Weyl symmetry of an orbital integral transform for symmetric superspaces (IHP – Tom Spencer – April 2012)

[Gruzberg, Ludwig, Mirlin & Z., PRL 2011]

– $H = -\frac{\hbar^2}{2m}\Delta + V$ random Schrödinger operator (on $\Lambda \subset \mathbb{Z}^d$)

– (finite-volume) eigenfunctions $H\psi_\alpha = E_\alpha \psi_\alpha$

– local density of states (LDOS):

$$r \equiv r(E,x) := \sum_\alpha |\psi_\alpha(x)|^2 \, \delta(E - E_\alpha)$$

– $0 \leqslant \xi := r / \langle r \rangle$ random variable with law $p(\xi)\,|d\xi|$

<u>Observation</u> (numerics, RMT, CFT, experiments):

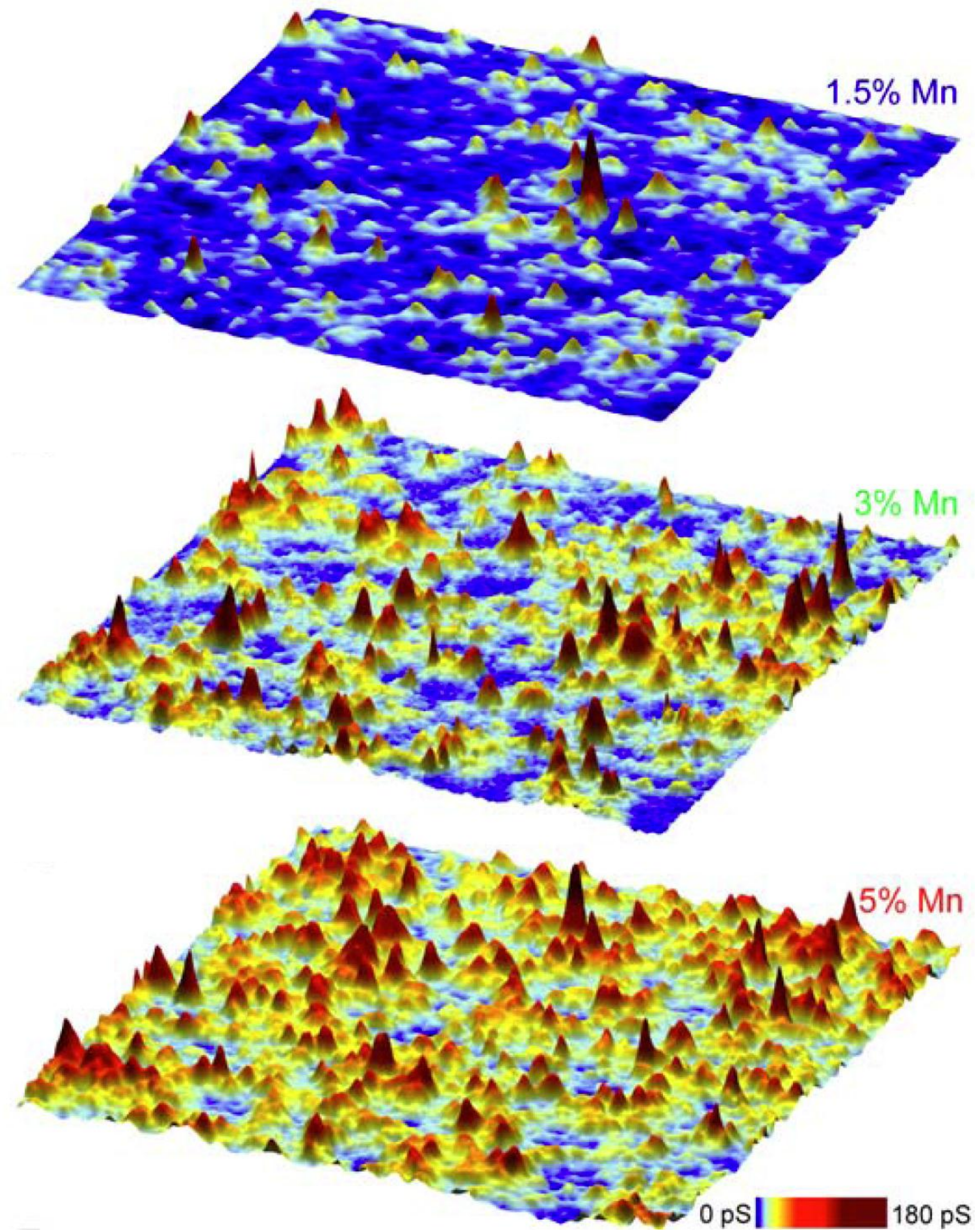
$$(*) \qquad \boxed{p(\xi)\,|d\xi| = \xi^{2\delta}\, p\!\left(\tfrac{1}{\xi}\right) \left|d\tfrac{1}{\xi}\right|} \qquad \text{"self-reciprocity"}$$

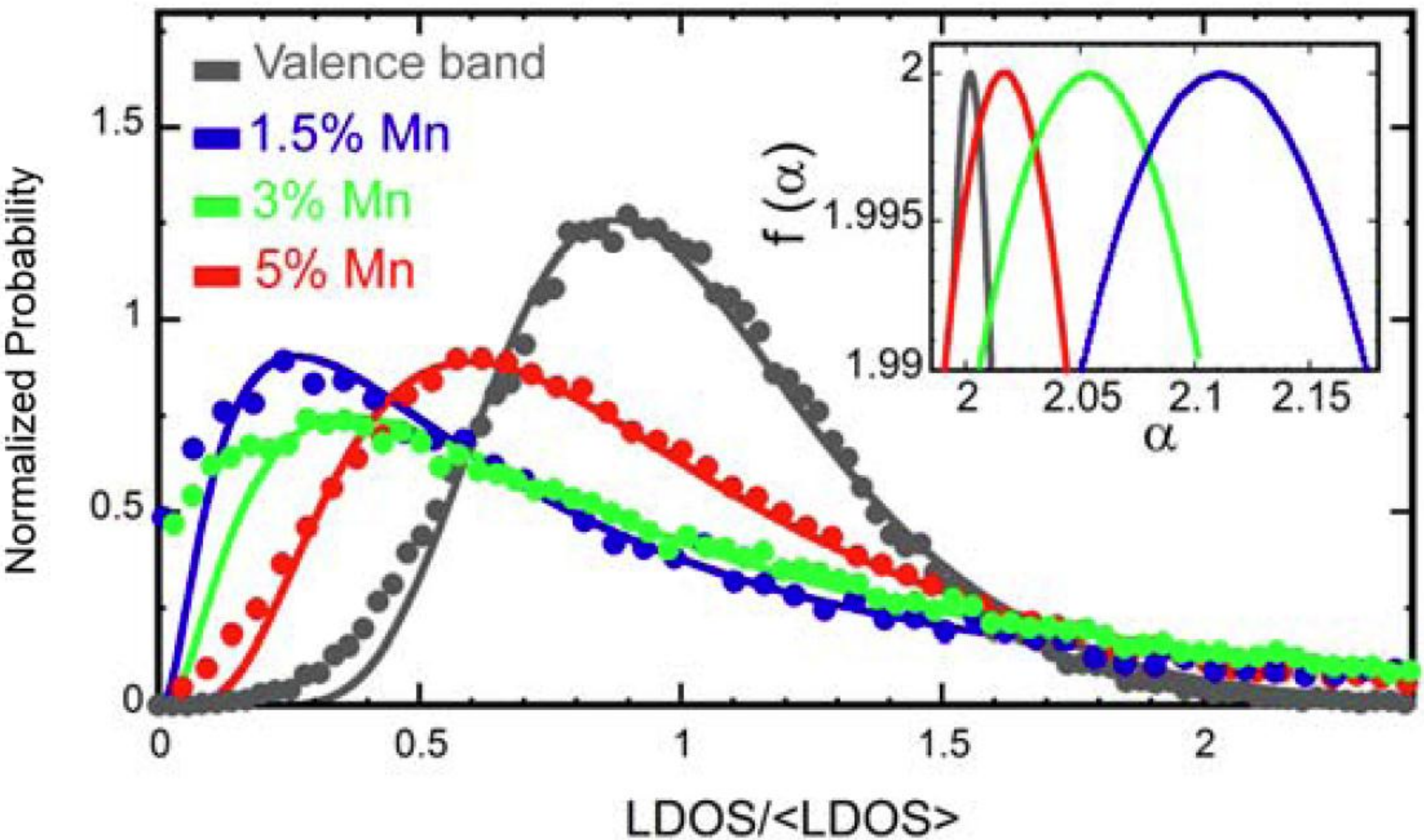
- p depends on dimension, disorder (metallic / critical / insulating)
- δ is universal (i.e., depends only on symmetry class)
- $(*)$ holds in the nonlinear sigma model approximation
- $(*)$ is true also for open systems (conducting boundary)

$Ga_{1-x}Mn_xAs$

spatial variations of the LDOS at the Fermi level

(data taken by a Princeton group using STM)





Valence band
1.5% Mn
3% Mn
5% Mn
Normalized Probability
LDOS/<LDOS>
0
0.5
1
1.5
2
f(α)
α
1.99
1.995
2
2.05
2.1
2.15

Mirlin, Fyodorov, Mildenberger, Evers (2006)

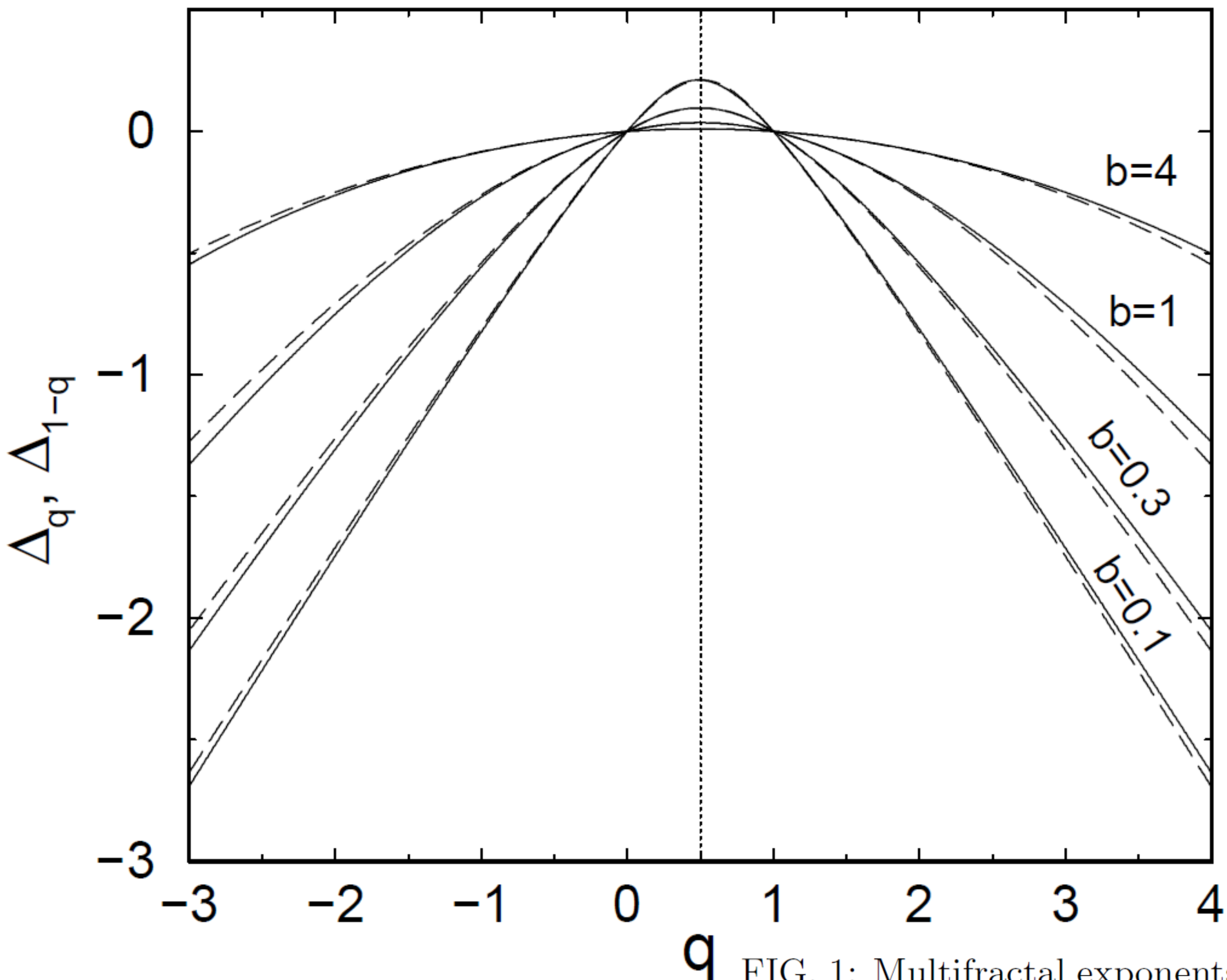


FIG. 1: Multifractal exponents Δ_q for the PRBM model with $b = 4, 1, 0.3, 0.1$. The symmetry (2) with respect to the point $q = 1/2$ is evident. A small difference between Δ_q (full line) and Δ_{1-q} (dashed) is due to numerical errors.

Power-law
Random
Band
Matrices

$\langle \xi^q \rangle \propto L^{-\Delta_q}$

Note:

$\langle \xi^q \rangle = \langle \xi^{-2\delta-q} \rangle$

$-2\delta = 1$

Analogy (model situation exhibiting the main features)

G noncompact semisimple Lie group,
$G = NAK$ Iwasawa decomposition.

Example: $SL_2(\mathbb{R}) \ni g = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = nak$

Note: $aNa^{-1} = N$.

Integral formula:

$$\int_{G/K} f(gK)\, d_{gK} = \int_A \left(\int_N f(nak)\, dn \right) e^{-2\delta(\ln a)} da,$$

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta_+} m_\alpha \alpha, \qquad \Delta = \Sigma\left(\mathrm{Lie}(G), \mathrm{Lie}(A)\right)$$

Theorem (Harish-Chandra).

Let $f \in C_c(G)$ be K-invariant: $f(g) = f(kgk^{-1})$.

Then $I_f(a) := e^{-\delta(\ln a)} \int_N f(na)\, dn$ satisfies the fctl eqn

$$I_f(a) = I_f(a^w) \quad \text{for all } w \in W = N_K(A) / Z_K(A).$$

Note: $kNk^{-1} \neq N$.

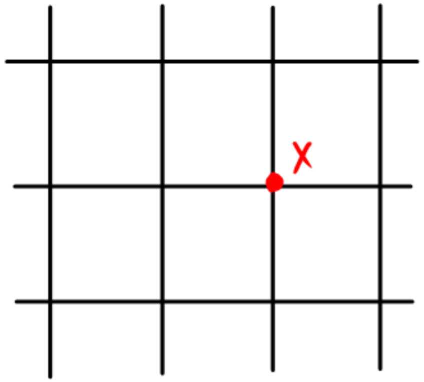
Example $(SL_2(\mathbb{R}))$: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Remark: dictionary $\xi \longleftrightarrow a^{-1}$

$$\xi^{-\delta} p(\xi)\, |d\xi| \longleftrightarrow e^{-\delta(\ln a)} \left(\int_N f(na)\, dn \right) da$$

$$= \xi^{\delta} p\left(\tfrac{1}{\xi}\right) \left|d\tfrac{1}{\xi}\right|$$

Bigger Picture



$\Lambda \subset \mathbb{Z}^d$

random Schrödinger operator

Wegner $\Downarrow$ Efetov

nonlinear sigma model

To be explained: $2\delta = \underset{(A,\,AI,\,AII)}{-1}\,,\quad \underset{(CI)}{-2}\,,\quad \underset{(C)}{-3}$

✓ $\delta < 0 \Longleftarrow$ SUSY (odd roots)

Problem: what to do with the compact sector?

Example (DSZ): $\langle f(t) \rangle = \langle f(-t)\, e^t \rangle$,

e.g. $\langle e^{qt} \rangle = \langle e^{(1-q)t} \rangle$.