FAKULTÄT FÜR PHYSIK

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Supersymmetry and Random Matrices: Avoiding the Saddle–Point Approximation

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SuSy and Random Matrices — in Honor of Tom Spencer

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Outline

- Gaussian Unitary Ensemble and Supersymmetric Itzykson–Zuber Integral
- Supersymmetric Harish-Chandra Integral (beyond the Unitary Case)
- Crossover Transitions and Dyson’s Brownian Motion in Superspace
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- Chiral Random Matrix Theory and Yet Another Group Integral
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Introduction

Efetov (early 80's): supersymmetric non–linear σ model for disordered systems, connection to RMT established

very broad range of applications

Efetov, Schwiete, Takahashi (2004): new foundation by superbosonization

Verbaarschot, Zirnbauer (1985): supersymmetric non–linear σ model for two–point function starting from RMT (zero dimensions)

weak disorder, large matrix dimension $\longrightarrow$ saddle–point approximation, Goldstone modes, coset manifold

some cases: direct and exact solution possible, finite matrix dimension, structural information, applications

Gaussian Unitary Ensemble and

Supersymmetric Itzykson–Zuber Integral

Generating and Correlation Functions

Gaussian ensemble ($\beta = 1, 2, 4$) of $N \times N$ random matrices H

k–level correlations $R_k^{(\beta)}(x_1, \ldots, x_k) = \left. \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k^{(\beta)}(x + J) \right|_{J=0}$

generating function obeys the identity

$$Z_k^{(\beta)}(x + J) = \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \frac{\det(H - x_p - J_p)}{\det(H - x_p^- + J_p)}$$

$$= \int d[\sigma] \exp(-\mathrm{str}\, \sigma^2) \mathrm{sdet}^{-N}(\sigma - x^- - J)$$

where σ is a $2k \times 2k$ or $4k \times 4k$ supermatrix

$\longrightarrow$ drastic reduction of dimensions

GUE Generator — Eigenvalues and Angles

diagonalization $\sigma = u^{-1} s u$ with $u \in \mathrm{U}(k/k)$
and $s = \mathrm{diag}\,(s_{11}, \ldots, s_{k1}, is_{12}, \ldots, is_{k2})$

$$d[\sigma] \;=\; B^2_{k/k}(s) d[s] \mathrm{d}\mu(u)\ , \qquad B_{k/k}(s) = \det\left[\frac{1}{s_{p1} - is_{q2}}\right]_{p,q=1,\ldots,k}$$

generating function, $r = x + J$

$$Z_k^{(2)}(r) \;=\; \int d[s] B^2_{k/k}(s) \int \mathrm{d}\mu(u) \exp\left(-\mathrm{str}\,(u^{-1}su + r)^2\right) \mathrm{sdet}\,^{-N} s^{-}$$

$$=\; 1 + \frac{1}{B_{k/k}(r)} \int d[s] B_{k/k}(s) \exp\left(-\mathrm{str}\,(s + r)^2\right) \mathrm{sdet}\,^{-N} s^{-}$$

everything compact, hyperbolic symmetry not needed

TG (1991)

Correlation Functions

J_p derivatives trivial

$$R_k(x_1, \ldots, x_k) \;=\; \det\left[C_N(x_p, x_q)\right]_{p,q=1,\ldots,k}$$

well–known kernel is found to be a double integral

$$C_N(x_p, x_q) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{ds_{p1}ds_{q2}}{s_{p1} - is_{q2}} \exp\left(-(s_{p1}+x_p)^2 + (is_{q2}+x_q)^2\right)\left(\frac{is_{q2}}{s_{p1}^{-}}\right)^N$$

$$= \exp\left(x_q^2 - x_p^2\right)\sum_{n=0}^{N-1}\varphi_n(x_p)\varphi_n(x_q)$$

determinant structure is a built–in feature of supersymmetry

SuSy Harish-Chandra–Itzykson–Zuber Integral

unitary supergroup $\mathrm{U}(k_1/k_2)$

$$\int d\mu(u) \exp(i \mathrm{str}\, u^{-1} s u r) = \frac{\det[\exp(i s_{p1} r_{q1})] \det[\exp(i s_{p2} r_{q2})]}{B_{k_1/k_2}(s) B_{k_1/k_2}(r)}$$

$$= \frac{\exp(i \mathrm{str}\, s r) + \text{permutations}}{B_{k_1/k_2}(s) B_{k_1/k_2}(r)}$$

with $B_{k_1/k_2}(s) = \frac{\Delta_{k_1}(s_1)\Delta_{k_2}(is_2)}{\prod_{p,q}(s_{p1} - is_{q2})}$, $\Delta_{k_1}(s_1) = \prod_{p<q}(s_{p1} - s_{q1})$

for $k_1/k_2 = N/0, 0/N$ result in ordinary space recovered

TG, J. Math. Phys. 32 (1991) 336
Alfaro, Medina, Urrutia, J. Math. Phys. 36 (1995) 3085, received 28 Nov. 1994, hep-th/9412012
TG, Commun. Math. Phys. 176 (1996) 555, received 22 Nov. 1994

Radial Laplacian and Separability

supersymmetric Itzykson–Zuber integral is eigenfunction of

$$\Delta_s \psi(s,r) \ = \ -\mathrm{str}\, r^2 \ \psi(s,r) \ , \qquad \text{with} \qquad \psi(s,r) = \psi(r,s)$$

and with radial Laplacian

$$\Delta_s = \sum_{p=1}^{k_1} \frac{1}{B^2_{k_1/k_2}(s)} \frac{\partial}{\partial s_{p1}} B^2_{k_1/k_2}(s) \frac{\partial}{\partial s_{p1}} + \sum_{p=1}^{k_2} \frac{1}{B^2_{k_1/k_2}(s)} \frac{\partial}{\partial s_{p2}} B^2_{k_1/k_2}(s) \frac{\partial}{\partial s_{p2}}$$

separability, generalizes ordinary case

$$\Delta_s \frac{f(s)}{B_{k_1/k_2}(s)} \ = \ \frac{1}{B_{k_1/k_2}(s)} \sum_{i=1}^{2} \sum_{p=1}^{k_i} \frac{\partial^2}{\partial s_{pi}^2} f(s)$$

eigenfunctions of flat Laplacian trivial

Supersymmetric Harish-Chandra Integral

(beyond the Unitary Case)

Ordinary Harish–Chandra Integral

$\mathcal{G}$ compact semi–simple Lie group, a, b fixed elements in Cartan subalgebra $\mathcal{H}_0$ of Lie algebra of $\mathcal{G}$

$$\int_{\mathcal{G}} \exp\left(\mathrm{tr}\, U^{-1} a U b\right) d\mu(U) = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \frac{\exp\left(\mathrm{tr}\, w(a) b\right)}{\Pi(a) \Pi(w(b))}$$

$\Pi(a)$ product of all positive roots of $\mathcal{H}_0$, $\mathcal{W}$ Weyl reflection group

everything stays in the space of the Lie group and its algebra !

Harish–Chandra (1957)

Supersymmetric Harish–Chandra Integral

supersymmetric Itzykson–Zuber integral: case of $\mathrm{U}(k_1/k_2)$

$\longrightarrow$ most interesting remaining case is $\mathrm{UOSp}(k_1/2k_2)$

conjecture: Serganova (1992) and Zirnbauer (1996)

proof: TG, Kohler (2002)

Laplacian Δ_A over Lie superalgebra $\mathrm{uosp}(k_1/2k_2)$

construct radial part Δ_a over Cartan subalgebra

identify Harish–Chandra integrals as eigenfunctions of Δ_a

Δ_a is separable ! $\longrightarrow$ solution of eigenequation trivial

proof also includes Lie groups in ordinary space

Crossover Transitions and Dyson's Brownian Motion in Superspace

Crossover Transitions

random matrix H drawn from GOE, GUE, GSE
(random) matrix $H^{(0)}$ with arbitrary $P^{(0)}(H^{(0)})$
interpolating ensemble with α “strength of chaos”

$$H(\alpha) \;=\; H^{(0)} + \alpha H$$

correlations on local scale of mean level spacing D depend on

$\xi_p = x_p/D$ and $\lambda = \alpha/D$

fictitious time $t = \alpha^2/2$ and locally $\tau = t/D^2 = \lambda^2/2$

Brownian motion transports initial $H^{(0)}$ into chaos, $\tau \to \infty$

Dyson, French, Pandey, Mehta, ...

Dyson's Brownian Motion in Superspace

generator for k–level correlations of initial condition

$$Z_k^{(0)}(s) \ = \ \int P^{(0)}(H^{(0)}) \ \mathrm{sdet}^{\,-1}\big(1 \otimes H^{(0)} - s \otimes 1\big) \, d[H^{(0)}]$$

generator for crossover correlations, $r = x + J$

diffusion $$\Delta_r Z_k(r,t) \ = \ \frac{\partial}{\partial t} Z_k(r,t)$$

convolution $$Z_k(r,t) \ = \ \int \Gamma_k(s,r,t) Z_k^{(0)}(s) B_{k/k}(s) d[s]$$

kernel $$\Gamma_k(s,r,t) \ = \ \int \exp\left(-\frac{1}{t}\mathrm{str}\,(u^{-1}su - r)^2\right) d\mu(u)$$

where $u \in \mathrm{U}(k/k)$ or $u \in \mathrm{UOSp}(2k/2k)$

diagonalizes hierarchic equations of French et al. (1988)

TG (1996)

Special Case: GUE plus External Field

for $P^{(0)} \sim \delta \quad \longrightarrow \quad H^{(0)} = \mathrm{diag}\,(E_1^{(0)}, \ldots, E_N^{(0)})$ fixed

transition to GUE, $H(t) = H^{(0)} + \sqrt{2t}H$, find immediately

$$R_k(x_1, \ldots, x_k) \ = \ \det\left[C_N(x_p, x_q)\right]_{p,q=1,\ldots,k}$$

$$C_N(x_p, x_q) = \frac{1}{2t} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \frac{ds_{p1} ds_{q2}}{s_{p1} - is_{q2}} \exp\left(-\frac{(s_{p1} + x_p)^2}{2t} + \frac{(is_{q2} + x_q)^2}{2t}\right) \prod_{n=1}^{N} \frac{is_{q2} - E_n^{(0)}}{s_{p1}^{-} - E_n^{(0)}}$$

cf. Brézin, Hikami (1996)

average over $H^{(0)}$ destroys determinant structure

General Case on Local Scale

local scale, unfolding $\xi = x/D$, $j = J/D$, $\tau = t/D^2$
and $\rho = \xi + j = r/D$, $s' = s/D$

$z_k(\rho,\tau) = \lim_{N\to\infty} Z_k(r,t)$ and $z_k^{(0)}(s') = \lim_{N\to\infty} Z_k^{(0)}(s)$

generator for local crossover correlations obeys same equations:

diffusion $$\Delta_\rho z_k(\rho,\tau) = \frac{\partial}{\partial\tau} z_k(\rho,\tau)$$

convolution $$z_k(\rho,\tau) = \int \Gamma_k(s',\rho,\tau) z_k^{(0)}(s') B_{k/k}(s') d[s']$$

diffusion process to chaos is scale invariant !

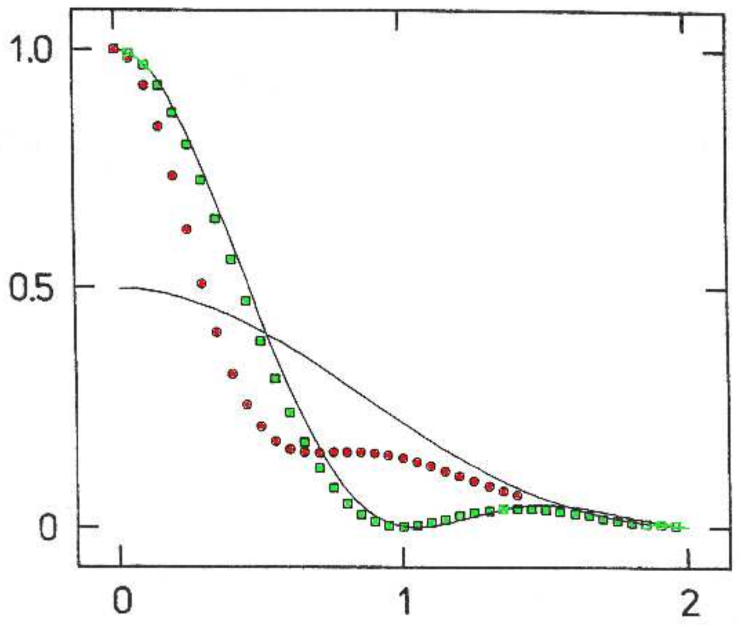
TG (1996)

Some Applications

Symmetry Breaking

ensemble $H(\alpha) = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} + \alpha H\,,$ all from GUE's

$Y_2(\omega, \lambda)$



ω

$$\omega = \xi_2 - \xi_1$$

$$\lambda = \alpha / D = 0, 0.3, 0.9, \infty$$

$$X_2(\omega, \lambda) = 1 - Y_2(\omega, \lambda)$$

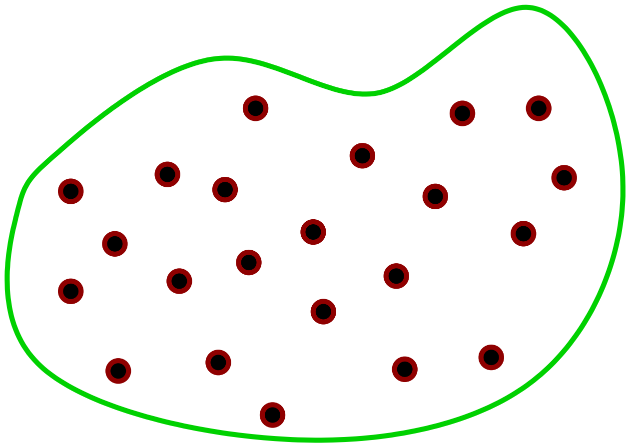
statistical enhancement

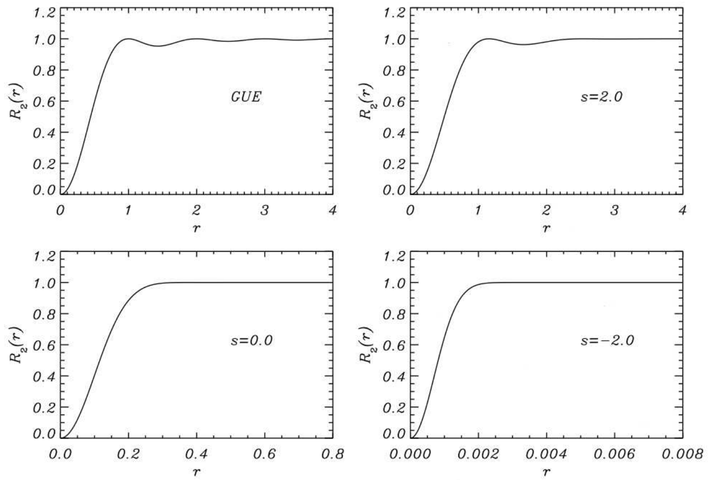
TG, Weidenmüller (90's)

Chaotic Billiard with Random Scatterers

unitary, L scatterers, strength α, generator of k-level correlations

$$Z_k(x+J) \;=\; \int \exp\left(-\mathrm{str}\,\sigma^2 + \mathrm{str}\,\sigma(x+J)\right) \frac{\mathrm{sdet}^{\,L}(1+\alpha\sigma)}{\mathrm{sdet}^{\,N}\sigma} d[\sigma]$$





GUE statistics for all λ in bulk, transition to Poisson for tail states

TG, Stöckmann (2004)

Chiral Random Matrix Theory

and Yet Another Group Integral

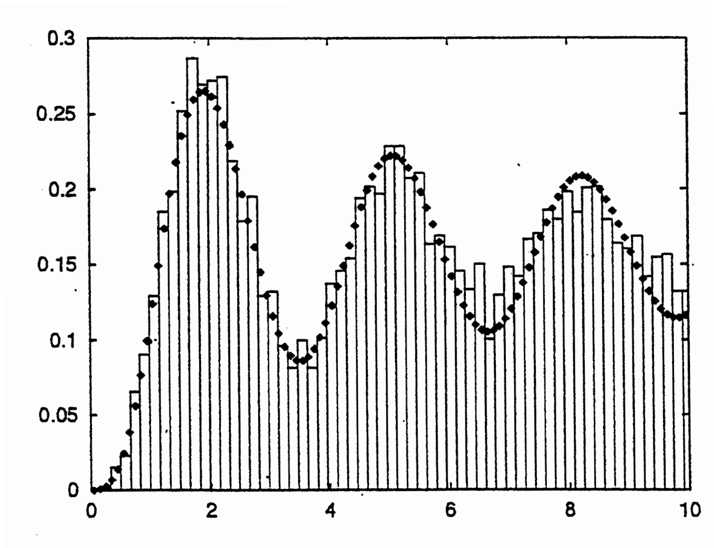
Paris, April 2012

Chiral Random Matrix Theory

Dirac operator has chiral symmetry, in chiral basis it is

$$i\not{D} = \begin{bmatrix} 0 & i\not{D}^c \\ (i\not{D}^c)^\dagger & 0 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}, \quad W \text{ random matrix}$$

compare chRMT with lattice gauge calculations



spectral density

hard edge

microscopic limit (zoom)

chiral condensate

Verbaarschot (1993)

General Case with Temperatures and Masses

random complex $N \times N$ matrix W, chiral condensate Σ

$$\exp\left(-N\Sigma\,\mathrm{tr}\,WW^\dagger\right)\prod_{f=1}^{N_f}\det\left(\begin{bmatrix} 0 & W+T \\ W^\dagger + T^\dagger & 0 \end{bmatrix} + im_f\right)$$

temperatures $T = \mathrm{diag}\,(T_1, \ldots, T_N)$ and quark masses $m_f,\ f = 1, \ldots, N_f$ with N_f number of flavors

various correlation functions for comparison with lattice gauge

sacling property: replace Σ with

$$\Theta = \Sigma\vartheta\,, \qquad \text{where} \qquad 1 = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(\Sigma T_n)^2 + \vartheta^2}$$

microscopic k–point correlations same as for $T = 0$

TG, Seif, Wettig, Wilke (late 90's)

SuSy Berezin–Karpelevich Integral

unitary supergroup, $u \in \mathrm{U}(k_1/k_2)$ and $v \in \mathrm{U}(k_1/k_2)/\mathrm{U}^{k_1+k_2}(1)$,
diagonal supermatrices s and r

$$\int d\mu(u) \int d\mu(v) \exp(i \mathrm{Re}\, \mathrm{str}\, usvr) \;=\; \frac{\det[J_0(s_{p1}r_{q1})]\, \det[J_0(s_{p2}r_{q2})]}{B_{k_1/k_2}(s^2) B_{k_1/k_2}(r^2)}$$

$\sigma = usv$ is polar decomposition of complex square supermatrix,
further generalization to complex rectangular supermatrices

for $k_1 = 0$ or $k_2 = 0$ result in ordinary space recovered

SuSy: TG, Wettig, J. Math. Phys. 37 (1996) 6395

ordinary case: Berezin, Karpelevich, Dok. Akad. Nauk SSSR 118 (1958) 9

ordinary case: Jackson, Şener, Verbaarschot, Phys. Lett. B387 (1996) 355

A Certain Class of Matrix Bessel Functions

Matrix Bessel Functions versus HC Integral

$N \times N$ matrices, x, k eigenvalues of H, K

$$\Phi_N^{(\beta)}(x,k) \;=\; \int d\mu(U)\exp(i\mathrm{tr}\,HK) \;=\; \int d\mu(U)\exp(i\mathrm{tr}\,U^{-1}xUk)$$

eigenfunctions of Laplacian in radial space

x, k in general not in Cartan subalgebra $\longrightarrow$ no separability !

$U \in \mathrm{O}(N)$	if H, K real symmetric	$\mathrm{U}(N)/\mathrm{O}(N)$	$\beta = 1$
$U \in \mathrm{U}(N)$	if H, K Hermitean	$\mathrm{U}(N)/1$	$\beta = 2$
$U \in \mathrm{USp}(2N)$	if H, K selfdual	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\beta = 4$

and in these symmetric superspaces

$\mathrm{U}(k_1/k_2)/1$ and $\mathrm{Gl}(k_1/2k_2)/\mathrm{OSp}(k_1/2k_2)$ (two forms)

Recursion Formula for Arbitrary $\beta > 0$

map group integral to recursion integral in the radial coordinates

$$\Phi_N^{(\beta)}(x,k) \;=\; \int d\mu(x') \, \exp\left(i \left(\sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n \right) k_N \right) \Phi_{N-1}^{(\beta)}(x', \widetilde{k})$$

where $\widetilde{k}$ denotes the variables $\quad k_n, \;\; n = 1, \ldots, (N-1)$

$x'_n, \;\; n = 1, \ldots, (N-1)$ integration variables, $\quad x_n \leq x'_n \leq x_{n+1}$

measure $\quad d\mu(x') \;=\; \dfrac{\Delta_{N-1}(x')}{\Delta_N^{\beta-1}(x)} \left(-\prod_{n,m}(x_n - x'_m) \right)^{(\beta-2)/2} \prod_{n=1}^{N-1} dx'_n$

$\longrightarrow$ Lie groups and more general structures !

observation: finite number of terms for all even β

TG, Kohler, JMP 43 (2002) 2707, `math-ph/0011007`

recent progress: Bergère, Eynard, JPA 42 (2009) 265201, `arXiv:0805.4482`

Explicit Example for $\beta = 4$ and $N = 4$

use recursion formula to calculate integral over $\mathrm{USp}(8)$

$$\begin{aligned}
\Phi_4^{(4)}(x,k) &= \int_{\mathrm{USp}(8)} d\mu(U) \exp(i \mathrm{tr}\, U^{-1} x U k) \\
&= \sum_{\mathrm{Weyl}} \frac{\exp(i \mathrm{tr}\, xk)}{\Delta_4^3(x)\Delta_4^3(k)} \left(\prod_{n<m} (1 - z_{nm}) \right. \\
&\qquad \left. + \sum_n \prod_{m \neq n} (1 - z_{nm}) + \sum_{n,m} (1 - z_{nm}) \right)
\end{aligned}$$

with composite variable $z_{nm} = i(x_n - x_m)(k_n - k_m)/2$

TG, Kohler, `math-ph/0011007`, JMP 43 (2002) 2707

used in: Brézin, Hikami, `math-ph/0103012`, CMP 223 (2001) 363

SuSy Matrix Bessel Function for $\mathrm{UOSp}(2/2)$

good news: recursion formula can be extended to superspaces

$$\int_{\mathrm{UOSp}(2/2)} d\mu(u) \exp(i\mathrm{str}\, u^{-1} s u r) \; = $$

$$\mathrm{e}^{ir_1(s_{11}+s_{21})+i2r_2s_2}\Big((s_{11}+s_{21}-i2s_2)(r_1-ir_2)$$

$$-2(s_{11}s_{21}-is_2(s_{11}+s_{21}))(r_1-ir_2)^2\Big)$$

with $s = \mathrm{diag}\,(s_{11}, s_{21}, is_2, is_2)$ and $r = \mathrm{diag}\,(r_1, r_1, ir_2, ir_2)$

$\longrightarrow$ everything you need to calculate all level densities

TG, Kohler, `math-ph/0012047`, JMP 43 (2002) 2741

SuSy Matrix Bessel Function for $\mathrm{UOSp}(4/4)$

$$
\mathrm{e}^{-i\mathrm{tr}\, r_2 s_2}\prod_{i,j}(r_{i1}-ir_{j2})\Bigg[\left(\frac{1}{\Delta_2^2(ir_2)\Delta_2^2(is_2)}+\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\right)
$$

$$
\left(\prod_{i=1}^{2}(r_{i1}-ir_{12})\prod_{j=1}^{4}(s_{j1}-is_{12})+\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{12})\left(r_{11}+r_{21}-ir_{12}-\frac{\partial^{\rightarrow}}{\partial s_{i1}}\right)\right)
$$

$$
\left(\prod_{i=1}^{2}(r_{i1}-ir_{22})\prod_{j=1}^{4}(s_{j1}-is_{22})+\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{22})\left(r_{11}+r_{21}-ir_{22}-\frac{\partial}{\partial s_{i1}}\right)\right)
$$

$$
-\frac{1}{\Delta_2^3(ir_2)\Delta_2^3(is_2)}\sum_{i=1}^{4}\prod_{j\neq i}^{4}(s_{j1}-is_{12})(s_{j1}-is_{22})
$$

$$
\left(r_{11}^2+r_{21}^2+r_{11}r_{21}-(ir_{12}+ir_{22})(r_{11}+r_{21})+ir_{12}ir_{22}+\mathrm{str}\, r\frac{\partial}{\partial s_{i1}}\right)
$$

$$
-\frac{1}{\Delta_2^3(ir_2)\Delta_2^4(is_2)}\sum_{\substack{i=1\\ j=1}}^{4}\prod_{l\neq i}^{4}(s_{l1}-is_{12})\prod_{l\neq j}^{4}(s_{l1}-is_{22})\left(\frac{\partial}{\partial s_{i1}}-\frac{\partial}{\partial s_{j1}}\right)\Bigg]\Phi_4^{(1)}(s_1,r_1)
$$

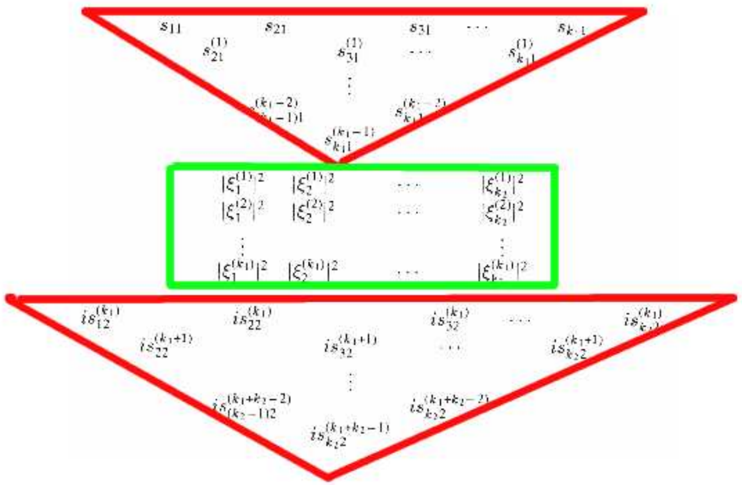
for all two–level correlations — well, in principle

TG, Kohler, `math-ph/0012047`, JMP 43 (2002) 2741

SuSy Gelfand–Tzetlin Coordinates

coset decomposition in ever smaller spheres, ordinary space:

$$\mathrm{U}(N;\beta) = \frac{\mathrm{U}(N;\beta)}{\mathrm{U}(N-1;\beta)} \otimes \frac{\mathrm{U}(N-1;\beta)}{\mathrm{U}(N-2;\beta)} \otimes \cdots \otimes \mathrm{U}(1;\beta)$$



- SuSy GT pattern
- here: $\mathrm{U}(k_1/k_2)$
- another coset structure
- representations ?
- "actions"
- role of Grassmannians

next: Guillemin–Sternberg collective integrability to superspaces

TG (1996) — TG, Kohler (2003)

“Supersymmetry without Supersymmetry”

Random Matrix Kernels

kernels of Random Matrix correlation functions given by lowest order averages of ratios of characteristic polynomials

$$K_N^{(\beta)}(x_q, x_p) \sim \operatorname{Im} \frac{Z_1^{(\beta)}(x_p, x_q) - Z_1^{(\beta)}(0,0)}{x_q - x_p}$$

$$Z_1^{(\beta)}(x_p, x_q) \sim \int \exp\left(-\frac{\beta}{2}\mathrm{tr}\, H^2\right)\left(\frac{\det(H - x_q)}{\det(H - x_p^-)}\right)^{|\gamma|} d[H]$$

$$\sim \int \exp\left(-\frac{\beta}{2|\gamma|}\mathrm{str}\,\sigma^2\right) \mathrm{sdet}^{\,-\beta N/2\gamma}(\sigma - x^-)d[\sigma]$$ with

$2 \times 2\ (\beta = 2),\ 4 \times 4\ (\beta = 1, 4)$ supermatrix σ

Grönqvist, TG, Kohler (2004)

Borodin, Strahov (2006): arbitrary order, factorizing probability density yields determinants/Pfaffians in terms of $Z_1^{(\beta)}(x_p, x_q)$

Supersymmetry without Mapping onto Superspace

factorization $P(E) = \prod_{n=1}^{N} \widetilde{P}(E_n)$, example: Hermitean matrices

$$Z_k(r) = \int P(H) \prod_{p=1}^{k} \frac{\det(H - r_{p2})}{\det(H - r_{p1})} d[H]$$

$$= \int \prod_{n=1}^{N} \left[\widetilde{P}(E_n) \prod_{p=1}^{k} \frac{E_n - r_{p2}}{E_n - r_{p1}} \right] \Delta_N^2(E) d[E]$$

Vandermonde determinant

$$\Delta_N(E) = \prod_{n,l} (E_n - E_l) = \det \left[E_n^{l-1} \right]_{n,l=1,\ldots,N}$$

Berezinian (Jacobian) in Hermitean Superspace

k eigenvalues s_{p1} in bosonic sector,
m eigenvalues s_{q2} in fermionic sector

$$s = \mathrm{diag}\,(s_1, s_2) = \mathrm{diag}\,(s_{11}, \ldots, s_{k1}, s_{12}, \ldots, s_{m2})$$

Berezinian is $B^2_{k/m}(s)$, for $m \geq k$ we find

$$B_{k/m}(s) = \frac{\Delta_k(s_1)\Delta_m(s_2)}{\prod_{p,q}(s_{p1} - s_{q2})} = \det \begin{bmatrix} \left[\dfrac{1}{s_{p1} - s_{q2}}\right]_{p=1,\ldots,k,q=1,\ldots,m} \\ \left[s_{q2}^{p-1}\right]_{q=1,\ldots,m,p=1,\ldots,m-k} \end{bmatrix}$$

this is a $m \times m$ determinant !

Cauchy–Vandermonde identity: Basor, Forrester (1994)

connection to SuSy: Kieburg, TG (2010)

Integrand is Ratio of Berezinians

$$r = \mathrm{diag}\,(r_{11}, \ldots, r_{k1}, r_{12}, \ldots, r_{k2})$$

crucial identity

$$\prod_{n=1}^{N} \prod_{p=1}^{k} \frac{E_n - r_{p2}}{E_n - r_{p1}} \Delta_N(E) \frac{B_{k/k}(r)}{B_{k/k}(r)} = \frac{B_{k/k+N}(r, E)}{B_{k/k}(r)}$$

makes integral elementary

$$Z_k(r) = \int \prod_{n=1}^{N} \widetilde{P}(E_n) \frac{B_{k/k+N}(r, E)}{B_{k/k}(r)} \Delta_N(E) d[E]$$

$$= \frac{1}{B_{k/k}(r)} \int \prod_{n=1}^{N} \widetilde{P}(E_n) E_n^{n-1} B_{k/k+N}(r, E) d[E]$$

Supersymmetry Implies Decomposition Formula

straightforward reordering of determinants gives

$$Z_k(r) = \frac{1}{B_{k/k}(r)} \det \left[\frac{Z_1(r_{p1}, r_{q2})}{r_{p1} - r_{q2}} \right]_{p,q=1,\ldots,k}$$

- reminiscent of Wick theorem: $1 \times k \to k \times 1$
- $Z_1(r_{p1}, r_{q2})$ explicitly known in terms of $\widetilde{P}(E_n)$
- specific form of $\widetilde{P}(E_n)$ never used
- clear separation of algebraic and analytic features
- applicability is very general

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Some Matrix Ensembles Yielding Determinants

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $g(z)$
Hermitian ensemble [57, 31, 62, 32, 34, 35]	$\widetilde{P}\,(\mathrm{tr}\,H^m, m \in \mathbb{N})$ $H = H^\dagger$	H	$P(x)\delta(y)$
circular unitary ensemble (unitary group) [37, 63, 14, 13, 38, 20, 64, 21]	$\widetilde{P}\,(\mathrm{tr}\,U^m, m \in \mathbb{N})$ $U^\dagger U = \mathbb{1}_N$	U and $U^\dagger$	$P\,(e^{\imath\varphi})\,\delta(r-1)$
Hermitian chiral (complex Laguerre) ensemble [65, 66, 67, 7]	$\widetilde{P}\,(\mathrm{tr}(AA^\dagger)^m, m \in \mathbb{N})$ A is a complex $N \times M$ matrix with $N \leq M$	$AA^\dagger$	$P(x)x^{M-N}\Theta(x)\delta(y)$
Gaussian elliptical ensemble [9, 10, 11, 36]; for $\tau = 1$ complex Ginibre ensemble	$\exp\left[-\frac{(\tau+1)}{2}\mathrm{tr}\,H^\dagger H\right]\times$ $\times\exp\left[-\frac{(\tau-1)}{2}\mathrm{Re}\;\mathrm{tr}\,H^2\right]$ H is a complex matrix; $\tau > 0$	H and $H^\dagger$	$\exp\left[-r^2\left(\sin^2\varphi + \tau\cos^2\varphi\right)\right]$
Gaussian complex chiral ensemble [12]	$\exp\left[-\mathrm{tr}\,A^\dagger A - \mathrm{tr}\,B^\dagger B\right]$ $C = \imath A + \mu B$ $D = \imath A^\dagger + \mu B^\dagger$ A and B are complex $N \times M$ matrices with $N \leq M$	CD and $D^\dagger C^\dagger$	$K_{M-N}\left(\frac{1+\mu^2}{2\mu^2}r\right)r^{M-N}\times$ $\times\exp\left(\frac{1-\mu^2}{2\mu^2}r\cos\varphi\right)$

Some Matrix Ensembles Yielding Pfaffians

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability densities $g(z_1, z_2)$ and $\tilde{g}(z_1, z_2)$	probability density $h(z)$
real symmetric matrices [31, 24, 18]	$\widetilde{P}\,(\mathrm{tr}\, H^m, m \in \mathbb{N})$ $H = H^T = H^*$	H	$P(x_1)P(x_2)\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)$	$P(x)\delta(y)$
circular orthogonal ensemble [4]	$\widetilde{P}\,(\mathrm{tr}\, U^m, m \in \mathbb{N})$ $U^\dagger U = \mathbb{1}_N$ and $U^T = U$	U and $U^\dagger$	$P(e^{\imath\varphi_1})P(e^{\imath\varphi_2})\times$ $\times\delta(r_1 - 1)\delta(r_2 - 1)\times$ $\times\Theta(\varphi_2 - \varphi_1)$	$P(e^{\imath\varphi})\delta(r - 1)$
real symmetric chiral (real Laguerre) ensemble [21, 32, 33, 34]	$\widetilde{P}\left(\mathrm{tr}(AA^T)^m, m \in \mathbb{N}\right)$ A is a real $N \times M$ matrix with $\nu = M - N \geq 0$	AA^T	$P(x_1)P(x_2)\times$ $\times(x_1x_2)^{(\nu-1)/2}\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)$	$P(x)\delta(y)x^{(\nu-1)/2}$
Gaussian real elliptical ensemble; for $\tau = 1$ real Ginibre ensemble [10, 15, 35, 16, 36, 25] [37, 38, 23]	$\exp\left[-\frac{(\tau+1)}{2}\,\mathrm{tr}\, H^T H\right]\times$ $\times\exp\left[-\frac{(\tau-1)}{2}\,\mathrm{tr}\, H^2\right]$ $H = H^*$; $\tau > 0$	H	$\prod\limits_{j\in\{1,2\}} \exp\left[-\tau x_j^2\right]\times$ $\times\sqrt{\mathrm{erfc}(\sqrt{2(1+\tau)}y_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)+$ $+2\imath\delta^2(z_1 - z_2^*)\Theta(y_1)]$	$\exp(-\tau x^2)\delta(y)$
Gaussian real chiral ensemble [9, 39]	$\exp\left[-\,\mathrm{tr}\, A^T A - \mathrm{tr}\, B^T B\right]$ $C = A + \mu B$ $D = -A^T + \mu B^T$ A and B are real $N \times M$ matrices with $\nu = M - N \geq 0$	CD	$\prod\limits_{j\in\{1,2\}} \exp\left[-2\eta_- z_j\right]\times$ $\times\lvert z_j\rvert^\nu\sqrt{f(2\eta_+ z_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)+$ $+2\imath\delta^2(z_1 - z_2^*)\Theta(y_1)]$	$x^{\nu/2}\exp\left[-2\eta_- x\right]\times$ $\times K_{\nu/2}(2\eta_+ x)\delta(y)$

Supersymmetry for

Arbitrary Invariant Probability Densities

SuSy for Arbitrary Invariant Probability Densities

$P(H)$ arbitrary and invariant $\longrightarrow$ generating function, $\kappa = x + J$

$$\int d[H] P(H) \prod_{p=1}^{k} \frac{\det(H - \kappa_{p2})}{\det(H - \kappa_{p1})} = \int d[\rho] \exp\left(-i \,\mathrm{str}\, \rho\kappa\right) \Omega(\rho) \mathrm{sdet}^{\,+N} \rho \Big|_{r_1 > 0}$$

$$= \int d[\sigma] Q(\sigma) \mathrm{sdet}^{\,-N} (\sigma - \kappa)$$

$$\Omega(K) = \int P(H) \exp\left(-i \,\mathrm{tr}\, KH\right) d[H] \ , \ \ Q(\sigma) = \int \Omega(\rho) \exp\left(+i \,\mathrm{str}\, \sigma\rho\right) d[\rho]$$

$$\Omega(\mathrm{tr}\, K, \mathrm{tr}\, K^2, \mathrm{tr}\, K^3, \ldots) = \Omega(\mathrm{str}\, \rho, \mathrm{str}\, \rho^2, \mathrm{str}\, \rho^3, \ldots)$$

TG (2006) — Littelmann, Sommers, Zirnbauer (2008) — Kieburg, Grönqvist, TG (2009) — Kieburg, Sommers, TG (2009)

Reduction to Eigenvalue Integrals in Unitary Case

apply SuSy Harish-Chandra–Itzykson–Zuber integral,
do derivatives with respect to source variables

$$R_k(x_1,\ldots,x_k) = \int d[r]\, B_k(r) \exp\left(-i\,\mathrm{trg}\, xr\right) \Omega(r) \mathrm{sdet}^{\,+N} r \Big|_{r_1>0}$$

with Berezinian (Jacobian) $B_k(r) = \det\left[\frac{1}{r_{p1} - i r_{q2}}\right]_{p,q=1,\ldots,k}$

if $\Omega(K)$ is known, correlations reduced to $2k$ integrals for arbitrary invariant $P(H)$, including those which do not factorize

Summary and Conclusions

- in certain cases, exact evaluation of SuSy model
- determinant structure built–in property of SuSy
- full control over matrix dimension advantageous
- SuSy Harish-Chandra, SuSy Berezin–Karpelevich: closed form
- recursion formulae for Matrix Bessel Functions: some closed form results
- SuSy Dyson's Brownian Motion: scale invariance
- some applications: crossovers, temperature scaling
- SuSy structures without mapping to superspaces
- SuSy for arbitrary invariant probability densities