

# A non-convex gradient model

notes for the talk at the conference “SuSy and Random Matrices”, Paris, 2012

(based on discussions with David Brydges and Tom Spencer)

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This talk is about 2 statistical mechanics problems in 2D.

**A)** A non-convex gradient model: consider a periodic box  $\{-L, \dots, L\}^2$  ( $L$  eventually tends to  $\infty$ ), and define a measure on functions  $\phi$  from the box to  $\mathbb{R}$  (with  $\phi(0) = 0$ ) by

$$\begin{aligned} \frac{d\mu(\phi)}{d\phi} &= \frac{1}{Z} e^{-H(\phi)} \\ &= \prod_{x \sim y} \left\{ e^{-\frac{\beta}{2}(\phi(x) - \phi(y))^2} + \delta e^{-\frac{\beta}{2}(\phi(x) - \phi(y) - 1)^2} + \delta e^{-\frac{\beta}{2}(\phi(x) - \phi(y) + 1)^2} \right\} \end{aligned}$$

(here  $\beta, \delta \geq 0$  are two parameters,  $\beta$  will be mapped to the inverse temperature under a transformation to be explained in the sequel).

The model is motivated by the recent work of Biskup–Kotecký, Biskup–Spohn, and Cotar–Deuschel–Müller on non-convex gradient models.

In this talk, we focus on the magnitude of fluctuations:

**Question.**  $\langle \phi(x)^2 \rangle \sim ?$  for large  $x \in \mathbb{Z}^2$

For  $\delta = 0$ , the model is the Gaussian Free Field (GFF). It is known that

$$\langle \phi(x)^2 \rangle_{\delta=0} \sim \frac{1}{2\pi\beta} \ln(1 + |x|).$$

For  $\delta > 0$ , a Mermin–Wagner-type argument of McBryan–Spencer ’76 yields

$$\langle \phi(x)^2 \rangle_{\delta>0} \geq \langle \phi(x)^2 \rangle_{\delta=0} \geq \frac{1}{C\beta} \ln(1 + |x|).$$

Therefore the question is whether there is a matching upper bound.

**Simple regime:**  $\beta \ll 1$ . Then  $H$  is convex for any  $\delta$ ,

$$\text{Hess } H \geq \frac{\beta}{C}(-\Delta),$$

therefore the Brascamp–Lieb inequality yields

$$\langle \phi(x)^2 \rangle \leq \frac{C'}{\beta} \ln(1 + |x|).$$

More sophisticated RG/homogeneization arguments yield

$$\langle \phi(x)^2 \rangle \sim \frac{C}{\beta} \ln(1 + |x|),$$

where  $C$  depends on  $\delta$  and  $\beta$  but stays bounded between two constants.

Therefore we shall focus on the **interesting regime**  $\beta \gg 1$ .

- $H$  is not convex for any  $\delta$  (and decimation does not help)
- for  $\delta \gtrsim \exp(-\beta^2/2)$ ,  $H$  is “three-well” (has three local minima).

For very small  $\delta$ ,  $H$  can be still regarded as a perturbation of GFF. Therefore the renormalisation group methods (Gawedzki–Kupiainen, Brydges) yield that for  $\delta \leq \delta_0(\beta)$

$$\langle \phi(x)^2 \rangle \sim \frac{C}{\beta} \ln(1 + |x|).$$

Again,  $C$  depends on  $\delta$  and  $\beta$  but stays bounded between two constants. David checked that  $\delta_0^{\text{Brydges}} \gg \exp(-\beta^2/2)$ , so his method covers part of the three-well regime.

We are after a larger range of parameters at the expense of getting less precise results.

**Theorem.** For  $\delta \leq \frac{1}{C_1 \beta^{C_2}}$ ,  $\langle \phi(x)^2 \rangle \leq \frac{C}{\beta} \ln(1 + |x|)$ .

We shall discuss the proof (based on the method of Fröhlich–Spencer '81) later.

**Question.** *What happens for larger  $\delta$ ?*

For  $\delta = \infty$  we get

$$\prod_{x \sim y} \left\{ e^{-\frac{\beta}{2}(\phi(x)-\phi(y)-1)^2} + e^{-\frac{\beta}{2}(\phi(x)-\phi(y)+1)^2} \right\} .$$

Setting also  $\beta = \infty$  leads us to the:

**B) Height model:**  $\phi : \{-L, \dots, L\}^2 \rightarrow \mathbb{Z}$  is chosen uniformly at random, with the constraints  $\phi(x) = \phi(y) \pm 1$  for  $x \sim y$  and  $\phi(0) = 0$ .

**Conjecture** (Probably folklore, appears e.g. in a survey of Oded Schramm). *The model scales to GFF, and in particular*

$$\langle \phi(x)^2 \rangle_{\mathbf{B}} \sim C \ln(1 + |x|) .$$

The only rigorous result I know of is due to Ron Peled, who showed that in dimension  $\geq 10000$  the fluctuations are bounded:  $\langle \phi(x)^2 \rangle \leq \text{const}$ . The same is expected to hold in any dimension  $\geq 3$ .

*Connection to six-vertex model:* drawing an arrow from  $x$  to  $y$  if  $x \sim y$  and  $\phi(y) = \phi(x) + 1$ , we obtain a configuration of arrows with the constraint that the flow around every face is zero. Turning every arrow by  $\pi/2$ , we get the square ice solved by Lieb 50 years ago (actually, the subspace of configurations with the number of “up” arrows equal to the number of “down” arrows in every row). The model is therefore integrable.

But how to recover information about the original model? I do not know how to deduce something like

$$\langle \phi(x)^2 \rangle_{\mathbf{B}} \leq C \ln(1 + |x|)$$

or

$$\langle \phi(x)^2 \rangle_{\mathbf{B}} \geq C \ln(1 + |x|)$$

from the results on square ice<sup>1</sup>

So back to **A**).

**Conjecture.**

1.  $\langle \phi(x)^2 \rangle \sim \alpha(\beta, \delta) \ln(1 + |x|)$  for “almost every”  $\beta, \delta$ .

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<sup>1</sup>in the talk, I drew some pictures on the blackboard at this point, but I did not manage to reproduce them in L<sup>A</sup>T<sub>E</sub>X.

2. (very speculative) for large  $\beta$ ,  $\alpha$  is discontinuous in  $\delta$  (first-order phase transition).<sup>2</sup>

Finally, *work in progress*: extend the theorem to a similar model in which the phase transition is less subtle:

$$\langle \phi(x)^2 \rangle \sim \begin{cases} C \ln(1 + |x|), & \delta \ll 1 \\ C|x|^2, & \delta \gg 1. \end{cases}$$

===End of Introduction===

**Technical part:** the ideas behind the proof of the theorem, based on Fröhlich–Spencer ’81. First, the model **A**) is “dipole gas” in disguise. Consider the following dipole gas:

$$\sigma : \text{edges} \rightarrow \{0, \pm 1\};$$

if we choose an arbitrary orientation of the edges,  $\sigma = \pm 1$  represents a dipole oriented with/against the orientation. Charges interact via Coulomb law ( $G = (-\Delta)^{-1}$ ), therefore dipoles interact via  $\nabla G \nabla^*$ . Hence the model is

$$\frac{1}{Z} e^{+\beta \nabla G \nabla^* \sigma \cdot \sigma} z^{\#\text{dipoles}}$$

( $z$  is the activity). The interaction is interesting since it decays as  $|x - y|^{-2}$  (which is not integrable, therefore the virial expansion may diverge).

Our model **A**) maps to DG via the sine-Gordon transformation:

$$\begin{aligned} & \prod_{x \sim y} \left\{ e^{-\frac{\beta}{2}(\phi(x) - \phi(y))^2} + \delta e^{-\frac{\beta}{2}(\phi(x) - \phi(y) - 1)^2} + \delta e^{-\frac{\beta}{2}(\phi(x) - \phi(y) + 1)^2} \right\} \\ &= \prod_{x \sim y} e^{-\frac{\beta}{2}(\phi(x) - \phi(y))^2} \left[ 1 + \tilde{\delta} \cosh((\beta(\phi(x) - \phi(y)))) \right] \quad (\tilde{\delta} = 2\delta e^{-\beta/2}) \\ &= \sum_{\sigma} \prod_{x \sim y} e^{-\frac{\beta}{2}(\phi(x) - \phi(y))^2 + \beta \sigma(x,y)(\phi(x) - \phi(y))} z^{\#\text{supp } \sigma} \quad (z = \delta e^{-\beta/2}); \end{aligned}$$

integrating over  $\phi$  we obtain the dipole gas.

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<sup>2</sup>Here again there was a picture – of the expected phase diagram – which I do not manage to  $\LaTeX$ -ify

We will not integrate over  $\phi$ , but we shall use the electrostatic interpretation of dipoles. We shall consider more general “dipole densities”  $\rho \cdot \nabla \phi$ , i.e. Boltzmann factors of the form

$$e^{-\frac{\beta}{2} \nabla \phi^2} \prod_{\rho} (1 + K(\rho) \cosh(\rho \cdot \nabla \phi)) .$$

The convexity of the corresponding  $H$  depends only on  $\|\sum \rho\|_{\infty}$ , not on  $K(\rho)$ .

The basic idea is to renormalise the dipoles: e.g. a dipole in the centre of a ball is equivalent (by Newton’s law) to a dipole density smeared over the ball. This transformation reduces the sup-norm by a factor equal to the area of the ball.

If the dipoles were well-separated, one such transformation would be sufficient. In reality, we need the multiscale analysis of Fröhlich–Spencer:

**Step 1:** After change of variables, we can consider

$$\prod_{e=(xy)} e^{-\frac{1}{2}(\phi(x)-\phi(y))^2} \left[ 1 + \tilde{\delta} \cosh(a\delta_e \cdot \nabla \phi(e)) \right] .$$

**Proposition** (combinatorial argument). *For any  $R > 0$ ,  $1 < \alpha < 2$ ,*

$$\begin{aligned} & \prod_{e=(xy)} \left[ 1 + \tilde{\delta} \cosh(a\delta_e \cdot \nabla \phi(e)) \right] \\ &= \sum_N C_N \prod_{N \ni \rho: \text{edges} \rightarrow 0, \pm a} [1 + K_N(\rho) \cosh(\rho \cdot \nabla \phi)] , \end{aligned} \tag{1}$$

where the coefficients  $C_N$ ,  $K_N(\rho)$  are positive, and for any  $N$

i. for any  $\rho_1, \rho_2 \in N$ ,

$$\text{dist}(\rho_1, \rho_2) \geq R \min^{\alpha}(\text{diam } \rho_1, \text{diam } \rho_2) ;$$

ii. for any  $\rho \in N$ ,

$$K_N(\rho) \leq (C_{\alpha} R^{C_{\alpha}} \tilde{\delta})^{\#\text{supp } \rho} .$$

Note that (1)+ii. is trivial since the left-hand side of (1) is already of this form

(with just one  $N$  in the sum). (1)+i. is also trivial, since one can apply the identity

$$\begin{aligned}
& (1 + K(\rho_1) \cosh(\rho_1 \cdot \nabla \phi))(1 + K(\rho_2) \cosh(\rho_2 \cdot \nabla \phi)) \\
&= \frac{1}{3}(1 + 3K(\rho_1) \cosh(\rho_1 \cdot \nabla \phi)) \\
&+ \frac{1}{3}(1 + 3K(\rho_2) \cosh(\rho_2 \cdot \nabla \phi)) \\
&+ \frac{1}{6}(1 + 3K(\rho_1)K(\rho_2) \cosh((\rho_1 + \rho_2) \cdot \nabla \phi)) \\
&+ \frac{1}{6}(1 + 3K(\rho_1)K(\rho_2) \cosh((\rho_1 - \rho_2) \cdot \nabla \phi)) .
\end{aligned}$$

until we end up with one density  $\rho$  in every  $N$ . The non-trivial part, proved by Fröhlich and Spencer, is that one can have i. and ii. together, and this is where the condition  $\alpha < 2$  is used.

From now on, we consider one fixed  $N$  from (1).

**Step 2** (Electrostatic/potential-theoretic argument).

Suppose a density  $\rho$  is supported in a ball of radius  $D$ , and the observable  $\phi(x)^2$  is supported outside the concentric ball of radius  $RD^\alpha$ . Then by Newton's law we can replace  $\rho$  with the density  $\bar{\rho}$  smeared (almost) uniformly in the larger ball.

The sup-norm of  $\rho$  is  $a$ , therefore the sup-norm of  $\bar{\rho}$  is at most

$$\frac{aD^2}{(RD^\alpha)^2} = \frac{a}{R^2D^{2(\alpha-1)}} .$$

Choosing  $R$  large enough,  $R \gg \sqrt{a}$ , we can make this  $\ll 1$ .

There are however smaller dipoles that are supported in the ball of radius  $RD^\alpha$  (condition i. does not rule this out). Therefore a less naïve computation would be to sum the contributions of all scales  $D = 2^j$  to the sup-norm:

$$\sum_j \frac{a}{R^2 2^{2j(\alpha-1)}} \leq \frac{Ca}{R^2} ,$$

and here we see the rôle of the condition  $\alpha > 1$ .

This estimate can be made rigorous. However, there are still densities  $\rho$  to which the argument can not be applied:

1. “giant” densities ( $D$  is comparable to  $L$ , therefore there is no larger ball in which we can renormalise  $\rho$ )

2. densities that are close to the support of the observable ( $\{x\}$ ).

For these, we shall need the entropy estimate ii.

**Step 3:** convexity argument. For every  $N$ , we write the corresponding measure as

$$\text{measure} = \underbrace{\text{GFF} \times \text{renormalised dipoles}}_{\text{convex, Hess}H \geq (1-\epsilon)(-\Delta)} \times \underbrace{\text{all the rest}}_{(1+K(\rho) \cosh(\rho \cdot \nabla \phi)), K(\rho) \text{ is small by ii.}}$$

The first term is a measure with convex  $H$ , so the Brascamp–Lieb inequality can be applied to it. We combine the second term with the observable; the Brascamp–Lieb inequality gives an estimate on the integral. The dipoles which were not renormalised do not have much effect, since there are few of them, and each has small  $K(\rho)$  due to ii.  $\square$