Combinatorial Soln to Ising Model

"Solution" in some sense as

Kac – Ward 1952 (det = Σ loops)

S. Sherman 1952
Vdovichenko 1965 } some,

M. Loebl 2004
Cimasoni 2010 } graphs

I am reporting on work by

$\boxed{\text{My PhD student Tyler Helmuth.}}$

soon to be posted on archive

Combinatorial interpretation of Mayer Expansion
Viennot Theory of Heaps.

Thanks also to

Christian Krattenthaler

Malek Abdesselam (preprint on my website)

H.T. Expansion (of Ising Model)

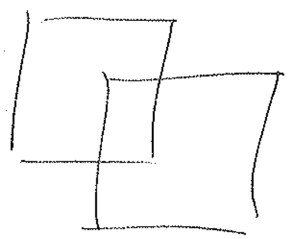
$G = (V, E)$ fixed finite graph

edge $xy \in E \longrightarrow$ weight K_{xy}

Subgraph $F \subset G$

$$K^F = \prod_{xy \in E(F)} K_{xy}$$

Even subgraph



all vertices even degree

$$Z = \sum_{\substack{F \subset G \\ \text{even}}} K^F$$

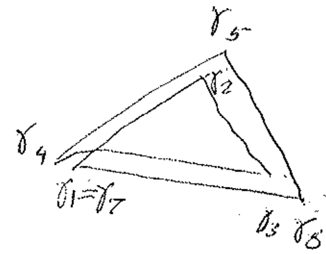
up to $(\cosh\beta)^G$ equals Ising model partition fn

$$\sum_{\sigma : V \to \{\pm 1\}} e^{-\beta \sum_{xy \in E} \sigma_x \sigma_y}$$

Cycle

$n = n(\gamma)$

$$\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_{n+1})$$



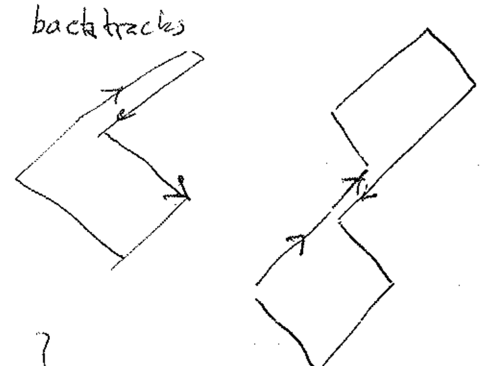
$\gamma_j \sim \gamma_{j+1}$ in G

/ cyclic permutation (keeps orientation)

Notes: γ is oriented.

Non-Backtracking

$$\gamma_{j+2} \neq \gamma_j$$



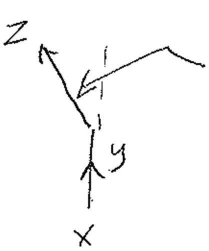
$$\Gamma_{nb} = \{\text{non-backtracking cycles}\}$$

is not edge simple

Embedding

$G \longrightarrow$ orientable surface (2d)

Turning angle (depends on embedding)



$\angle(x, y, z)$

Results

Let

$$\omega(\gamma) = \prod_{j=1}^{n(\gamma)} K_{\gamma_j \gamma_{j+1}} e^{\frac{i}{2} \angle (\gamma_j, \gamma_{j+1}, \gamma_{j+2})}$$

Then, if $G \longrightarrow \mathbb{R}^2$,

$$\log Z = -\tfrac{1}{2} \sum_{\gamma \in \Gamma_{nb}} \omega(\gamma) / n(\gamma)$$

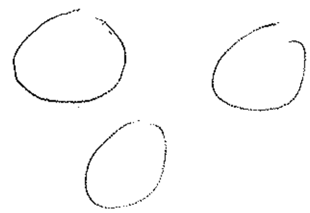
because γ has two orientations

There are also formulas for the two point function in terms of non backtracking walks from one point to other.

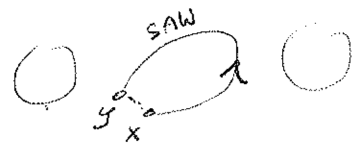
Proof for Special Case

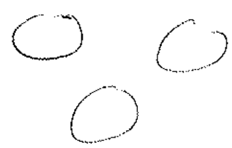
Every vertex in G is degree 3, planar

$$\sum_{\substack{F \subset G \\ \text{even}}} K^F = \sum_{\substack{\text{disjoint} \\ \text{edge simple cycles}}}$$



$$K_{xy} \frac{d}{dK_{xy}} \log Z = \frac{1}{2} \frac{\sum}{\sum}$$





Call this (Ratio)

The right hand side is the quotient of two polynomials in the variables (K_{xy}). We are about to generalise

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

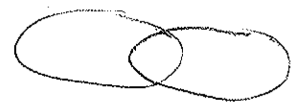
$x^n \leftrightarrow$ object appears n times, -1 per object.

Theory of heaps: alternative form of Mayer expansion
= how to divide ratio of polynomials

In the quotient formal series terms are labeled by heaps.

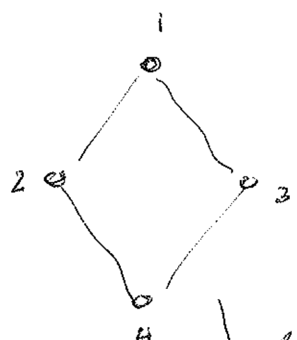
Heaps

Two cycles C_1, C_2 are incompatible if share edge $C_1 \not R C_2$



Want to describe collections of cycles which are allowed to be incompatible or have repeated cycles.

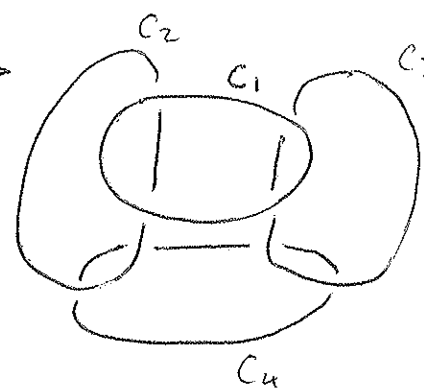
Example of $(\mathcal{P}, \leq, \ell)$, called a heap



$\mathcal{P} = \{1, 2, 3, 4\}$ ($\mathcal{P}$ poset)

$1 \geq 2$, $1 \geq 3$, $2 \geq 4$, $3 \geq 4$

$\ell : \mathcal{P} \to \mathcal{B}$ $\quad \mathcal{B} = \{\text{simple cycles in } \Gamma_{nb}\}$



This is a heap visualised as loops of rope lying on top of each other

Condition if $C_i \not R C_j$ then $i \geq j$ or $j > i$

$\mathcal{H}$ = set of heaps $(\mathcal{B}, \mathcal{R})$

$\mathcal{H}(R\ SAW) = \{H \in \mathcal{H} :$ maximal pieces R SAW$\}$

Theorem

Ratio on p I-5

$$\stackrel{(1)}{=} \sum_{SAW} \sum_{H \in \mathcal{H}(RSAW)} (-K)^H$$

$$\stackrel{(2)}{=} -\sum_{SAW} \sum_{H \in \mathcal{H}(RSAW)} \omega(H)$$

Xavier Viennot 1986 proved a general theorem of which this is an application

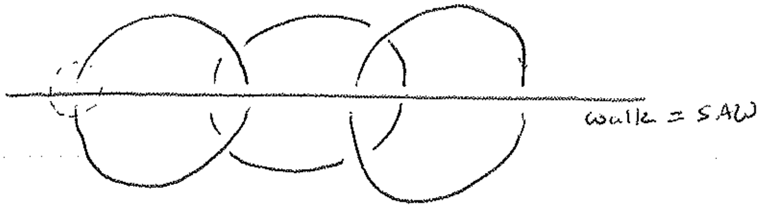
Integrate w.r.t $\frac{dK_{xy}}{K_{xy}}$ to get $\log Z$

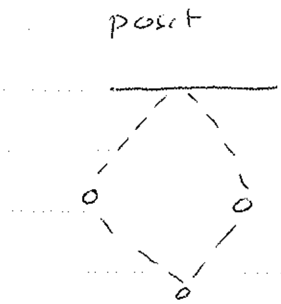
(1) Each cycle labeling H contributes (-1)

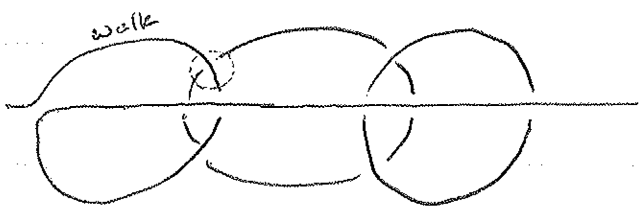
(2) $\underbrace{\prod e^{\frac{i}{2}\text{ turning angle}}}_{\text{in } \omega(H)} = -1$ for each cycle including the SAW which should not have (-1) hence overall (-1).

Bijection heap ⟷ walk

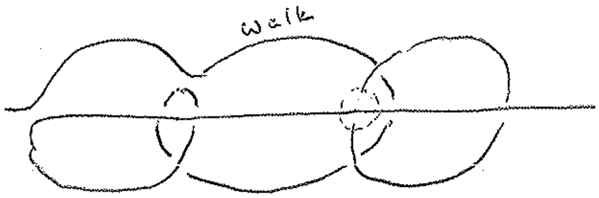
Start with heap



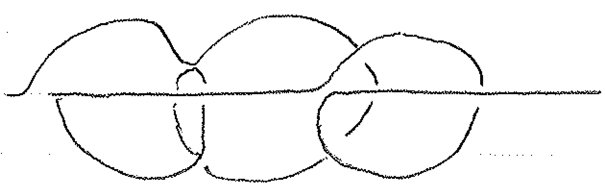




insert first cycle at first common edge (The inserted cycle is always below, in each step)

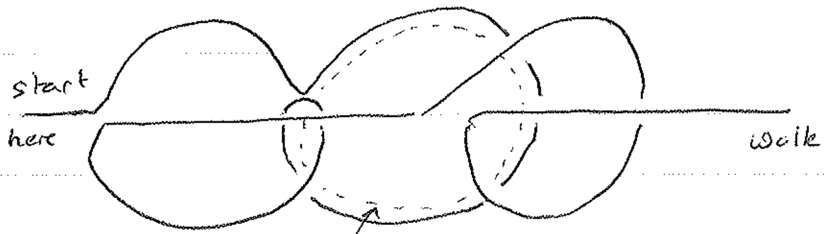


insert first cycle below walk at first common edge

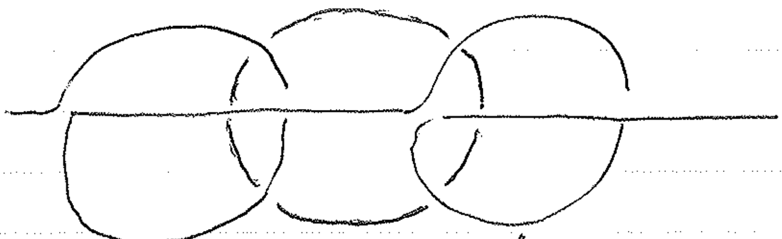


until no cycles

The inverse of heaps $\rightarrow$ walks
is erasing loops and letting them
fall into a heap

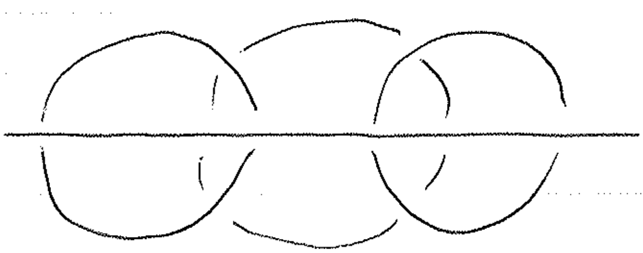


first loop
completed by walk:
erase it from walk,
let it fall to bottom of
the heap we will grow

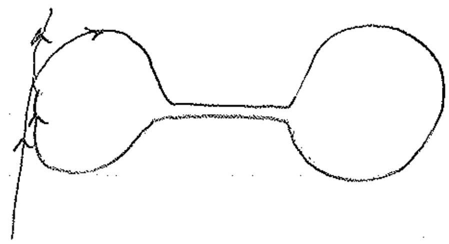


now this is
the first loop,
erase, let it fall

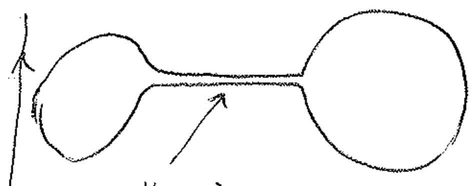
then this becomes
the first loop,
erase it, let it fall



but the bijection is not onto the set of non backtracking walks if we define heaps with simple cycles:



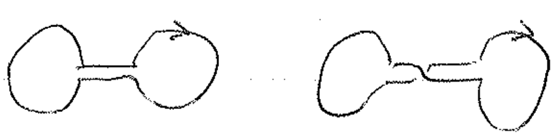
Erase the loop at the first edge in the walk at least twice with common orientation



this is
not an edge simple cycle

To obtain all non backtracking walks we must allow into the heap, cycles which are "oriented edge simple"

These cycles have zero w weight because



cancel.

turning angle makes one cancel the other