Delocalization for Random Band Matrices

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dedicated to Tom Spencer

Joint with Antti Knowles, Horng-Tzer Yau and Jun Yin

INTRODUCTION

Universality conjecture for disordered quantum systems (vague):

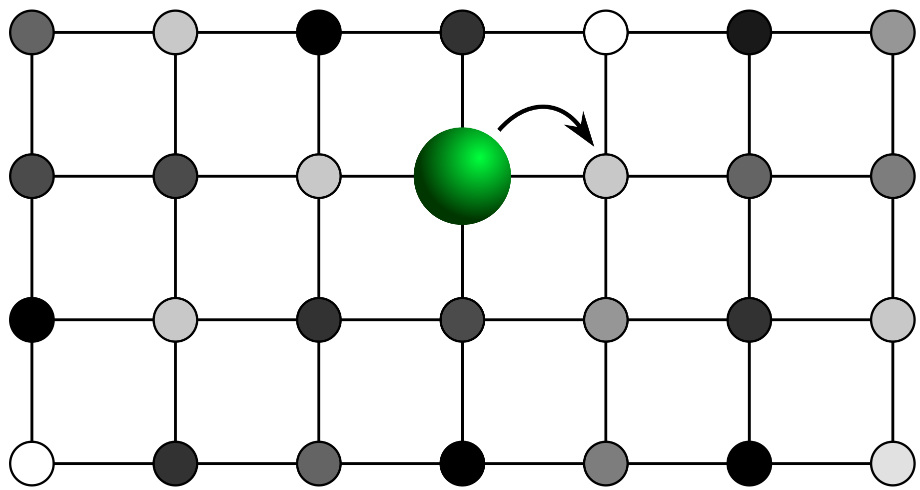
There are two regimes, depending on disorder strength:

i) Strong disorder: localization and Poisson local spectral statistics

ii) Weak disorder: delocalization and random matrix (GUE, GOE) local statistics (RMT). Local statistics depends only on symmetry but is otherwise independent of the details of the system.

Two well studied models

• Random Schrödinger operators: (on-site randomness, short range hopping); represented by a narrow band matrix with nonzero elements at finite distance from the diagonal (E.g. $d = 1$, $-\Delta + \lambda V$ is tridiagonal).



• Wigner random matrices: $H = (H_{xy})_{x,y\in\Lambda}$, with H_{xy} centered i.i.d. up to symmetry constraint ($H = H^*$ or $H = H^t$). They model a mean-field hopping mechanism with random quantum transition rates. No spatial structure (d is irrelevant).

In this talk we consider an intermediate model, the random band matrices (RBM) with band width W in a d-dimensional box $\Lambda \subset \mathbb{Z}^d$. H_{xy} are independent, centered, with variance

$$s_{xy} = \mathbb{E}|H_{xy}|^2, \qquad \sum_y s_{xy} = 1 \quad \forall y$$

such that

$$s_{xy} = 0 \qquad \text{for} \quad |x-y| \geqslant W,$$

More generally,

$$s_{xy} = \frac{1}{W} f\Big(\frac{|x-y|_N}{W}\Big), \qquad f \in \mathcal{S}(\mathbb{R}), \;\; \int f = 1$$

($W = O(1) \sim$ Random Schrödinger; $W = \Lambda$, $d = 1$ is Wigner)

$$s_{xy} = \frac{1}{W} f\Big(\frac{|x-y|_N}{W}\Big),$$

In $d = 1$ the matrix H and its variance matrix S are really banded:

$$H = \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ 0 & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \end{pmatrix}, \qquad S = \begin{pmatrix} {\color{red}*} & {\color{red}*} & {\color{red}*} & 0 & 0 & 0 & 0 \\ {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & 0 & 0 & 0 \\ {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & 0 & 0 \\ 0 & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & 0 \\ 0 & 0 & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} \\ 0 & 0 & 0 & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} \\ 0 & 0 & 0 & 0 & {\color{red}*} & {\color{red}*} & {\color{red}*} \end{pmatrix}$$

$* = h_{ij}$, ${\color{red}*} = s_{ij}$, $\Lambda = \{1,2,3,4,5,6,7\}$, $W = 3$, $N = |\Lambda| = 7$.

ANDERSON TRANSITION FOR BAND MATRICES

$W = O(1)$ [$\sim$ Random Schrödinger]

In $d = 1$ always localized [Goldsheid-Molchanov-Pastur]
In $d > 1$ large energy and band edge localization [Fröhlich-Spencer...]
Poisson statistics [Minami, Klopp-Germinet, ...]

$W = |\Lambda|$ [Wigner ensemble]

Always delocalized [E-Schlein-Yau]
RMT statistics [Dyson-Mehta-Gaudin, E-Schlein-Yau-Yin]

Varying $1 \ll W \ll |\Lambda| = N$ can test the transition

RBM's interpolate between random Schrödinger and Wigner.

PHYSICAL PICTURE FOR BAND MATRICES

The system exhibits metal-insulator transition:

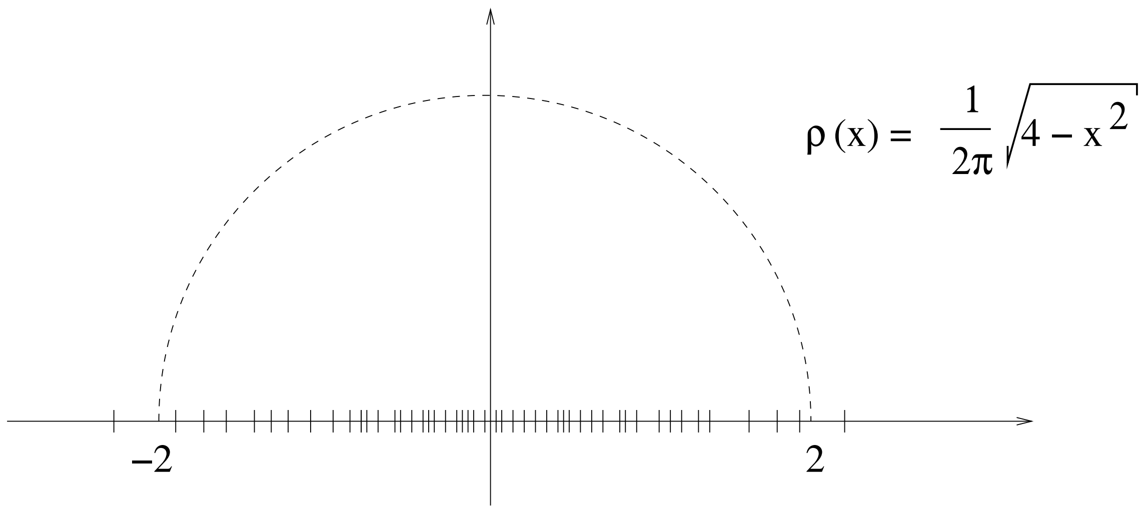
- In $d = 1$ the localization length is $\ell \sim W^2$.
  Complete delocalization and RMT statistics for $N \ll W^2$
  Poisson statistics for $N \gg W^2$

- In $d = 2$ the localization length is ℓ is exponential in W

- In $d \geqslant 3$ the localization length is $\ell \sim L$ (system size, $L^d = N$)
  Complete delocalization, RMT.

Based on SUSY Fyodorov-Mirlin (91) in $d = 1$
and on RG scaling arguments by Abrahams *et. al* (79) in $d = 2$
See: Tom Spencer's overview article/lecture notes on band matrices.

LOCAL SEMICIRCLE LAW



Limiting density of the eigenvalues is $\varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$

$$m(z) = \frac{1}{N} \operatorname{Tr} \frac{1}{H-z} = \frac{1}{N} \operatorname{Tr} G(z), \qquad m_{sc}(z) = \int \frac{\varrho_{sc}(x)}{x-z} \mathrm{d}x$$

FACT: Suppose for some fixed $\eta > 0$ and any E we have

$$|m(z) - m_{sc}(z)| \leqslant \varepsilon, \qquad z = E + i\eta$$

then the local density in spectral windows of size η about E is given by $\varrho_{sc}(E)$ up to a precision ε. We work with G and m.

Theorem [E-Yau-Yin, 2011]. Suppose the rescaled matrix elements $H_{xy}/\sqrt{s_{xy}}$ have subexp decay. Then the local semicircle law holds up to $\eta = \text{Im} z \gg W^{-1}$:

$$|m(z) - m_{sc}(z)| \lesssim \frac{1}{W\eta}, \qquad |G_{xy}(z) - \delta_{xy} m_{sc}(z)| \lesssim \frac{1}{(W\eta)^{1/2}}$$

(with very high probability and modulo log corrections)

Related results

- Global semicircle law for the expectation $\mathbb{E}m$, uniform in η, error W^{-2}, ($d = 3$, Gaussian, with a special covariance). [Disertori-Pinsker-Spencer, 2002] SUSY

- Local semicircle law for the expectation $\mathbb{E}m$ at $\eta = W^{-0.99}$ (in $d = 1$, Bernoulli distr) [Sodin, 2011] Chebysev-expansion

For $\mathbb{E}m$ one needs to compute $\mathbb{E}\text{Tr}G$ and not $\mathbb{E}\text{Tr}G\text{Tr}G^*$ or $\mathbb{E}G_{xx}$

FROM RESOLVENT TO LOWER BOUND ON LOC. LENGTH

Corollary (of local sc law) [E-Yau-Yin]: $\ell \geqslant W^1$. (nontrivial!)

Proof: $|u_\alpha(x)|^2 \le \eta \, \mathrm{Im} G_{xx} \le C\eta \qquad$ if $\eta \ge W^{-1}$

For $\ell \gg W^1$ without control for small η, we need offdiag estimate.
Lemma Suppose for some L and for some $W^{-1} \ll \eta \ll 1$ we have

$$\sup_E \max_{x \neq y} |G_{xy}(E + i\eta)|^2 \lesssim \frac{1}{\eta L}.$$

Then the localization length of most eigenfunctions is at least L.

Proof: Fix $x = 0$. By Ward identity and local semicircle law

$$\mathrm{Im} m_{sc} \le \mathrm{Im} G_{00} = \sum_y \eta |G_{0y}|^2 \lesssim \frac{1}{L} |\mathrm{Supp}(G_{0x})|$$

Thus $\eta |G_{0y}|^2$ has a spread of at least size L. By spectral theorem this would contradict a strong localization on scale $\ell \ll L$:

$$|u_\alpha(0) u_\alpha(y)| \lesssim e^{-|y|/\ell}$$

DELOCALIZATION

Theorem [E-Knowles-Yau-Yin, '12] Let $N \leqslant W^{5/4}$, $\eta \geqslant W^{-1/2}$, then

$$|G_{xy}(z) - \delta_{xy} m_{sc}(z)|^2 \leqslant \frac{1}{N\eta} + \frac{1}{W\sqrt{\eta}}$$

Consequently, we have delocalization ($\ell \sim N$) if $N \leqslant W^{5/4}$.

New method: Self-consistent equation for $\mathbb{E}|G_{xy}|^2$.

Previous results

- Control of the resolvent for $\eta \geqslant W^{-1/3}$ (e^{itH} for $t \leqslant W^{1/3}$). Localization length $\ell \geqslant W^{7/6}$ (via Chebyshev) [E-Knowles, 2010]

- Upper bound for localization length, $\ell \leqslant W^8$ [Schenker, 2008]

RESOLVENT PROFILE

Expect: the system exhibits diffusion on scale W until the localization length is achieved, $\sqrt{t}W \leqslant \ell = W^2$, i.e. up to time $t \leqslant W^2$.

Equivalently, for $\eta \geqslant W^{-2}$, the resolvent should behave as

$$|G_{xy}(z)|^2 \sim T^{\text{det}}_{x-y} := \int \frac{1}{D(Wp)^2 + \eta} e^{ip(x-y)} \mathrm{d}p \approx \frac{\operatorname{Im} m}{N\eta} + \frac{1}{W\sqrt{\eta}} e^{-\sqrt{\frac{\eta}{D}}\frac{|x-y|}{W}}$$

with diff. coeff. $\quad D = D(E) = \frac{1}{2}\sqrt{4 - E^2}D_0, \qquad D_0 = \int f(x)x^2 \mathrm{d}x$

First term is from $p = 0$ mode:

$$\frac{1}{N^2}\sum_{xy} |G_{xy}|^2 = \frac{1}{N\eta}\frac{1}{N}\operatorname{Im}\operatorname{Tr} G \approx \frac{\operatorname{Im} m}{N\eta}$$

The profile is visible only if $\frac{1}{N\eta} \leqslant \frac{1}{W\sqrt{\eta}}$, i.e. $\eta \geqslant (W/N)^2$ corresponding to time $t \leqslant (N/W)^2$, i.e. before $\sqrt{t}W$ reaches N.

Ladder diagram in diagrammatic perturbation theory shows [Spencer]:

$$\mathbb{E}|G_{xy}|^2 \sim \int \frac{S(p)}{1-|m_{sc}|^2 S(p)}\, e^{ip(x-y)}\, \mathrm{d}p \qquad (1)$$

Taylor expansion

$$S(p) := \sum_k e^{ikp} s_{0k} \approx \widehat{f}(Wp) \approx 1 - D_0 (Wp)^2 + \ldots$$

$$|m_{sc}(z)| = 1 - \alpha\eta + O(\eta^2), \qquad \alpha = \alpha(E) = \frac{2}{\sqrt{4-E^2}}$$

thus the small p behaviour is

$$\frac{S(p)}{1-|m_{sc}|^2 S(p)} \approx \frac{1}{D_0(Wp)^2 + \alpha\eta}$$

We can rigorously prove (1) in a certain regime.

Theorem [E-Knowles-Yau-Yin, '12] Let $N \leqslant W^{5/4}$, $\eta \geqslant (W/N)^2$ and

$$s_{xy}(= \mathbb{E}|H_{xy}|^2) = \frac{1}{W} f\Big(\frac{|x-y|_N}{W}\Big), \qquad f \in \mathcal{S}(\mathbb{R}), \quad \int f = 1$$

Let $\mathbb{E}_x$ = expectation in the entries in the x-column of H. Then

$$\begin{aligned} \mathbb{E}_x|G_{xy}|^2 &= T^{\text{det}}_{x-y} + \delta_{xy}|m|^2 + O\Big(\frac{1}{N\eta} + \frac{\delta_{xy}}{W\sqrt{\eta}}\Big) \\ \sum_z s_{xz}|G_{zy}|^2 &= T^{\text{det}}_{x-y} + O\Big(\frac{1}{N\eta}\Big) \\ |G_{xy} - \delta_{xy} m| &\leqslant T^{\text{det}}_{x-y} \end{aligned}$$

All bounds hold with high probability and up to W^{ε} corrections.

Related results: (i) Exponential decay of the analogue of $\mathbb{E}G_{xy}$ and localization in a related lattice SUSY σ-model. [Disertori-Spencer]

(ii) Diffusion up to $t \leq W^{\frac{1}{3}}$ [E-Knowles]: $t = W^{\frac{1}{3}}T$, $x = W\sqrt{W^{\frac{1}{3}}}X$

$$\varrho(t,x) := \mathbb{E}\Big|\langle x|e^{-itH/2}|0\rangle\Big|^2 \sim \int_0^1 \mathrm{d}\lambda\, \frac{4}{\pi}\frac{\lambda^2}{\sqrt{1-\lambda^2}} G(\lambda T, X)$$

LOCAL SPECTRAL UNIVERSALITY

H is a band matrix, U is an indep. standard GUE/GOE. Define

$$H_\varepsilon := (1-\varepsilon)^{1/2} H + \sqrt{\varepsilon} U, \qquad 0 < \varepsilon \ll 1.$$

Matrix of variances

$$S_\varepsilon = (1-\varepsilon) S + \varepsilon |\mathbf{e}\rangle\langle\mathbf{e}|, \qquad \mathbf{e} = N^{-1/2}(1,1,\dots 1)$$

The small GUE component effectively regularizes η and

$$\frac{S(p)}{1 - |m_{sc}|^2 S(p)} \approx \frac{1}{D(Wp)^2 + \varepsilon + \alpha\eta}$$

Theorem [E-Knowles-Yau-Yin, 2012]
(i) For $\varepsilon \gg (N/W^2)^{2/3}$ we have delocalization of evectors.
(ii) For $\varepsilon \gg (N/W^2)^{1/3}$ local spectral stat. coincide with GUE/GOE.

Proof. (i) Resolvent control (as above).
(ii) Part (i) gives the necessary estimate to speed up the Dyson Brownian Motion.

SELF-CONSISTENT EQUATION

Let $P_a = \mathbb{E}_a$, $Q_a = I - P_a$ projections. Define

$$T_{xy} := \sum_a s_{xa}|G_{ay}|^2 = \sum_a s_{xa}P_a|G_{ay}|^2 + \mathcal{E}_{xy}, \qquad \mathcal{E}_{xy} := \sum_a s_{xa}Q_a|G_{ay}|^2$$

Perform P_a by expanding G_{ay} in a:

$$G_{ay} = G_{aa}\sum_p h_{ap}G^{(a)}_{py}, \qquad G^{(a)}(z) = (H^{(a)} - z)^{-1}$$

$$G_{aa} = m_{sc} + Z_a + \ldots, \qquad Z_a \sim \sum_{xy} h_{ax}G^{(a)}_{xy}h_{ya}$$

[$H^{(a)}$ is the minor after removing a-th row/column]. Then

$$|G_{ay}|^2 = |m_{sc}|^2\Big[\delta_{ay} + \sum_{p,q} h_{ap}G^{(a)}_{py}G^{(a)}_{yq}h_{qa} + \ldots\Big]$$

$$P_a|G_{ay}|^2 = |m_{sc}|^2\Big[\delta_{ay} + \sum_{p} s_{ap}|G^{(a)}_{py}|^2 + \ldots\Big]$$

Expansion is in the small parameter $\Lambda := \max_{xy}|G_{xy} - \delta_{xy}m_{sc}|$

$$P_a|G_{ay}|^2 = |m_{sc}|^2\Big[\delta_{ay}+\sum_p s_{ap}|G^{(a)}_{py}|^2+\dots\Big], \qquad \Lambda := \max_{xy}|G_{xy}-\delta_{xy}m_{sc}|$$

Remove upper index by

$$G^{(a)}_{py} = G_{py} - \frac{G_{pa}G_{ay}}{G_{aa}} = G_{py} + O(\Lambda^2)$$

Thus

$$P_a|G_{ay}|^2 = |m_{sc}|^2\Big[\delta_{ay} + \sum_p s_{ap}|G_{py}|^2 + \dots\Big] = |m_{sc}|^2\Big[\delta_{ay} + T_{ap} + \dots\Big],$$

thus

$$T_{xy} = \sum_a s_{xa}P_a|G_{ay}|^2 + \mathcal{E}_{xy} = |m_{sc}|^2\Big[s_{xy} + \sum_a s_{xa}T_{ap}\Big] + \mathcal{E}_{xy}$$

In matrix form

$$T = |m|^2[S + ST] + \mathcal{E} \quad \Longrightarrow \quad T = \frac{|m|^2 S}{1-|m|^2 S} + \frac{|m|^2}{1-|m|^2 S}\mathcal{E}$$

Main term is correct. For the error, we need $\mathcal{E} = O(\Lambda^4)$ and to control the inverse by spectral gap of S which is $\eta + (W/N)^2$.

FLUCTUATION AVERAGING THEOREM

We need to control the fluctuation term

$$\mathcal{E}_{xy} = \sum_a s_{xa} Q_a |G_{ay}|^2 = \sum_a s_{xa}(1 - P_a)|G_{ay}|^2$$

in terms of $\Lambda = \max_{xy} |G_{xy} - \delta_{xy} m_{sc}|$.

Naive size of $\mathcal{E}_{xy}$ is $O(\Lambda^2)$ [neglect W^{-1} from $a = y$ summand]

But $\mathbb{E}\mathcal{E} = 0$; need to exploit a cancellation, like CLT.

Main difficulty: the correlation between $|G_{ay}|^2$ and $|G_{a'y}|^2$ is not sufficiently small for any CLT type argument to work.

We use a detailed expansion for the high moments and identify correlation structure hierarchically.

We will need to control general monomials.

Theorem [Special cases] ($x, y, z, \ldots$ are fixed, "external")
blue = naive size, red = gain:

$$\sum_a s_{xa} G_{ay} \prec \Lambda^{1+1}, \qquad \sum_a s_{xa} Q_a G_{ay} \prec \Lambda^{1+2}$$

$$\sum_a s_{xa} G_{ya} G_{az} \prec \Lambda^{2+1}, \qquad \sum_a s_{xa} G_{ya} G^*_{ay} \prec \Lambda^{2+0}$$

$$\sum_a s_{xa} Q_a \Big[G_{ya} G_{az} \Big] \prec \Lambda^{2+1}, \qquad \sum_a s_{xa} Q_a \Big[G_{ya} G^*_{ay} \Big] \prec \Lambda^{2+2}$$

$$\sum_{ab} s_{xa} s_{yb} G_{za} G_{ab} G^*_{bu} \prec \Lambda^{3+1}, \qquad \sum_{ab} s_{xa} s_{yb} Q_a \Big[G_{za} G_{ab} G^*_{bu} \Big] \prec \Lambda^{3+1},$$

$$\sum_{ab} s_{xa} s_{yb} Q_b \Big[G_{za} G_{ab} G^*_{bu} \Big] \prec \Lambda^{3+2}, \qquad \sum_{ab} s_{xa} s_{yb} Q_a Q_b \Big[G_{za} G_{ab} G^*_{bu} \Big] \prec \Lambda^{3+4},$$

"Good" indices: that connect GG or G^*G^*:

$$G_{xa} G_{ay} \qquad \text{or} \qquad G^*_{xa} G^*_{ay}$$

Gains come either from Q's or from "good" indices.
Sometimes not from both (a good index with Q may be useless)

Theorem [General version, informally]
Denote $\mathbf{a} = (a_1, a_2, \dots a_s)$ the set of summation labels
Let $\mathcal{F} \subset \{1, 2, \dots s\}$ be the set of (indices of) Q-labels.

$$\mathsf{Av}_{a_1,a_2,\dots a_s}\Big(\prod_{j\in F} Q_{a_j}\Big)(\text{monomial of } G_{a_i a_j} \text{ and } G^*_{a_i a_j}) \prec \Lambda^{d+|\mathcal{F}|+|\mathcal{G}|}$$

where

$d := \#\Big\{\text{offdiag. factors}\Big\}$ ("naive size"), $\mathcal{G} :=$ set of "good" indices

Definition of "good" : an index $j \in \mathcal{G}$ if

either $j \in \mathcal{F}$ and $|\nu_i - \nu_i^*| \neq 2$, or $j \notin \mathcal{F}$ and $\nu_i \neq \nu_i^*$.

(ν_i is the number a_i's appearing in any G, ν_i^* is the same for G^*).

Gain from $\mathcal{F}$: Averaging the fluctuation (like CLT, but more subtle)
Gain from $\mathcal{G}$: It has a stable self-consistent equation

Mechanism of the gain from $\mathcal{F}$ (presence of Q's)

Decomposition into a sum of hierarchically classified terms in the spirit of "size versus independence."

$$\mathbb{E}\Big|\sum_a Q_a|G_{ax}|^2\Big|^2 = \mathbb{E}\sum_{ab} Q_a|G_{ax}|^2\, Q_b|G_{bx}|^2$$

If G_{bx} were independent of a (meaning, of the a-th column of H) then this would be zero, since for any general X and a-indep $Y^{(a)}$

$$\mathbb{E}\Big[Q_a(X)\cdot Y^{(a)}\Big] = \mathbb{E}\Big[Q_a(XY^{(a)})\Big] = \mathbb{E}P_aQ_a(XY^{(a)}) = 0$$

Decomposition formula: $$G_{bx} = \underbrace{G^{(a)}_{bx}}_{\text{indep of a}} + \underbrace{\frac{G_{ba}G_{ax}}{G_{aa}}}_{\text{one order smaller}}$$

Such decomposition is done recursively for all resolvent factors up to high order independence wrt. all summation indices:

$$G = G^{(abc)} + G^{(ab)}G + G^{(a)}G^{(c)} + \ldots + G^{(a)}GG + \ldots + GGGG$$

Mechanism of the gain from $\mathcal{G}$ ("good" index)

The quantity $R_{xy} = \sum_a s_{xa} G_{ya} G_{ay}$ satisfies a similar self-consistent equation as $T_{xy} = \sum_a s_{xa} G_{ya} G^*_{ay}$ did before, but

$$R = m^2 \Big[S + SR \Big] + \mathcal{E}, \qquad T = |m|^2 \Big[S + ST \Big] + \mathcal{E}$$

$$\Longrightarrow \qquad R = \frac{m^2 S}{1 - m^2 S}\,\mathcal{E}, \qquad T = \frac{m^2 S}{1 - |m|^2 S}\,\mathcal{E}.$$

$\text{Im}\, m = \text{Im}\, m_{sc}(z) > 0$, $|m|^2 = 1 - O(\eta)$ and S has a small gap, so

$$\left\| \frac{1}{1 - m^2 S} \right\| \leqslant \frac{1}{\text{Im}\, m} \leqslant C, \qquad \left\| \frac{1}{1 - |m|^2 S}\Big|_{1^\perp} \right\| \leqslant \frac{1}{\eta + \left(\frac{W}{N}\right)^2}$$

The complete proof is a complex expansion (bookkept by Feynman graphs) to exploit both effects up to a very high order precision.

SUMMARY

- Localization length $\ell \geqslant W^{7/6}$ and delocalization for $N \leqslant W^{5/4}$.
- Diffusion up to $t \leqslant W^{1/3}$.
- Identified resolvent profile for $N \leqslant W^{5/4}$, $\eta \geqslant (W/N)^2 (\geqslant W^{-1/2})$
- RMT universality for Gaussian convolution of size $\varepsilon \geqslant (N/W^2)^{1/3}$
- General fluctuation averaging mechanism for the Green function.

MAJOR OPEN QUESTIONS:

- Improve $W^{7/6} \leqslant \ell \leqslant W^8$ for the loc. length (closer) to $\ell \sim W^2$; improve $N \leqslant W^{5/4}$ to $N \leqslant W^2$ for deloc.
- Control resolvent for $\eta \ll W^{-1}$.
- RMT univ. without Gaussian component in the deloc. regime.

TIME EVOLUTION: DIFFUSION

Our previous result considered the quantum evolution directly.

Let $x, y \in \Lambda_N = [0, L]^d \subset \mathbb{Z}^d$ label H with $\mathbb{E}\, H_{xy} = 0$ and variance

$$\sigma_{xy}^2 := \mathbb{E}\, |H_{xy}|^2 = \frac{1}{W^d} f\Big(\frac{|x-y|_L}{W}\Big)$$

s.t. $\int f = 1$ and covariance $\Sigma_{ij} := \int_{\mathbb{R}^d} x_i x_j f(x) \mathrm{d}x$.

Define the quantum transition probability from 0 to x in time t by

$$\varrho(t, x) := \mathbb{E} \Big| \langle x | e^{-itH/2} | 0 \rangle \Big|^2,$$

clearly $\varrho(t, \cdot)$ is a probability density on Λ. Goal: $t \gg 1$.

This is like controlling $\mathbb{E} G_{0x}(z) G_{x0}^*(z')$, for $z = E + i\eta$, $z' = E' + i\eta$ with small $\eta \sim 1/t$. Note the expectation and star.

Theorem (Quantum diffusion) [E-Knowles, 2010] Fix $0 < \kappa < 1/3$. For any $T_0 > 0$ and any testfunction $\varphi \in C_b(\mathbb{R}^d)$ we have

$$\lim_{W\to\infty} \sum_{x\in\Lambda_N} \rho(W^{d\kappa}T, x)\, \varphi\Big(\frac{x}{W^{1+d\kappa/2}}\Big) \;=\; \int_{\mathbb{R}^d} \mathrm{d}X\; L(T,X)\,\varphi(X)\,, \qquad (2)$$

uniformly in $N \geqslant W^{1+d/6}$ and $0 \leqslant T \leqslant T_0$. Here

$$L(T,X) := \int_0^1 \mathrm{d}\lambda\, \frac{4}{\pi}\frac{\lambda^2}{\sqrt{1-\lambda^2}}\, G(\lambda T, X) \qquad (3)$$

is a superposition of heat kernels

$$G(T,X) := \frac{1}{(2\pi T)^{d/2}\sqrt{\det\Sigma}}\, e^{-\frac{1}{2T}\, X\cdot\Sigma^{-1}X}\,,$$

$\lambda \in [0,1]$ in (3) represents the fraction of the macroscopic time T that the particle spends moving effectively; the remaining fraction $1-\lambda$ of T represents the time the particle “wastes” in backtracking. Backtracking is due to a self-energy renormalization.

Method: Chebyshev + classification of Feynman diagrams.