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THE mKdV EQUATION ON THE HALF-LINE

A. BOUTET DE MONVEL[1], A. S. FOKAS[2] AND D. SHEPELSKY[3]

[1]*Institut de Mathématiques de Jussieu, Case 7012, Université de Paris 7, 2 Place Jussieu, 75251 Paris, France* (aboutet@math.jussieu.fr)
[2]*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK* (T.Fokas@damtp.cam.ac.uk)
[3]*Mathematical Division, Institute for Low Temperature Physics, 47 Lenin Avenue, 61103 Kharkiv, Ukraine* (shepelsky@yahoo.com)

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Abstract An initial boundary-value problem for the modified Korteweg–de Vries equation on the half-line, $0 < x < \infty$, $t > 0$, is analysed by expressing the solution $q(x,t)$ in terms of the solution of a matrix Riemann–Hilbert (RH) problem in the complex k-plane. This RH problem has explicit (x,t) dependence and it involves certain functions of k referred to as the *spectral functions*. Some of these functions are defined in terms of the initial condition $q(x,0) = q_0(x)$, while the remaining spectral functions are defined in terms of the boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, and $q_{xx}(0,t) = g_2(t)$. The spectral functions satisfy an *algebraic global relation* which characterizes, say, $g_2(t)$ in terms of $\{q_0(x), g_0(t), g_1(t)\}$. It is shown that for a particular class of boundary conditions, *the linearizable boundary conditions*, all the spectral functions can be computed from the given initial data by using algebraic manipulations of the global relation; thus, in this case, the problem on the half-line can be solved as efficiently as the problem on the whole line.

Keywords: nonlinear integrable equation; boundary-value problem; Riemann–Hilbert problem; long-time asymptotics

AMS 2000 *Mathematics subject classification:* Primary 37K15; 35Q53
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1. Introduction

In this paper, the general method announced in [**4**] for solving boundary-value problems for two-dimensional linear and integrable nonlinear partial differential equations is applied to the modified Korteweg–de Vries (mKdV) equation on the half-line.

This method, which was further developed in [**5**–**7**, **9**], is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It expresses the solution in terms of the solution of a matrix Riemann–Hilbert (RH) problem formulated in the complex plane of the spectral parameter. The spectral functions determining the RH problem are expressed in terms of the boundary values of the solution. The fact that these boundary values are, in general, related can be expressed in a simple way in terms of a global relation satisfied by the corresponding spectral functions.

The rigorous implementation of the method to the nonlinear Schrödinger equation on the half-line is presented in [**9**]. The application to the sine-Gordon equation in laboratory coordinates and to the Korteweg–de Vries equation with dominant surface tension is presented in [**7**]. In the present paper, this methodology is applied to the mKdV equation on the half-line.

The paper is organized as follows.

In § 2, we review the general methodology and study the direct spectral problem formulated in terms of the simultaneous spectral analysis of the associated Lax pair: we define appropriate eigenfunctions and spectral functions and study their properties.

In § 3, the inverse spectral problem is formulated as a matrix RH problem, the solution of which gives the solution of the mKdV equation with prescribed initial and boundary data provided that the spectral functions satisfy an algebraic relation, the 'global relation'.

In § 4, we show that for particular boundary conditions it is possible to solve the problem on the half-line with the same effectiveness as the problem on the whole line (solved by the inverse-scattering transform method).

In § 5, following the nonlinear steepest descent method [**1**, **3**], we obtain the long-time asymptotics of the solution.

2. Lax pair, eigenfunctions and spectral functions

2.1. Problem formulation. Lax pair and closed 1-form

The mKdV equation

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1 \tag{2.1}$$

admits the Lax pair formulation

$$\psi_x - \mathrm{i}k\sigma_3\psi = Q(x,t)\psi, \tag{2.2 a}$$

$$\psi_t + 4\mathrm{i}k^3\sigma_3\psi = \tilde{Q}(x,t,k)\psi, \tag{2.2 b}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda q(x,t) & 0 \end{pmatrix}, \tag{2.3 a}$$

$$\tilde{Q}(x,t,k) = \begin{pmatrix} -2\mathrm{i}\lambda q^2 k & -4qk^2 + 2\mathrm{i}q_x k - 2\lambda q^3 + q_{xx} \\ \lambda(-4qk^2 - 2\mathrm{i}q_x k - 2\lambda q^3 + q_{xx}) & 2\mathrm{i}\lambda q^2 k \end{pmatrix}. \tag{2.3 b}$$

Let $\hat{\sigma}_3 = \operatorname{ad}\sigma_3$ denote the commutator with respect to σ_3,

$$\hat{\sigma}_3 A := [\sigma_3, A] = \sigma_3 A - A\sigma_3.$$

Then

$$\mathrm{e}^{\hat{\sigma}_3}A = \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3}$$

for any 2×2 matrix A. Let

$$\mu := \psi\mathrm{e}^{\mathrm{i}(-kx+4k^3t)\sigma_3}.$$

Then (2.2 *a*) becomes

$$\mu_x - \mathrm{i}k\hat{\sigma}_3\mu = Q(x,t)\mu, \tag{2.4 a}$$

$$\mu_t + 4\mathrm{i}k^3\hat{\sigma}_3\mu = \tilde{Q}(x,t,k)\mu. \tag{2.4 b}$$

These equations can be rewritten as

$$\mathrm{d}(\mathrm{e}^{\mathrm{i}(-kx+4k^3t)\hat{\sigma}_3}\mu) = W, \tag{2.5}$$

where W is the exact 1-form defined by

$$W(x,t,k) = \mathrm{e}^{\mathrm{i}(-kx+4k^3t)\hat{\sigma}_3}(Q\mu\,\mathrm{d}x + \tilde{Q}\mu\,\mathrm{d}t). \tag{2.6}$$

The problem we are dealing with is the initial boundary-value problem for the mKdV equation in the domain $\{0<x<\infty,\ 0<t<T\}$, $T\leqslant\infty$. We use the following steps.

Step 1. Assuming that the solution of the mKdV equation, $q(x,t)$, exists, express it via the solution of a matrix RH problem. For this purpose, we make use of the following.

(a) Define proper solutions of (2.5) (eigenfunctions) analytic and bounded (in k) in domains forming a partition of the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$.

(b) Define spectral functions $s(k)$, $S(k)$ such that:

  (1) they determine an RH problem;

  (2) $s(k)$ is determined by the initial conditions $q(x,0) = q_0(x)$, $0<x<\infty$;

  (3) $S(k)$ is determined by the boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$, $0<t<T$;

  (4) they satisfy an algebraic *global relation*, expressing the fact that $q_0(x)$, $g_0(t)$, $g_1(t)$ and $g_2(t)$ being the initial and boundary values for the mKdV equation, cannot be chosen arbitrary.

Step 2. Given $s(k)$ and assuming that $g_0(t)$, $g_1(t)$ and $g_2(t)$ are such that the associated $S(k)$, together with $s(k)$, satisfy the global relation, prove that the solution of the RH problem constructed from $s(k)$ and $S(k)$ generates the solution of the initial boundary-value problem for the mKdV with initial data $q(x,0) = q_0(x)$ and boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$.

Step 3. Find a class of boundary conditions, for which $S(k)$ can be calculated explicitly in terms of the given initial data.

Step 4. Study the long-time asymptotics of the solution $q(x,t)$.

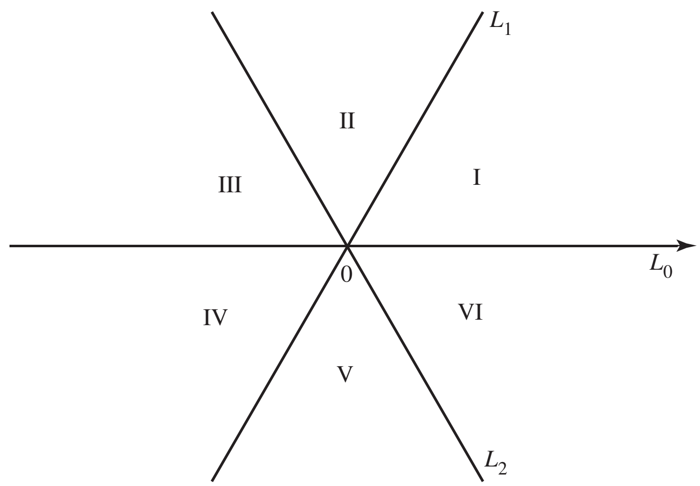

Figure 1. Domains of analyticity and boundedness of eigenfunctions.

2.2. Eigenfunctions and spectral functions

We assume that there exists a real-valued function $q(x,t)$ with sufficient smoothness and decay satisfying (2.1) in $\{0<x<\infty,\ 0<t<T\}$, $T\leqslant\infty$.

Define $\mu_n(x,t,k)$, $n=1,2,3$ as 2×2-matrix-valued solutions of the integral equations

$$\mu_n(x,t,k)=I+\int_{(x_n,t_n)}^{(x,t)}\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}W(y,\tau,k), \tag{2.7}$$

where $(x_1,t_1)=(0,T)$, $(x_2,t_2)=(0,0)$, $(x_3,t_3)=(\infty,t)$, and the paths of integration are chosen to be parallel to the x- and t-axes,

$$\begin{aligned}\mu_1(x,t,k)=I&+\int_0^x\mathrm{e}^{\mathrm{i}k(x-y)\hat\sigma_3}(Q\mu_1)(y,t,k)\,\mathrm{d}y\\&-\mathrm{e}^{\mathrm{i}kx\hat\sigma_3}\int_t^T\mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat\sigma_3}(\tilde Q\mu_1)(0,\tau,k)\,\mathrm{d}\tau,\end{aligned} \tag{2.8 a}$$

$$\begin{aligned}\mu_2(x,t,k)=I&+\int_0^x\mathrm{e}^{\mathrm{i}k(x-y)\hat\sigma_3}(Q\mu_2)(y,t,k)\,\mathrm{d}y\\&+\mathrm{e}^{\mathrm{i}kx\hat\sigma_3}\int_0^t\mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat\sigma_3}(\tilde Q\mu_2)(0,\tau,k)\,\mathrm{d}\tau,\end{aligned} \tag{2.8 b}$$

$$\mu_3(x,t,k)=I-\int_x^\infty\mathrm{e}^{\mathrm{i}k(x-y)\hat\sigma_3}(Q\mu_3)(y,t,k)\,\mathrm{d}y. \tag{2.8 c}$$

The domains where the exponentials are bounded are separated by the three lines such that $L_0\cup L_1\cup L_2=\{k\in\mathbb{C}\mid\operatorname{Im}k^3=0\}$. The relevant domains in the k-plane are denoted by I, …, VI (see figure 1).

Let the columns of a 2×2 matrix μ be denoted as $(\mu^{(1)}\ \mu^{(2)})$. Then the columns of μ_n are analytic and bounded in the following domains in the complex k-plane (determined by the domains of boundedness of the exponentials involved in the relative integral equations):

eigenfunction	domain of analyticity and boundedness
$\mu_1^{(1)}(x,t,k)$	$\{k \mid \operatorname{Im} k \leqslant 0,\ \operatorname{Im} k^3 \leqslant 0\} = \mathrm{IV} \cup \mathrm{VI}$
$\mu_1^{(2)}(x,t,k)$	$\{k \mid \operatorname{Im} k \geqslant 0,\ \operatorname{Im} k^3 \geqslant 0\} = \mathrm{I} \cup \mathrm{III}$
$\mu_2^{(1)}(x,t,k)$	$\{k \mid \operatorname{Im} k \leqslant 0,\ \operatorname{Im} k^3 \geqslant 0\} = \mathrm{V}$
$\mu_2^{(2)}(x,t,k)$	$\{k \mid \operatorname{Im} k \geqslant 0,\ \operatorname{Im} k^3 \leqslant 0\} = \mathrm{II}$
$\mu_3^{(1)}(x,t,k)$	$\{k \mid \operatorname{Im} k \geqslant 0\} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$
$\mu_3^{(2)}(x,t,k)$	$\{k \mid \operatorname{Im} k \leqslant 0\} = \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}$

μ_1 (for $T < \infty$) and μ_2 are entire functions of k. Thus, in each domain $\mathrm{I}, \ldots, \mathrm{VI}$, one has a 2×2-matrix-valued eigenfunction, analytic and bounded, consisting of the appropriate vectors $\mu_n^{(k)}$, $n = 1, 2, 3$, $k = 1, 2$.

For particular values of x or t, the domains of boundedness of the eigenfunctions are larger than above. Particularly, for $t = 0$, the domain of boundedness of μ_2 are

eigenfunction	domain of boundedness
$\mu_2^{(1)}(x,0,k)$	$\{k \mid \operatorname{Im} k \leqslant 0\} = \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}$
$\mu_2^{(2)}(x,0,k)$	$\{k \mid \operatorname{Im} k \geqslant 0\} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$

and for $x = 0$ the domain of boundedness of μ_1 and μ_2 are

eigenfunctions	domain of boundedness
$\mu_1^{(1)}(0,t,k)$ and $\mu_2^{(2)}(0,t,k)$	$\{k \mid \operatorname{Im} k^3 \leqslant 0\} = \mathrm{II} \cup \mathrm{IV} \cup \mathrm{VI}$
$\mu_1^{(2)}(0,t,k)$ and $\mu_2^{(1)}(0,t,k)$	$\{k \mid \operatorname{Im} k^3 \leqslant 0\} = \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$

Since the μ_j are solutions of the system of differential equations (2.4), they are simply related (in the domains where they are defined),

$$\mu_3(x,t,k) = \mu_2(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat{\sigma}_3}\mu_3(0,0,k), \tag{2.9 a}$$

$$\begin{aligned} \mu_1(x,t,k) &= \mu_2(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat{\sigma}_3}\mu_1(0,0,k) \\ &= \mu_2(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat{\sigma}_3}(\mathrm{e}^{4\mathrm{i}k^3T\hat{\sigma}_3}\mu_2(0,T,k))^{-1}. \end{aligned} \tag{2.9 b}$$

Introduce the spectral (2×2-matrix-valued) functions

$$s(k) := \mu_3(0,0,k), \tag{2.10 a}$$

$$S(k) = S(k;T) := \mu_1(0,0,k) = (\mathrm{e}^{4\mathrm{i}k^3T\hat{\sigma}_3}\mu_2(0,T,k))^{-1}. \tag{2.10 b}$$

In what follows, $s(k)$ and $S(k)$ will be used to construct a matrix RH problem (more precisely, a family of RH problems parametrized by (x,t)), the solution of which gives the eigenfunctions μ_n and hence $q(x,t)$, the solution of the mKdV problem. On the other hand, from (2.8 c), (2.10), it follows that $s(k)$ is determined by the initial values of $q(x,t)$, whereas $S(k)$ is determined by the boundary values, namely,

$$s(k) = I - \int_0^\infty \mathrm{e}^{-\mathrm{i}ky\hat\sigma_3}(Q\mu_3)(y,0,k)\,\mathrm{d}y, \tag{2.11 a}$$

$$S(k;T) = \left(I + \int_0^T \mathrm{e}^{4\mathrm{i}k^3\tau\hat\sigma_3}(\tilde Q\mu_2)(0,\tau,k)\,\mathrm{d}\tau\right)^{-1}, \tag{2.11 b}$$

where $\mu_l(x,0,k)$, $l=1,2,3$ are the solutions of the integral equations

$$\mu_1(0,t,k) = I - \int_t^T \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat\sigma_3}(\tilde Q\mu_1)(0,\tau,k)\,\mathrm{d}\tau, \tag{2.12 a}$$

$$\mu_2(0,t,k) = I + \int_0^t \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat\sigma_3}(\tilde Q\mu_2)(0,\tau,k)\,\mathrm{d}\tau, \tag{2.12 b}$$

$$\mu_3(x,0,k) = I - \int_x^\infty \mathrm{e}^{\mathrm{i}k(x-y)\hat\sigma_3}(Q\mu_3)(y,0,k)\,\mathrm{d}y. \tag{2.12 c}$$

Note that $Q(x,0)$ is determined by $q(x,0)$, whereas $\tilde Q(0,t,k)$ is determined by $q(0,t)$, $q_x(0,t)$ and $q_{xx}(0,t)$.

2.3. Symmetry properties

Since $q(x,t)$ is real valued, the matrices $U(x,t,k)$ and $V(x,t,k)$ in the Lax pair (2.2 a) written in the form

$$\psi_x = U\psi, \qquad \psi_t = V\psi$$

satisfy the following symmetry relations,

$$U(-k) = \overline{U(\bar k)}, \qquad V(-k) = \overline{V(\bar k)} \tag{2.13}$$

and

$$\sigma_1\overline{U(\bar k)}\sigma_1 = U(k), \quad \sigma_1\overline{V(\bar k)}\sigma_1 = V(k) \quad \text{for } \lambda = 1, \tag{2.14 a}$$

$$\sigma_2\overline{U(\bar k)}\sigma_2 = U(k), \quad \sigma_2\overline{V(\bar k)}\sigma_2 = V(k) \quad \text{for } \lambda = -1, \tag{2.14 b}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

In turn, relations (2.13) and (2.14) imply

$$\psi(-k) = \overline{\psi(\bar k)} \tag{2.15}$$

and

$$\sigma_1\overline{\psi(\bar{k})}\sigma_1 = \psi(k) \quad \text{for } \lambda = 1, \tag{2.16 a}$$

$$\sigma_2\overline{\psi(\bar{k})}\sigma_2 = \psi(k) \quad \text{for } \lambda = -1. \tag{2.16 b}$$

Particularly, it follows from (2.16) that the spectral matrices $s(k)$ and $S(k)$ can be written as

$$s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \lambda\overline{b(\bar{k})} & a(k) \end{pmatrix}, \qquad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \lambda\overline{B(\bar{k})} & A(k) \end{pmatrix}. \tag{2.17}$$

2.4. Global relation

The initial and boundary values of a solution of the mKdV equation (taken as the traces of q, q_x and q_{xx} at $x = 0$) are not independent. It turns out that the relations between the initial and boundary values of the solution can be expressed in a surprisingly simple form in terms of the corresponding spectral functions.

Evaluating (2.9 *a*) at $x = 0$, $t = T$, we find

$$\mu_3(0, T, k) = \mu_2(0, T, k)\mathrm{e}^{-4\mathrm{i}k^3T\hat{\sigma}_3}\mu_3(0, 0, k). \tag{2.18}$$

Writing $\mu_3(0, 0, k)$, $\mu_2(0, T, k)$ in terms of $s(k)$, $S(k)$ (see (2.10)) and using (2.8 *c*) to evaluate $\mu_3(0, T, k)$, equation (2.18) becomes

$$-I + S^{-1}(k;T)s(k) + \mathrm{e}^{4\mathrm{i}k^3T\hat{\sigma}_3}\int_0^\infty \mathrm{e}^{-\mathrm{i}ky\hat{\sigma}_3}(Q\mu_3)(y, T, k)\,\mathrm{d}y = 0. \tag{2.19}$$

The $(1, 2)$ coefficient of this equation is

$$a(k)B(k) - A(k)b(k) = \mathrm{e}^{8\mathrm{i}k^3T}c^+(k), \quad \operatorname{Im} k \leqslant 0, \tag{2.20}$$

for $T < \infty$, where

$$c^+(k) = c^+(k;T) = \int_0^\infty \mathrm{e}^{-2\mathrm{i}ky}(Q\mu_3)_{12}(y, T, k)\,\mathrm{d}y$$

is a function analytic for $\operatorname{Im} k < 0$ which is $O(1/k)$ as $k \to \infty$.

In the case $T = \infty$, equation (2.19) becomes

$$a(k)B(k) - A(k)b(k) = 0, \quad k \in \mathrm{V}. \tag{2.21}$$

Equations (2.20) and (2.21) are algebraic relations between the spectral functions; we call them the global relations, because they express, in the spectral terms, the relations between the initial and boundary values of a solution of the mKdV equation.

2.5. Properties of spectral functions

The spectral map

$$\mathbb{S} : \{q_0(x)\} \mapsto \{a(k), b(k)\} \tag{2.22}$$

is defined following (2.10 a), (2.17), by the second column of the solution $\mu_3(x,0,k)$ of equation (2.12 c), where $Q = Q(x,0)$ is determined by (2.3 a) with $q(x,t)$ replaced by $q_0(x)$. Thus this map is the same as in the inverse-scattering theory on the whole line for the Dirac equation

$$\psi_x - \mathrm{i}k\sigma_3\psi = \begin{pmatrix} 0 & q_0(x) \\ \lambda q_0(x) & 0 \end{pmatrix}\psi$$

restricted to the potentials with support on the half-line. The analysis of the linear Volterra integral equation (2.12 c) gives the following properties of $a(k)$ and $b(k)$:

- $a(k)$ and $b(k)$ are analytic and bounded for $\operatorname{Im} k < 0$;
- $a(k) = 1 + O(1/k)$, $b(k) = O(1/k)$, $k \to \infty$.

The fact that Q is traceless, together with the behaviour as $k \to \infty$ given above, imply that $\det s(k) = 1$, which, in terms of $a(k)$ and $b(k)$, reads

- $|a(k)|^2 - \lambda|b(k)|^2 = 1$, $k \in \mathbb{R}$.

The symmetry relation (2.15) yields

- $a(-k) = \overline{a(\bar{k})}$, $b(-k) = \overline{b(\bar{k})}$.

The inverse map $\mathbb{Q} : \{a(k), b(k)\} \mapsto \{q_0(x)\}$ is again the same as in the inverse-scattering theory on the whole line; it is defined as

$$q_0(x) = -2\mathrm{i} \lim_{k\to\infty} k(M^{(x)}(x,k))_{12}, \tag{2.23}$$

where $M^{(x)}(x,k)$ is the solution of the following RH problem.

- $M^{(x)}(x,k)$ is a sectionally meromorphic function relative to the real axis in the complex k-plane.
- The limits $M^{(x)}_{\pm}(x,\zeta)$ of $M^{(x)}(x,k)$ as k approaches the real axis, $k = \zeta \mp \mathrm{i}0$, are related by the jump matrix $J^{(x)}(x,\zeta)$,

  $$M^{(x)}_{-}(x,\zeta) = M^{(x)}_{+}(x,\zeta)J^{(x)}(x,\zeta), \quad \zeta \in \mathbb{R}, \tag{2.24}$$

  where

  $$J^{(x)}(x,k) = \begin{bmatrix} 1 & -\dfrac{b(k)}{\bar{a}(k)}\mathrm{e}^{2\mathrm{i}kx} \\ \lambda\dfrac{\bar{b}(k)}{a(k)}\mathrm{e}^{-2\mathrm{i}kx} & 1 - \lambda\left|\dfrac{b(k)}{\bar{a}(k)}\right|^2 \end{bmatrix}, \quad k \in \mathbb{R}. \tag{2.25}$$

- $M^{(x)}(x,k) = I + O(1/k)$, $k \to \infty$.

- We assume that if $\lambda = -1$, $a(k)$ can have n simple zeros $\{k_j\}_1^n$, $n = n_1 + 2n_2$, where $\{k_j\}_1^{n_1} \in \mathrm{V}$, $\{k_j\}_{n_1+1}^{n_1+n_2} \in \mathrm{IV}$, $k_{n_1+n_2+j} = -\bar{k}_{n_1+j} \in \mathrm{VI}$, $j = 1, \dots, n_2$.

- If $\lambda = -1$, the first column of $M^{(x)}(x,k)$ can have simple poles at $k = k_j$, $j = 1, \dots, n$ and the second column of $M^{(x)}(x,k)$ can have simple poles $k = \bar{k}_j$, $j = 1, \dots, n$, where $\{k_j\}_1^n$ are the simple zeros of $a(k)$, $\operatorname{Im} k < 0$. The associated residues are given by

$$\operatorname{Res}_{k_j}[M^{(x)}(x,k)]^{(1)} = \frac{\mathrm{e}^{-2\mathrm{i}k_j x}}{\dot{a}(k_j)b(k_j)}[M^{(x)}(x,k_j)]^{(2)}, \tag{2.26 a}$$

$$\operatorname{Res}_{\bar{k}_j}[M^{(x)}(x,k)]^{(2)} = \frac{\lambda \mathrm{e}^{2\mathrm{i}\bar{k}_j x}}{\overline{\dot{a}(k_j)b(k_j)}}[M^{(x)}(x,\bar{k}_j)]^{(1)}. \tag{2.26 b}$$

Apart from the symmetry $-k \mapsto \bar{k}$, the properties of the spectral map $\mathbb{S}$ and its inverse are identical to those described in [**9**], where the reader can find the details.

The spectral map

$$\tilde{\mathbb{S}} : \{g_0(t), g_1(t), g_2(t)\} \mapsto \{A(k), B(k)\} \tag{2.27}$$

is defined following (2.10 b), (2.17), by the second column of the solution $\mu_2(0,t,k)$ of equation (2.12 b), where $\tilde{Q} = \tilde{Q}(0,t,k)$ is determined by (2.3 b) with q, q_x and q_{xx} replaced by $g_0(t)$, $g_1(t)$ and $g_2(t)$, respectively,

$$\begin{pmatrix} -\mathrm{e}^{-8\mathrm{i}k^3 T} B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \mu_2^{(2)}(0,T,k). \tag{2.28}$$

The symmetry relation (2.15) gives

- $A(-k) = \overline{A(\bar{k})}$, $B(-k) = \overline{B(\bar{k})}$.

From the Volterra integral equations (2.12 b) and (2.12 a), it follows that

- if $T < \infty$, $A(k)$ and $B(k)$ are entire functions bounded in $\mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$; if $T = \infty$, $A(k)$ and $B(k)$ are defined only in $\mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$ being analytic and bounded there.

To analyse the large-k behaviour of $A(k)$ and $B(k)$, one cannot use directly the Volterra equations (2.12 b) or (2.12 a), because $\tilde{Q}$ contains terms of order k^2 and k. Instead, one can proceed as follows. First, we determine a polynomial (in k^{-1}) matrix-valued function $\tilde{m}(t,k)$ that solves (2.4 b) approximately,

$$\tilde{m}(t,k) := I + \frac{\tilde{m}_1(t)}{k} + \frac{\tilde{m}_2(t)}{k^2} + \frac{\tilde{m}_3(t)}{k^3}, \tag{2.29}$$

such that

$$4\mathrm{i}k^3[\sigma_3, \tilde{m}(t,k)] = \tilde{Q}(t,k)\tilde{m}(t,k) + R(t,k), \tag{2.30}$$

where $R(t,k) = O(1/k)$ as $k \to \infty$.

Substituting $\tilde{m}(t,k)$ in the form of (2.29) into (2.30) shows that (2.30) can be satisfied by

$$\tilde{m}_1 = \tfrac{1}{2}\mathrm{i}q\begin{pmatrix}0 & 1\\ -\lambda & 0\end{pmatrix}, \tag{2.31 a}$$

$$\tilde{m}_2 = \tfrac{1}{4}q_x\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}, \tag{2.31 b}$$

$$\tilde{m}_3 = -\tfrac{1}{8}(2\mathrm{i}q^3 + \mathrm{i}q_{xx})\begin{pmatrix}0 & 1\\ -\lambda & 0\end{pmatrix}. \tag{2.31 c}$$

Notice that the existence of the approximate polynomial solution (2.29) is provided by the special structure of the matrix $\tilde{Q}$ (cf. [**14**]).

Then, for $\phi(t,k) := \tilde{m}^{-1}(t,k)\mu_2(0,t,k)$, one has the differential equation

$$\phi'_t + 4\mathrm{i}k^3[\sigma_3, \phi(t,k)] = R_1(t,k)\phi(t,k), \tag{2.32}$$

where $R_1(t,k) = -\tilde{m}^{-1}(t,k)R(t,k) = O(1/k)$. The solution of (2.32) is given by the solution of the Volterra equation

$$\phi(t,k) = \mathrm{e}^{-4\mathrm{i}k^3 t\hat{\sigma}_3}\tilde{m}^{-1}(0,k) + \int_0^t \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(R_1\phi)(\tau,k)\,\mathrm{d}\tau. \tag{2.33}$$

Now the asymptotics of ϕ can be obtained from (2.33); particularly, one has

$$\phi^{(1)}(t,k) = \begin{pmatrix}1\\0\end{pmatrix} + O\left(\frac{1}{k}\right) \quad \text{for } k \in \mathrm{I}\cup\mathrm{III}\cup\mathrm{V}, \tag{2.34 a}$$

$$\phi^{(2)}(t,k) = \begin{pmatrix}0\\1\end{pmatrix} + O\left(\frac{1}{k}\right) \quad \text{for } k \in \mathrm{II}\cup\mathrm{IV}\cup\mathrm{VI}. \tag{2.34 b}$$

To study the behaviour of the columns of $\phi(t,k)$ in domains where they are not bounded, one can consider the Volterra equation for $\tilde{\phi}(t,k) := \phi(t,k)\mathrm{e}^{-8\mathrm{i}k^3 t\sigma_3}$,

$$\tilde{\phi}(t,k) = \mathrm{e}^{-4\mathrm{i}k^3 t\sigma_3}\tilde{m}^{-1}(0,k)\mathrm{e}^{-4\mathrm{i}k^3 t\sigma_3} + \int_0^t \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\sigma_3}(R_1\tilde{\phi})(\tau,k)\mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\sigma_3}\,\mathrm{d}\tau, \tag{2.35}$$

which gives

$$\tilde{\phi}^{(1)}(t,k) = \begin{pmatrix}\mathrm{e}^{-8\mathrm{i}k^3 t}\\0\end{pmatrix} + O\left(\frac{1}{k}\right) \quad \text{for } k \in \mathrm{II}\cup\mathrm{IV}\cup\mathrm{VI}, \tag{2.36 a}$$

$$\tilde{\phi}^{(2)}(t,k) = \begin{pmatrix}0\\ \mathrm{e}^{8\mathrm{i}k^3 t}\end{pmatrix} + O\left(\frac{1}{k}\right) \quad \text{for } k \in \mathrm{I}\cup\mathrm{III}\cup\mathrm{V}. \tag{2.36 b}$$

Finally, from (2.34) and (2.36), one obtains

$$\phi^{(1)}(t,k) = \begin{pmatrix}1\\0\end{pmatrix} + O\left(\frac{1}{k}\right) + O\left(\frac{\mathrm{e}^{8\mathrm{i}k^3 t}}{k}\right), \tag{2.37 a}$$

$$\phi^{(2)}(t,k) = \begin{pmatrix}0\\1\end{pmatrix} + O\left(\frac{1}{k}\right) + O\left(\frac{\mathrm{e}^{-8\mathrm{i}k^3 t}}{k}\right), \tag{2.37 b}$$

which implies

$$A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{\mathrm{e}^{8\mathrm{i}k^3 T}}{k}\right), \tag{2.38 a}$$

$$B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{\mathrm{e}^{8\mathrm{i}k^3 T}}{k}\right), \quad k \to \infty. \tag{2.38 b}$$

Now the fact that $\tilde{Q}$ is traceless and the asymptotics (2.38) imply

$$A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1, \quad k \in \mathbb{C} \quad (\text{and } \operatorname{Im} k^3 = 0 \text{ if } T = \infty). \tag{2.39}$$

The map

$$\tilde{\mathbb{Q}} : \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t), g_2(t)\}$$

inverse to $\tilde{\mathbb{S}}$ is defined via the solution of an appropriate RH problem as follows (the details are similar to those in the case of the NLS equation (see [**9**])),

$$g_0(t) = -2\mathrm{i}(M_1^{(t)})_{12}(t), \tag{2.40 a}$$

$$g_1(t) = 4(M_2^{(t)})_{12}(t) - 2\mathrm{i}g_0(t)(M_1^{(t)})_{22}(t), \tag{2.40 b}$$

$$g_2(t) = \lambda g_0^3(t) + 8\mathrm{i}(M_3^{(t)})_{12}(t) + 4g_0(t)(M_2^{(t)})_{22}(t) - 2\mathrm{i}g_1(t)(M_1^{(t)})_{22}(t), \tag{2.40 c}$$

where

$$M^{(t)}(t,k) = I + \frac{M_1^{(t)}(t)}{k} + \frac{M_2^{(t)}(t)}{k^2} + \frac{M_3^{(t)}(t)}{k^3} + O\left(\frac{1}{k^4}\right), \quad k \to \infty,$$

with $M^{(t)}(t,k)$ the solution of the following RH problem.

- $M^{(t)}(t,k)$ is a sectionally meromorphic function relative to the lines $\operatorname{Im} k^3 = 0$ in the complex k-plane.
- Let $\Omega_+ = \mathrm{I}\cup\mathrm{III}\cup\mathrm{V}$, $\Omega_- = \mathrm{II}\cup\mathrm{IV}\cup\mathrm{VI}$. Denote by $M_\pm^{(t)}(t,\zeta)$ the limits of $M^{(t)}(t,k)$ as k approaches $\{\zeta \mid \zeta^3 = 0\}$ from $\Omega_\pm$. Then

$$M_-^{(t)}(t,\zeta) = M_+^{(t)}(t,\zeta)J^{(t)}(t,\zeta), \quad \operatorname{Im}\zeta^3 = 0, \tag{2.41}$$

  where

$$J^{(t)}(t,k) = \begin{pmatrix} 1 & -\dfrac{B(k)}{\overline{A(\bar{k})}}\mathrm{e}^{-8\mathrm{i}k^3 t} \\ \dfrac{\lambda\overline{B(\bar{k})}}{A(k)}\mathrm{e}^{8\mathrm{i}k^3 t} & \dfrac{1}{A(k)\overline{A(\bar{k})}} \end{pmatrix}. \tag{2.42}$$

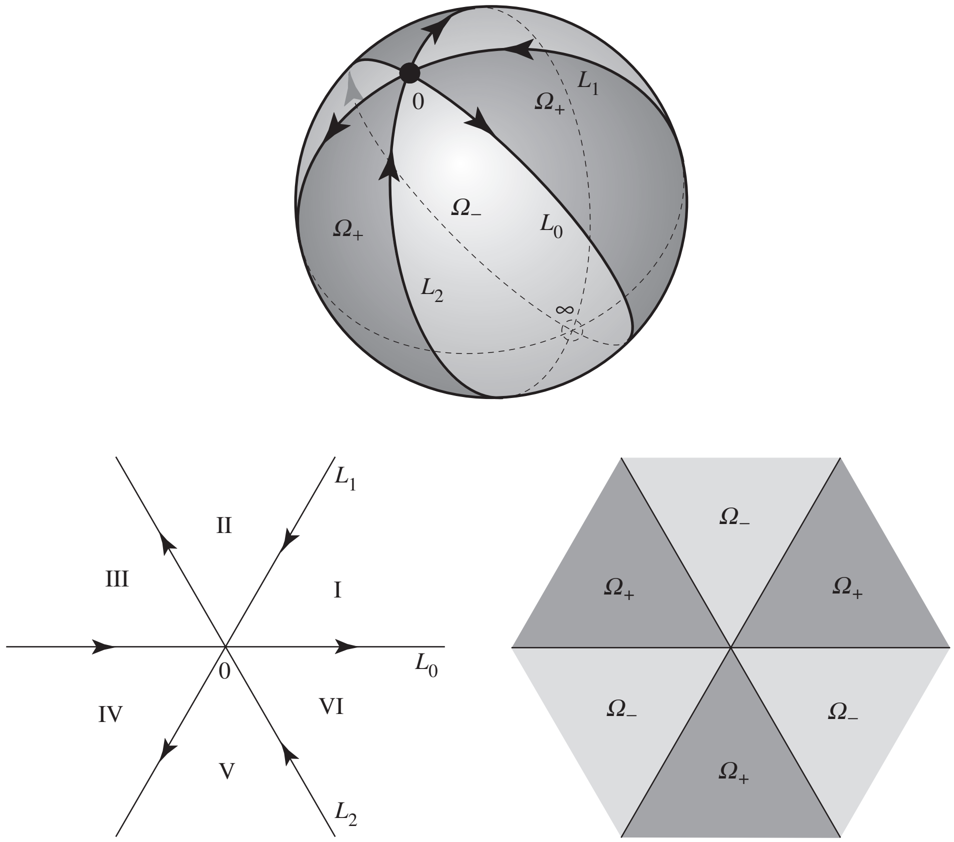

Figure 2. Domains Ω_+ and Ω_-.

- $M^{(t)}(t,k) = I + O(1/k)$, $k \to \infty$.
- We assume that $A(k)$ can have N simple zeros $K_1, \dots, K_N$, $K_j \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$. The first column of $M^{(t)}(t,k)$ has simple poles at $k = K_j$, and the second column of $M^{(t)}(t,k)$ has simple poles at $k = \bar{K}_j$. The associated residues are given by

$$\operatorname{Res}_{K_j}[M^{(t)}(t,k)]^{(1)} = \frac{\mathrm{e}^{8\mathrm{i}K_j^3 t}}{\dot{A}(K_j)B(K_j)}[M^{(t)}(t,K_j)]^{(2)}, \tag{2.43 a}$$

$$\operatorname{Res}_{\bar{K}_j}[M^{(t)}(t,k)]^{(2)} = \frac{\lambda \mathrm{e}^{-8\mathrm{i}\bar{K}_j^3 t}}{\overline{\dot{A}(K_j)}\,\overline{B(K_j)}}[M^{(t)}(t,\bar{K}_j)]^{(1)}. \tag{2.43 b}$$

3. The RH problem

Relating the vector solutions of (2.4) (analytic in domains I, …, VI) by using (2.9) and the definitions of the spectral functions (2.11), we find

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \qquad \operatorname{Im} k^3 = 0, \tag{3.1}$$

where $M_\pm(x,t,k)$ are the limit values (as k approaches $\{k \mid \operatorname{Im} k^3 = 0\}$ from $\Omega_\pm$) of a sectionally meromorphic function $M(x,t,k)$ defined as follows,

$$M = \begin{cases} \left(\mu_3^{(1)}, \dfrac{\mu_1^{(2)}}{\overline{d(\bar{k})}}\right), & k \in \mathrm{I} \cup \mathrm{III}, \\ \left(\mu_3^{(1)}, \dfrac{\mu_2^{(2)}}{\overline{a(\bar{k})}}\right), & k \in \mathrm{II}, \\ \left(\dfrac{\mu_1^{(1)}}{d(k)}, \mu_3^{(2)}\right), & k \in \mathrm{IV} \cup \mathrm{VI}, \\ \left(\dfrac{\mu_2^{(1)}}{a(k)}, \mu_3^{(2)}\right), & k \in \mathrm{V}, \end{cases} \tag{3.2}$$

where

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})}, \quad k \in \mathrm{IV} \cup \mathrm{VI}, \tag{3.3}$$

and

$$J = \begin{cases} \begin{pmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})}\mathrm{e}^{-2\mathrm{i}\theta} \\ 0 & 1 \end{pmatrix}, & \arg k = \frac{1}{3}\pi, \frac{2}{3}\pi, \\ \begin{pmatrix} 1 & 0 \\ \Gamma(k)\mathrm{e}^{2\mathrm{i}\theta} & 1 \end{pmatrix}, & \arg k = \frac{4}{3}\pi, \frac{5}{3}\pi, \\ \begin{pmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})}\mathrm{e}^{-2\mathrm{i}\theta} \\ 0 & 1 \end{pmatrix} & \\ \quad \times \begin{pmatrix} 1-\lambda|\gamma(k)|^2 & \gamma(k)\mathrm{e}^{-2\mathrm{i}\theta} \\ -\lambda\bar{\gamma}(k)\mathrm{e}^{2\mathrm{i}\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Gamma(k)\mathrm{e}^{2\mathrm{i}\theta} & 1 \end{pmatrix}, & \arg k = 0, \pi, \end{cases} \tag{3.4}$$

with

$$\left.\begin{aligned} \gamma(k) &= \frac{b(k)}{\bar{a}(k)}, \quad k \in \mathbb{R}, \\ \Gamma(k) &= \frac{\lambda\overline{B(\bar{k})}}{a(k)d(k)}, \quad \arg k = 0, \pi, \tfrac{4}{3}\pi, \tfrac{5}{3}\pi, \\ \theta(x,t,k) &= -kx + 4k^3 t. \end{aligned}\right\} \tag{3.5}$$

$\Gamma(k)$ is, in general, a meromorphic function in IV and VI.

The matrix function $M_\pm(x,t,k)$ defined by (3.2) is, in general, a sectionally meromorphic function of k, with possible poles at the zeros of $a(k)$, $d(k)$ and at the complex conjugates of these zeros. We have assumed that

- $a(k)$ can have $n = n_1 + 2n_2$ simple zeros in $\{k \mid \operatorname{Im} k < 0\}$: $\{k_j\}_1^{n_1} \in \mathrm{V}$, $\{k_j\}_{n_1+1}^{n_1+n_2} \in \mathrm{IV}$, $k_{n_1+n_2+j} = -\bar{k}_{n_1+j} \in \mathrm{VI}$, $j = 1, \ldots, n_2$.

We also assume that

- $d(k)$ can have $\Lambda = 2\Lambda_1$ simple zeros: $\{\lambda_j\}_1^{\Lambda_1} \in \mathrm{IV}$, $\lambda_{\Lambda_1+j} = -\bar{\lambda}_j \in \mathrm{VI}$, $j = 1, \ldots, \Lambda_1$;
- none of the zeros of $a(k)$ in $\mathrm{IV} \cup \mathrm{VI}$ coincide with a zero of $d(k)$.

The associated residue formulae are the following:

$$\operatorname{Res}_{k_j}[M(x,t,k)]^{(1)} = \frac{\mathrm{e}^{2\mathrm{i}\theta(k_j)}}{\dot{a}(k_j)b(k_j)}[M(x,t,k_j)]^{(2)}, \quad j = 1, \ldots, n_1, \tag{3.6 a}$$

$$\operatorname{Res}_{\bar{k}_j}[M(x,t,k)]^{(2)} = \frac{\lambda \mathrm{e}^{-2\mathrm{i}\theta(\bar{k}_j)}}{\overline{\dot{a}(k_j)b(k_j)}}[M(x,t,\bar{k}_j)]^{(1)}, \quad j = 1, \ldots, n_1, \tag{3.6 b}$$

and

$$\operatorname{Res}_{\lambda_j}[M(x,t,k)]^{(1)} = \frac{\lambda \overline{B(\bar{\lambda}_j)}\mathrm{e}^{2\mathrm{i}\theta(\lambda_j)}}{\dot{d}(\lambda_j)a(\lambda_j)}[M(x,t,\lambda_j)]^{(2)}, \quad j = 1, \ldots, \Lambda, \tag{3.7 a}$$

$$\operatorname{Res}_{\bar{\lambda}_j}[M(x,t,k)]^{(2)} = \frac{B(\bar{\lambda}_j)\mathrm{e}^{-2\mathrm{i}\theta(\bar{\lambda}_j)}}{\overline{\dot{d}(\lambda_j)a(\lambda_j)}}[M(x,t,\bar{\lambda}_j)]^{(1)}, \quad j = 1, \ldots, \Lambda. \tag{3.7 b}$$

Note that the only zeros of $a(k)$, which generate poles of M, are those in V.

The main result on the inverse spectral problem is the following.

Theorem 3.1. *Let $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$. Suppose that the set $\{g_0(t), g_1(t), g_2(t)\}$ of smooth functions is such that the associated spectral functions $s(k)$ and $S(k)$ satisfy the global relation (2.20) for $T < \infty$ and (2.21) for $T = \infty$ (in this case, it is assumed that $g_0(t)$, $g_1(t)$, $g_2(t)$ belong to $\mathcal{S}(\mathbb{R}^+)$), where $c^+(k)$ is analytic in $\{k \in \mathbb{C} \mid \operatorname{Im} k < 0\}$. Assume that the following hold.*

- *If $\lambda = -1$, then $a(k)$ has at most $n = n_1 + 2n_2$ simple zeros in $\{k \mid \operatorname{Im} k < 0\}$:*

$$\{k_j\}_{j=1}^{n_1} \in \mathrm{V}, \quad \{k_j\}_{j=n_1+1}^{n_1+n_2} \in \mathrm{IV}, \quad k_{n_1+n_2+j} = -\bar{k}_{n_1+j} \in \mathrm{VI}, \quad j = 1, \ldots, n_2.$$

- *If $\lambda = -1$, the function $d(k)$ has at most $\Lambda = 2\Lambda_1$ simple zeros:*

$$\lambda_j \in \mathrm{IV}, \quad \lambda_{\Lambda_1+j} = -\bar{\lambda}_j \in \mathrm{VI}, \quad j = 1, \ldots, \Lambda_1.$$

  If $\lambda = 1$, the function $d(k)$ has no zeros in $\mathrm{IV} \cup \mathrm{VI}$.

- *None of the zeros of $a(k)$ in $\mathrm{IV} \cup \mathrm{VI}$ coincides with a zero of $d(k)$.*

Let $M(x,t,k)$ be a solution of the following 2×2 matrix RH problem.

- *M is sectionally meromorphic in $k \in \mathbb{C} \setminus \{k \mid \operatorname{Im} k^3 = 0\}$.*
- *The first column of M has simple poles at k_j, $j = 1, \ldots, n_1$ and at λ_j, $j = 1, \ldots, \Lambda$. The associated residues satisfy (3.6) and (3.7).*

- *At* $\{k \mid \operatorname{Im} k^3 = 0\}$, *M satisfies the jump conditions* (3.1), *where the jump matrix J is defined in terms of the spectral functions* a, b, A *and* B *by* (3.4) *and* (3.5).
- *As* $k \to \infty$,

$$M(x,t,k) = I + O\biggl(\frac{1}{k}\biggr). \tag{3.8}$$

Then we have the following.

(1) $M(x,t,k)$ *exists and is unique.*

(2) $q(x,t)$ *defined in terms of* $M(x,t,k)$ *by*

$$q(x,t) = -2\mathrm{i} \lim_{k\to\infty} (kM(x,t,k))_{12} \tag{3.9}$$

satisfies the mKdV equation (2.1).

(3) $q(x,t)$ *satisfies the initial and boundary conditions*

$$q(x,0) = q_0(x),$$
$$q(0,t) = g_0(t), \qquad q_x(0,t) = g_1(t), \qquad q_{xx}(0,t) = g_2(t).$$

Proof. The proof follows the same lines as in the case of the NLS equation (see [**9**]), so here we just describe the main steps.

If $\lambda = 1$, the RH problem is 'regular' ($a(k) \neq 0$ because, in this case, the x-differential operation in (2.2 a) is self-adjoint and $d(k) \neq 0$ by assumption; so M is sectionally holomorphic). Its unique solvability is a consequence of a 'vanishing lemma' for the associated RH problem with the vanishing condition at infinity $M = O(1/k)$, $k \to \infty$ (see [**13**]). If $\lambda = -1$, the 'singular' RH problem (with poles at zeros of $a(k)$ and $d(k)$) can be mapped to a 'regular' RH problem coupled with a system of algebraic equations uniquely solvable due to the symmetry properties of J (see [**8**]).

The proof that the constructed $q(x,t)$ *solves the mKdV equation* is straightforward and follows the proof in the case of a whole-line problem (see [**11**, **12**]).

The proof that q *satisfies the initial condition* $q(x,0) = q_0(x)$ follows from the fact that it is possible to map the RH problem for $M(x,0,k)$ to that for $M^{(x)}(x,k)$; namely, $M^{(x)}(x,k)$, $\operatorname{Im} k < 0$ is the analytic continuation of $M(x,0,k)$ from II and V into $\{k \mid \operatorname{Im} k < 0\}$ and $\{k \mid \operatorname{Im} k > 0\}$, respectively, because $J(x,0,k)$ is, in fact, analytic and bounded in $\mathrm{I} \cup \mathrm{III}$ and $\mathrm{IV} \cup \mathrm{VI}$. In this way, the jump conditions and the residue relations for $M(x,0,k)$ are mapped to those for $M^{(x)}(x,k)$.

The proof that q *satisfies the boundary conditions* $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$ is, in turn, based on the map $M(0,t,k) \mapsto M^{(t)}(t,k)$. In this case, the fact that the factors relating $M(0,t,k)$ and $M^{(t)}(t,k)$ in the different sections $\mathrm{I}, \ldots, \mathrm{VI}$ are analytic and bounded, is a consequence of the global relation. For instance, in the domain V,

$$M^{(t)}(t,k) = M(0,t,k) \begin{pmatrix} a/A & (aB - Ab)\mathrm{e}^{-8\mathrm{i}k^3 t} \\ 0 & A/a \end{pmatrix},$$

and it is the global relation (2.20) that provides the boundedness of $(aB - Ab)\mathrm{e}^{-8\mathrm{i}k^3 t}$ for $t \leqslant T$, $k \in \mathrm{V}$.

Since the solution of the mKdV equation for $0 < t < T$ depends only on the boundary data taken for $0 < t < T$, the RH problems corresponding to T and T^*, $T^* < T$ must be related for $0 < t < T^*$. □

Theorem 3.2. *Let $A^*(k)$, $B^*(k)$ be defined by (2.28), with T replaced by $T^* < T$, $J^*(x,t,k)$ denote the corresponding jump matrix of the RH problem constructed as in (3.4), with $A(k)$, $B(k)$ replaced by $A^*(k)$, $B^*(k)$. Let $M^*(x,t,k)$ be the solution of this RH problem. Then, for $0 < t < T^*$, the solutions of the corresponding RH problems, M and M^*, are related as follows,*

$$M(x,t,k) = M^*(x,t,k) \qquad \text{for } k \in \mathrm{II} \cup \mathrm{V}, \tag{3.10 a}$$
$$M(x,t,k) = M^*(x,t,k)(J^*(x,t,k)J^{-1}(x,t,k)) \qquad \text{for } k \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{VI}, \tag{3.10 b}$$

where J^*J^{-1} for $k \in \mathrm{I} \cup \mathrm{III}$ and $k \in \mathrm{IV} \cup \mathrm{VI}$ are the analytic continuations of J^*J^{-1} from $\arg k = \frac{1}{3}\pi, \frac{2}{3}\pi$ and $\arg k = \frac{4}{3}\pi, \frac{5}{3}\pi$, respectively.

Proof. In the solitonless case, the only fact to be proved is the boundedness and the asymptotic behaviour $I + O(1/k)$ of J^*J^{-1} in the respective domains. In $\mathrm{I} \cup \mathrm{III}$, one has

$$J^*(x,t,k)J^{-1}(x,t,k) = \begin{pmatrix} 1 & \lambda(\overline{\Gamma(\bar{k})} - \overline{\Gamma^*(\bar{k})})\mathrm{e}^{-2\mathrm{i}\theta(x,t,k)} \\ 0 & 1 \end{pmatrix}, \tag{3.11}$$

where Γ^* is given as in (3.5), with $A(k)$, $B(k)$ replaced by $A^*(k)$, $B^*(k)$. The $(1,2)$ entry can be written as

$$\lambda(\overline{\Gamma(\bar{k})} - \overline{\Gamma^*(\bar{k})})\mathrm{e}^{-2\mathrm{i}\theta} = \frac{B(k)A^*(k) - B^*(k)A(k)}{\overline{d(\bar{k})d^*(\bar{k})}}\mathrm{e}^{2\mathrm{i}kx}\mathrm{e}^{-8\mathrm{i}k^3 t}. \tag{3.12}$$

Lemma 3.3. *For $k \in \mathrm{I} \cup \mathrm{III}$, $B(k)A^*(k) - B^*(k)A(k) = O(\mathrm{e}^{8\mathrm{i}k^3 T^*}/k)$.*

Proof. Note first that $B(k)A^*(k) - B^*(k)A(k)$ is the $(1,2)$ entry of the matrix product $S^{-1}(k;T)S(k;T^*)$, which, in view of (2.10 b), can be written as follows:

$$S^{-1}(k;T)S(k;T^*) = \mathrm{e}^{4\mathrm{i}k^3 T\sigma_3}\mu_2(0,T,k)\mathrm{e}^{4\mathrm{i}k^3(T^*-T)\sigma_3}(\mu_2(0,T^*,k))^{-1}\mathrm{e}^{-4\mathrm{i}k^3 T^*\sigma_3}. \tag{3.13}$$

The solution $\mu_2(0,t,k)$ of (2.4 b) for $x = 0$ can be expressed as

$$\mu_2(0,t,k) = \mathbb{M}(t,T,k)\mathrm{e}^{-4\mathrm{i}k^3(t-T)\hat{\sigma}_3}\mu_2(0,T,k), \tag{3.14}$$

where $\mathbb{M}(t,T,k)$ is the solution of (2.4 b) ($x = 0$) with respect to t satisfying the initial condition $\mathbb{M}(T,T,k) = I$. Substituting

$$\mu_2(0,T^*,k) = \mathbb{M}(T^*,T,k)\mathrm{e}^{-4\mathrm{i}k^3(T^*-T)\hat{\sigma}_3}\mu_2(0,T,k)$$

into (3.13) gives

$$S^{-1}(k;T)S(k;T^*) = \mathrm{e}^{4\mathrm{i}k^3T^*\sigma_3}\mathbb{M}^{-1}(T^*,T,k)\mathrm{e}^{-4\mathrm{i}k^3T^*\sigma_3}. \tag{3.15}$$

Taking into account that

$$\mathrm{e}^{4\mathrm{i}k^3(T^*-T)\sigma_3}\mathbb{M}^{-1}(T^*,T,k) = \mathbb{M}(T,T^*,k)\mathrm{e}^{4\mathrm{i}k^3(T^*-T)\sigma_3},$$

one finally gets

$$S^{-1}(k;T)S(k;T^*) = \mathrm{e}^{4\mathrm{i}k^3T\sigma_3}\mathbb{M}(T,T^*,k)\mathrm{e}^{-4\mathrm{i}k^3T\sigma_3}. \tag{3.16}$$

The same reasoning as the one used in obtaining the estimates (2.37), yields

$$\mathbb{M}_{12}(T,T^*,k) = O\left(\frac{\mathrm{e}^{-8\mathrm{i}k^3(T-T^*)}}{k}\right), \tag{3.17}$$

which, together with (3.16), proves the lemma. □

Finally, assuming the solitonless case, the desired behaviour of J^*J^{-1} for $k \in \mathrm{I} \cup \mathrm{III}$ follows from the lemma and the fact that $\overline{d(\bar{k})d^*(\bar{k})} = I + O(1/k)$, $\mathrm{e}^{2\mathrm{i}kx}$ and $\mathrm{e}^{8\mathrm{i}k^3(T^*-t)}$ are bounded (for $x > 0$, $0 < t < T^*$). In the soliton case, one can directly verify that the above map transforms correctly the residue conditions.

The analogous facts for $k \in \mathrm{IV} \cup \mathrm{VI}$ follows from symmetry consideration. □

4. Linearizable boundary conditions

Since

$$\Gamma(k) = \lambda\frac{\overline{B(\bar{k})}}{a(k)d(k)} = \lambda\frac{(\overline{B(\bar{k})}/\overline{A(\bar{k})})}{a(k)(a(k) - \lambda b(k)(\overline{B(\bar{k})}/\overline{A(\bar{k})}))}, \tag{4.1}$$

the jump data for the RH problem (3.1) are, in general, determined by

(1) $\gamma(k) = b(k)/\overline{a(\bar{k})}$, $k \in \mathbb{R}$;

(2) $a(k)$ and $b(k)$, $\arg k = \frac{4}{3}\pi, \frac{5}{3}\pi$;

(3) $\dfrac{B(k)}{A(k)}$, $\arg k = \frac{1}{3}l\pi$, $l = 0, 1, 2, 3$.

The spectral functions $a(k)$ and $b(k)$, $\operatorname{Im} k \leqslant 0$ are determined by the initial data $q_0(x)$, $x > 0$, whereas $A(k)$ and $B(k)$ are determined by the boundary values $g_0(t)$, $g_1(t)$ and $g_2(t)$. These functions cannot be given as three independent boundary conditions for the mKdV equation; they are related, in the spectral domain, by the global relation. The analysis of the algebraic global relation shows that a proper subset of $g_0(t)$, $g_1(t)$ and $g_2(t)$, characterizes the unknown part of the boundary conditions via the solution of nonlinear Volterra integral equation (see [**7**, **9**]).

There exists a class of boundary conditions for which, due to an additional symmetry, one can compute explicitly all spectral data necessary to construct the RH problem from the initial data. To describe this class, we will follow the approach proposed in [**7**, **9**].

Suppose that the coefficient matrix $V(t,k)=\tilde{Q}(0,t,k)-4\mathrm{i}k^3\sigma_3$, appearing in the second equation of the Lax pair (2.2 a), $\psi_t'=V\psi$, satisfies the symmetry relation

$$V(t,\nu(k))=N(k)V(t,k)N^{-1}(k), \tag{4.2}$$

where $N(k)$ is a t-independent non-singular matrix-valued function of k. Then

$$\psi_2(0,T,\nu(k))=N(k)\psi_2(0,T,k)N^{-1}(k), \tag{4.3}$$

where

$$\psi_2(0,T,k)=\mu_2(0,T,k)\mathrm{e}^{-4\mathrm{i}k^3T\sigma_3}.$$

Hence, in view of the definition of $S(k)$ (2.10 b),

$$S(\nu(k))=N(k)S(k)\mathrm{e}^{4\mathrm{i}k^3T\sigma_3}N^{-1}(k)\mathrm{e}^{-4\mathrm{i}\nu^3(k)T\sigma_3}. \tag{4.4}$$

This equation implies that it is possible to calculate $S(k)$ in a particular domain of the complex k-plane from the values of $S(k)$ in another domain.

A necessary condition for the existence of N is $\det V(t,\nu(k))=\det V(t,k)$. Since

$$\det V(t,k)=2(8k^6-2k^2\{3q^4(0,t)-2\lambda q(0,t)q_{xx}(0,t)+\lambda q_x^2(0,t)\}-\lambda(2\lambda q^3-q_{xx})^2)$$

and since a non-trivial choice of $\nu(k)$ is

$$\nu(k)=\omega k \quad \text{or} \quad \nu(k)=\omega^2 k \quad \text{with } \omega=\tfrac{2}{3}\pi\mathrm{i},$$

it follows that a necessary condition is the following relation between $q(0,t)$, $q_x(0,t)$ and $q_{xx}(0,t)$:

$$3q^4(0,t)-2\lambda q(0,t)q_{xx}(0,t)+\lambda q_x^2(0,t)=0. \tag{4.5}$$

To satisfy (4.5), on one hand, and to ensure that $N(k)$ is t independent, on the other hand, we set

$$q(0,t)=0, \qquad q_x(0,t)=0. \tag{4.6}$$

This implies $N(k)\equiv I$ and, consequently, $S(\omega k)=S(\omega^2 k)=S(k)$. In terms of $A(k)$ and $B(k)$,

$$A(\omega k)=A(\omega^2 k)=A(k), \quad B(\omega k)=B(\omega^2 k)=B(k), \quad k\in \mathrm{I}\cup\mathrm{III}\cup\mathrm{V}. \tag{4.7}$$

This means, in particular, that if $B(k)/A(k)$ is known for $k\in\mathrm{V}$, then $B(k)/A(k)$ is also known for $k\in\mathrm{I}\cup\mathrm{III}$.

In the case $T=\infty$, the global relation (2.21) implies

$$\frac{B(k)}{A(k)}=\frac{b(k)}{a(k)}, \quad k\in\mathrm{V}, \tag{4.8}$$

which, together with (4.7), yields $\Gamma(k)$ entirely in terms of $a(k)$ and $b(k)$,

$$\Gamma(k)=\begin{cases}\lambda\dfrac{\overline{b(\omega\bar{k})}}{a(k)\Delta(k,\omega)}, & \arg k=\pi,\,\frac{4}{3}\pi,\\[2ex] \lambda\dfrac{\overline{b(\omega^2\bar{k})}}{a(k)\Delta(k,\omega^2)}, & \arg k=0,\,\frac{5}{3}\pi,\end{cases}\tag{4.9}$$

where

$$\Delta(k,\omega)=a(k)\overline{a(\omega\bar{k})}-\lambda b(k)\overline{b(\omega\bar{k})}.\tag{4.10}$$

Now we will show that in the case $T<\infty$, one can also use (4.8) and (4.7) to determine $B(k)/A(k)$. To do this, we proceed as in the proof of Theorem 3.2. Namely, we will show that the RH problems determined by $B(k)/A(k)=b(k)/a(k)$, on one hand, and by the ratio $B(k;T)/A(k;T)$ corresponding to some $T<\infty$, are equivalent, in the sense that their solutions give the same solution of the mKdV equation for $t<T$. Similarly to (3.11), one has to show that for $k\in \mathrm{I}\cup\mathrm{III}$, $(\overline{\Gamma(\bar{k};T)}-\overline{\Gamma(\bar{k})})\exp\{-2\mathrm{i}\theta\}$ is bounded and is $O(1/k)$ as $k\to\infty$; here, $\Gamma(k;T)$ is defined as in (3.5), with $B(k)$ and $A(k)$ replaced by $B(k;T)$ and $A(k;T)$, respectively. One has (cf. (3.12))

$$\lambda(\overline{\Gamma(\bar{k};T)}-\overline{\Gamma(\bar{k})})=\frac{A(k;T)A(k)}{d(k;T)d(k)}\left(\frac{B(k;T)}{A(k;T)}-\frac{B(k)}{A(k)}\right).\tag{4.11}$$

The global relation (2.20) implies that, for $\{k\mid \operatorname{Im}k<0\}$ (particularly, for $k\in\mathrm{V}$),

$$\frac{B(k;T)}{A(k;T)}=\frac{b(k)}{a(k)}-\frac{c^+(k;T)\mathrm{e}^{8\mathrm{i}k^3T}}{a(k)A(k;T)}.\tag{4.12}$$

Therefore,

$$\left(\frac{B(k;T)}{A(k;T)}-\frac{B(k)}{A(k)}\right)\mathrm{e}^{-8\mathrm{i}k^3t}=O\left(\frac{1}{k}\right)\mathrm{e}^{8\mathrm{i}k^3(T-t)},\quad k\in\mathrm{V}.\tag{4.13}$$

The symmetry relations

$$\frac{B(\omega k;T)}{A(\omega k;T)}=\frac{B(\omega^2 k;T)}{A(\omega^2 k;T)}=\frac{B(k;T)}{A(k;T)},$$

together with the analogous relations for $B(k)/A(k)$, yield (4.13) for $k\in\mathrm{I}\cup\mathrm{III}$, which, in turn, implies

$$\lambda(\overline{\Gamma(\bar{k};T)}-\overline{\Gamma(\bar{k})})\mathrm{e}^{-2\mathrm{i}\theta(x,t,k)}=O\left(\frac{1}{k}\right),\quad k\in\mathrm{I}\cup\mathrm{III},\quad 0<t<T.$$

Consider now the residue conditions (3.6), (3.7). If $a(\omega\bar{k})\neq 0$ for $k\in\mathrm{IV}$ (which, due to the global relation, implies $A(\omega\bar{k})\neq 0$), then (cf. (4.9))

$$d(k)=\frac{\overline{A(\omega\bar{k})}}{\overline{a(\omega\bar{k})}}\Delta(k,\omega).\tag{4.14}$$

If $a(\omega\bar{k}) = 0$, formula (4.14) must be replaced by

$$d(k) = \frac{(\mathrm{d}/\mathrm{d}k)\overline{A(\omega\bar{k})}}{(\mathrm{d}/\mathrm{d}k)\overline{a(\omega\bar{k})}}\Delta(k,\omega). \tag{4.15}$$

Therefore, the zero sets of $d(k)$ and $\Delta(k,\omega)$ coincide in IV. Expressing $\dot{d}(k)$ at a zero of $d(k)$ via $\dot{\Delta}(k,\omega)$ by using (4.14) or (4.15) gives the following modification of the residue conditions (3.7). For $\lambda_j \in \mathrm{IV}$,

$$\operatorname{Res}_{\lambda_j}[M(x,t,k)]^{(1)} = \frac{\lambda\overline{b(\omega\bar{\lambda}_j)}\mathrm{e}^{2\mathrm{i}\theta(\lambda_j)}}{\dot{\Delta}(\lambda_j,\omega)a(\lambda_j)}[M(x,t,\lambda_j)]^{(2)}, \quad j = 1,\dots,\Lambda_1. \tag{4.16}$$

Similarly, for $\lambda_j \in \mathrm{VI}$,

$$\operatorname{Res}_{\lambda_j}[M(x,t,k)]^{(1)} = \frac{\lambda\overline{b(\omega^2\bar{\lambda}_j)}\mathrm{e}^{2\mathrm{i}\theta(\lambda_j)}}{\dot{\Delta}(\lambda_j,\omega^2)a(\lambda_j)}[M(x,t,\lambda_j)]^{(2)}, \quad j = \Lambda_1+1,\dots,2\Lambda_1. \tag{4.17}$$

Consequently, the residue conditions in III take the form, for $\bar{\lambda}_j \in \mathrm{III}$,

$$\operatorname{Res}_{\bar{\lambda}_j}[M(x,t,k)]^{(2)} = \frac{b(\omega\bar{\lambda}_j)\mathrm{e}^{-2\mathrm{i}\theta(\bar{\lambda}_j)}}{\overline{\dot{\Delta}(\lambda_j,\omega)}\,\overline{a(\lambda_j)}}[M(x,t,\bar{\lambda}_j)]^{(1)}, \quad j = 1,\dots,\Lambda_1. \tag{4.18}$$

Similarly, in I, for $\bar{\lambda}_j \in \mathrm{I}$,

$$\operatorname{Res}_{\bar{\lambda}_j}[M(x,t,k)]^{(2)} = \frac{b(\omega^2\bar{\lambda}_j)\mathrm{e}^{-2\mathrm{i}\theta(\bar{\lambda}_j)}}{\overline{\dot{\Delta}(\lambda_j,\omega^2)}\,\overline{a(\lambda_j)}}[M(x,t,\bar{\lambda}_j)]^{(1)}, \quad j = \Lambda_1+1,\dots,2\Lambda_1. \tag{4.19}$$

Theorem 4.1. *Let $q(x,t)$, $x > 0$, $t > 0$ satisfy the mKdV equation (2.1), the initial condition*

$$q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}^+), \quad 0 < x < \infty,$$

with $q_0(0) = q_0'(0) = 0$, and the boundary conditions

$$q(0,t) = 0, \quad q_x(0,t) = 0, \quad t > 0.$$

Furthermore, if $\lambda = -1$, assume that

- *$a(k)$ defined in (2.22) has a finite number of simple zeros for $\operatorname{Im} k < 0$;*
- *$\Delta(k,\omega)$ defined by (3.3) has a finite number of simple zeros in IV, none of which coincide with the possible zeros of $a(k)$.*

Then the solution $q(x,t)$ can be constructed via (3.9), where M satisfies the RH problem (3.1). This RH problem involves $\Gamma(k)$, which is given by (4.9); the residue conditions are given by (3.6) and (4.16)–(4.19).

Remark 4.2. The initial boundary-value problem for the mKdV equation with the boundary conditions (4.6) has been considered in [**10**] under the assumption that there exist no discrete eigenvalues.

Remark 4.3. The boundary conditions

$$q(0,t) = C_0, \qquad q_x(0,t) = C_1, \qquad q_{xx}(0,t) = C_2,$$

where C_0, C_1 and C_2 are constants related by

$$3C_0^4 - 2C_0C_2 + C_1^2 = 0,$$

are also linearizable (see (4.5)) in the sense that, *assuming* that the solution of the corresponding initial boundary problem for the mKdV equation exists, this solution can be constructed via an RH problem whose jump matrix can be given explicitly in terms of $a(k)$ and $b(k)$. However, in the case of the boundary conditions

$$q(0,t) = 0, \quad q_x(0,t) = 0, \quad t > 0,$$

one can prove the following stronger version of Theorem 4.1.

Theorem 4.4. *Consider the mKdV equation (2.1) with the initial condition*

$$q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}^+), \quad 0 < x < \infty,$$

with $q_0(0) = q_0'(0) = 0$, *and the boundary conditions*

$$q(0,t) = 0, \quad q_x(0,t) = 0, \quad t > 0.$$

Furthermore, assume the following.

(1) *If* $\lambda = -1$,

- $a(k)$ *defined in (2.22) has a finite number of simple zeros for* $\operatorname{Im} k < 0$*;*
- $\Delta(k,\omega)$ *defined by (3.3) has a finite number of simple zeros in* IV, *none of which coincide with the possible zeros of* $a(k)$.

(2) *If* $\lambda = 1$,

$$|a(k)|^2 - |b(k)|^2 > 0, \quad \arg k = \tfrac{4}{3}\pi, \tfrac{5}{3}\pi.$$

Then this initial boundary-value problem has a unique solution such that $q(x,t) \to 0$ *as* $x \to \infty$, *which can be constructed via (3.9), where* M *satisfies the RH problem (3.1). This RH problem involves* $\Gamma(k)$, *which is given by (4.9); the residue conditions are given by (3.6) and (4.16)–(4.19).*

Proof. In view of Theorems 3.1 and 4.1, it is sufficient to show that given $a(k)$ and $b(k)$, one can construct $A(k)$ and $B(k)$ satisfying the global relation (2.21). Equivalently, one has to construct the function $f(k)$ holomorphic and having no zeros for $k \in \mathrm{V}$, such that

$$A(k) = \frac{a(k)}{f(k)}, \quad B(k) = \frac{b(k)}{f(k)}, \quad k \in \mathrm{V}.$$

Due to the global relation,

$$d(k)=\frac{a(k)}{A(k)},\quad \arg k=\tfrac{4}{3}\pi,\tfrac{5}{3}\pi,$$

where $d(k)$ is defined by (3.3). Thus the function $f(k):=a(k)/A(k)$, $k\in\mathrm{V}$, can be analytically continued to the half-plane $\operatorname{Im}k<0$ as $f(k):=d(k)$ for $k\in\mathrm{IV}\cup\mathrm{VI}$.

For $\arg k=\frac{4}{3}\pi$, equation (4.14) gives

$$\Delta(k,\omega)=d(k)\frac{\overline{a(\omega\bar k)}}{\overline{A(\omega\bar k)}}=f(k)\frac{\overline{a(k)}}{\overline{A(k)}}=|f(k)|^2, \tag{4.20}$$

or, in terms of a and b,

$$|f(k)|^2=|a(k)|^2-\lambda|b(k)|^2,\quad \arg k=\tfrac{4}{3}\pi.$$

Analogously, for $\arg k=\frac{5}{3}\pi$, one has

$$\Delta(k,\omega^2)=|f(k)|^2. \tag{4.21}$$

On the other hand, for $\arg k=\pi$, equation (4.14) gives

$$|\Delta(k,\omega)|=|f(k)|\times|f(\omega k)|=|f(k)|\times|\Delta(\omega k,\omega^2)|^{1/2}, \tag{4.22}$$

where the last equality follows from (4.21). Therefore, we can express $|f(k)|$ for $\arg k=\pi$, in terms of $a(k)$ and $b(k)$, $\operatorname{Im}k\leqslant 0$,

$$|f(k)|=\frac{|\Delta(k,\omega)|}{|\Delta(\omega k,\omega^2)|^{1/2}}=\frac{|a(k)\overline{a(\omega k)}-\lambda b(k)\overline{b(\omega k)}|}{(|a(\omega k)|^2-\lambda|b(\omega k)|^2)^{1/2}},\quad k\leqslant 0. \tag{4.23}$$

In a similar way, for $\arg k=0$, one has

$$|\Delta(k,\omega^2)|=|f(k)|\times|f(\omega^2 k)|=|f(k)|\times|\Delta(\omega^2 k,\omega)|^{1/2}, \tag{4.24}$$

so that

$$|f(k)|=\frac{|\Delta(k,\omega^2)|}{|\Delta(\omega^2 k,\omega)|^{1/2}}=\frac{|a(k)\overline{a(\omega^2 k)}-\lambda b(k)\overline{b(\omega^2 k)}|}{(|a(\omega^2 k)|^2-\lambda|b(\omega^2 k)|^2)^{1/2}},\quad k\geqslant 0. \tag{4.25}$$

Equations (4.23) and (4.25) give the values of $|f(k)|$ for $\operatorname{Im}k=0$. Since the zeros of $f(k)$ coincide with the zeros of $d(k)$ in IV and VI (which, in turn, are the zeros of $\Delta(k,\omega)$ and $\Delta(k,\omega^2)$), $f(k)$ is uniquely determined for $\operatorname{Im}k<0$ as follows:

$$\begin{aligned}
f(k)&=\prod_j\frac{k-\lambda_j}{k-\bar\lambda_j}\exp\biggl\{-\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\ln|f(\mu)|}{\mu-k}\,\mathrm{d}\mu\biggr\}\\
&=\prod_{j=1}^{\Lambda_1}\frac{(k-\lambda_j)(k+\bar\lambda_j)}{(k-\bar\lambda_j)(k+\lambda_j)}\exp\biggl\{-\frac{k}{\pi\mathrm{i}}\int_0^{\infty}\frac{2\ln|\Delta(k,\omega^2)|-\ln\Delta(\omega^2 k,\omega)}{\mu^2-k^2}\,\mathrm{d}\mu\biggr\}.
\end{aligned}$$

□

5. Long-time asymptotics: solitons

The formalism of the RH problem with $T = \infty$ is well suited for the study of the large-time behaviour of the solution. Since the jump matrix $J(x, t, k)$ depends explicitly on x and t,

$$J(x, t, k) = \mathrm{e}^{(\mathrm{i}kx - 4\mathrm{i}k^3 t)\sigma_3} J(0, 0, k) \mathrm{e}^{-(\mathrm{i}kx - 4\mathrm{i}k^3 t)\sigma_3},$$

it is possible to apply the nonlinear steepest descent method of [**1**,**3**] to study the large-t behaviour of the oscillatory RH problem (3.1). This method has been applied for the analogous problem on the line in [**1**].

In the case of the half-line, by using a rational approximation of $\Gamma(k)$, we first deform the RH problem (3.1) to an approximate RH problem formulated on the line $\operatorname{Im} k = 0$ only. Then one can directly apply the nonlinear steepest descent method of [**1**].

Assuming $\Gamma(k)$ to be rational, we define $\tilde{M}(x, t, k)$ as follows:

$$\tilde{M} = \begin{cases} M, & k \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{VI}, \\ M \begin{pmatrix} 1 & \lambda \overline{\Gamma(\bar{k})} \mathrm{e}^{-2\mathrm{i}\theta} \\ 0 & 1 \end{pmatrix}, & k \in \mathrm{II}, \\ M \begin{pmatrix} 1 & 0 \\ -\Gamma(k) \mathrm{e}^{2\mathrm{i}\theta} & 1 \end{pmatrix}, & k \in \mathrm{V}. \end{cases} \tag{5.1}$$

Then, in view of (3.1) and (3.4), $\tilde{M}$ is analytic in $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$ and satisfies the jump condition

$$\tilde{M}_-(x, t, k) = \tilde{M}_+(x, t, k) J(x, t, k), \quad \operatorname{Im} k = 0, \tag{5.2}$$

where $J(x, t, k)$ is the same as in (3.4) for $\operatorname{Im} k = 0$. Now the asymptotics of $\tilde{M}$ (and, therefore, of $q(x, t)$) can be obtained following [**1**] as in the case of the whole-line problem.

In the regular case ($N = 0$), one has

$$M(x, t, k) = (I + O(t^{-1/2})) \begin{pmatrix} \delta(k) & 0 \\ 0 & \delta^{-1}(k) \end{pmatrix}, \quad t \to \infty, \tag{5.3}$$

uniformly for $|\operatorname{Im} k| \geqslant \varepsilon > 0$, $0 < \beta_- \leqslant x/t \leqslant \beta_+ < \infty$, where $\delta(k)$ is analytic in $\mathbb{C} \setminus \{(-\infty, k_0] \cup [k_0, \infty)\}$ and satisfies

$$\left.\begin{aligned} &\delta_-(k) = \delta_+(k)(1 + |\gamma(k) - \lambda \overline{\Gamma(\bar{k})}|^2), \\ &\delta_\pm(k) = \delta(k \pm \mathrm{i}0), \quad k \in (-\infty, k_0) \cup (k_0, \infty). \end{aligned}\right\} \tag{5.4}$$

The solution of (5.4) is

$$\delta(k) = \exp\left\{ -\frac{1}{2\pi \mathrm{i}} \left(\int_{-\infty}^{-k_0} + \int_{k_0}^{\infty} \right) \frac{\ln(1 + |\gamma(k') - \lambda \overline{\Gamma(k')}|^2)}{k' - k} \, \mathrm{d}k' \right\}. \tag{5.5}$$

In the singular case ($\lambda = -1$, $N \neq 0$), the dressing method (see, for example, [**8**]) gives

$$\left.\begin{aligned} q(x,t) = q_N(x,t) + O(t^{-1/2}), \quad t \to \infty, \\ 0 < \beta_- \leqslant x/t \leqslant \beta_+ < \infty, \end{aligned}\right\} \tag{5.6}$$

where $q_N(x,t)$ is the pure N-soliton solution of the mKdV equation, with parameters $\{k_j\}_1^N$ and $c_j\delta^{-2}(k_j)$, and the c_j are the coefficients in the residue relations

$$\operatorname{Res}_{k_j}[M(x,t,k)]^{(1)} = c_j[M(x,t,k_j)]^{(2)}.$$

Therefore, we arrive at the following result.

Theorem 5.1. *Suppose that the conditions of Theorem 3.1 are satisfied. Denote*

$$\begin{aligned} \tilde k_{2l-1} &= \lambda_l, & l &= 1,\dots,\Lambda_1, \\ \tilde k_{2l} &= \lambda_{\Lambda_1+l}, & l &= 1,\dots,\Lambda_1, \\ \tilde k_{2\Lambda_1+j} &= k_j, & j &= 1,\dots,n_1. \end{aligned}$$

Let $\tilde k_j = \xi_j + \mathrm{i}\eta_j$. Suppose that

$$3\xi_{2l-1}^2 - \eta_{2l-1}^2 \neq \xi_{2m-1}^2 - \eta_{2m-1}^2 \quad \text{if } l \neq m \text{ for } l,m = 1,\dots\Lambda_1.$$

Without loss of generality, we assume that $3\xi_{2l-1}^2 - \eta_{2l-1}^2 > \xi_{2m-1}^2 - \eta_{2m-1}^2$ if $l < m$.

Then there are Λ_1 directions in the (x,t)-plane, $j = 2l-1$, $l = 1,\dots,\Lambda_1$, namely,

$$\frac{x}{4t} = 3\xi_j^2 - \eta_j^2 + O\left(\frac{1}{t}\right), \quad t \to \infty,$$

along which the asymptotics of the solution $q(x,t)$ of the mKdV equation on the half-line is given by the one-breather formula,

$$q(x,t) = 4\eta_j\xi_j \frac{\xi_j \cosh(\nu_j^{(2)} + \Delta_j)\sin(\nu_j^{(1)} + \phi_j) - \eta_j \sinh(\nu_j^{(2)} + \Delta_j)\cos(\nu_j^{(1)} + \phi_j)}{\xi_j^2 \cosh^2(\nu_j^{(2)} + \Delta_j) + \eta_j^2 \cos^2(\nu_j^{(1)} + \phi_j)} + O(t^{-1/2}), \tag{5.7}$$

where

$$\begin{aligned} \nu_j^{(1)}(x,t) &= -2\xi_j x + 8\xi_j(\xi_j^2 - 3\eta_j^2)t, \\ \nu_j^{(2)}(x,t) &= 2\eta_j x - 8\eta_j(3\xi_j^2 - \eta_j^2)t, \end{aligned}$$

and the phase shifts Δ_j and ϕ_j are described by the following equations,

$$\left.\begin{aligned} \Delta_j &= \log 2 - \ln\left|\frac{\xi_j}{\eta_j}\right| + \log|\tilde k_j| - \log|c_j| \\ &\quad - \frac{2}{\pi}(\eta_j\chi_j + \xi_j\kappa_j) - 2\sum_{l=2j+1}^{N} \log\left|\frac{\tilde k_j - \tilde k_l}{\tilde k_j + \tilde k_l}\right|, \\ \phi_j &= -\arg\tilde k_j + \arg c_j + \frac{2}{\pi}(-\xi_j\chi_j + \eta_j\kappa_j) + 2\sum_{l=2j+1}^{N} \arg\left|\frac{\tilde k_j - \tilde k_l}{\tilde k_j + \tilde k_l}\right|, \end{aligned}\right\} \tag{5.8}$$

with

$$\chi_j = \operatorname{Re}\int_{\sqrt{x/12t}}^{\infty} \frac{\log(1+|\gamma(\mu)+\overline{\mathit{\Gamma}(\mu)}|^2)}{\mu^2-\tilde{k}_j^2}\,\mathrm{d}\mu$$
$$= \int_{\sqrt{x/12t}}^{\infty} \frac{\log(1+|\gamma(\mu)+\overline{\mathit{\Gamma}(\mu)}|^2)(\mu^2-\xi_j^2+\eta_j^2)}{(\mu^2-\xi_j^2+\eta_j^2)^2-4\xi_j^2\eta_j^2}\,\mathrm{d}\mu, \tag{5.9 a}$$

$$\kappa_j = \operatorname{Im}\int_{\sqrt{x/12t}}^{\infty} \frac{\log(1+|\gamma(\mu)+\overline{\mathit{\Gamma}(\mu)}|^2)}{\mu^2-\tilde{k}_j^2}\,\mathrm{d}\mu$$
$$= -2\xi_j\eta_j\int_{\sqrt{x/12t}}^{\infty} \frac{\log(1+|\gamma(\mu)+\overline{\mathit{\Gamma}(\mu)}|^2)}{(\mu^2-\xi_j^2+\eta_j^2)^2-4\xi_j^2\eta_j^2}\,\mathrm{d}\mu, \tag{5.9 b}$$

$$c_j = -\frac{\overline{B(\bar{\tilde{k}}_j)}}{\dot{d}(\tilde{k}_j)a(\tilde{k}_j)}. \tag{5.9 c}$$

Remark 5.2. The soliton-type asymptotics (5.7)–(5.9) are of the same *form* as in the case of the whole line. But the detailed expression is influenced by the boundary conditions as follows.

(a) The spectral functions $A(k)$ and $B(k)$ participate in the determination of all the parameters of the solitons; particularly, the soliton directions are determined by the zeros of $d(k)$, instead of the zeros of $a(k)$ (which is what happens in the case of the whole line).

(b) The zeros of $a(k)$ in V, although being the poles of the basic RH problem, do not generate solitons, since the corresponding solitons on the whole line move to the left and, therefore, leave the domain $\{(x,t) \mid x>0,\ t>0\}$. However, the zeros of $a(k)$ (together with part of the continuous spectrum described in terms of $\gamma(k)$ and $\mathit{\Gamma}(k)$ for $|k|>\sqrt{x/12t}$) participate in the soliton asymptotics via the phase shifts Δ_j and ϕ_j.

Remark 5.3. One can obtain more precise asymptotics by using a more detailed analysis of the RH, namely, by constructing the exact solutions of the RH problem near $k=\pm k_0$ (see [**1**, **2**] for the case of the whole line).

Remark 5.4. In the case of the linearizable boundary conditions, all the parameters in the asymptotic formulae can be expressed in terms of the initial data only (via the spectral functions $a(k)$ and $b(k)$). In this case,

$$c_j = \frac{-\overline{b(\omega\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}(\lambda_j,\omega)}, \quad j=1,\dots,\mathit{\Lambda}_1,$$

$$\mathit{\Gamma}(k) = \begin{cases} -\dfrac{\overline{b(\omega\bar{k})}}{a(k)\Delta(k,\omega)}, & \arg k = \pi,\ \frac{4}{3}\pi, \\[2ex] -\dfrac{\overline{b(\omega^2\bar{k})}}{a(k)\Delta(k,\omega^2)}, & \arg k = 0,\ \frac{5}{3}\pi. \end{cases}$$

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