Random hyperbolic 3-manifolds

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Motivation

For 2-manifolds...

For 3-manifolds...

Thurston’s geometrization conjecture

$S^3, \mathbb{E}^3, \mathbb{H}^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}, \tilde{SL}(2, \mathbb{Z}), \text{Nil, Sol}$

Growth of their geometric invariants
One question to ask

Given a random hyperbolic 3-manifold...

of large volume...

what is the probability that it has 2000 closed geodesics of length at most 10?
Random manifold

\{ \text{Set of manifolds} \} + \{ \text{Probability measure} \} = (\Omega, \mathbb{P})

\Downarrow

⇒ What is the probability that a random manifold has a certain property?
CONSTRUCTING RANDOM 3-MANIFOLDS
Random triangulation

Introduced by Bram Petri and Jean Raimbault (2020).

**General idea**: To construct manifolds by randomly gluing polyhedra together along their faces.

1st attempt:
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1st attempt:

This doesn’t work!

The neighbourhoods of the vertices are not typically homeomorphic to $\mathbb{R}^3$. 

The model $M_N$

Solution:

By gluing them along their hexagonal faces, we obtain:

$\Rightarrow$ A compact 3-manifold with boundary $M_N$, where $N$ is the number of tetrahedra.
The model $Y_N$

If we do this transformation for every tetrahedra in $M_N$, we obtain:

$\Rightarrow$ A new 3-manifold with boundary $Y_N$, made of octahedra.
We endow each octahedron in $Y_N$ with the hyperbolic metric of an ideal right-angled octahedron in $\mathbb{H}^3$.

Thus, $Y_N$ becomes a complete finite volume hyperbolic 3-manifold with totally geodesic boundary.
Our question

Given a random hyperbolic 3-manifold

of large volume

$Y_N$

what is the probability that it has 2000 closed geodesics of length at most 10?
COUNTING CLOSED GEODESICS
THE LENGTH SPECTRUM

**Definition**
The length spectrum $L(M)$ of a hyperbolic manifold $M$ is the set of lengths of closed geodesics in $M$.

> We encode $L(Y_N)$ through the function:

$$L \mapsto C_L(Y_N) = \#\{\text{closed geodesics of length } \leq L \text{ on } Y_N\},$$

where $L > 0$ and $C_L(Y_N)$ is a random variable.
Poisson distribution

$\mathbb{P}[\text{there are } k \text{ events happening in a specified interval } [0, L]]^*$

* provided that they are independent and occur with a known constant mean rate.

**Definition**

A random variable $Z : \Omega \rightarrow \mathbb{N}$ follows a *Poisson distribution of parameter* $\lambda > 0$ if for any $k \in \mathbb{N}$,

$$P[Z = k] = \frac{\lambda^k e^{-\lambda}}{k!}.$$
Main result

Theorem (Roig Sanchis)

As \( N \to \infty \), \( C_L(Y_N) \) converges in distribution to a Poisson random variable \( C_L(Y) \) with explicit parameter \( \lambda(L) \).
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\[
\lim_{N \to \infty} P[C_L(Y_N) = k] = \frac{\lambda(L)^k e^{-\lambda(L)}}{k!}.
\]
Main result

**Theorem (Roig Sanchis)**

As $N \to \infty$, $C_L(Y_N)$ converges in distribution to a Poisson random variable $C_L(Y)$ with explicit parameter $\lambda(L)$.

\[
\lim_{N \to \infty} P[C_{10}(Y_N) = 2000] = \frac{\lambda(10)^{2000} e^{-\lambda(10)}}{2000!} \ll \epsilon.
\]
Main result

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As $N \to \infty$, $C_L(Y_N)$ converges in distribution to a Poisson random variable $C_L(Y)$ with explicit parameter $\lambda(L)$.

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Thank you!