this imposes $g_C \geq 3$ hence $n(r,d) \geq 4$, whence, $2 \leq d \leq \frac{n(r,d)}{2}$. Under this numerical condition, it follows from [JKrLM25, Thm. 7.3] that, for any $t \in S$, $\mathcal{Y}_{\bar{t}} \simeq \mathrm{Sym}^d(C)$ is a symmetric power of a curve if and only if $G(\mathcal{P}|_{\mathcal{Y}_{\bar{t}}})^{\circ\,\mathrm{der}}$ acting on $\omega(\mathcal{P}|_{\mathcal{X}_{\bar{t}}})$ identifies with the image of $\mathrm{SL}_{n\overline{\mathbb{Q}}_\ell}$ acting on a wedge power $\wedge^d\mathrm{Std}_{n(r,d)}$ of the standard representation of $\mathrm{SL}_{n(r,d)\overline{\mathbb{Q}}_\ell}$. The assertion thus follows, again, from Corollary 1.5 (2) (b). ☐

Remark 5.5. Using [KrM25, Thm. 6.1] instead of [JKrLM25, Thm. 7.3], one could probably relax the assumption that $\mathcal{Y} \to S$ is smooth.

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