# A supersymmetric non linear sigma model for quantum diffusion

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- the general setting : Anderson localization, random matrices and sigma models

- a toy model for quantum diffusion

# Disordered conductors

**Anderson localization**: disorder induced localization for conducting electrons

## The framework

quantum mechanics: lattice field model, **finite volume** $\Lambda \subset \mathbb{Z}^d$

Hamiltonian describing the system: **matrix** $H \in \mathcal{M}_\Lambda(\mathbb{C})$, $H^* = H$
$\psi$ **eigenvector** with $\|\psi\|_2 = 1$: $|\psi_j|^2 = P(\text{electron at } j)$

$$\text{then} \quad \begin{array}{lcl} |\psi_j| \sim const. \forall j & \rightarrow & \text{quantum diffusion (conductor)} \\ |\psi_j| \sim \delta_{j j_0} & \rightarrow & \text{localization (insulator)} \end{array}$$

**disorder** $\rightarrow$ $H$ random with some probability law $P(H)dH$
transition local/ext

# Some models

| Random Schrödinger | Band Matrix |
|---|---|
| $H=-\Delta+\lambda\hat{V}$<br>$\hat{V}_{ij}=\delta_{ij}V_j$   $V_j$ **i.i.d.r.v.**<br>$\lambda\in\mathbb{R}$   $\Delta$ discete Lapl. | $H^*=H$, $H_{ij}$ **i.r.v. not i.d.**<br>$H_{ij}\sim\begin{cases}\mathcal{N}_{\mathbb{C}}(0,J_{ij})\\ \mathcal{N}_{\mathbb{R}}(0,J_{ii})\end{cases}$ $J_{ij}\sim\begin{cases}1/W & \lvert i-j\rvert\leq W\\ 0 & \lvert i-j\rvert>W\end{cases}$ |
| $\lambda=0$   $H=-\Delta$: ext.<br>$\lambda\gg1$   $H\simeq\lambda\hat{V}$: local. | $W\geq\lvert\Lambda\rvert$   $H\sim$ GUE: ext.<br>$W\sim0$   $H\sim$ diagonal: local. |
| $0<\lambda\ll1$, $\Lambda\to\mathbb{Z}^d$<br>$d=1,2$   local.<br>$d=3$   ext. | $W\gg1$, $\Lambda\to\mathbb{Z}^d$<br>$d=1,2$   local.<br>$d=3$   ext. |

# signatures of quantum diffusion

**Green's Function**: $G_{\varepsilon,\Lambda}(E,x,y) = (E + i\varepsilon - H_\Lambda)^{-1}(x,y)$, $x,y \in \Lambda$, $E \in \mathbb{R}$, $\varepsilon > 0$

we have to study $\langle |G_{\epsilon,\Lambda}(E;x,y)|^2 \rangle_H$

## Localization

**(a)** $|x-y| \gg 1$: $\left\langle |G_{\varepsilon,\Lambda}(E,x,y)|^2 \right\rangle \leq \frac{K}{\varepsilon} e^{-|x-y|/\xi_E}$ uniformly in $\varepsilon,\Lambda$

**(b)** $x=y$: $\left\langle |G_{\varepsilon,\Lambda}(E,x,x)|^2 \right\rangle \geq \frac{K}{\varepsilon}$ uniformly in $\varepsilon,\Lambda$

## Diffusion

**(a)** $|x-y| \gg 1$: $\lim_{\Lambda \to \mathbb{Z}^d} \left\langle |G_{\varepsilon,\Lambda}(E,x,y)|^2 \right\rangle \simeq (-\Delta + \varepsilon)^{-1}_{x,y}$

**(b)** $x=y$: $\left\langle |G_{\varepsilon,\Lambda}(E,x,x)|^2 \right\rangle \leq K,$ $\forall \ \varepsilon|\Lambda| = 1$

## Some interesting quantities in physics

- **Conductivy (Kubo formula)**

$$\sigma_{ij}(E) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \lim_{\Lambda\to\infty} \sum_{x\in\Lambda} x_i x_j \; \varepsilon^2 \; \left\langle |G_{\varepsilon,\Lambda}(E,0,x)|^2 \right\rangle \to \begin{cases} = 0 & \text{insulator (loc.))} \\ > 0 & \text{conductor (diff.)} \end{cases}$$

- **Inverse participation ratio**

$$\|\psi^E\|_4^4 = \sum_{x\in\Lambda} |\psi^E(x)|^4 \sim \frac{1}{|\text{supp}(\psi^E)|} \sim \begin{cases} 1 & \psi^E \text{ localized} \\ \frac{1}{|\Lambda|} & \psi^E \text{extended} \end{cases}$$

$$P_\Lambda(E) = \frac{\left\langle \rho_\Lambda(E) \|\psi^E\|_4^4 \right\rangle}{\langle \rho_\Lambda(E)\rangle} = \lim_{\varepsilon\to 0} \frac{\sum_{x\in\Lambda} \varepsilon \left\langle |G_{\varepsilon,\Lambda}(E,x,x)|^2 \right\rangle}{\pi |\Lambda| \langle \rho_\Lambda(E)\rangle} \xrightarrow[\Lambda\to\mathbb{Z}^d]{} \begin{cases} K>0 & \text{insulator} \\ 0 & \text{conductor} \end{cases}$$

# Techniques

multiscale analysis, cluster expansion, renormalization : good in the localization regime

transfert matrix : applies to 1 dimension

delocalization regime : no general technique

# Supersymmetric approach

F. Wegner, K. Efetov

1. change of representation: algebraic operations involving **fermionic** and **bosonic** variables

$$\left\langle |G_\epsilon(E;x,y)|^2 \right\rangle_H \overset{\text{SUSY}}{=} \int d\mu(\{Q_j\}) \mathcal{O}(Q_x, Q_y)$$

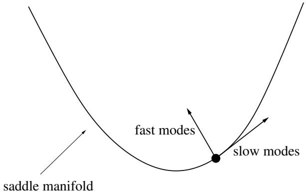
   - $Q_j$ $j \in \Lambda$ (small) matrix containing both fermionic and bosonic elements
   - $d\mu(\{Q_j\})$ strongly correlated $\rightarrow$ saddle analysis

2. restriction to the saddle manifold $\rightarrow$ non linear sigma model
3. control the fluctuations around the saddle manifold

# Saddle analysis: analytic tools

## new integration variables

- slow modes along the saddle manifold $\rightarrow$ **non linear sigma model**
- fast modes away from the saddle manifold




NLSM is believed to contaqin the low energy physics

# non linear sigma model

$$d\mu(Q) \to d\mu^{saddle}(Q) = \left[\prod_{j\lambda\Lambda} dQ_j \delta(Q_j^2 - \mathrm{Id})\right] e^{-F(\nabla Q)} e^{-\varepsilon M(Q)}$$

## features

- saddle is non compact
- no mass: $\varepsilon = \frac{1}{|\Lambda|} \to 0$ as $\Lambda \to \mathbb{Z}^d$
- internal symmetries (from SUSY structure)

**main problem:** obtain the correct $\varepsilon$ behavior
hard to exploit the symmetries $\to$ try something “easier”

## A nice SUSY model for quantum diffusion

vector model (no matrices), Zirnbauer (1991) $\rightarrow$ expected to have same features of exact SUSY model for random band matrix

### main advantages

- after integrating out Grassman variables measure is **positive**
- symmetries are simpler to exploit

$\Rightarrow$ good candidate to develop techniques to treat quantum diffusion

## The model after integrating out the Grassman variables

$$d\mu(t) = [\prod_j dt_j e^{-t_j}] \quad e^{-\mathcal{B}(t)} \quad \det{}^{1/2}[D(t)] \qquad\qquad t_j \in \mathbb{R},\ j \in \Lambda$$

- $\mathcal{B}(t) = \beta \sum_{<j,j'>} (\cosh(t_j - t_{j'}) - 1) + \varepsilon \sum_{j\in\Lambda} (\cosh \ t_j - 1)$
  $= (t, (-\beta\Delta + \varepsilon)t) +$ higher order terms
- $D(t) > 0$ **positive quadratic form**:

$$(f,\ D(t)\ f) = \beta \sum_{<j,j'>} (f_j - f_{j'})^2 \quad e^{t_j + t_{j'}} + \varepsilon \sum_j \ f_j^2 \quad e^{t_j}$$

**Observable**: current-current correlation $\mathcal{O}_{xy} = e^{t_x} D(t)^{-1}_{xy} e^{t_y}$

# Qualitative behavior of the observable

- $\beta \gg 1 \rightarrow$ the field is constant $t_j \simeq t\ \forall j$

$$\mathcal{O}(t)_{xy}=e^{tx+ty}D(t)^{-1}_{xy} \simeq [-\beta\Delta+\epsilon e^{-t}]^{-1}(x,y)$$

- saddle analysis to determine $t$:
  - in $d=1$, $\varepsilon e^{-t} \sim 1/\beta$
  - in $d=2$, $\varepsilon e^{-t} \sim e^{-\beta}$
  - in $d=3$, $t \sim 0$.

so in 1d and 2d we obtain a mass :

$$\mathcal{O}(t)_{xy} \simeq \frac{1}{-\beta\Delta+m_\beta}(x,y) \leq e^{-|x-y|\sqrt{m_\beta}}$$

# Results

phase transition in $d = 3$. Localization $\forall\beta$ at $d = 1$ and for $\beta \ll 1$ at $d = 2$

- **Diffusion** $(d = 3,\ \beta \gg 1)$ $\begin{cases} \text{Thm 1} & t_x \sim \text{constant } \forall x \\ \text{Thm 2} & t_x \sim 0\ \forall x \\ \text{Thm 3} & \int \mathcal{O}(t)_{xy} d\mu(t) \sim (-\Delta+\varepsilon)^{-1}_{xy} \end{cases}$

  No analogous result for random Schrödinger.

- **Localization**

  Thm 4 $\int \mathcal{O}(t)_{xy} d\mu(t) \leq \frac{1}{\varepsilon} e^{-|x-y|/\xi_\beta}$ for $\begin{cases} d = 1 & \forall\beta \\ d = 2, 3 & \beta \ll 1 \end{cases}$

  Same result as in random Schrödinger.

# Fluctuations of $t$: $t_x \sim const\ \forall x$

**Thm 1** **M. Disertori, T. Spencer, M. Zirnbauer 2010**

*For* $\beta \gg 1$

$$\int [\cosh(t_x - t_y)]^m \, d\mu_{\Lambda,\beta}(t) \;\leq\; 2$$

*uniformly in* $\Lambda$ *and* $\varepsilon$. *This bound holds*

- $\forall x, y \in \Lambda$, *and* $\forall m \leq \beta^{1/8}$ *in* $d = 3$ *(published)*
- $\forall \|x - y\| < e^{\beta^{1/3}}$ *and* $\forall m \leq \beta^{1/3}$ *in* $d = 2$ *(unpublished)*
- $\forall \|x - y\| < \frac{\beta}{\ln \beta}$ *and* $\forall m \leq \frac{\beta}{|x-y|}$ *in* $d = 1$ *(unpublished)*

## $t_x \sim 0$

**Thm 2** **M. Disertori, T. Spencer, M. Zirnbauer 2010**

*For $\beta \gg 1$ and $d = 3$ the field $t_x$ remains near zero $\forall x$. More precisely*

$$\int [\cosh(t_x)]^p \, d\mu_{\Lambda,\beta,\varepsilon}(t) \;\leq\; 2 \qquad \forall p \leq 4 \;\forall x \in \Lambda$$

*uniformly in $\Lambda$ and $\varepsilon \geq \frac{1}{|\Lambda|^{1-\alpha}}$, with $\alpha = 1/\ln\beta$.*

(optimal value would be $\alpha = 0$).

# Diffusion

$$\text{Set: } \langle \mathcal{O}_{xy} \rangle = \int \mathcal{O}(t)_{xy} d\mu(t) = \int e^{t_x + t_y} D(t)^{-1}_{xy} d\mu(t).$$

**Thm 3 M. Disertori, T. Spencer, M. Zirnbauer 2010**

*For $d = 3$ and $\beta \gg 1$ we have*

$$\langle \mathcal{O} \rangle \sim (-\beta \Delta_\Lambda + \varepsilon)^{-1}.$$

*More precisely the exists constant $C > 0$ such that*

$$\frac{1}{C}[f; (-\beta\Delta_\Lambda + \varepsilon)^{-1} f] \;\leq\; [f;\; \langle \mathcal{O} \rangle\, f] \;\leq\; C\, [f; (-\beta\Delta_\Lambda + \varepsilon)^{-1} f]$$

$\forall f : \Lambda \to \mathbb{R}^+$., *uniformly in* $\Lambda$ *and* $\varepsilon \geq \frac{1}{|\Lambda|^{1-\alpha}}$.

# Localization

**Thm 4** **M. Disertori, T. Spencer 2010**

*The correlation between $t_x$ and $t_y$ decays exponentially*

- $\forall \beta > 0$ *at* $d = 1$,
- *pour* $\beta \ll 1$ *at* $d = 2, 3$.

$$\langle \mathcal{O}_{xy} \rangle = \int e^{t_x + t_y} D(t)^{-1}_{xy} d\mu(t) \;\leq\; \frac{1}{\varepsilon} e^{-\frac{|x-y|}{l_\beta}} \quad \forall x, y \in \Lambda.$$

*uniformly in* $\Lambda$ *and* $\varepsilon > 0$.

# Bound on the $t$ fluctuations : sketch of the proof

- bound on short scale fluctuations: Ward identites
- conditional bound on large scale fluctuations: Ward identites
- unconditional bound on large scale fluctuations: previous bounds plus induction on scales

## Ward identities

$$\text{SUSY} \;\Rightarrow\; 1 \;=\; \left\langle \cosh^m(t_x - t_y) \left(1 - \frac{m}{\beta} C_{xy}\right) \right\rangle$$

- $0 \;<\; C_{xy} \;:=\; (\delta_x - \delta_y)\, \frac{1}{M(t)}\, (\delta_x - \delta_y)$
- $(f,\; M(t)\; f) \;=\; \sum_{<j,j'>} (f_j - f_{j'})^2 \quad A_{xy}(jj')$
- local conductance:

$$A_{xy}(jj') \;=\; e^{t_j + t_{j'} - t_x - t_y} \cosh(t_x - t_y) > 0$$

**Problem:** $A_{xy}(jj')$ can be very small!

# Short scale fluctuations: $|x-y| \leq 10$

## $C_{xy} \leq 1$ for all $t$ configurations:

$$1 = \left\langle \cosh^m(t_x - t_y) \left(1 - \frac{m}{\beta} C_{xy}\right) \right\rangle \geq \langle \cosh^m(t_x - t_y) \rangle \left(1 - \frac{m}{\beta}\right)$$

$$\Rightarrow \quad \langle \cosh^m(t_x - t_y) \rangle \leq \frac{1}{1 - \frac{m}{\beta}} \leq 2$$

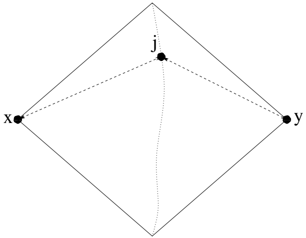
as long as $m \leq \beta/2$

# Large scale fluctuations $|x-y|=l \gg 1$

**no uniform bound on $C_{xy}$: $A_{xy}(jj')$ arbitrarily small**

**In 3 dimensions**: $C_{xy} \leq const$ uniformly in $x, y$ if

$A_{xy}(jj') \geq \frac{1}{|j-x|^\alpha} + \frac{1}{|j-y|^\alpha} \quad \forall j, j' \in R_{xy}$ $\quad R_{xy}$ 3d diamond region




- enough to have a lower bound on $A$ **inside a region** $R_{xy}$
- $A_{xy}(j, j')$ may become small far from $x, y$

# Conditional expectation

$$\bar{\chi}_{jj'} = \begin{cases} 1 & A_{xy}(jj') \geq \frac{1}{|j-x|^\alpha} + \frac{1}{|j-y|^\alpha} \\ 0 & \text{oth} \end{cases} \qquad \bar{\chi}_{xy} = \prod_{j,j' \in R_{xy}} \bar{\chi}_{jj'}$$

then

$$1 \neq \left\langle \cosh^m(t_x - t_y)\ \bar{\chi}_{xy} \left(1 - \frac{m}{\beta} C_{xy}\right) \right\rangle \geq \langle \cosh^m(t_x - t_y)\ \bar{\chi}_{xy} \rangle \left(1 - \frac{m}{\beta}\right)$$

**problem:** $\bar{\chi}_{xy}$ breaks the symmetry$\rightarrow$ make it supersymmetric

$$1 \geq \left\langle \cosh^m(t_x - t_y)\ \bar{\chi}_{xy} \left(1 - \frac{m}{\beta} C_{xy}\right) \right\rangle \geq \langle \cosh(t_x - t_y)^m\ \bar{\chi}_{xy} \rangle \left(1 - \frac{mC}{\beta}\right)$$

$$\Rightarrow \langle \cosh(t_x - t_y)\ \bar{\chi}_{xy} \rangle \leq \frac{1}{\left(1 - \frac{mC}{\beta}\right)}$$

# Unconditional expectation

$$\langle \cosh^m(t_x - t_y) \rangle = \langle \cosh^m(t_x - t_y)\ \bar{\chi}_{xy} \rangle + \text{remainder}$$

$$\text{remainder} = \langle \cosh^m(t_x - t_y)\ \bar{\chi}^c_{xy} \rangle$$

prove that the remainder is small: **induction on scales**

$$\bar{\chi}^c_{xy} : b \text{ first point where } \bar{\chi}_{xb} \text{ fails}$$

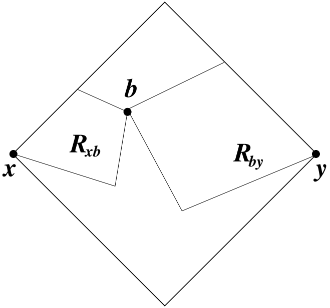
**two ingredients:**

- $\cosh(t_x - t_y) \leq 2\cosh(t_x - t_b)\ \cosh(t_b - t_y)$
- $\bar{\chi}^c_{xb} \leq \left(\frac{\cosh(t_x - t_b)}{|x-b|^\alpha}\right)^p$

$$\langle \cosh^m(t_x - t_y)\ \bar{\chi}^c_{xb} \rangle \leq \frac{2^m}{|x-b|^{p\alpha}} \langle \cosh^{m+p}(t_x - t_b)\ \cosh^m(t_b - t_y) \rangle \leq \frac{2^m 2^2}{|x-b|^{p\alpha}}$$

## iteration over scales: two problems

- $R_{xb}$, $R_{by}$ are no longer diamonds: may collapse
- $m + p > m$: may diverge



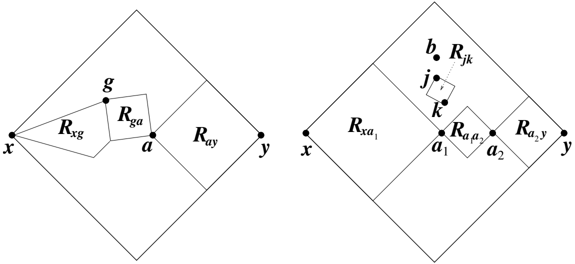

solution: the conditional bound is better

$$\left\langle \cosh^{m+p}(t_x - t_y)\ \bar{\chi}_{xy} \right\rangle \leq 2 \qquad \begin{cases} \text{larger exponent} \\ R_{xy} \text{ need not be a diamond} \end{cases}$$

## Strategy: two cases

**1.** $\exists$ a point $g$ near $b$ good at all scales up to $|x-g|$: then we have a factor $\bar{\chi}_{xg}$ and the diamond $R_{xg}$ can be deformed

**2.** there is no good point near $b$. All points near $b$ are bad <u>at some scale</u>. Test large scales first, then smaller ones.



# Conclusions

## phase transition for the vector model

- toymodel for real symmetric band matrices
- real model! It can be seen as a vertex reinforced jump process. It is also connected with edge reinforced random walk (the $\beta$ parameter is made random and edge dependent)

## open problems

- prove localization for $\beta$ large in 2d (or even in a strip)
- generalize this technique to the real band matrix model (the fermionic term is more complicated, the measure is no longer real)