Analysis of the Global Relation for the Nonlinear Schrödinger Equation on the Half-line

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Abstract. It has been shown recently that the unique, global solution of the Dirichlet problem of the nonlinear Schrödinger equation on the half-line can be expressed through the solution of a 2 × 2 matrix Riemann–Hilbert problem. This problem is specified by the spectral functions \(f_a(k), b(k)\) which are defined in terms of the initial condition \(q(x; 0) = q_0(x)\), and by the spectral functions \(A(k), B(k)\) which are defined in terms of the specified boundary condition \(q(0, t) = g_0(t)\) and the unknown boundary value \(q_x(0, t) = g_1(t)\). Furthermore, it has been shown that given \(q_0\) and \(g_0\), the function \(g_1\) can be characterized through the solution of a certain ‘global relation’ coupling \(q_0, g_0, g_1, \Phi(t, k)\), where \(\Phi\) satisfies the \(t\)-part of the associated Lax pair evaluated at \(x = 0\). We show here that, by using a Gelfand–Levitan–Marchenko triangular representation of \(\Phi\), the global relation can be explicitly solved for \(g_1\).


Key words. Gelfand–Levitan–Marchenko representation, global relation, half-line, initial-boundary value problem, nonlinear Schrödinger equation.

1. Introduction

Let the complex-valued function \(q(x; t)\) satisfy the nonlinear Schrödinger equation

\[iqt + q_{xx} - 2|q|^2q = 0, \quad \rho = \pm 1, \quad 0 < x < \infty, \quad 0 < t < T,\]

and the initial and boundary conditions

\[q(x, 0) = q_0(x) \in \mathscr{S}(\mathbb{R}^+), \quad 0 < x < \infty, \quad (2a)\]

\[q(0, t) = g_0(t), \quad 0 < t < T, \quad (2b)\]

where \(T\) is a given fixed constant, \(\mathscr{S}\) denotes the space of Schwartz functions, and the function \(g_0(t)\) has sufficient smoothness. The solution of this initial boundary value (IBV) problem can be constructed as follows ([3, 6]):
Given \( q_0(x) \) construct the spectral functions \( \{a(k), b(k)\} \). These functions are defined by
\[
a(k) = \phi_2(0, k), \quad b(k) = \phi_1(0, k),
\]
where the vector
\[
\phi(x, k) = \begin{pmatrix} \phi_1(x, k) \\ \phi_2(x, k) \end{pmatrix}
\]
is the following solution of the \( x \)-problem of the associated Lax pair evaluated at \( t = 0 \):
\[
\phi_\epsilon + ik\sigma_3 \phi = Q_0(x) \phi, \quad 0 < x < \infty, \quad \text{Im} \ k \geq 0,
\]
\[
\phi(x, k) = e^{ikx} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \quad \text{as} \quad x \to \infty,
\]
with \( \sigma_3 \) and \( Q_0(x) \) defined by
\[
\sigma_3 = \text{diag} [1, -1], \quad Q_0(x) = \begin{pmatrix} 0 & q_0(x) \\ \rho q_0(x) & 0 \end{pmatrix}.
\]

Given \( q_0(x) \) and \( g_0(t) \) characterize \( g_1(t) \) by the requirement that the spectral functions \( \{A(t, k), B(t, k)\} \) satisfy the global relation
\[
a(k)B(t, k) - b(k)A(t, k) = e^{i k^2 \epsilon t (t, k)}, \quad t \in [0, T], \quad k \in \mathcal{D},
\]
where \( \mathcal{D} \) denotes the closure of the first quadrant
\[
D = \{ k \in \mathbb{C} \mid \text{Re} k > 0, \quad \text{Im} \ k > 0 \},
\]
and \( \epsilon(t, k) \) is analytic in \( k \in \mathcal{D} \) and is \( O(1/k) \) as \( k \to \infty \). The spectral functions \( A(t, k), B(t, k) \) are defined by
\[
A(t, k) = e^{2ik^2 \Theta_2 (t, k)}, \quad B(t, k) = -e^{2ik^2 \Theta_1 (t, k)},
\]
where the vector
\[
\Theta(t, k) = \begin{pmatrix} \Theta_1 (t, k) \\ \Theta_2 (t, k) \end{pmatrix}
\]
is the following solution of the \( t \)-problem of the associated Lax pair evaluated at \( x = 0 \):
\[
\Theta_\epsilon + 2ik^2 \sigma_3 \Theta = (2kQ(t) + R(t)) \Theta, \quad 0 < t < T, \quad k \in \mathbb{C},
\]
\[
\Theta(0, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
with \( Q(t) \) and \( R(t) \) defined by
\[
Q(t) = \begin{pmatrix} 0 & g_0(t) \\ \rho g_0(t) \ & 0 \end{pmatrix}, \quad R(t) = i \rho \begin{pmatrix} -|g_0(t)|^2 & \rho g_3(t) \\ -\bar{g}_1(t) \ & |g_0(t)|^2 \end{pmatrix}.
\]
Given \( a(k), b(k), A(k) = A(T, k), B(k) = B(T, k) \), define a \( 2 \times 2 \) matrix Riemann–Hilbert (RH) problem. This problem has the distinctive feature that its jump has explicit \((x, t)\)-dependence in the exponential form \( \exp[i k x + 2 ik^2 t] \). Determine \( q(x, t) \) in terms of the solution of this RH problem. The function \( q(x, t) \) solves the NLS equation with initial-boundary conditions \( q(x, 0) = q_0(x), q(0, t) = g_0(t) \).

The most complicated step in the above construction is the characterization of \( g_1 \). In the particular case \( q_0(x) \equiv 0 \), it can be shown that the global relation is equivalent to the integral relation

\[
\int_{\partial D} k \int_0^t e^{ik^2(t' - t) + ip} \int_0^{[g_0]^2} \{2kg_0(t') + ig_1(t')\} \Phi_2(t', k) \, dt' \, dk = 0, \tag{10}
\]

where \( \partial D = (\infty, 0) \cup (0, \infty) \) is the oriented boundary of \( D \). Given \( g_0(t) \), the vector equation (8a) and the scalar equation (10) are two equations for the unknown vector-valued and scalar-valued functions \( \Phi(t, k) \) and \( g_1(t) \), respectively. It is shown in [6] that these equations constitute a system of coupled nonlinear Volterra integral equations for \( \Phi \) and \( g_1 \).

1.1. A FORMULATION IN TERMS OF \( \{M_j(t, s), L_j(t, s)\}_{j=1,2} \)

In this Letter we will show that the above system decouples. Indeed, we will show that the global relation (6) can be solved explicitly for \( g_1(t) \) in terms of \( g_0(t) \) and \( \Phi(t, k) \). For this purpose we will use the Gelfand–Levitan–Marchenko integral representation for \( \Phi(t, k) \); namely we will express \( \Phi(t, k) \) in terms of four scalar functions \( \{M_j(t, s), L_j(t, s)\}_{j=1,2} \), see Proposition 2. The function \( \Phi \) satisfies Equation (8) if and only if these four functions satisfy Equations (23), (24). Furthermore using definitions (7), it follows that \( A(t, k) \) and \( B(t, k) \) can be expressed in terms of these four functions by

\[
A(t, k) = 1 + \int_0^t e^{ik^2} [2 \tilde{L}_2(t, t - 2 \tau) - ipg_0(t) \tilde{M}_1(t, t - 2 \tau) + 2k \tilde{M}_2(t, t - 2 \tau)] \, d\tau, \tag{11a}
\]

\[
B(t, k) = - \int_0^t e^{ik^2} [2L_1(t, 2 \tau - t) - ig_0(t) M_2(2 \tau - t) + 2k M_1(2 \tau - t)] \, d\tau. \tag{11b}
\]

It turns out that if we replace \( A \) and \( B \) in the global relation (6) by the above expressions, then the global relation can be solved in closed form for \( g_1(t) \) in terms of \( g_0(t) \) and \( \{M_j, L_j\}_{j=1,2} \), see Proposition 3. In the particular case \( q_0(x) \equiv 0 \), the following proposition is valid.

PROPOSITION 1. Let \( B(t, k) \) satisfy the global relation corresponding to the case of zero initial condition, \( q_0(x) \equiv 0 \), i.e., Equation (6) with \( a \equiv 1, b \equiv 0 \):

\[
B(t, k) = e^{ik^2} c(t, k), \quad k \in \tilde{B}, \text{ i.e. Re} \, k \geq 0, \text{ Im} \, k \geq 0. \tag{12}
\]
Let the solution $\Phi(t, k)$ of Equation (8) be expressed in terms of the functions $M_j(t, s)$, $L_j(t, s), j = 1, 2, \text{see Proposition 2}$. Then Equation (12) can be solved explicitly for $g_1(t)$ in terms of $g_0(t)$, $M_1$, and $M_2$:

$$g_1(t) = g_0(t)M_2(t, t) + \frac{4i}{\pi} \int_{\partial D} \left[ 2k^2 \int_0^t e^{4ik^2(t-\tau)} M_1(t, 2\tau - t) d\tau - \frac{g_0(t)}{2i} \right] dk. \quad (13)$$

**Proof.** We multiply the global relation (12) by $k \exp[-4ik^2t']$, $t' < t$, and integrate along $\partial D$. Using the fact that $k c(k, t)$ is $O(1)$ as $k \to \infty$, and that $\exp[4ik^2(t-t')]$ is bounded in $D$. Jordan’s lemma implies that the right-hand side of the resulting equation vanishes. Using for $B(t, k)$ the expression (11b) we find

$$-\int_{\partial D} \left[ \int_0^t e^{4ik^2(t-\tau)} (2kL_1(t, 2\tau - t) - ig_0(t)kM_2(t, 2\tau - t) + 2k^2M_1(t, 2\tau - t)) d\tau \right] dk = 0. \quad (14)$$

The term involving $M_1$ can be written as

$$-\int_{\partial D} \left[ \int_0^t e^{4ik^2(t-\tau)} 2k^2M_1(t, 2\tau - t) d\tau - \frac{M_1(t, 2t' - t)}{2i} \right] dk$$

$$-\int_{\partial D} \left[ \int_0^t e^{4ik^2(t-\tau)} 2k^2M_1(t, 2\tau - t) d\tau + \frac{M_1(t, 2t' - t)}{2i} \right] dk.$$

Integration by parts implies that the bracket appearing in the second integral is bounded in $D$; since $\tau > t'$, this integral vanishes.

In order to evaluate the term in (14) involving $L_1$, we note that the contour $\partial D$ involves an integral from 0 to $\infty$ which can be mapped to an integral from 0 to $-\infty$ by replacing $k$ with $-k$, thus $\partial D$, can be replaced by $\partial \tilde{D}$, where $\tilde{D}$ denotes the second quadrant of the complex $k$-plane. Hence, this term can be written as

$$-\int_{\partial \tilde{D}} \left[ \int_0^t e^{4ik^2(t-\tau)} 2kL_1(t, 2\tau - t) d\tau - \frac{L_1(t, 2t' - t)}{2i} \right] dk - \frac{L_1(t, 2t' - t)}{2i} \int_{\partial \tilde{D}} \frac{dk}{k}$$

$$-\int_{\partial \tilde{D}} \left[ \int_0^t e^{4ik^2(t-\tau)} 2kL_1(t, 2\tau - t) d\tau + \frac{L_1(t, 2t' - t)}{2i} \right] dk + \frac{L_1(t, 2t' - t)}{2i} \int_{\partial \tilde{D}} \frac{dk}{k},$$

where the superscript zero in $\partial \tilde{D}$ and $\partial D^o$ indicates that we have deformed the contours to avoid $k = 0$. The first and the third integrals vanish, each of the brackets is bounded and analytic in $\tilde{D}$ and $D^o$, respectively. The remaining two integrals equal

$$-\frac{L_1(t, 2t' - t)}{2i} \int_0^\pi i d\theta = -\frac{\pi}{2} L_1(t, 2t' - t).$$
Using the same method for the term in (14) involving \(M_2\), we finally find

\[
-\frac{\pi}{2} L_1(t, 2t' - t) + \frac{i\pi}{4} g_0(t) M_2(t, 2t' - t) - \int_{\partial D} \left[ \int_0^{t'} e^{4ik^2(t' - \tau)} 2k^2 M_1(t, 2\tau - t) \, d\tau - \frac{M_1(t, 2t' - t)}{2i} \right] \, dk = 0.
\]

Letting \(t' \to t\) and expressing \(L_1(t, t)\) in terms of \(g_1(t)\), see (33a), Equation (15) becomes (13).

\[\Box\]

1.2. A FORMULATION IN TERMS OF \((\mu_j(t, k), \lambda_j(t, k))_{j=1,2}\)

Equation (13) expresses \(g_1\) in terms of \(M_1(t, s)\) and \(M_2(t, t)\), where the functions \(M_j(t, s), L_j(t, s), j = 1, 2\) satisfy Equations (25), (26). It is more convenient to express these latter functions in terms of the functions \((\mu_j(t, k), \lambda_j(t, k))_{j=1,2}\) defined by the equations \((j = 1, 2)\)

\[
\mu_j(t, k) = \int_{-t}^{t} e^{2ik^2(s-t)} M_j(t, s) \, ds, \quad t \geq 0, \quad k \in \mathbb{C},
\]

\[
\lambda_j(t, k) = \int_{-t}^{t} e^{2ik^2(s-t)} L_j(t, s) \, ds, \quad t \geq 0, \quad k \in \mathbb{C}.
\]

Using these functions, the expressions for \(A(t, k)\) and \(B(t, k)\), given by Equations (11), become

\[
A(t, k) = 1 + \left( \lambda_2(t, k) - \frac{i\rho}{2} g_0(t) \mu_1(t, k) + k \bar{\mu}_2(t, k) \right),
\]

\[
B(t, k) = -e^{4ik^2t} \left( \lambda_1(t, k) - \frac{i}{2} g_0(t) \mu_2(t, k) + k \bar{\mu}_1(t, k) \right).
\]

Furthermore, rewriting Equations (13) and (25), (26), in terms of the new variables \(\mu_j(t, k), \lambda_j(t, k), j = 1, 2\) we arrive at the following result.

**THEOREM 1.** Let \(q(x, t)\) satisfy the NLS equation on the half-line \(0 < x < \infty, \ t > 0\) with the initial and boundary conditions

\[
q(x, 0) = 0, \quad 0 < x < \infty,
\]

\[
q(0, t) = g_0(t), \quad t > 0,
\]

where \(g_0(t)\) is a smooth function satisfying \(g_0(0) = 0\).

Then \(g_1(t) := q_x(0, t)\) can be expressed explicitly in terms of \(\mu_1(t, k)\) and \(\mu_2(t, k)\) by the equation

\[
g_1(t) = \frac{2g_0(t)}{\pi} \int_{\partial D} k \mu_2(t, k) \, dk + \frac{4i}{\pi} \int_{\partial D} \left[ k^2 \mu_1(t, k) - \frac{g_0(t)}{2i} \right] \, dk.
\]
where the functions \( \{ \mu_j(t, k), \lambda_j(t, k) \}_{j=1,2} \) for \( t > 0, k \in \mathbb{C} \), satisfy the system of equations

\[
\begin{align*}
\lambda_{1t} + 4ik^2 \lambda_1 &= ig_1(t)\lambda_2 + \alpha(t)\mu_1 + \beta(t)\mu_2 + \gamma(t), \\
\lambda_{2t} &= -ip\tilde{g}_1(t)\lambda_1 - \alpha(t)\mu_2 + \rho\tilde{\beta}(t)\mu_1, \\
\mu_{1t} + 4ik^2 \mu_1 &= 2g_0(t)\lambda_2 + ig_1(t)\mu_2 + 2g_0(t), \\
\mu_{2t} &= 2\rho\tilde{g}_0(t)\lambda_1 - ip\tilde{g}_1(t)\mu_1,
\end{align*}
\]

with

\[
\alpha(t) = \frac{\rho}{2}(g_0\tilde{g}_1 - \tilde{g}_0\tilde{g}_1), \quad \beta(t) = \frac{1}{2} \left( \frac{dg_0}{dt} + \rho|g_0|^2g_0 \right),
\]

and the initial conditions

\[
\lambda_j(0, k) = \mu_j(0, k) = 0, \quad j = 1, 2.
\]

We note that, in the framework of the Dirichlet problem, \( M_2(t, t) \) is an unknown function; that is why we prefer to use the function \( \mu_j(t, k) \) in the formulation of Theorem 1, Equation (18), as well as in Theorem 2 below.

Remark 1. Replacing \( g_1(t) \) in Equations (19) by the explicit expression (18) we obtain a system of nonlinear Volterra integral equations for \( \{ \mu_j, \lambda_j \}_{j=1,2} \) in terms of the given function \( g_0(t) \). The rigorous analysis of this system remains open.

Remark 2. Suppose that \( q(x, t) \) satisfies the linearized Schrödinger equation \( iq_t + q_{xx} = 0 \). In this case the global relation is (4)

\[
\hat{g}(k, t) = -\hat{g}_0(k) + e^{ik^2t}g(k, t), \quad t \in [0, T], k \in D,
\]

where \( \hat{g}_0(k) \) is the Fourier transform of \( q_0(x) \), and \( \hat{g}(k, t) \) is defined by

\[
\hat{g}(k, t) = \int_0^t e^{ik^2(t - \tau)}(i\hat{g}_1(\tau) - k\hat{g}_0(\tau))\,d\tau.
\]

Actually it can be shown that for small \( q \),

\[
a \to 1, \quad A \to 1, \quad b \to -\hat{q}_0(k), \quad B \to \hat{g},
\]

and the global relation (6) yields Equation (22). The analysis of the latter equation plays a key role in the solution of the linearized Schrödinger equation. This analysis makes crucial use of the particular explicit \( k \)-dependence of \( \hat{g}(k, t) \). Using this dependence one can either

1. utilise the transformation \( k \mapsto -k \) to eliminate the term containing the unknown function \( g_1(t) \),

or

2. obtain explicitly \( g_1 \) in terms of \( g_0 \) by multiplying Equation (22) with \( \exp(ik^2t) \) and integrating w.r.t. \( k \) over \( \partial D \).
It is remarkable, that if one uses the Gelfand–Levitan–Marchenko representation for \( \Phi \), then the global relation (6) has precisely the same \( k \) dependence as Equation (20), see the expressions (11) for \( A, B \). Thus again one can obtain explicitly \( g_1 \) using the same procedure as in (ii) above.

**Remark 3.** The first attempt to characterize the spectral functions was made in [5] and led to a formal nonlinear RH problem. A similar formulation was presented in [2], where a different formulation was also presented based on an attempt to express \( g_1 \) explicitly in terms of \( \Phi \) using certain analyticity arguments. However, all these formal attempts yield a system of nonlinear Fredholm integral equations for the spectral functions. This is to be contrasted with the formulation of [6] as well as with the simplified formulation presented here, which yield a system of nonlinear Volterra integral equations. The insurmountable problem with the former formulations is that, since the spectral functions are characterised to within an equivalent class (\( c(t, k) \) in Equation (6) is arbitrary), one cannot rigorously establish solvability for the associated Fredholm equations.

**Remark 4.** For economy of presentation we have concentrated on the Dirichlet boundary value problem. This analysis applies mutatis-mutandis to the Neumann boundary value problem as well.

**Organization of the Letter.** In Section 2 we present the Gelfand–Levitan–Marchenko representation for \( \Phi \) [1], and derive Equations (19). In Section 3, we present the analogue of Theorem 1 when \( q(x, 0) \neq 0 \). In Section 4 we discuss linearizable boundary conditions; namely, it has been shown in [3, 6] that for some particular boundary conditions it is possible to bypass the nonlinear Volterra equations satisfied by \( g_1 \) and \( \Phi \), and to define the spectral functions \( \{A(k), B(k)\} \) using only algebraic manipulations. These particular cases can also be analysed using the present formulation.

### 2. The Gelfand–Levitan–Marchenko Representation of \( \Phi(t, k) \)

**PROPOSITION 2 ([1]).** Let the 2-vector function \( \Phi(t,k) \) satisfy (8). Then:

(i) \( \Phi(t,k) \) can be represented in the form

\[
\Phi(t, k) = \begin{pmatrix} 0 \\ e^{i k^2 t} \end{pmatrix} + \int_{-t}^{t} \begin{pmatrix} L_3(t,s) - i \tilde{g}_0(t) M_3(t,s) + k M_1(t,s) \\ L_2(t,s) + i \tilde{g}_0(t) M_1(t,s) + k M_2(t,s) \end{pmatrix} e^{2i k^2 s} \, ds, \tag{24}
\]

where the four functions \( L_1, L_2, M_1, \) and \( M_2 \) satisfy the differential equations with \( t > 0, -t < s < t \)

\[
L_1 - L_3 = i \tilde{g}_1(t) L_2 + z(t) M_1 + \beta(t) M_2, \tag{25a}
\]
\[ L_{21} + L_{22} = -ip \tilde{g}_1(t)L_1 - z(t)M_2 + \rho \tilde{\beta}(t)M_1, \]  
(25b)

\[ M_{11} - M_{12} = 2g_0(t)L_2 + ig_1(t)M_2, \]  
(25c)

\[ M_{21} + M_{22} = 2\rho \tilde{g}_0(t)L_1 - ip\tilde{g}_1(t)M_1, \]  
(25d)

as well as the boundary conditions

\[ L_1(t, t) = \frac{i}{2} g_1(t), \quad L_2(t, -t) = 0, \]  
(26a)

\[ M_1(t, t) = g_0(t), \quad M_2(t, -t) = 0, \]  
(26b)

with \( z \) and \( \beta \) defined by (20).

(ii) Let \( \{\mu_j(t, k), \lambda_j(t, k)\}_{j=1, 2} \) be defined in terms of \( \{M_j(t, s), L_j(t, s)\}_{j=1, 2} \) by Equations (14). Then Equations (19) are valid.

**Proof.** (i) Let the \( 2 \times 2 \) matrix-valued function \( \Psi(t, k) \) satisfy Equation (8a) and the initial condition \( \Psi(0, k) = I \). Using the representation

\[ \Psi(t, k) = e^{-2ik^2 \sigma_3} + \int_{-t}^{t} (N(t, s) + kM(t, s))e^{-2ik^2 \sigma_3} \, ds \]  
(27)

into Equation (8a) and integrating by parts in order to eliminate integral terms containing \( k^2e^{-2ik^2 \sigma_3} \) and \( k^3e^{-2ik^2 \sigma_3} \), we find

\[ \ldots \]  

\[ + \int_{-t}^{t} \ldots \]  

\[ + \int_{-t}^{t} \ldots \]  

\[ + \int_{-t}^{t} \ldots \]  

\[ + \int_{-t}^{t} \ldots \]  

Hence, each of the above brackets \( \ldots \) vanishes, which gives

\[ M(t, t) - \sigma_3M(t, t)\sigma_3 = 2Q(t), \]  
(28a)

\[ N(t, t) - \sigma_3N(t, t)\sigma_3 = iQ(t)M(t, t)\sigma_3 + R(t), \]  
(28b)

\[ M(t, -t) + \sigma_3M(t, -t)\sigma_3 = 0, \]  
(28c)

\[ N(t, -t) + \sigma_3N(t, -t)\sigma_3 = iQ(t)M(t, -t)\sigma_3 = 0, \]  
(28d)

\[ M_j(t, s) + \sigma_3M_j(t, s)\sigma_3 = 2Q(t)N(t, s) + R(t)M(t, s), \]  
(28e)

\[ N_j(t, s) + \sigma_3N_j(t, s)\sigma_3 = -iQ(t)M_j(t, s)\sigma_3 + R(t)N(t, s). \]  
(28f)

Denote by \( A_{\text{diag}} \) and \( A_{\text{off}} \) the diagonal and off-diagonal parts of a matrix \( A \), respectively. Equation (28a) is consistent with \( Q \) being off-diagonal and it gives

\[ M_{\text{off}}(t, t) = Q(t). \]  
(29)
The diagonal part of (28b) reads
\[ R_{\text{diag}}(t) + iQ^2(t)\sigma_3 = 0, \]
which is consistent with the form of \( R(t) \), see (9), whereas the off-diagonal part of (28b) gives
\[ N_{\text{off}}(t, t) = \frac{i}{2} R_{\text{off}}(t) + \frac{i}{2} Q(t)\sigma_3 M_{\text{diag}}(t, t). \]
Equations (28c) and (28d) give
\[ M_{\text{diag}}(t, -t) = 0 \]
and
\[ N_{\text{diag}}(t, -t) = \frac{i}{2} Q(t)\sigma_3 M_{\text{off}}(t, -t), \]
respectively.

Equations (30), (32), and (28f) suggest the introduction of a new function \( L \), to be used instead of \( N \):
\[ L(t, s) = N(t, s) - \frac{i}{2} Q(t)\sigma_3 M(t, s). \]
Then the boundary conditions (30) and (32) simplify to
\[ L_{\text{off}}(t, t) = \frac{1}{2} R_{\text{off}}(t), \]
\[ L_{\text{diag}}(t, -t) = 0. \]
Writing the differential equations (28e) and (28f) in terms of \( L \) and \( M \) we find
\[ M_1(t, s) + \sigma_3 M_2(t, s)\sigma_3 = 2Q(t)L(t, s) + R_{\text{off}}(t)M(t, s), \]
\[ L_2(t, s) + \sigma_3 L_1(t, s)\sigma_3 = R_{\text{off}}(t)L(t, s) + W(t)M(t, s), \]
where
\[ W = \frac{i}{2} (R_{\text{off}}Q + QR_{\text{off}})\sigma_3 - \frac{i}{2} \frac{dQ}{dt}\sigma_3 - \frac{1}{2} Q^3. \]
Taking into account the particular form of \( Q \) and \( R \) (see (9)) and writing the matrices \( L \) and \( M \) as
\[ L = \begin{pmatrix} \bar{L}_2 & L_1 \\ \rho \bar{L}_1 & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} \bar{M}_2 & M_1 \\ \rho \bar{M}_1 & M_2 \end{pmatrix}, \]
the matrix differential equations (34) reduce to the system of four scalar equations (23), whereas the boundary conditions (29), (30), and (32) reduce to (26). Equations (25) and (26) constitute a well-posed Goursat problem, the solution of which can be obtained via the solution of the associated system of linear Volterra integral equations.
(ii) Multiplying (25) by $e^{2ikz}$, integrating with respect to $s$ from $-t$ to $t$, and using the definitions (16) and the boundary conditions (26), we derive Equations (19). For this derivation we note that integration by parts yields

$$\int_{-t}^{t} e^{2ikz} M_j(t,s) ds = e^{2ikz} M_j(t,t) - e^{-2ikz} M_j(t,-t) - 2ik^2 e^{2ikz} \mu_j(t,k).$$

(36)

Also, differentiating $\mu_j$ with respect to $t$ we find

$$\int_{-t}^{t} e^{2ikz} M_j(t,s) ds = -e^{2ikz} M_j(t,t) - e^{-2ikz} M_j(t,-t) + (e^{2ikz} \mu_j)(t,k).$$

(37)

The functions $L_j(t,s)$ satisfy similar equations. It is important to note that although the functions $M_1(t,-t)$, $M_2(t,t)$, $L_1(t,-t)$, and $L_2(t,t)$ are unknown, these functions cancel when we use Equations (36), (37), and the analogous equations involving $L_j$, in Equations (25):

$$\int_{-t}^{t} e^{2ikz(t-s)}[M_{1t} - M_{1s}] ds = -2g_0(t) + \mu_{1t} + 4ik^2 \mu_{1},$$

$$\int_{-t}^{t} e^{2ikz(t-s)}[L_{1t} - L_{1s}] ds = -ig_{1}(t) + \lambda_{1t} + 4ik^2 \lambda_{1},$$

$$\int_{-t}^{t} e^{2ikz(t-s)}[M_{2t} + M_{2s}] ds = \mu_{2t},$$

$$\int_{-t}^{t} e^{2ikz(t-s)}[L_{2t} + L_{2s}] ds = \lambda_{2t}. $$

□

3. Solution of the Global Relation

**PROPOSITION 3.** Let $A(t,k)$ and $B(t,k)$ satisfy the global relation (6). Let the solution $\Phi(t,k)$ of Equation (8a) be expressed in terms of the functions $\{M_j(t,s), L_j(t,s)\}_{j=1,2}$, see Proposition 2. Then Equation (6) can be solved explicitly for $g_1(t)$ in terms of $g_0(t)$, $M_1$, $M_2$, and $L_2$:

$$g_1(t) = g_0(t) \left\{ M_2(t,t) + \frac{4i}{\pi} \oint_{\partial D} k R(k) e^{4ikz(t-t)} M_1(t,t-2\tau) d\tau dk \right\} +$$

$$+ \frac{4i}{\pi} \oint_{\partial D} k R(k) e^{4ikz(t-t)} dk + \frac{4i}{\pi} \oint_{\partial D} \left[ 2k^2 \int_{0}^{t} e^{4ikz(t-\tau)} M_1(t,2\tau-t) d\tau - \frac{g_0(t)}{2i} \right] dk +$$

$$+ \frac{8i}{\pi} \oint_{\partial D} k R(k) \int_{0}^{t} e^{4ikz(t-\tau)} [L_2(t,t-2\tau) + k M_2(t,t-2\tau)] d\tau dk,$$

(38)

where $R(k) = b(k)/a(k)$ and, if $a(k)$ has zeros in $D$, the contour $\partial D$ has to be deformed to pass ‘above’ all the zeros.
Proof. Write the global relation (6) in the form

\[ B(t, k) = R(k)A(t, k) + e^{4ik^2t} \frac{c(k)}{a(k)}, \]  

(39)
multiply it by \( k \exp[-4ik^2t] \), and integrate along \( \partial D \). Since \( kR(k)A(t, k) \) is \( O(1) \) as \( k \to \infty \), the integral involving this term is well-defined. Then the proof of Proposition 3 follows the same lines as the proof of Proposition 1.

In terms of the functions \( \{ \mu_j, \lambda_j \}_{j=1,2} \) the analogue of Proposition 3 is the following theorem.

**THEOREM 2.** Let \( q(x, t) \) satisfy the NLS equation on the half-line \( 0 < x < \infty \), \( t > 0 \) with the initial and boundary conditions

\[
q(x, 0) = q_0(x) \in \mathscr{S}(\mathbb{R}^+), \quad 0 < x < \infty,
\]

\[
q(0, t) = g_0(t), \quad t > 0,
\]

where \( g_0(t) \) is a smooth function satisfying \( g_0(0) = q_0(0) \). Then \( g_1(t) := q_1(0, t) \) can be expressed in terms of the functions \( \mu_1(t, k), \mu_2(t, k), \) and \( \lambda_2(t, k) \) by the equation

\[
g_1(t) = \frac{4i}{\pi} \int_{\partial D} e^{-4ik^2t}R(k) \, dk + \frac{2g_0(t)}{\pi} \int_{\partial D} k \left[ \mu_2(t, k) + pe^{-4ik^2t}R(k)\mu_1(t, k) \right] \, dk + \frac{4i}{\pi} \int_{\partial D} k^2 \mu_1(t, k) - \frac{g_0(t)}{2i} + e^{-4ik^2t}R(k) \left[ \lambda_2(t, k) + k\mu_2(t, k) \right] \, dk,
\]

(40)
where \( R(k) = b(k)/a(k) \) and the functions \( \{ \mu_j(t, k), \lambda_j(t, k) \}_{j=1,2} \) satisfy Equations (19), (21). The contour \( \partial D \) is the boundary of the first quadrant of the complex \( k \)-plane; if \( a(k) \) has zeros in \( D \), \( \partial D \) has to be deformed to pass `above' all the zeros.

**Remark 5** (The linear limit). In the approximation of small \( q_0, g_0, g_1 \) (or small \( q_0 \) and \( t \)), from (25) and (26) we find

\[
L_1(t, s) \approx \frac{i}{2} g_1 \left( \frac{t+s}{2} \right), \quad M_1(t, s) \approx g_0 \left( \frac{t+s}{2} \right),
\]

(41a)
\[
L_2(t, s) \approx 0, \quad M_2(t, s) \approx 0,
\]

(41b)
as well as

\[
a(k) \approx 1, \quad R(k) \approx b(k) \approx -\int_0^\infty q_0(y)e^{2ky} \, dy.
\]

(42)
Using (41) and (42) in (38), we retrieve the formula relating the initial and boundary values for the linearized NLS equation:

\[
g_1(t) \approx -\frac{4i}{\pi} \int_{\partial D} \left[ ke^{-4ik^2t} \left\{ \int_0^\infty q_0(y)e^{2ky} \, dy - 2k \int_0^t g_0(\tau)e^{4kt^2} \, d\tau \right\} + \frac{g_0(t)}{2i} \right] \, dk.
\]

(43)
4. Linearizable Boundary Conditions

PROPOSITION 4 (Linearizable cases). Let \( q(x, t) \) satisfy the NLS equation, the initial condition \( q(x, 0) = q_0(x) \), and one of the following boundary conditions, either

(i) \( q(0, t) = 0 \),

or

(ii) \( q_x(0, t) - \chi q(0, t) = 0 \), where \( \chi \) is a real constant.

Let the solution \( F(t, k) \) of Equation (8a) be expressed in terms of the functions \( m_j(t, k) \), \( l_j(t, k) \), \( j = 1, 2 \), see Proposition 2.

Then the spectral functions \( A, B \) can be represented as follows:

(i)

\[
A(t, k) = 1 + \frac{\lambda_2(t, k)}{2}, \quad B(t, k) = -e^{4ik^2t_2}l_1(t, k).
\]

(ii)

\[
A(t, k) = 1 + \left( \frac{\lambda_2(t, k)}{2} - \frac{i\rho}{2} g_0(t) \mu_1(t, k) \right), \quad B(t, k) = -\left( \frac{i\rho}{2} + k \right) e^{4ik^2t_2} \mu_1(t, k).
\]

Equations (44) and (45) imply, respectively

\[
A(t, k) = A(t, -k), \quad B(t, k) = B(t, -k),
\]

\[
A(t, k) = A(t, -k), \quad \frac{B(t, k)}{i\rho + 2k} = \frac{B(t, -k)}{i\rho - 2k}.
\]

Proof. Let us show that the linearizable boundary conditions correspond to particular reductions of (25), (26), such that a part of this system constitutes a closed system of homogeneous Volterra integral equations, whose only solution is the trivial solution.

First, note that in the linearizable cases, \( \tilde{g}_0 g_1 - \tilde{g}_1 g_0 = 0 \), which, in terms of (25), reads \( \lambda(t) = 0 \).

Case (i) \( q(0, t) = 0 \). In this case we also have \( \beta(t) = 0 \) and \( M_1(t, t) = 0 \), so that \( M_1 \) and \( M_2 \) satisfy the system of equations

\[
M_1(t, t) = i\mu_1 M_2, \quad M_2(t, t) = -i\rho \tilde{g}_1 M_1,
\]

\[
M_1(t, -t) = 0, \quad M_2(t, -t) = 0.
\]
The unique solution of (48) is trivial: \( M_1(t, s) = M_2(t, s) \equiv 0 \). Hence, (11a) and (11b) become (44a) and (44b), respectively.

*Case (ii)* \( q_x(0, t) - \gamma q(0, t) = 0 \). Introduce \( P(t, s) \) by
\[
P(t, s) = L_1(t, s) - \frac{i\gamma}{2} M_1(t, s).
\]
Then, in terms of \( P \) and \( M_2 \), we again obtain from (25), (26) a homogeneous system of equations
\[
\begin{align*}
P_t - P_s &= \left( \beta + \frac{\gamma^2}{2} g_0 \right) M_2, \\
M_{21} + M_{22} &= 2\rho g_0 P, \\
P(t, t) &= 0, \\
M_2(t, -t) &= 0,
\end{align*}
\]
whose only solution is the trivial solution \( P(t, s) = M_2(t, s) \equiv 0 \). In view of (49), (11a) and (11b) become (45a) and (45b), respectively.

Using the fact that \( \lambda_j \) and \( \mu_j \) are even functions of \( k \) (see (16)), Equations (44) and (45) imply Equations (46) and (47).

**Remark 6.** Using the symmetry relations (46), (47), the global relation (6), and the fact that the solution of the NLS equation for \( 0 < t < t^* \) does not depend on \( g_0(t) \), \( g_1(t) \), for \( t > t^* \), it can be shown [3] that the ratio \( B(k)/A(k) \) (which is needed for the relevant RH problem) can be expressed in terms of \( b(k)/a(k) \) and \( \chi \).

**Remark 7.** In the case of the linearizable boundary condition \( g_0(t) \equiv 0 \), we find the following system characterizing \( g_1(t) \):
\[
\begin{align*}
\lambda_{11}(t, k) + 4ik^2 \lambda_1(t, k) &= ig_1(t)\lambda_2(t, k) + ig_1(t), \\
\lambda_{21}(t, k) &= -ipg_1^2(t)\lambda_1(t, k), \\
g_1(t) &= \frac{4i}{\pi} \int_{\partial D} e^{-ikz^2}k R(k) \left( 1 + \frac{\lambda_2(t, \bar{k})}{\lambda_2(t, k)} \right) dk.
\end{align*}
\]

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