Special subvarieties of non-arithmetic ball quotients
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Arithmetic of Shimura varieties over global fields

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Lattices

Let G be a semisimple real group without compact factors and $X = G/K_\infty$ the associated symmetric space. A discrete subgroup Γ of G is a *lattice* if $\Gamma\backslash G$ has finite G-invariant measure. Let $S := \Gamma\backslash X$.

- Lattices are Zariski dense (Borel);
- They contain a finite index subgroup which is torsion free (Selberg);
- If Γ is discrete and $\Gamma\backslash G$ is compact, then Γ is a lattice.

Arithmetic subgroups

- A subgroup $\Gamma \subset G$ is *arithmetic* if there exists a semisimple linear algebraic group $\mathbf{G}/\mathbb{Q}$ and a surjective morphism with compact kernel $p : \mathbf{G}(\mathbb{R}) \to G$ such that Γ lies in the commensurability class of $p(\mathbf{G}(\mathbb{Z}))$;
- Arithmetic subgroups are lattices;
- **Example** A subgroup Γ of $G = \mathrm{SL}_2(\mathbb{R})$ is arithmetic iff there exists a totally real number field F, a quaternion algebra $\mathbb{B}$ over F which is split at one archimedean place and non split at the others such that Γ is commensurable to $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{B}^{*,1}(\mathbb{Z})$ (where $\mathbb{B}^{*,1}$ denotes the units of reduced norm 1 in $\mathbb{B}$).

Where can we find irreducible non-arithmetic lattices? After Margulis (1974)

- (Margulis) A lattice $\Gamma \subset G$ is arithmetic if and only if it has infinite index in its commensurator $\mathrm{Comm}(\Gamma)$:

$$\{g \in G : \Gamma_g := \Gamma \cap g\Gamma g^{-1} \text{ has finite index in both } \Gamma \text{ and } g\Gamma g^{-1}\}$$

- Any $g \in \mathrm{Comm}(\Gamma)$ defines a Hecke correspondence $\Gamma_g \backslash X \subset \Gamma\backslash X \times \Gamma\backslash X$. There is a link between arithmeticity of Γ and special subvarieties of $\Gamma\backslash X \times \Gamma\backslash X$.
- Non-arithmetic lattices can only exist in real rank one (Margulis).
- Actually only in $\mathrm{SO}(1,n)$ and $\mathrm{PU}(1,n)$ (Margulis, Corlette, Gromov-Schoen).

Associated symmetric space: $X_{\mathrm{SO}(1,n)}$ is the *real* ball, $X_{\mathrm{PU}(1,n)}$ is the *complex* ball. So $\Gamma\backslash X$ can be a complex algebraic variety only in the latter case...

Examples of non-arithmetic lattices

- $\mathrm{SO}(1,n)$. There are non-arithmetic lattices for any $n \geq 2$ (Gromov–Piatetski-Shapiro);
- $\mathrm{PU}(1,n)$. Known examples only for $n = 1,2,3$ by the work of Deligne, Mostow, Deraux, Parker and Paupert. For $n = 2$ there are 22 commensurability known classes. For $n = 3$ only 2 (and they are not cocompact). Most of them are related to the monodromy of hyper-geometric functions...

For each $n > 1$ how many non-arithmetic lattices in $\mathrm{PU}(1,n)$ are there? How can arithmeticity be detected?

Motivation: generalising the theory of Shimura varieties

- Γ a (torsion free) lattice in $G = \mathrm{PU}(1, n)$, $n > 1$;
- $X = \mathbb{B}^n$ the Hermitian space associated to G;
- S_Γ be the n-dimensional ball quotient $\Gamma\backslash X$. It has a unique structure of complex algebraic variety, with nice compactifications (Baily-Borel, Mumford, Mok);

By looking at special correspondences in $S_\Gamma \times S_\Gamma$, we can detect arithmeticity. Can we do it in S_Γ?
What is a *special subvariety* of S_Γ?

Main result

Theorem (Baldi–U.)

If S_Γ contains infinitely many <u>maximal</u> complex totally geodesic subvarieties, then $\Gamma \subset G$ is arithmetic.

Bader, Fisher, Miller and Stover proved the real and complex hyperbolic version of the theorem using some superrigidity theorems and results on equidistribution from homogeneous dynamics.
A similar strategy applied to $S_\Gamma \times S_\Gamma$ gives.

Theorem (Baldi–U.)

New proof of Margulis commensurator theorem for lattices in $\mathrm{PU}(1, n)$ (for $n > 1$).

Main steps of the strategy

Goal: parametrise maximal special subvarieties by a countable and definable set (in some o-minimal structure)!

1. Realise S_Γ inside a Shimura variety/a period domain for some $\mathbb{Z}$-VHS (that we construct);
2. Totally geodesic subvarieties = unlikely intersection;
3. Prove the geometric part of (Klingler's generalised) Zilber–Pink.

Main tools:

1. Simpson's theory;
2. Monodromy/Mumford–Tate computations (André-Deligne monodromy's theorem);
3. Ax-Schanuel for $\mathbb{Z}$-VHS of Bakker-Tsimerman.

Constructing a $\mathbb{Z}$-Variation of Hodge structures

Theorem (Baldi–U.)

Every element in Γ has trace in the ring of integers of a totally real number field K. Up to conjugation by some $g \in G$, Γ lies in $\mathbf{G}(\mathcal{O}_K)$. Moreover the natural $\mathbb{C}$-VHS $\mathbb{V}$ induces a $\mathbb{Z}$-variation of Hodge structures $\widehat{\mathbb{V}}$ on S_Γ.

Rigidities for lattices (after Calabi, Vesentini, Weil, Garland and Raghunathan):

- Infinitesimal rigidity: $H^1(\Gamma, \mathrm{Ad}) = 0$.

It follows $K := \mathbb{Q}\{\mathrm{tr}\,\mathrm{Ad}\,\gamma : \gamma \in \Gamma\}$ is a number field (rather than a f.g. field, not true for $n = 1$)! As Γ is Zariski dense $\rightsquigarrow \exists \mathbf{G}/K$, K-form of G and

$\Rightarrow$ $\Gamma \subset \mathbf{G}(K)$.

Simpson's theory and Weil restriction

Let S be a smooth quasi-projective variety. Starting point: every representation of $\pi_1(S)$ can be deformed to a $\mathbb{C}$-VHS.

Conjecture (Simpson)

Rigid semisimple representations of $\pi_1(S)$ with quasi-unipotent monodromy at infinity come from geometry.

- Recent progress of Esnault and Groechenig: "nice" *cohomologically* rigid representations with quasi-unipotent monodromy at ∞ admit an integral structure;
- Infinitesimal $\Rightarrow$ cohomologically rigidity;
- Explicit toroidal compactification of S_Γ shows that $\mathbb{V}$ has quasi-unipotent monodromy at ∞, so we can apply EG;
- Twists by $\sigma : K \to \mathbb{R}$ preserve infinitesimal rigidity $\Rightarrow \mathbb{V}^\sigma$ is a VHS;
- Eventually $\bigoplus_\sigma \mathbb{V}^\sigma$ has a natural structure of $\mathbb{Z}$-VHS.

Fundamental commutative diagram

- $\widehat{\mathbb{V}} := \bigoplus_i \mathbb{V}^{\sigma_i}$, $\sigma_1 = \mathrm{id}, \ldots, \sigma_r : K \to \mathbb{R}$;
- $\widehat{\mathbf{G}}$:= Weil restriction from K to $\mathbb{Q}$ of $\mathbf{G}$.

Griffiths theory of period domains and period maps gives a commutative diagram in the complex analytic category:

$$\begin{array}{ccc} X & \xhookrightarrow{\tilde{\psi}} & D = D_{\widehat{G}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ {S_\Gamma}^{\mathrm{an}} & \xrightarrow{\psi} & \widehat{\mathbf{G}}(\mathbb{Z})\backslash D \end{array}$$

It may happen that $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ is a Shimura variety. But in general $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ is not algebraic, and ψ is just an holomorphic map.

More about $D_{\widehat{G}}$ and $\tilde{\psi}$

- $D_{\widehat{G}}$ is a $\widehat{G} = \widehat{\mathbf{G}}(\mathbb{R}) = \prod G_{\sigma_i}$-orbit of one of the HSs constructed above, and the stabiliser is compact (in general not a maximal compact subgroup) ;
- All the G_{σ_i} are isomorphic over $\mathbb{C}$, so they are $\mathrm{PU}(p_{\sigma_i}, q_{\sigma_i})$, $p_{\sigma_i} + q_{\sigma_i} = n + 1$;
- We can write $D_{\widehat{G}} = X \times X'$ where X' is homogeneous under $\prod_{i>1}^{r} G_{\sigma_i}$;
- $\tilde{\psi}$ is holomorphic and Γ-equivariant:

$$\tilde{\psi}(\gamma.x) = (\gamma.x, \sigma_2(\gamma).x_{\sigma_2}, \ldots, \sigma_r(\gamma).x_{\sigma_r}),$$

  where x_{σ_i} is the fibre of $\mathbb{V}^{\sigma_i}$ at x;
- $\tilde{\psi}$ detects arithmeticity: Γ is arithmetic iff X' is a point, i.e G_{σ_i} is compact for any $i \geq 2$ (Mostow-Vinberg).

Γ-world vs $\mathbb{Z}$-world

Two ways for constructing *irreducible* algebraic subvarieties of S_Γ:

- Γ. Take a subgroup $H \subset G$, s.t. $\Gamma_H := \Gamma \cap H$ a lattice and $H.x \subset X$ a sub-Hermitian domain, then $W = \pi(H.x) \subset S_\Gamma$ is algebraic. We say that such a W is Γ-special;
- $\mathbb{Z}$. Take a $\mathbb{Q}$-subgroup $\mathbf{M} \subset \widehat{\mathbf{G}}$, which is the Mumford-Tate group of some element $\hat{x} \in D_{\widehat{G}}$, then $\psi^{-1}(\mathbf{M}(\mathbb{Z})\backslash\mathbf{M}(\mathbb{R}).\hat{x})$ is algebraic (Cattani-Deligne-Kaplan). Algebraic subvarieties (=Hodge locus) constructed in this way are called $\mathbb{Z}$-special.

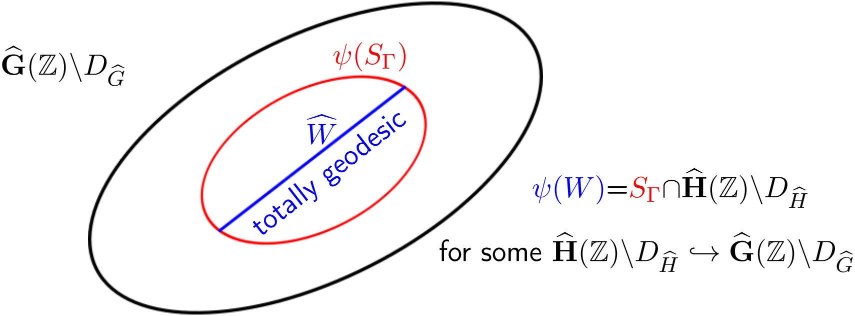
What makes a subvariety special?

Theorem (Baldi–U.)

Let $W \subset S_\Gamma$ be an irreducible algebraic subvariety. The following are equivalent:

1. *W is totally geodesic;*
2. *W is bi-algebraic: some (equivalently any) analytic component of the preimage of W along $\pi : X \to S_\Gamma$ is algebraic;*
3. *W is Γ-special;*
4. *W is $\mathbb{Z}$-special;*
5. *W is a component of $\psi^{-1}(\pi(Y))$ for some algebraic subvariety Y of $D^\vee$.*

A picture



Main idea: If Γ is not arithmetic, W is an unlikely intersection!

Ax-Schanuel for $\mathbb{Z} - VHS \implies$ (Geometric part of) Zilber-Pink $\implies$ the main theorem

Ax-Schanuel implies the geometric part of Zilber-Pink

- If $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is a Shimura variety: Daw-Ren proved that "Ax-Schanuel $\implies$ the geometric part of Zilber-Pink" generalising the proof that "Ax-Lindeman implies the geometric part of André-Oort".
- We adapted the proof of such a statement generalising the method for ball quotients.
- Recently with (Baldi and Klingler) we proved the geometric part of Zilber-Pink for general $\mathbb{Z} - VHS$.
- All these proofs uses functional transcendence results to parametrize the set of maximal "atypical intersections" by a countable definable set in some o-minimal theory.

Example

- $\Gamma \subset G = \mathrm{PU}(1,2)$ non arithmetic, with trace field K of degree 2 over $\mathbb{Q}$. So $\widehat{G} = G \times G$.
- $W \subset S_\Gamma$ special subvariety (associated to $\mathbf{H} \subset \mathbf{G}/K$);
- Suppose that $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is a Shimura variety;
- Write $W = S_\Gamma \cap \widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}}$;
- $\mathrm{codim}_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}} S_\Gamma = 2$;
- $\mathrm{codim}_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}} \widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}} = 2$;
- $\mathrm{codim}_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}} W = 3$.

Two objects of codimension 2 in a 4-dimensional space, should intersect in a finite number of points, not in a curve!...

Ax-Schanuel For Period domain

Denote by $D^\vee = D_{\widehat{G}}{}^\vee$ the compact dual of $D = D_{\widehat{G}}$.

Theorem (Hodge Ax-Schanuel) Bakker-Tsimerman

Let $\widehat{W} \subset D^\vee \times S_\Gamma$ be an algebraic subvariety. Let $\widehat{U}$ be an irreducible component of $\widehat{W} \cap D \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} S_\Gamma$ such that

$$\operatorname{codim} \widehat{U} < \operatorname{codim} \widehat{W} + \operatorname{codim} D \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} S_\Gamma,$$

the codimension being in $D^\vee \times S_\Gamma$. Then the projection of $\widehat{U}$ to S_Γ is contained in a strict weak Mumford–Tate subvariety of S_Γ.

Dimension counting- Atypical Intersection

- $D_{\widehat{G}} = X \times D_{\sigma_2} \times \cdots \times D_{\sigma_r}$
- Let $d_i := \dim D_{\sigma_i}$, $d_1 = \dim X$.
- Let $S' \simeq \Gamma_{\mathbf{H}} \backslash H.x_0 = \Gamma_H \backslash X_H \subset S_\Gamma$ be a totally geodesic subvariety.
- Let $\widehat{\mathbf{H}} = \mathrm{Res}_{K/\mathbb{Q}} \mathbf{H}$ and $D_{\widehat{H}} = \widehat{H}.\tilde{\psi}(x_0) = X_H \times D_{H_{\sigma_2}} \times \cdots \times D_{H_{\sigma_r}}$.
- Then $S' = \Gamma_H \backslash X_H = \psi^{-1}(\pi(D_{\widehat{H}} \cap \tilde{\psi}(X)))$.
- Let $d_{H_i} := \dim D_{H_{\sigma_i}}$, $d_{H_1} = \dim X_{H_1}$.
- Take $\widehat{W} = D^{\vee}_{\widehat{H}} \times S_\Gamma$. Then $\widehat{U} = \mathrm{graph}(\tilde{\psi}(X) \cap D_{\widehat{H}} \longrightarrow S_\Gamma)$.
- Then $\dim \widehat{W} = d_1 + \sum_{i=1}^r d_{H_i}$, $\dim \widehat{U} = d_{H_1}$
- $\mathrm{codim}\, \widehat{U} - \mathrm{codim}\, \widehat{W} - \mathrm{codim}\, D \times_{\widehat{\mathbf{G}}(\mathbb{Z}) \backslash D} S_\Gamma = \sum_{i=2}^r (d_{H_i} - d_i) < 0$ unless for all $i \geq 2$, $d_{H_i} = d_i = 0$. Which occures iff Γ is arithmetic.

Internal Ax-Schanuel

Theorem (Baldi–U.)

Let $W \subset X \times S_\Gamma$ be an algebraic subvariety and $\Pi \subset X \times S_\Gamma$ be the graph of $\pi : X \to S_\Gamma$. Let U be an irreducible component of $W \cap \Pi$ such that

$$\operatorname{codim} U < \operatorname{codim} W + \operatorname{codim} \Pi,$$

the codimension being in $X \times S_\Gamma$ or, equivalently,

$$\dim W < \dim U + \dim S_\Gamma.$$

If the projection of U to S_Γ is not zero dimensional, then it is contained in a strict totally geodesic subvariety of S_Γ.

Sketch of Proof of the Main theorem

We want to prove that maximal totally geodesic subvarieties are parametrised by a countable and definable set (in $\mathbb{R}_{\mathrm{an,exp}}$):

- Let $S' \subsetneq S_\Gamma$ be a special subvariety of S_Γ of maximal dimension;
- S' is associated to a K-subgroup $\mathbf{H} \subset \mathbf{G}$.
  $S' = \Gamma_{\mathbf{H}} \backslash H.x_0 = \psi^{-1}\pi(\tilde{\psi}(X) \cap \widehat{H}.\psi(x_0))$
- $\mathcal{F} \subset X$ definable fundamental domain for Γ. The set

  $$\Pi_0(\mathbf{H}) := \{(x, \hat{g}) \in \mathcal{F} \times \widehat{G} : \mathrm{Im}(\tilde{\psi}(x) : \mathbb{S} \to \widehat{G}) \subset \hat{g}\widehat{H}\hat{g}^{-1}\},$$

  is definable.
- Given $(x, \hat{g}) \in \Pi_0(\mathbf{H})$, when is

  $$S_{x,\hat{g}} := \psi^{-1}\pi(\tilde{\psi}(X) \cap \hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)) \subset {S_\Gamma}^{an}$$

  a special subvariety? By definition $S_{x_0,\hat{1}} = S'$ is special.

Sketch of Proof of the Main theorem

Consider the set

$$\Sigma = \{\hat{g}\widehat{H}\hat{g}^{-1} : (x,\hat{g}) \in \Pi_0(\mathbf{H}) \text{ for a } x \text{ and } \dim S_{x,\hat{g}} \geq \dim S' = \dim S_{x_0,\hat{1}}\};$$

- It is definable and we will deduce from "Hodge Ax-Schanuel" that it parametrises special subvariety of S_Γ (of dimension $\dim(S')$). We only have to prove that it is countable (then induction);
- Claim: each $\hat{g}\widehat{H}\hat{g}^{-1} \in \Sigma$ is a $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}}$;
- Set $\widehat{W} := \left(\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)\right) \times S_\Gamma$. It is algebraic in $D \times S_\Gamma$;
- Let $\widehat{U}$ be a component at $\tilde{\psi}(x)$ of the intersection

$$\widehat{W} \cap D \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} S_\Gamma,$$

such that the projection of $\widehat{U}$ to S_Γ contains $S_{x,\hat{g}}$

Sketch of Proof of the Main theorem

Proposition

If Γ is non-arithmetic, then $\widehat{U}$ is an atypical intersection. That is

$$\operatorname{codim}_{D\times S_\Gamma}\widehat{U} < \operatorname{codim}_{D\times S_\Gamma}\widehat{W} + \operatorname{codim}_{D\times S_\Gamma}\left(D \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} S_\Gamma\right).$$

- The proof is the dimensional computation we did, when you realise that $\widehat{U}$ for $S_{x,\hat{g}}$ is even more atypical that the analogue for the special subvariety $S_{x_o,\hat{1}}$ when you use the property

$$\dim S_{x,\hat{g}} \geq \dim S' = \dim S_{x_0,\hat{1}}.$$

Sketch of Proof of the Main theorem

- By "Hodge Ax-Schanuel", $S_{x,\hat{g}}$ is contained in a strict special subvariety. By maximality $S_{x,\hat{g}}$ is special.
- So the set Σ is definable (in $\mathbb{R}_{an,\exp}$) and countable and therefore finite.
- *End of the proof*: Up to $G(\mathbb{R})$-conjugacy class, you have only finitely many H to consider.
- Induction to obtain the finiteness of the maximal totally geodesics subvarieties of S_Γ, of maximal possible dimension which are not contained in the algebraic set consisting of totally geodesic subvarieties of maximal dimension.

THANKS FOR YOUR ATTENTION!