Anna Szumowicz

Research Statement

My first research interest is the representation theory of reductive p-adic groups. The main motivation behind my PhD thesis project is to find new ways to parametrize the types on p-adic reductive groups and use them to find a direct proof of the inertial Local Langlands correspondence. To shed more light on the problem let me recall the definition of a cuspidal type on the general linear group. Let F be a nonarchimedean local field and let $\mathcal{O}$ be its ring of integers. Let π be an irreducible cuspidal representation of $GL_n(F)$. By $\mathfrak{I}(\pi)$ we will denote the inertial support of π.[1] A **cuspidal type (on $K = GL_n(\mathcal{O})$)** for $\mathfrak{I}(\pi)$ is an irreducible representation λ of $GL_n(\mathcal{O})$ which satisfies the following condition: an irreducible representation π_1 of $GL_n(F)$ contains λ if and only if the inertial support of π_1 coincides with that of π. Guy Henniart studied cuspidal types on $GL_2(\mathcal{O})$ in [4]. Paskunas in [13] showed the existence and uncity of cuspidal types on $GL_n(\mathcal{O})$. More precisely he proved that for any π irreducible cuspidal representation of $GL_n(F)$ there exists λ an irreducible representation of K depending only on $\mathfrak{I}(\pi)$ which is a cuspidal type on K. Moreover λ is unique up to isomorphism. Using that and the local Langlands correspondence he deduced an inertial Langlands correspondence. In rough terms the inertial Langlands correspondence is a correspondence between certain representations of $GL_n(\mathcal{O})$ and certain irreducible representations of the inertia subgroup of F. The regular representations of $GL_n(\mathcal{O})$ were introduced by Shintani ([15]). Those are in certain sense the best behaved representations of $GL_n(\mathcal{O})$, in particular they can be classified [17]. Parametrizing all irreducible representations of $GL_n(\mathcal{O})$ is known to be very hard. It contains the matrix pair problem (see [2]). In my PhD I am determining which cuspidal types on $GL_p(\mathcal{O})$ (where p is a prime number) are regular. In order to do so I am using tools from Clifford theory, the classification of irreducible cuspidal representations of $GL_p(\mathcal{O})$ and the properties of the actions of reductive p-adic groups on their Bruhat-Tits buildings. For my PhD I studied (amongs other things) [5], [6] and Bruhat-Tits buildings.

I am also interested in theory the of integer-valued polynomials, simultaneous $\mathfrak{p}$-orderings and equidistribution in number fields. The notion of simultaneous $\mathfrak{p}$-ordering comes from Bhargava's work on generalization of the factorial function to all Dedekind domains [3]. Roughly speaking a simultaneous $\mathfrak{p}$-ordering (for $\mathfrak{p}$ a prime ideal in the ring of integers in some number field) is a sequence of elements from the ring of integers in a number field which is equidistributed modulo every power of every prime ideal in $\mathcal{O}_k$ as well as possible. Manjul Bhargava in [3] asked which subsets of Dedekind domains admit simultaneous $\mathfrak{p}$-ordering. Together with Mikołaj Fraczyk we proved [10] that the only number field k where the ring of integers $\mathcal{O}_k$ has a simultaneous $\mathfrak{p}$-ordering is $\mathbb{Q}$.

Contents

[1] $\mathfrak{I}(\pi) = \{\pi_2 \mid \pi_2 \cong \pi \otimes \chi \circ \det\}$ for χ an unramified character of $F^{\times}$.

1 Past and current research

1.1 Cuspidal types

In this section I am explaining how I use the language from Clifford theory to describe cuspidal types on K.

Any irreducible smooth representation π of $GL_n(\mathcal{O})$ factors through a finite group $GL_n(\mathcal{O}/\mathfrak{p}_F^r)$ where r is a natural number bigger or equal 1 and $\mathfrak{p}_F$ is the maximal ideal in $\mathcal{O}$. The minimal natural number r with this property is called the conductor of the representation π. Let ρ be an irreducible smooth representation of $GL_n(\mathcal{O})$ with conductor $r > 1$. Sometimes it will be convenient to see ρ as a representation of $GL_n(\mathcal{O}/\mathfrak{p}_F^r)$. In this case we will denote it by $\bar{\rho}$. Let $l = \lfloor \frac{r+1}{2} \rfloor$ and let K^l be the kernel of the projection from $GL_n(\mathcal{O}/\mathfrak{p}_F^r)$ onto $GL_n(\mathcal{O}/\mathfrak{p}_F^l)$. Note that K^l/K^r is an abelian group. Fix an additive character $\psi : F \to \mathbb{C}^\times$. Denote by $M_n(\mathcal{O})$ the set of all $n \times n$ - matrices with entries in $\mathcal{O}$. By Clifford theorem

$$\bar{\rho} \mid_{K^l} = m \bigoplus_{\bar{\alpha} \sim \bar{\alpha}_1} \bar{\psi}_{\bar{\alpha}}, \tag{1}$$

where $\bar{\alpha}, \bar{\alpha}_1 \in M_n(\mathcal{O}/\mathfrak{p}_F^{r-l})$, the equivalence classes of $\sim$ are $GL_n(\mathcal{O}/\mathfrak{p}^{r-l})$-conjugacy classes, $m \in \mathbb{N}$ and the characters $\bar{\psi}_{\bar{\alpha}} : K^l \to \mathbb{C}^\times$ are defined as follows: $\bar{\psi}_{\bar{\alpha}}(1 + x) = \psi(\mathrm{tr}\hat{\alpha}\hat{x})$ for some lifts of $x, \bar{\alpha}$ to elements in $M_n(\mathcal{O})$. The definition of $\bar{\psi}_{\bar{\alpha}}$ does not not depend on the choice of lifts. If a matrix $\alpha \in M_n(\mathcal{O})$ is such that its image in $M_n(\mathcal{O}/\mathfrak{p}_F^{r-l})$ appears in the decomposition (1) we say that α **is in the orbit of** ρ. We say that a representation is regular if its orbit contains a matrix whose image in $M_n(\mathcal{O}/\mathfrak{p}_F)$ has abelian centralizer. Roi Krakovski, Uri Onn and Pooja Singla [12] constructed all such representations under the condition that the characteristic of the residue field of F is different than 2. Alexander Stasinski and Shaun Stevens in [17] constructed all regular representations of $GL_n(\mathcal{O})$. In [16] Stasinski asked which cuspidal types are regular. I gave a full descritpion of cuspidal types on $GL_2(\mathcal{O})$ in terms of orbits. For big conductors they behave as one can predict but for small conductors some new phenomena appear. Denote by k the residue field of F. For a character $\bar{\psi}_{\bar{\alpha}}$ on K^l by $\mathrm{Stab}_K\bar{\psi}_{\bar{\alpha}}$ we denote the preimage of $\mathrm{Stab}_{GL_2(\mathcal{O}/\mathfrak{p}^r)}\bar{\psi}_{\bar{\alpha}}$ through the canonical projection $K \twoheadrightarrow GL_2(\mathcal{O}/\mathfrak{p}^r)$. The following gives a full description of cuspidal types on $GL_2(\mathcal{O})$ in terms of orbits.

Theorem 1 *([18]) A cuspidal type on K is exactly a one-dimensional twist of one of the following*

1. *a representation inflated from some irreducible cuspidal representation of $GL_2(k)$,*

2. *a representation whose orbit contains a matrix which characteristic polynomial is irreducible modulo $\mathfrak{p}_F$*

3. *a representation whose orbit is equivalent to an orbit containing a matrix β whose characteristic polynomial is Eisenstein and which satisfies one of the following conditions:*

   (a) *has conductor bigger than 3*

   (b) *has conductor 2 and is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_K(\bar{\psi}_{\bar{\beta}})}^{K}\theta$ for certain θ containing a lift of $\bar{\psi}_{\bar{\beta}}$ to $Stab_K(\bar{\psi}_{\bar{\beta}})$ and such that θ is nontrivial when restricted to $\begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix}$ and where $Stab_K(\bar{\psi}_{\bar{\alpha}})$ denotes the preimage of $Stab_{GL_2(\mathcal{O}/\mathfrak{p}_F^r)}(\bar{\psi}_{\bar{\alpha}})$ under the projection $K \to GL_2(\mathcal{O}/\mathfrak{p}_F^r)$*

   (c) *has conductor 3 and is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_K(\bar{\psi}_{\bar{\beta}})}^{K}\theta$ where θ contains a lift of $\bar{\psi}_{\bar{\beta}}$ to $Stab_K(\bar{\psi}_{\bar{\beta}})$ and it does not contain the trivial character of $\begin{pmatrix} 1 & \mathfrak{p}_F \\ 0 & 1 \end{pmatrix}$.*

Using similar methods currently I am working on a generalization of my result to $GL_p(\mathcal{O})$ where p is prime and determining which of them are regular. Let $\mathfrak{I}$ be the chain order consisting of matrices that are upper triangular modulo $\mathfrak{p}_F$, let $U_{\mathfrak{I}}$ be the group of invertible elements of $\mathfrak{I}$ and let $\mathfrak{P}_{\mathfrak{I}}$ be the Jacobson radical in $\mathfrak{I}$. I prove the following result:

Theorem 2 *Let $G = GL_p(\mathcal{O})$. There exists $m \in \mathbb{N}$ satisfying the following property. Up to tensoring by a one-dimensional representation, every cuspidal type λ on $K = GL_p(\mathcal{O})$ of conductor at least m is of one of the following forms:*

1. *a representation inflated from an irreducible cuspidal representation of $GL_p(k)$,*

2. *a representation whose orbit contains a matrix β such that its characteristic polynomial is irreducible modulo $\mathfrak{p}_F$*

3. *a representation whose orbit contains a matrix of the form $\Pi^j B$ where Π generates the Jacobson radical $\mathfrak{P}_{\mathfrak{I}}$, $0 < j < p$ and B is an element of $U_{\mathfrak{I}}$.*

Moreover in point 3. all $0 < j < p$ can be achieved in this way.

In particular this implies that for big enough m a cuspidal type on $GL_p(\mathcal{O})$ of conductor $r \geq m$ is regular if and only if its orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$ or a matrix whose characteristic polynomial is Eisenstein. In particular for $p > 2$ even for big conductors there are cuspidal types which are not regular.

1.2 Optimal rate of equidistribution in number fields

Let k be a number field. Denote by $\mathcal{O}_k$ its ring of integers and let S be a subset of $\mathcal{O}_k$. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_k$. We say that $(a_i)_{i\in\mathbb{N}}$ is a **$\mathfrak{p}$-ordering** in S if for every $n \in \mathbb{N}$

$$v_S(\mathfrak{p}, n) := v_{\mathfrak{p}}\left(\prod_{i=0}^{n-1}(a_i - a_n)\right) = \min_{s\in S} v_{\mathfrak{p}}\left(\prod_{i=0}^{n-1}(a_i - s)\right),$$

where $v_{\mathfrak{p}}$ is $\mathfrak{p}$-adic additive valuation in $\mathcal{O}_k$. The value $v_S(\mathfrak{p}, n)$ does not depend on the choice of $\mathfrak{p}$-ordering. Bhargava defined the generalized factorial as the ideal $n!_S = \prod_{\mathfrak{p}} \mathfrak{p}^{v_S(\mathfrak{p},n)}$ where $\mathfrak{p}$ runs over all prime ideals in $\mathcal{O}_k$. A sequence is called a simultaneous $\mathfrak{p}$-ordering if it is a $\mathfrak{p}$-ordering for all prime ideals $\mathfrak{p}$ in $\mathcal{O}_k$. Simultaneous $\mathfrak{p}$-ordering are also called Newtotnian sequences [9]. Bhargava in [3] asked which subsets of Dedekind domains contain a simultaneous $\mathfrak{p}$-ordering. Melanie Wood in [19] has proved that there are no simultaneous $\mathfrak{p}$-orderings in $\mathcal{O}_k$ for k imaginary quadratic number field. This was generalized by David Adam and Paul-Jean Cahen in [1] to all quadratic number fields $\mathbb{Q}(\sqrt{d})$ besides $d = 2, 3, 5$ and $d \equiv 1 \bmod 8$. In a joint work with Mikolaj Fraczyk we have proved that this true for any number field [10]. The precise statement we prove is stronger. To state it properly we need to define n-optimal sets. A finite set $S \subseteq \mathcal{O}_k$ is **almost uniformly distributed modulo $\mathfrak{p}$** if for every $a, b \in \mathcal{O}_k$ we have

$$| \{s \in S \mid \ s - a \in \mathfrak{p}\} | - | \{s \in S \mid \ s - b \in \mathfrak{p}\} | \in \{-1, 0, 1\}.$$

We say that a finite subset $S \subseteq \mathcal{O}_k$ with $n + 1$ elements is **n-optimal** if it is almost uniformly equidistributed modulo every power of $\mathfrak{p}$ for every prime ideal $\mathfrak{p}$. If $(a_i)_{i\in\mathbb{N}}$ is a simultaneous $\mathfrak{p}$-ordering, then $\{a_i \mid \ 0 \leqslant i \leqslant n\}$ forms an n-optimal set. In particular the non-existence of n-optimal sets for n big enough implies non-existence of simultaneous $\mathfrak{p}$-orderings. The idea for study n-optimal sets comes from the theory of integer valued polynomials. We say that a polynomial $f \in k[x]$ is **integer valued** if $f(\mathcal{O}_k) \subseteq \mathcal{O}_k$ (see [8]). The n-optimal sets are in some sense the smallest testing sets for finding such polynomials. Let $n \in \mathbb{N}$. An equivalent definition of n-optimal sets is the following: a set S is n-optimal if and only if for every polynomial $f \in k[x]$ of degree at most n the following condition is satisfied: $f(S) \subseteq \mathcal{O}_k$ implies that f is integer valued. My first work on n-optimal sets was done in [7] together with Mikolaj Fraczyk and Jakub Byszewski where we have proved that there are no n-optimal sets in the ring of integers of quadratic imaginary number fields. In our recent preprint together with Mikołaj Fraczyk [10] we have proved that there are no n-optimal sets in $\mathcal{O}_k$ for k any number field different that $\mathbb{Q}$. Let me explain very briefly what is the idea of the proof. We assume the contrary, i.e. that there exists $\mathcal{S}_n$ a sequence of n-optimal sets with n tending to the infinity. Let $V = k \otimes_{\mathbb{Q}} \mathbb{R}$. First we show that for every n there exists a cylinder in V of the volume $O(n)$ which contains $\mathcal{S}_n$. Something similar was proved in the case of imaginary quadratic number field in [7] but the general case was much more complicated because the norm in this case is not always convex. In the proof of that we used some number theoretical input, for example Ikehara's Tauberian theorem and Baker-Wüstholz's theorem. Then we deduce that there exists a compact set Ω and sequences $(s_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ in V with some restriction on the norm of s_n such that sets $s_n^{-1}(\mathcal{S}_n - t_n)$ are contained in Ω. Define $\mu_n = \frac{1}{n} \sum_{x\in\mathcal{S}_n} \delta_{s_n^{-1}(x-t_n)}$. Since Ω is compact we consider weak -* limits of μ_n. We call them limit measures. They provide information about geometry of large n-optimal sets. We study properties of n-optimal sets to show that limit measures cannot exist.

2 Future projects

2.1 Construction of types on $GL_n(\mathcal{O})$

To the best of my knowledge regular representations of $GL_p(\mathcal{O})$ form the biggest group of irreducible smoth representations of $GL_p(\mathcal{O})$ which has been classified so far. My results of the description of types in terms of orbits suggest that even though the types may not be regular they still form quite manageable subset of all irreducible representations of $GL_p(\mathcal{O})$. I would like to extend the description of regular representations of $GL_p(\mathcal{O})$ given in [17] to similar description of cuspidal represenations which are not regular.

Let me recall the precise statement of the inertial Langlands correspondence. Denote by W_F the Weil group of F and by I_F the inertia subgroup. For an infinitely dimensional smooth representation π_1 of $GL_n(F)$ we denote by $WD(\pi_1)$ the Weil-Deligne representation of W_F which corresponds to π through the local Langlands correspondence. Paskunas in [13] proved the following result (**the inertial Langlands correpondence**): for ρ a smooth n-dimensional representation of I_F which extends to a smooth irreducible Frobenius semisimple representation of W_F there exists a unique up to isomorphism smooth irreducible representation λ of $GL_n(\mathcal{O})$ which satisfies the following condition: for every irreducible smooth infinitely dimensional representation π of $GL_n(F)$ we have: π contains λ if and only if $WD(\pi)\mid_{I_F}$ is isomorphic to ρ. Moreover λ has multiplicity at most one in π.

Having a description of cuspidal types which are regular it would also be interesting to look at the representations of I_F corresponding to regular cuspidal types on $GL_p(\mathcal{O})$ under the inertial Langlands correspondence.

2.2 Bernstein-Zelevinsky decomposition for $GL_n(\mathcal{O})$

In this project I would like to develop an analog of the Bernstein-Zelevinsky decomposition for $GL_n(\mathcal{O})$. It is partly motivated by the work of Schneider and Zink ([14]) on the K-types for the tempered components. In order to do that I need to construct generalized Steinberg representations for $GL_n(\mathcal{O})$. Hill in [11] defined the Steinberg representation for $GL_n(\mathcal{O}/\mathfrak{p}^r)$ as the unique regular representation of $\mathrm{Ind}_B^{GL_n(\mathcal{O}/\mathfrak{p}^r)} 1_B$ and where B denotes the subgroup of upper triangular matrices in $GL_n(\mathcal{O}/\mathfrak{p}^r)$. In general instead of B I need to consider parabolic subgroups of upper block triangular matrices and instead of trivial representation 1_B cuspidal representation of "Levi subgroup" of P and I need replace induction with parabolic induction. I suspect that in this induced representation there will be still a unique regular representation.

2.3 Simultaneous $\mathfrak{p}$-orderings

Using similar methods as in [10] together with Mikolaj Fraczyk we would like to extend our result on non-existence of simultaneous $\mathfrak{p}$-ordering to S-integers in a number field as well as subsets of $\mathcal{O}_k$ of positive upper density. The problem of existence of simultaneous $\mathfrak{p}$-orderings remains open for the rings of integers in the global fields of positive characteristic. In [3] Bhargava showed that there exists a simulatenous $\mathfrak{p}$-ordering in $\mathbb{F}_q[t]$ where $\mathbb{F}_q$ is a finite field. We want to study if his example can be extended to $\mathcal{O}_E$ for E a finite extension of $\mathbb{F}_q(t)$.

As opposed to the case of number fields this is a problem in algebraic geometry, more precisely about the geometry of curves over a finite field. We expect that it can lead to interesting new questions in that setting.

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