A proof of the Uniform Mordell–Lang Conjecture

(joint with Vesselin Dimitrov, Philipp Habegger; Tangli Ge, Lars Kühne)

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Part 0. Faltings's Theorem

Let $g \geq 0$ be an integer. Let C be an irreducible smooth projective curve of genus g, defined over a number field K.
In 1983, Faltings proved the Mordell Conjecture.

Theorem (Faltings 1983)

When $g \geq 2$, the set $C(K)$ is finite.

- Faltings's 1983 proof does not give a good upper bound on $\#C(K)$.
- The cardinality $\#C(K)$ must depend on g and $[K : \mathbb{Q}]$ (not hard to see by examples).

Part 0. In search of an upper bound on $\#C(K)$

Here is a very ambitious bound.

Question

Is it possible to find a number $B(g, [K : \mathbb{Q}]) > 0$ *such that*

$$\#C(K) \leq B?$$

This question has an affirmative answer if one assumes Lang's conjecture (Caporaso–Harris–Mazur, Pacelli).

- ✎ Two divergent opinions towards this conditional result: either this ambitious bound is true, or one could use this to disprove Lang's conjecture.

Part 0. Classical result on $\#C(K)$

In early 90s, Vojta gave a second proof to Faltings's Theorem. The proof was simplified and generalized by Faltings, and further simplified by Bombieri. This new proof (BFV) gives an upper bound, which was later on made explicit by de Diego, David–Philippon, and Rémond.

Theorem (Vojta, Faltings, Bombieri, de Diego, David–Philippon, Rémond)

$$\#C(K) \leq c(g, [K:\mathbb{Q}], h_{\mathrm{Fal}}(J))^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}$$

where $J =$Jacobian of C, and $h_{\mathrm{Fal}}(J)$ is the stable Faltings height of J.

Roughly speaking, the number $h_{\mathrm{Fal}}(J)$ measures the "complexity" of the coefficients of the equations defining the curve C.

Part 0. Bound on $\#C(K)$

Theorem (Dimitrov-G'-Habegger, 2021)

If $g \geq 2$, then

$$\#C(K) \leq c(g, [K:\mathbb{Q}])^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}$$

where J is the Jacobian of J. Moreover, $c(g, [K:\mathbb{Q}])$ grows at most polynomially in $[K:\mathbb{Q}]$.

- The proof is based on Vojta's proof of the Mordell Conjecture (Faltings's theorem).
- Compared with the classical bound (by Vojta, Faltings, Bombieri, de Diego, David–Philippon, Rémond), our bound does not depend on the height of J.
- This proves a conjecture of Mazur (1986, 2000).
- We showed that the dependence of c on $[K:\mathbb{Q}]$ can be removed assuming the Relative Bogomolov Conjecture. More recently, this is achieved unconditionally by Kühne.

Part 0. Previously known results on this bound

- By Diophantine method, based on BFV,
  - David–Philippon 2007: when $J \subset E^n$.
  - David–Nakamaye–Philippon 2007: for some particular families of curves.
  - Alpoge 2018: average number of $\#C(K)$ when $g = 2$.
- By the Chabauty–Coleman method,
  - Stoll 2015: hyperelliptic curves when $\mathrm{rk} J(K) \leq g - 3$.
  - Katz–Rabinoff–Zureick-Brown 2016: when $\mathrm{rk} J(K) \leq g - 3$.

Part 0. Uniform Mordell–Lang for curves

Our method allows to prove a more general result, replacing $J(K)$ by any finite rank subgroup $\Gamma \rightsquigarrow$ Uniform Mordell–Lang Conjecture for curves.

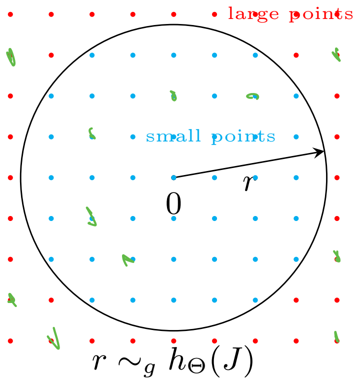
Theorem (Dimitrov-G'-Habegger)

Let $P_0 \in C(\overline{\mathbb{Q}})$ and $J =$Jacobian of C. Let $C - P_0$ be the image of the Abel–Jacobi embedding of C in J based at P_0. Let Γ be a finite rank subgroup of $J(\overline{\mathbb{Q}})$. If $h_{\mathrm{Fal}}(J) \geq \delta(g)$, then

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c(g)^{1+\mathrm{rk}\Gamma}.$$

- In particular, $\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c(g, [K : \mathbb{Q}])^{1+\mathrm{rk}\Gamma}$ if C is defined over a number field K and $P_0 \in C(K)$.
- Kühne recently proved the result for curves with $h_{\mathrm{Fal}}(J) < \delta(g)$.
- Torsion points: If we take $\Gamma = J(\overline{\mathbb{Q}})_{\mathrm{tor}}$, then this becomes the *uniform Manin-Mumford conjecture* (Kühne). Katz–Rabinoff–Zureick-Brown 2016: assuming some good reduction behavior. DeMarco–Krieger–Ye 2018: $g = 2$ bi-elliptic.

Part 0. Review of the BFV method



On $J(\overline{\mathbb{Q}})$, there is a function $\hat{h}_L \colon J(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ vanishing precisely on $J(\overline{\mathbb{Q}})_{\mathrm{tor}}$.

$\rightsquigarrow$ $\hat{h}_L \colon J(K) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}_{\geq 0}$.

$\rightsquigarrow$ "Normed Euclidean space" $(J(K) \otimes_{\mathbb{Z}} \mathbb{R}, \hat{h}_L)$, and $J(K)$ becomes a lattice in it.

Theorem (Bombieri, de Diego, Alpoge)

$\#$large points $\leq c(g)1.872^{\mathrm{rk}J(K)}$.

Up to replacing 1.872 by 7, this theorem is a consequence of Mumford's Gap Principle and Vojta's Inequality.

Part 0. A New Gap Principle

Theorem (New Gap Principle, Dimitrov–G'–Habegger + Kühne)

Each $P \in C(\overline{\mathbb{Q}})$ satisfies

$$\#\left\{Q \in C(\overline{\mathbb{Q}}) : \hat{h}_L(Q-P) \leq c_1 \max\{h_{\mathrm{Fal}}(J), 1\}\right\} \leq c_2$$

for some constants $c_1 > 0$ and $c_2 > 0$ depending only on g.

- This theorem says (roughly) that algebraic points in $C(\overline{\mathbb{Q}})$ are in general far from each other in a quantitative way. In particular, this holds true for rational points in C.

Part 0. Key new ingredients

- ✎ Key notion: non-degenerate subvarieties of any given abelian scheme (Habegger 2013). Two basic tools to define non-degeneracy are the Betti map (Corvaja, Masser, Zannier, Bertrand, André) and the Betti form (N.Mok).
- ✎ What are used in the proof?
  - ➢ Criterion of non-degeneracy and constructions (G' 2020),
  - ➢ A height inequality on any given non-degenerate subvariety (Dimitrov–G'–Habegger 2021),
  - ➢ An equidistribution result on any given non-degenerate subvariety (Kühne 2021) + Ullmo–Zhang approach.

Kühne's proof of the equidistribution result also uses the height inequality.

Part 1. Bi-algebraic geometry: algebraic tori

Classical transcendence:

Conjecture (Schanuel's Conjecture)

Assume $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent. Then

$$\mathrm{trdeg}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n.$$

In particular, $\mathrm{trdeg}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n) + \mathrm{trdeg}_{\mathbb{Q}}(e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n$.

- ➢ $n = 2$, $\alpha_1 = 1$, $\alpha_2 = \pi i \Rightarrow e$ and π are algebraically independent (open).
- ➢ $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$: Lindemann–Weierstraß Theorem.

Bi-algebraic system:

✎ $u \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $u = (\exp, \cdots, \exp)$.

- ➢ Both $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ are algebraic varieties;
- ➢ u is transcendental.

Part 1. Bi-algebraic geometry: algebraic tori

To state the functional analogue, it is more convenient to introduce the bi-algebraic subvarieties.

Proposition (-Definition)

Take an irreducible algebraic subvariety $W \subseteq \mathbb{C}^n$. Then $u(W)$ is algebraic if and only if $u(W)$ is a coset of an algebraic subtorus of $(\mathbb{C}^*)^n$.

Now we are ready to state the functional transcendence result.

Theorem (Ax, weak version)

Let $u = (\exp, \cdots, \exp) \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. Let Z be a complex analytic irreducible subvariety of $\mathbb{C}^n$. Assume Z is not contained in any proper bi-algebraic subvariety of $\mathbb{C}^n$. Then

$$\dim Z^{\mathrm{Zar}} + \dim u(Z)^{\mathrm{Zar}} \geq n + \dim Z.$$

Part 1. Bi-algebraic geometry: algebraic tori

An equivalent formulation is

Theorem (Ax, weak version)

Let $u = (\exp, \cdots, \exp) \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. Let Z be a complex analytic irreducible subvariety of $\mathbb{C}^n$. Then

$$\dim Z^{\mathrm{Zar}} + \dim u(Z)^{\mathrm{Zar}} \geq \dim Z^{\mathrm{biZar}} + \dim Z.$$

Here Z^{biZar} is the smallest bi-algebraic subvariety of $\mathbb{C}^n$ which contains Z.

Of course, $\dim Z^{\mathrm{biZar}} = \dim u(Z)^{\mathrm{biZar}}$ because $u(Z^{\mathrm{biZar}}) = u(Z)^{\mathrm{biZar}}$.

Bi-algebraic system:

- $u \colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $u = (\exp, \cdots, \exp)$.
  - Both $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ are algebraic varieties;
  - u is transcendental.

Part 1. Bi-algebraic geometry: abelian variety

Bi-algebraic system:

- ✎ $u\colon \mathbb{C}^g \to A$ with A a complex abelian variety.
  - ➢ Both $\mathbb{C}^g$ and A are algebraic varieties;
  - ➢ u is transcendental.

One can show that bi-algebraic subvarieties of A for this system are precisely cosets of A (= translate of an abelian subvariety by a closed point).

Theorem (Ax, weak version)

Consider $u\colon \mathbb{C}^g \to A$. Let Z be a complex analytic irreducible subvariety of $\mathbb{C}^g$. Then

$$\dim Z^{\mathrm{Zar}} + \dim u(Z)^{\mathrm{Zar}} \geq \dim u(Z)^{\mathrm{biZar}} + \dim Z.$$

Here $u(Z)^{\mathrm{biZar}}$ is the smallest bi-algebraic subvariety of A which contains $u(Z)$.

Part 1. Bi-algebraic geometry: hyperbolic case

$\mathbb{A}_g$ moduli space of pp abelian varieties, $\mathfrak{H}_g$ Siegel upper half space.
Bi-algebraic system:

- ✎ $u\colon \mathfrak{H}_g \to \mathbb{A}_g$ uniformizing map.
  - ➢ $\mathfrak{H}_g \subseteq \mathbb{C}^{g(g+1)/2}$ open semi-algebraic $\rightsquigarrow$ algebraic structure on $\mathfrak{H}_g$: $W \subseteq \mathfrak{H}_g$ is algebraic if W is a component of $W^\vee \cap \mathfrak{H}_g$ for $W^\vee \subseteq \mathbb{C}^{g(g+1)/2}$ algebraic.
  - ➢ $\mathbb{A}_g$ has a canonical structure of algebraic variety (Baily–Borel).
  - ➢ u is transcendental.

Bi-algebraic subvarieties of $\mathbb{A}_g$ are characterized by Ullmo–Yafaev, precisely the totally geodesic subvarieties.
Functional transcendence in this case: Mok–Pila–Tsimerman (Pila, Ullmo, Yafaev, Tsimerman, Klingler...)

Part 1. Bi-algebraic geometry: mixed case

What we need is for the universal abelian variety $\mathfrak{A}_g$ over $\mathbb{A}_g$ (mixed Shimura variety).

Bi-algebraic system:

- ✎ $u\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g$ uniformizing map.
  - ➢ Both $\mathbb{C}^g \times \mathfrak{H}_g$ and $\mathfrak{A}_g$ have natural algebraic structures;
  - ➢ u is transcendental.

What are the bi-algebraic varieties of $\mathfrak{A}_g$?

Part 1. Bi-algebraic geometry: mixed case

Let $Y \subseteq \mathfrak{A}_g$ be an irreducible subvariety.

$$
\begin{array}{ccccc}
\mathbb{C}^g \times \mathfrak{H}_g & \xrightarrow{u} & \mathfrak{A}_g & \supseteq \mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y) & \supseteq Y \\
\downarrow & & \downarrow \pi & \downarrow & \\
\mathfrak{H}_g & \longrightarrow & \mathbb{A}_g & \supseteq \pi(Y). &
\end{array}
$$

Then $\mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y) = \pi^{-1}(\pi(Y)) \to \pi(Y)$ itself is an abelian scheme of relative dimension g. Set $\mathcal{C}$ to be its isotrivial part, *i.e.* the largest isotrivial abelian subscheme $\rightsquigarrow$ constant sections.

One can apply this to Y^{biZar}.

$$
\begin{array}{ccccc}
\mathbb{C}^g \times \mathfrak{H}_g & \xrightarrow{u} & \mathfrak{A}_g & \supseteq \mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y)^{\mathrm{biZar}} & \supseteq Y^{\mathrm{biZar}} \\
\downarrow & & \downarrow \pi & \downarrow & \\
\mathfrak{H}_g & \longrightarrow & \mathbb{A}_g & \supseteq \pi(Y)^{\mathrm{biZar}}. &
\end{array}
$$

Part 1. Bi-algebraic geometry: mixed case

$$\begin{array}{ccccc}
\mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y) & \subseteq & \mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y)^{\mathrm{biZar}} & \supseteq & Y^{\mathrm{biZar}} \\
\downarrow & & \downarrow \pi & \swarrow & \\
\pi(Y) & \subseteq & \pi(Y)^{\mathrm{biZar}}. & &
\end{array}$$

Proposition (G')

Y^{biZar} is the translate of an abelian subscheme of $\mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y)^{\mathrm{biZar}} \to \pi(Y)^{\mathrm{biZar}}$ of a torsion section and then by a constant section.

Better, geometric meaning of $\dim Y^{\mathrm{biZar}} - \dim \pi(Y)^{\mathrm{biZar}}$.

Proposition (G')

$(\mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y)) \cap \pi(Y)^{\mathrm{biZar}}$ is the translate of an abelian subscheme of $\mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y) \to \pi(Y)$ of a torsion section and then by a constant section, smallest among such translates which contain Y.

Part 1. Bi-algebraic geometry: mixed case

Bi-algebraic system $u\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g$.

Theorem (weak Ax–Schanuel for $\mathfrak{A}_g$, G')

Let Z be a complex analytic irreducible subset of $\mathbb{C}^g \times \mathfrak{H}_g$. Then

$$\dim Z^{\mathrm{Zar}} + \dim u(Z)^{\mathrm{Zar}} \geq \dim Z + \dim u(Z)^{\mathrm{biZar}}.$$

Part 2. Application to Betti map and non-degeneracy

Bi-algebraic system $u\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g$.

- Betti map:
  - Identification $\mathbb{R}^{2g} \times \mathfrak{H}_g \xrightarrow{\sim} \mathbb{C}^g \times \mathfrak{H}_g$, $(a, b, Z) \mapsto (a + Zb, Z)$.

Definition (Corvaja, Masser, Zannier, Bertrand, André)

The Betti map is defined to be the natural projection $b\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}$. *It is a real-analytic map with complex fibers.*

- Betti form:
  - The 2-form $2da \wedge db$ descends to a $(1,1)$-form ω on $\mathfrak{A}_g$ which is semi-positive (N.Mok).

Part 2. Application to Betti map and non-degeneracy

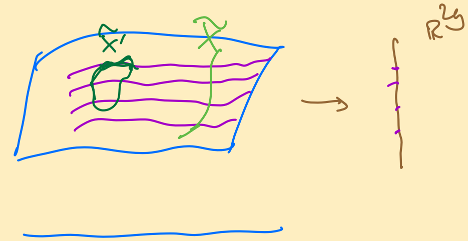
Bi-algebraic system $u\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g$.
$b\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}$.

Definition

A subvariety X of $\mathfrak{A}_g$ is said to be non-degenerate if

$$\mathrm{rk}_{\mathbb{R}} db|_{\tilde{X}} = 2\dim X.$$

Here $\tilde{X}$ is an irreducible component of $u^{-1}(X^{\mathrm{sm}})$.



In particular, X is always degenerate if $\dim X > g$ (naive degenerate subvarieties).

Remark

If one uses the Betti form ω on $\mathfrak{A}_g$, then X is non-degenerate if and only on $\omega|_X^{\wedge \dim X} \not\equiv 0$. $\rightsquigarrow$ non-degeneracy is in some way a "bigness" condition.

Part 2. Application to Betti map and non-degeneracy

For the notation

$$\begin{array}{ccc} \mathbb{C}^g \times \mathfrak{H}_g \cong \mathbb{R}^{2g} \times \mathfrak{H}_g & \xrightarrow{u} & \mathfrak{A}_g \\ \downarrow & & \downarrow \pi \\ \mathcal{H}_g & \longrightarrow & \mathbb{A}_g \end{array}$$

with $X \subseteq \mathfrak{A}_g$ and $\widetilde{X}$ a component of $u^{-1}(X)$, we have

$$X \text{ is degenerate}$$

$$\Leftrightarrow \widetilde{X} \text{``}=\text{''} \bigcup_{r \in \mathbb{R}^{2g}, \widetilde{C} \text{ curve in } \mathcal{H}_g} (\{r\} \times \widetilde{C})$$

$$\Leftrightarrow X \text{``}=\text{''} \bigcup_{r \in \mathbb{R}^{2g}, \widetilde{C} \text{ curve in } \mathcal{H}_g} u(\{r\} \times \widetilde{C})$$

Here "$=$" means the RHS contains a non-empty open of the LHS.

Part 2. Application to Betti map and non-degeneracy

So X is degenerate $\Rightarrow X = \bigcup_{r \in \mathbb{R}^{2g}, \widetilde{C} \text{ curve in } \mathfrak{H}_g} \overline{u(\{r\} \times \widetilde{C})}^{\mathrm{Zar}}$.
Now let us study

$$Y := \overline{u(\{r\} \times \widetilde{C})}^{\mathrm{Zar}}$$

Apply mixed Ax-Schanuel to $Z = \{r\} \times \widetilde{C}$ (version of G'). We get

$$\dim \overline{Z}^{\mathrm{Zar}} + \dim \overline{u(Z)}^{\mathrm{Zar}} \geq \dim Z + \dim \overline{u(Z)}^{\mathrm{biZar}}.$$

It then becomes

$$\dim(\{r\} \times \widetilde{C}^{\mathrm{Zar}}) + \dim Y > \dim \overline{Y}^{\mathrm{biZar}}.$$

As $\dim \widetilde{C}^{\mathrm{Zar}} \leq \dim \overline{\pi(Y)}^{\mathrm{biZar}}$, we have

$$\dim Y > \dim \overline{Y}^{\mathrm{biZar}} - \dim \overline{\pi(Y)}^{\mathrm{biZar}}.$$

Part 2. Application to Betti map and non-degeneracy

So we have

$$X \text{ is degenerate} \Leftrightarrow X = \bigcup_{\dim Y > \dim \overline{Y}^{\mathrm{biZar}} - \dim \overline{\pi(Y)}^{\mathrm{biZar}}} Y.$$

The previous slide showed ⇒ using mixed Ax-Schanuel. The direction ⇐ follows directly from the description of bi-algebraic subvarieties in $\mathfrak{A}_g$ (each member in the union of RHS is a naive degenerate subvariety).

We will show that the union on the right hand side is actually a finite union, and get a criterion to degeneracy from this!

Part 3. Handle the unlikely intersection problem

Theorem (Bogomolov, 1981)

Let A be an abelian variety and let X be a subvariety. There are only finitely many abelian subvarieties B of A satisfying:
(1) $\dim B > 0$ *and* $a + B \subseteq X$ *for some* $a \in A$;
(2) B is maximal for the property described in (1).

Generalization of this theorem, all by using o-minimality.

- Ullmo (2014) proved the corresponding result for pure Shimura varieties, for the purpose of studying the André-Oort conjecture.
- Inspired by Rémond, Habegger–Pila (2016) introduced the notion of *weakly optimal* subvarieties when studying the more general Zilber-Pink conjecture. They also proved the corresponding finiteness result for the case $Y(1)^N$.
- Daw–Ren (2018) proved the finiteness result for pure Shimura varieties.
- G' (2020) for $\mathfrak{A}_g$.

Part 3. Criterion to non-degeneracy

- $\pi\colon \mathcal{A} \to S$ abelian scheme;
- $X \subseteq \mathcal{A}$ subvariety, dominant to S;
- WMA X not contained in a proper subgroup scheme.

Theorem (G', 2020)

X is degenerate if and only if there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$ such that $\dim X - \dim \iota(X) > g - g'$ *(i.e. a generic fiber is "naive degenerate").*

$$
\begin{array}{ccccc}
\iota\colon \mathcal{A} & \longrightarrow & \mathcal{A}/\mathcal{B} & \longrightarrow & \mathfrak{A}_{g'} \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
S & \xrightarrow{=} & S & \longrightarrow & \mathbb{A}_{g'}
\end{array}
$$

Part 3. Non-degeneracy: a construction

Abelian scheme $\mathcal{A} \to S$, modular map $\iota\colon \mathcal{A} \to \mathfrak{A}_g$.

Theorem (G', 2020)

Let X be a subvariety of $\mathcal{A}$ which dominates S. Assume furthermore

(a) $\dim X > \dim S$.

(b) *No proper subgroup of $\mathcal{A}_s$ contains X_s, for some $s \in S(\mathbb{C})$.*

(c) *On the geometric generic fiber $\mathcal{A}_{\overline{\eta}}$, the stabilizer of $X_{\overline{\eta}}$ is finite.*

Then as subvarieties of $\mathcal{A}^{[M]} := \mathcal{A} \times_S \cdots \times_S \mathcal{A}$ (M-copies), we have

(i) $X^{[M]}$ *is non-degenerate if $\iota^{[M]}$ is generically finite and $M \geq \dim S$.*

(ii) $\mathcal{D}_M(X^{[M+1]})$ *is non-degenerate if ι is quasi-finite and $M \geq \dim X$.*

- Part (ii) can be easily deduced from part (i) by doing a base change.
- This theorem turns the question of non-degeneracy into an algebro-geometry question. Part (i) can be applied to maximal moduli.

Part 3. Application to universal curve

$$\begin{array}{ll} \mathfrak{C}_g & \text{universal curve} \\ \downarrow & \\ \mathbb{M}_g & \text{moduli space of curves of genus } g \text{ with level-3-structure} \end{array}$$

Over each $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$, the fiber is precisely the curve parametrized by s.

$$\begin{array}{ccc} \mathfrak{C}_g^{[M+1]} & \xrightarrow{\mathcal{D}_M} & \mathrm{Jac}(\mathfrak{C}_g/M_g)^{[M]} \\ & \searrow & \downarrow \pi \\ & & \mathbb{M}_g \end{array}$$

while $\mathcal{D}_M$ is defined fiberwise as $(P_0, P_1, \ldots, P_M) \mapsto (P_1 - P_0, \ldots, P_M - P_0)$.

Theorem

$\mathcal{D}_M(\mathfrak{C}_g^{[M+1]})$ is non-degenerate if $M \geq 3g - 2$.

Part 3. Application to Hilbert scheme

Let $r \geq 1$ and $d \geq 1$. Fix $\mathfrak{A}_g \subseteq \mathbb{P}^N_{\mathbb{A}_g}$.

Set $H = Hilb_{d,r}(\mathfrak{A}_g/\mathbb{A}_g)^\circ$, parametrizing all integral subschemes of relative dimension r and degree d. Finitely many irreducible components because finitely many choices for the Hilbert polynomial!

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\subseteq} & H \times_{\mathbb{A}_g} \mathfrak{A}_g & \xrightarrow{\iota} & \mathfrak{A}_g \\ & \searrow & \downarrow & \lrcorner & \downarrow \\ & & H & \longrightarrow & \mathbb{A}_g \end{array}$$

For each irreducible subvariety $S \subseteq H$, $\iota^{[M]}|_{\mathcal{X}^{[M]} \times_H S}$ is generically finite for $M \gg 1$ (maximal moduli). So:

Theorem

$\mathcal{X}^{[M]} \times_H S$ is a non-degenerate subvariety of $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S$ for $M \gg 1$.

This leads to the proof of Uniform Mordell–Lang.

Part 4. Uniform Mordell–Lang.

- A abelian variety;
- L an ample line bundle;
- X irreducible subvariety;
- Γ a finite rank subgroup of $A(\overline{\mathbb{Q}})$.

Theorem (Mordell–Lang Conjecture, Falting 1991 + Hindry 1988)

$$(X(\overline{\mathbb{Q}}) \cap \Gamma)^{\mathrm{Zar}} = \bigcup_{i=1}^{n} (x_i + B_i) \bigcup S$$

with B_i abelian subvariety $\dim B_i > 0$ *and S a finite set.*

Rémond (2000) proved a bound $n + \#S \leq c(g, \deg_L X, \deg_L A, h_{\mathrm{Fal}}(A))^{1+\mathrm{rk}\Gamma}$.

Part 4. Uniform Mordell–Lang.

Uniform Mordell–Lang, conjectured by David–Philippon:

Theorem (G'–Ge–Kühne, 2021 preprint)

$n + \#S \leq c(g, \deg_L X)^{1+\mathrm{rk}\Gamma}$.

A key notion to study Mordell–Lang is the Ueno locus

$$\bigcup_{x+B \subseteq X,\ \dim B > 0} (x + B).$$

Set X° to be its complement. It is known that X° is Zariski open in X.

Part 4. Uniform Mordell–Lang.

Based on results of Rémond (generalized Mumford and Vojta Inequalities + ε), the theorem is reduced to:

Theorem (Generalized New Gap Principle, G'–Ge–Kühne, 2021 preprint)

Assume X generates A, then

$$\{P \in X^\circ(\overline{\mathbb{Q}}) : \hat{h}_L(P) \leq c_1 \max\{1, h_{\mathrm{Fal}}(A)\}\}$$

is contained in a proper Zariski closed $X' \subsetneq X$ with $\deg_L X' < c_2$. Here $c_1 = c_1(g, \deg_L X) > 0$ and $c_2 = c_2(g, \deg_L X) > 0$.

Case of $X \subsetneq A$ being $C \subsetneq J$ (Abel–Jacobi) is precisely the New Gap Principle for curves by Dimitrov–G'–Habegger + Kühne.

- Impossible to completely remove the hypothesis "X generates A" or to directly get a bound on the cardinality, due to $h_{\mathrm{Fal}}(A)$.

From curve to high dim subvarieties

Sketch of the proof:

- Reduce to (A, L) principally polarized;
- Construct the desired families using Hilbert schemes;
- Construct/Prove non-degenerate subvariety;
- Apply the height inequality of DGH (for A with large height) and equidistribution of Kühne (for A with small height); one needs to run a family version of the Ullmo–Zhang approach to obtain a bound from equidistribution.

Part 5. Height Inequality

For $\pi : \mathcal{A} \to S$ abelian scheme and $X \subseteq \mathcal{A}$ a subvariety. For $\mathcal{L}$ relatively very ample on $\mathcal{A}/S$ and $\mathcal{M}$ very ample on $\overline{S}$, a compactification of S.

Then $\mathcal{L} + \mathcal{M} := \mathcal{L} \otimes \pi^* \mathcal{M}|_S$ gives a compactification $\overline{\mathcal{A}} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ of $\mathcal{A}$.

$$\begin{array}{ccccc} \overline{\mathcal{A}_N} & \xrightarrow{\overline{[N]}} & \overline{\mathcal{A}} & & \subseteq \mathbb{P}^n \times \mathbb{P}^m \\ \uparrow & & \uparrow & & \\ \mathcal{A} & \xrightarrow{[N]} & \mathcal{A} & & \end{array}$$

In practice, need to work with the graphs of [N].

To prove the desired height inequality $\hat{h}_{\mathcal{L}}(x) \geq c h_{\overline{S},\mathcal{M}}(\pi(x)) - c'$, it suffices to prove: for each $N = 2^l$ large enough, there exists a Zariski open dense $U_N \subseteq X$ on which

$$h_{\mathcal{L}+\mathcal{M}}([N]x) \geq c_1 N^2 h_{\overline{S},\mathcal{M}}(\pi(x)) - c_2(N).$$

$$\rightsquigarrow h_{\overline{[N]}^* \mathcal{O}(1,1)}(x) \geq c_1 N^2 h_{\mathcal{O}(0,1)}(x) - c_2(N).$$

So want $\overline{[N]}^* \mathcal{O}(1,1) \otimes \mathcal{O}(0,1)^{\otimes -c_1 N^2}$ is big on $\overline{X_N}$.

Part 5. Height Inequality

Want $\overline{[N]}^*\mathcal{O}(1,1)\otimes\mathcal{O}(0,1)^{\otimes -c_1N^2}$ is big on $\overline{X_N}$. Write $d=\dim X=\dim\overline{X_N}$.

By a criterion of Siu, we need to bound $(\overline{[N]}^*\mathcal{O}(1,1))^{\cdot d}\cdot[\overline{X_N}]$ from below, and $(\overline{[N]}^*\mathcal{O}(1,1))^{\cdot(d-1)}\mathcal{O}(0,1)\cdot[\overline{X_N}]$ from above.

Proposition

- *If X is non-degenerate, then $\overline{[N]}^*\mathcal{O}(1,1)^{\cdot d}\cdot[\overline{X_N}]\gg_X N^{2d}$ for all N.*
- *If $N=2^l$, then $\mathcal{O}(1,1)^{\cdot(d-1)}\mathcal{O}(0,1)\cdot[\overline{X_N}]\ll_X N^{2(d-1)}$.*

For $(\overline{[N]}^*\mathcal{O}(1,1))^{\cdot d}\cdot[\overline{X_N}]$: take $U\subseteq S^{\mathrm{sm,an}}$ compact. Then restricted to $\pi^{-1}(U)$ we have $\omega_{\mathrm{FS}}>c_U\omega$ for some $c_U>0$. So

$$
\begin{aligned}
(\overline{[N]}^*\mathcal{O}(1,1))^{\cdot d}\cdot[\overline{X_N}] &= \int_{\overline{X_N}}\overline{[N]}^*\omega_{\mathrm{FS}}^{\wedge d}\geq\int_{\overline{X_N}\cap\pi^{-1}(U)}\overline{[N]}^*\omega_{\mathrm{FS}}^{\wedge d}\\
&> c_U^d\int_{X\cap\pi^{-1}(U)}[N]^*\omega^{\wedge d}=c_U^d\int_{X\cap\pi^{-1}(U)}(N^2\omega)^{\wedge d}\\
&= c_U^dN^{2d}\int_{X\cap\pi^{-1}(U)}\omega^{\wedge d}.
\end{aligned}
$$

If X is non-degenerate, there exists U such that the last integral is >0.

Thanks!