# Cubulation of Gromov-Thurston manifolds

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**Abstract**

In this article we prove that the fundamental group of certain manifolds, introduced by Gromov and Thurston [GT87] and obtained by branched cyclic covering over arithmetic manifolds, acts geometrically on a $CAT(0)$ cube complex. We show in particular that these groups are linear over $\mathbb{Z}$.

## 1 Introduction.

The main result of this paper concerns the fundamental group of some negatively curved manifolds, introduced by Gromov and Thurston in [GT87]. Gromov and Thurston constructed infinitely many manifolds which can be equipped with a Riemannian metric with negative sectional curvature less or equal to $-1$, arbitrarily close to $-1$, but do not admit any Riemannian metric of constant curvature. For the construction, they consider cyclic ramified coverings over a certain hyperbolic manifold $V$, constructed by arithmetic considerations, called a "simple type" arithmetic manifold. The branch locus is a totally geodesic submanifold $K \subset V$ of codimension 2, obtained by intersecting two transverse submanifolds of codimension 1. It is important to consider arbitrary large degrees of ramified covering to make sure that among those manifolds, infinitely many do not have a constant curvature metric. On the other hand, the more the injectivity radius of $K$ in $V$ is large, the more the sectional curvature of the metric will be close to $-1$. Then, they consider the ramifications on finite covers of $V$ to control the injectivity radius and so the sectional curvature. We will call such a manifold a Gromov-Thurston manifold.

Our first result is the following:

**Theorem 1.** *Let $\widehat{V}$ be a Gromov-Thurston manifold. Then $\pi_1(\widehat{V})$ is cubical.*

A group is *cubical* if it acts geometrically on a $CAT(0)$ cube complex (see Definition 6 below). Furthermore, a group is *special*, as defined by Haglund and Wise in [HW08], if the quotient of the $CAT(0)$ cube complex by the group is *special*, i.e. does not have certain global pathologies (see Section 4). Note that being a virtually special group has many consequences: such a group injects in $\mathrm{GL}(n, \mathbb{Z})$ for a certain $n \in \mathbb{Z}$, has separable quasi-convex subgroups (see [HW08]), is large (see [HW08]), contains a finite index bi-ordonnable subgroup (see [HW08] and [DT92]), etc. Recently Agol [AGM12] proved that every hyperbolic cubical group is virtually special. Since the fundamental group of Gromov-Thurston manifolds is hyperbolic, using Agol Theorem [AGM12], Theorem 3 below is a corollary of Theorem 1. Nevertheless, we will prove virtual specialness without using Agol Theorem.

To prove Theorem 1, we will use ramified coverings of cube complexes (see Definition 13). We will see the following:

**Theorem 2.** *All cyclic ramified coverings of special cube complexes are special cube complexes.*

We will deduce our main result from Theorems 1 and 2:

**Theorem 3.** *Let $\widehat{V}$ be a Gromov-Thurston manifold. Then $\pi_1(\widehat{V})$ is virtually special.*

Many examples of compact hyperbolic manifolds with constant curvature equal to $-1$ have virtually special fundamental groups. For example, hyperbolic surfaces are virtually special. In dimension 3, the works of Kahn-Markovic, Wise and Agol prove that fundamental groups of all compact hyperbolic 3-manifolds are special [AGM12]. Moreover, in every dimension $n \geq 4$, Bergeron, Haglund and Wise [BHW11] showed that "simple type" arithmetic hyperbolic manifolds also have a cubical virtually special fundamental group. We will use this to prove Theorem 1. As a consequence of Theorem 3 we can deduce the following:

**Corollary 4.** *Fundamental groups of Gromov-Thurston manifolds are linear (over $\mathbb{Z}$), contain a finite index bi-ordonnable subgroup, have separable quasi-convex subgroups, are large, etc.*

Moreover, we will prove a more general version of Theorem 1 (Theorem 5 below). Let $V = \Gamma\backslash\mathbb{H}^n$ be a hyperbolic manifold, with $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$. Denote by $p : \mathbb{H}^n \to V$ the covering map. Suppose that there exist two transverse totally geodesic submanifolds $V_1$ and $V_2$ of $V$, and suppose that both of them separate $V$. We can construct a cyclic ramified covering in the following way: Each submanifold $V_i$, $i = 1, 2$, splits $V$ into two disjoint submanifolds $V_i^+$ and $V_i^-$ of $V$, with boundary $V_i$ with two different orientations. The intersection $W = V_1^+ \cap V_2$ is a hypersurface of $V$ with boundary. Let $\overline{V}$ be the manifold obtained by cutting $V$ along $W$. The boundary of $\overline{V}$ is a disjoint union of two copies of $W$ with opposite orientations. For $k \in \mathbb{N}^*$, consider the manifold $\widehat{V}$ obtained by cyclically gluing $k$ copies of $\overline{V}$ along the copies of $W$ according to their orientations. Then the natural projection $\widehat{V} \to V$ is a *cyclic ramified covering* of degree $k$ of $V$ above $\partial W = V_1 \cap V_2$.

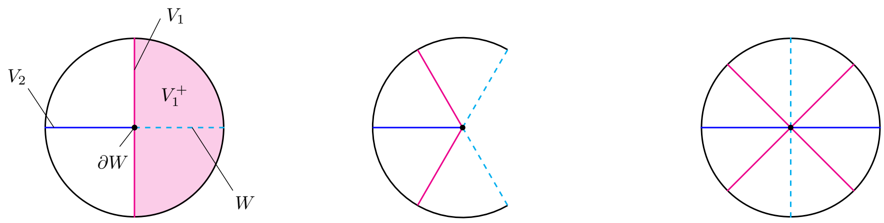



Figure 1: A cyclic ramified covering of degree 2 above $\partial W$.

In Gromov-Thurston's construction, $V$ is obtained by using arithmetic, as well as $V_1$ and $V_2$. By passing to a finite cover, these two submanifolds satisfy the properties described above. Then Gromov and Thurston construct $\widehat{V}$, a cyclic ramified covering of degree $k$ above the intersection of $V_1$ and $V_2$.

An arithmetic manifold such as $V$ contains many immersed compact hypersurfaces. Choose a finite number of them and denote by $\mathcal{H}$ the collection of preimages of theses hypersurfaces in $\mathbb{H}^n$. One can associate a dual $CAT(0)$ cube complex acted on by $\pi_1(V)$ to the wallspace given by $\mathcal{H}$ on $\mathbb{H}^n$. One can ensure that the action of $\pi_1(V)$ on this cube complex is proper by carefully choosing immersed compact hypersurfaces of $V$. The cube complex quotient by $\pi_1(V)$ is compact. We will give more details on this construction in Section 2. In general, for a compact hyperbolic manifold $V$, with such a collection $\mathcal{H}$ of hyperplane of $\mathbb{H}^n$, we will say that $V$ is $\pi_1$*-cubulated* by $\mathcal{H}$.

A collection of hyperplanes is *generic* if every pair of hyperplanes in the collection is transverse and the intersection between three different hyperplanes is either empty or of codimension 3.

**Theorem 5.** *Let $V$ be an oriented hyperbolic compact manifold, and let $V_1$ and $V_2$ be totally geodesic separating, transverse submanifolds of $V$. Let $k \geq 1$ be an integer, and $\widehat{V}$ be the ramified covering of $V$ of degree $k$ above $V_1 \cap V_2$. Furthermore, we assume that $V$ is $\pi_1$-cubulated by a collection $\mathcal{H}$ of hyperplanes in $\mathbb{H}^n$, such that the collection of hyperplanes of $\mathcal{H}$ and preimages of $V_1$ and $V_2$ under $p$ is generic. Then the fundamental group $\pi_1(\widehat{V})$ is cubical.*

The construction of a dual cube complex in the case described above is given in the next section. In Section 3 we prove Theorem 5, in Section 4 we prove Theorem 2. Finally, in Section 5 we prove Theorems 1 and 3 by using Theorems 5 and 2.

# 2 Cubulation

**Definition 6.** *A group is* cubical *if it acts* geometrically*, i.e.* properly *and* cocompactly*, on a $CAT(0)$ cube complex.*

**Definition 7.** *A* cube complex *is a CW-complex, such that each cell is a metric Euclidean cube $[0, 1]^n$, and gluing maps are isometries between subcubes, i.e. cubes obtained by restricting certain coordinates to 0 or 1.*

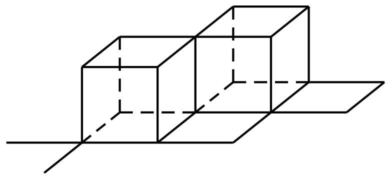


Figure 2: A cube complex.

The construction of a $CAT(0)$ cube complex dual to a wallspace is the generalisation of an idea from to Sageev [Sag95]. Haglund and Paulin defined the general notion of wallspace (see [HP98]). Then Nica [Nic04], Chatterji and Niblo [CN05] described the cube complex dual to those wallspaces. If a group acts on a wallspace (defined in [HP98]) then it also acts on the $CAT(0)$ cube complex dual to the wallspace. We remind this construction in a particular case.

Let $V$ be a compact hyperbolic manifold. Let $W_1, W_2, \dots, W_\ell$ be immersed compact submanifolds of $V$ of codimension 1. Assume that these manifolds intersect each other in a generic way. Any lift of $W_i$ to $\mathbb{H}^n$ is a hyperplane of $\mathbb{H}^n$, splitting $\mathbb{H}^n$ into two connected components. Let $\mathcal{H}$ be the collection of such hyperplanes of $\mathbb{H}^n$, and let $\mathcal{S}$ be the set of connected components of the space $\mathbb{H}^n \setminus \bigcup_{H\in\mathcal{H}} H$. Each hyperplane $H$ of $\mathcal{H}$ provide a natural bipartition (a so-called *wall*) of $\mathcal{S}$, and each element of this partition will be called a *halfspace* of $H$.

We construct a cube complex $Y$ as follows:

A vertex $\sigma$ of $Y$ is a collection of halfspaces of $\mathcal{S}$ such that:

- $\forall H \in \mathcal{H}$, exactly one of the two halfspaces of $H$ belongs to $\sigma$.
- $\forall A, B$ two halfspaces, $A \subset B$ and $A \in \sigma \Rightarrow B \in \sigma$.

Put an edge between two vertices $\sigma$ and $\tau$ iff $|\sigma\Delta\tau| = 2$, i.e. $\sigma$ and $\tau$ share exactly the same halfspace for every hyperplane of $\mathcal{H}$ except for one. Add a cube of any dimension every time one sees the 1-skeleton of a cube to get the cube complex $Y$.

Although $Y$ can be complicated, one can immediately see a part of this cube complex. Each connected component $x$ of $\mathcal{S}$ provide a vertex of $Y$: for every hyperplane of $\mathcal{H}$, take the halfspace which contains $x$. It satisfies the Definition above. Two such vertices are adjacent if and only if the associated connected components are separated by a unique hyperplane of $\mathcal{H}$. Let $X' \subset Y$ be the 2-complex completed from the graph above by gluing a square every time one can see the 1-skeleton. Note that the cube complex $X'$ is connected.

Let $X$ be the connected component of $X'$ in $Y$, $X$ is the cube complex dual to the collection $\mathcal{H}$ of walls of $\mathbb{H}^n$.

**Definition 8.** *Let $P_{\mathbb{H}^n}$ be the 2-complex dual to the cellulation of $\mathbb{H}^n$ by $\mathcal{H}$: choose a vertex in each connected component of $\mathcal{S}$, join every pair of vertices of two adjacent components by a geodesic segment, and for every face $K$ of codimension 2, glue a 2-disc along the geodesic segments associated with the codimension 1 faces of the cellulation which contains $K$.*

As hyperplanes of $\mathcal{H}$ intersect generically, each intersection between hyperplanes of $\mathcal{H}$ produces an infinite number of squares in the 1-skeleton of $P_{\mathbb{H}^n}$. So $P_{\mathbb{H}^n}$ is a square complex. In fact:

**Lemma 9.** *The square complexes $P_{\mathbb{H}^n}$ and $X'$ are combinatorially equivalent.*

*Proof.* In both cases, every vertex associated with a connected component of $\mathcal{S}$, every edge in $X'$ is given by a unique hyperplane of $\mathcal{H}$ between the two vertices. Finally, $X'$ is obtained by adding a square every time one sees its 1-skeleton, i.e. when there are four connected component of $\mathcal{S}$ $a, b, c$ and $d$, and two hyperplanes $H$ and $K$ of $\mathcal{H}$ such that $a$ and $b$, $c$ and $d$ are separated by $H$, and $b$ and $c$, $d$ and $a$ are separated by $K$. Then the intersection of $H$ and $K$ in $\mathbb{H}^n$ gives a face of codimension 2 of the cellulation of $\mathbb{H}^n$ by $\mathcal{H}$. Conversely, because of the genericity of intersections of hyperplanes of $\mathcal{H}$, a face of codimension 2 gives a square of $X'$. ☐

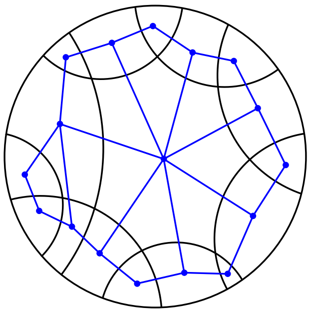


Figure 3: A cube complex.

As the fundamental group $\Gamma$ of $V$ acts on $\mathcal{H}$, it acts on $X$ too. In this specific construction, the action is always cocompact. There exists a general criterion on the set of hyperplanes $W_1, \ldots, W_\ell$ so that $\Gamma$ acts properly on $X$ (see [Duf12], Chapter I):

**Lemma 10.** *The action of $\Gamma$ on $X$ is proper if there exists a number $m$ such that every pair of points $x, y$ of $\mathbb{H}^n$ where $d(x,y) \geqslant m$ are separated by a hyperplane of $\mathcal{H}$.*

In this case, the criterion above is an equivalence.

Let $C$ be the quotient of $X$ by $\Gamma$. The group $\Gamma$ also acts on $P_{\mathbb{H}^n}$. Let $P_V$ be the quotient of $P_{\mathbb{H}^n}$ by $\Gamma$.

As $P_{\mathbb{H}^n}$ is the 2-skeleton of the complex dual to the cellulation of $\mathbb{H}^n$ by $\mathcal{H}$, the complex $P_{\mathbb{H}^n}$ is simply connected. Then:

**Lemma 11.** *The square complex $P_V$ injects combinatorially into $C$, and this injection induces an isomorphism of fundamental groups.*

*Proof.* By construction, the combinatorial equivalence $P_{\mathbb{H}^n} \simeq X'$ proved in lemma 9 is $\Gamma$-equivariant. The complex $P_{\mathbb{H}^n}$ identifies $\Gamma$-equivariantly with a subcomplex of $X$. Quotienting by $\Gamma$ identifies $P_V$ with a subcomplex of $C$. Since $P_{\mathbb{H}^n}$ is simply connected, the inclusion $P_V \hookrightarrow C$ induces an isomorphism of fundamental groups. □

**Definition 12.** *A* midcube *of a $k$-cube $[0,1]^k$ is a $(k-1)$-cube given by fixing one of the coordinates at $\frac{1}{2}$. In a $CAT(0)$ cube complex $X$, a* hyperplane *$H$ is a connected subspace of $X$ such that the intersection of $H$ with every cube of $X$ is a* midcube *or is empty. In a non positively curved cube complex $C$, a* hyperplane *is the projection of a hyperplane of the universal cover of $C$ onto $C$. It immerses in $C$.*

Hyperplanes of a subcomplex of a $CAT(0)$ cube complex are the trace of hyperplanes of the complex on the subcomplex.

There exists a natural correspondence between the collection of hyperplanes $\mathcal{H}$ and hyperplanes of $X$. A hyperplane of $\mathcal{H}$ can be associated with a hyperplane of the square complex $P_{\mathbb{H}^n}$, and to a hyperplane of $X'$ using the isomorphism between $P_{\mathbb{H}^n}$ and $X'$. Finally one can extend this hyperplane to a hyperplane of $X$. This function is well defined and is a natural bijection between hyperplanes of $\mathcal{H}$ and hyperplanes of $X$.

# 3 Proof of Theorem 5.

## 3.1 Construction.

To prove Theorem 5 we will first construct a ramified covering $\widehat{C}$ over the cube complex $C$, then in Section 3.2 we will prove that fundamental groups of $\widehat{V}$ and $\widehat{C}$ are isomorphic and finally we will show in Lemma 25 that $\widehat{C}$ is locally $CAT(0)$.

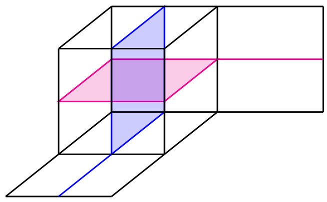


Figure 4: Two hyperplanes in a cube complex.

**Definition 13.** *Let $C$ be a cube complex and $k$ be an integer. Assume that $C$ has two separating subspaces $C_1$ and $C_2$, both of them are unions of disjoint hyperplanes of $C$. Each $C_i$, for $i = 1, 2$ splits $C$ into two parts: fix a base point $x_0$ in $C \setminus (C_1 \cup C_2)$, then $C_i^+$ (resp $C_i^-$) is the set of all elements $x$ of $C$ such that every path from $x$ to $x_0$ cuts $C_i$ an even (resp. odd) number of times. Let $\overline{C}$ be $C$ cut along $Int(C_1^+ \cap C_2)$. We define $\widehat{C}$ by cyclically gluing $k$ copies of $\overline{C}$ along copies of $Int(C_1^+ \cap C_2)$. The complex $\widehat{C}$ is called the* ramified covering *of degree $k$ of $C$.*

A ramified covering as described above is not a cube complex. Nevertheless, the same ramified coverings above the cubical barycentric subdivision of $C$ have a cube complex structure.

**Definition 14.** *Let $C_k = [0, 1]^k$ be a Euclidean cube, the* cubical barycentric subdivision *of $C^k$ is the subdivision of $C^k$ along its hyperplanes. The cube $C^k$ then becomes a cube complex composed of $2^k$ $k$-cubes glued together along hyperplanes of $C^k$. In general, the* cubical barycentric subdivision *of a cube complex $C$ is the cube complex described as the union of the* cubical barycentric subdivision *for each cube of $C$.*

*Proof of Theorem 5.* The manifold $V$ is $\pi_1$-cubulated by a finite collection of immersed cocompact submanifolds $W_1, W_2, \ldots, W_\ell$ of codimension 1. Let $\mathcal{H}'$ be the cofinite set of lifts of every $W_i$ in $\mathbb{H}^n$. Let $\mathcal{H}$ be the collection of hyperplanes of $\mathcal{H}'$ and of lifts of connected components of $V_1$ and $V_2$ to $\mathbb{H}^n$. As $V_i$ is a cocompact codimension 1 submanifold of $V$, for $i = 1, 2$, then $\mathcal{H}$ is a collection of walls of $\mathbb{H}^n$ as seen in the last section. Let $X$ be the cube complex dual to this wallspace and $\Gamma$ be the fundamental group of $V$. The group $\Gamma$ acts cocompactly on $X$, and according to Lemma 10, if we add a $\Gamma$-cofinite collection of hyperplanes in $\mathcal{H}$, the action of $\Gamma$ on the dual cube complex $X$ is still proper. Denote by $C$ the quotient of $X$ by $\Gamma$.

To construct a ramified covering $\widehat{C}$ of $C$, as in Definition 13, we need to define $C_1$ and $C_2$, two unions of hyperplanes of $C$. The bijection described at the end of Section 2 between hyperplanes of $\mathcal{H}$ and hypeprlanes of $X$ induces a bijection between hyperplanes of $V$ and $C$. Then we define $C_i$ for $i = 1, 2$ as the union of hyperplanes of $C$ in bijection with connected components of $V_i$. We will now show that these unions of hyperplanes $C_1$ and $C_2$ separate $C$.

The subspace $C_1$ separates $C$ if and only if $X \setminus p_c^{-1}(C_1)$ has a $\Gamma$-invariant bicoloration, where $p_c : X \to C$ is the covering map given by the quotient of $X$ by $\Gamma$. Denote by $p$ the projection $\mathbb{H}^n \to V$. As $V_1$ separates $V$, then $\mathbb{H}^n \setminus p^{-1}(V_1)$ has a $\Gamma$-invariant bicoloration, and so does the complex $P_{\mathbb{H}^n} \setminus (P_{\mathbb{H}^n} \cap p^{-1}(V_1))$. This complex can be seen as a subcomplex of $X \setminus p_c^{-1}(C_1)$. For each point $x$ of $X \setminus p_c^{-1}(C_1)$ consider an element $x_0 \in P_{\mathbb{H}^n}$ and a path $xx_0$ in $X$. Choose for $x$ the color of $x_0$ if the path $xx_0$ crosses $p^{-1}(C_1)$ an even number of times, and the other color if it crosses $p^{-1}(C_1)$ an odd number of times. This choice is well defined because $p^{-1}(C_1)$ is a union of hyperplanes of $X$, which separates $X$. So does $p^{-1}(C_1)$. By the same argument, $C_2$ separates $C$. To complete the proof of Theorem 5, we still need to show the following:

1. The groups $\pi_1(\widehat{V})$ and $\pi_1(\widehat{C})$ are isomorphic (see Proposition 15).

2. The ramified covering $\widehat{C}$ of $C$ is locally $CAT(0)$ (see Proposition 25).

□

## 3.2 Fundamental groups

The goal of this Section is to prove the following Proposition:

**Proposition 15.** *The groups $\pi_1(\widehat{V})$ and $\pi_1(\widehat{C})$ are isomorphic.*

To calculate the fundamental group $\pi_1(\widehat{V})$ (respectively $\pi_1(\widehat{C})$) we will use a different construction of $\widehat{V}$ (respectively $\widehat{C}$).

Let $N_o(V_1 \cap V_2)$ be an open tubular neighborhood of $V_1 \cap V_2$ in $V$. Let $V^0 = V \setminus N_o(V_1 \cap V_2)$. $V^0$ is a submanifold of $V$ with a boundary isomorphic to $(V_1 \cap V_2) \times \mathbb{S}^1$. Consider

$$\theta_V : \pi_1(V^0) \to \mathbb{Z},$$

such that for any loop $l$ of $V^0$, $\theta_V(l)$ is the algebraic number of intersections between $l$ and $V_1^+ \cap V_2$ (considering the orientation), and let $\pi$ be projection $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ where $k$ is the degree of the ramification $\widehat{V}$ over $V$. Denote by $\widehat{V}^0$ the covering of $V^0$ associated with the group $\mathrm{Ker}(\pi \circ \theta_V)$. In restriction to the boundary $(V_1 \cap V_2) \times \mathbb{S}^1$, this covering is a $k$-cyclic covering on the first factor and trivial on the second one. Then the manifold $\widehat{V}$ is obtained by gluing a product $(V_1 \cap V_2) \times D$ to $\widehat{V}^0$, where $D$ is a disk, along the boundary isomorphic to $(V_1 \cap V_2) \times \mathbb{S}^1$.

We will compute the fundamental group of $\widehat{V}$ using this construction. It will be a quotient of the subgroup $\mathrm{Ker}(\pi \circ \theta_V)$ of $\pi_1(V^0)$. Fix a base point $x_0$ in $V^0$. To any connected component $K_i$ of $V_1 \cap V_2$ we associate a loop $\gamma_i$ as follows: choose a path $p_i$ from $x_0$ to a point $x_i$ in the boundary of $K_i$then choose a loop $l_i$ based in $x_i$ which turns once around $K_i$. Define $\gamma_i$ as the concatenation $p_i l_i p_i^{-1}$. Note that $\theta_V(\gamma_i) = \pm 1$. Each $\gamma_i$ represents an element of $\mathrm{Ker}(\pi \circ \theta_V)$ (again abusively denoted $\gamma_i$). By the Seifert-Van Kampen Theorem recursively applied to the union of $\widehat{V}^0$ and for each $K_i \times D$ one obtains:

$$\pi_1(\widehat{V}) = \mathrm{Ker}(\pi \circ \theta_V)/\langle\langle \gamma_1^k, \ldots, \gamma_p^k \rangle\rangle.$$

We can construct $\widehat{C}$ in the same way. In the cube complex $C$ define a *tubular neighborhood* $N_o(C_1 \cap C_2)$ of $C_1 \cap C_2$ as the interior of the union of every cube which has a non trivial intersection with $C_1 \cap C_2$. Remark that it is isomorphic to the product $(C_1 \cap C_2) \times \square$, with $\square$ the interior of a square. Let $C^0 = C \setminus N_o(C_1 \cap C_2)$. The complex $(C_1 \cap C_2) \times \partial\square$ is called the boundary of $C^0$. Define $\theta_C : \pi_1(C^0) \to \mathbb{Z}$ in analogy with $\theta_V$ by counting intersections of a loop with $C_1^+ \cap C_2$. Let $\widehat{C}^0$ be the covering corresponding to the subgroup $\mathrm{Ker}(\pi \circ \theta_C)$. The preimage of the boundary of $C^0$ under this map is isomorphic to a product $(C_1 \cap C_2) \times C_{4k}$ where $C_{4k}$ is a cyclic graph with $4k$ edges. Call this complex the boundary of $\widehat{C}^0$. The complex $\widehat{C}$ is obtained by gluing the product of $(C_1 \cap C_2)$ with a $4k$-gon to $(C_1 \cap C_2) \times C_{4k}$.

Now we can calculate the fundamental group of $\widehat{C}$ in analogy with the calculation of the fundamental group of $\widehat{V}$. We simply need to define loops passing once around every connected component of $N_o(C_1 \cap C_2)$.

The link between the two complexes will be the subspace $P_V^0$ described below. The cellulation of $V$ by the $W_i$'s and by $V_1$ and $V_2$ induces a cellulation of $V^0$ by restricting the $W_i$'s, $V_1$ and $V_2$ to $V^0$ and by adding several $n-2$ cells on the boundary. Then we define $P_V^0$ as the cube complex dual to this cellulation of $V^0$. It can also be described as the complex obtained by removing every square intersecting $V_1 \cap V_2$ from $P_V$, indeed the 1-skeleton of the two complexes dual to the cellulations of $V$ and $V^0$ is the same, the only cells of codimension 2 that are in $V$ and not in $V^0$ are the ones given by $V_1 \cap V_2$. Finally according to Lemma 11 $P_V^0$ identifies simultaneously with a subspace of $V^0$ and of $C^0$.

*Proof of Proposition 15.* The square complex $P_V^0$ is a subspace of $V^0$ and the inclusion induces an isomorphism of fundamental groups by definition. We will prove in Proposition 16 below that the inclusion of $P_V^0$ in $C^0$ induces an isomorphism of fundamental groups. Note that $V^0$ and $C^0$ have isomorphic fundamental groups. For every connected component of $V_1 \cap V_2$, choose $\gamma_i$ included in $P_V^0$ and denote by $\theta$ the restriction of $\theta_V$ and $\theta_C$ to $P_V^0$. Therefore

$$\mathrm{Ker}(\pi \circ \theta_V) = \mathrm{Ker}(\pi \circ \theta) = \mathrm{Ker}(\pi \circ \theta_C),$$

and

$$\pi_1(\widehat{V}) = \mathrm{Ker}(\pi \circ \theta_V)/\langle\langle \gamma_1^k, \ldots, \gamma_p^k \rangle\rangle = \mathrm{Ker}(\pi \circ \theta_C)/\langle\langle \gamma_1^k, \ldots, \gamma_p^k \rangle\rangle = \pi_1(\widehat{C}).$$

□

**Proposition 16.** *The inclusion of $P_V^0$ in $C^0$ induces an isomorphism of fundamental groups.*

Denote by $p_c : X \to C$ and by $X^0$ the preimage of $C^0$ by $p_c$.

*Proof.* Let $P^0_{\mathbb{H}^n}$ be the 2-complex dual to the cellulation of $\mathbb{H}^n \setminus p^{-1}(V_1 \cap V_2)$ by hyperplanes of $\mathcal{H}$. It can be seen as a subcomplex of $P_{\mathbb{H}^n}$. The inclusion of $P_{\mathbb{H}^n}$ in $X$ is $\Gamma$-equivariant, and to prove the proposition it suffices to show that the inclusion of $P^0_{\mathbb{H}^n}$ in $X^0$ is a $\pi_1$-isomorphism.

$$\begin{array}{ccc} P^0_{\mathbb{H}^n} & \hookrightarrow & X^0 \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ P^0_V & \hookrightarrow & C^0 \end{array}$$

□

**Proposition 17.** *The inclusion of $P^0_{\mathbb{H}^n}$ in $X^0$ induces an isomorphism of fundamental groups.*

*Proof.* Let $x_0 \in P^0_{\mathbb{H}^n}$. The fundamental group of $\mathbb{H}^n \setminus p^{-1}(V_1 \cap V_2)$ is an infinite free group generated by a loop for every connected component of $p^{-1}(V_1 \cap V_2)$. Consider the following system of generators: one can choose the loops in $P^0_{\mathbb{H}^n}$ because it is the 2-skeleton of the dual cellulation. For every connected component $K_i$ of $p^{-1}(V_1 \cap V_2)$ let $l_i$ be a loop of $P^0_{\mathbb{H}^n}$ described as a boundary of a square of $P_{\mathbb{H}^n} \setminus P^0_{\mathbb{H}^n}$ associated with $K_i$. For each vertex of $l_i$ there exists a path from $x_0$ to this vertex which does not cross the same hyperplane of $X$ twice, as described in Lemma 18, and for one of the four vertices $y$ of $l_i$ the path does not cross either the two hyperplanes of $P_{\mathbb{H}^n}$ which form $K_i$. Denote by $a$ this path and take $\alpha_i = a l_i a^{-1}$ as a generator of $\pi_1(P^0_{\mathbb{H}^n})$ associated with $K_i$. We prove in Proposition 20 that the fundamental group of $X^0$ is an infinite free group generated by the $\alpha_i$. Then this inclusion induces a $\pi_1$-isomorphism. □

We will use combinatorial loops on the 1-skeleton of $X^0$. The combinatorial loops can be seen as loops of the 1-skeleton of $X$. If we choose a vertex, this loop is uniquely determined by the sequence of hyperplanes successively dual to the edges of this loop. As $X$ is $CAT(0)$, a path in the 1-skeleton of $X$ is a geodesic if and only if the associated sequence of hyperplanes of $X$ does not contain the same hyperplane twice.

**Lemma 18.** *For each pair of vertices $(x, y)$ of $P^0_{\mathbb{H}^n}$ there exists an edge path from $x$ to $y$ which crosses each hyperplane of $P_{\mathbb{H}^n}$ at most once.*

*Proof.* To a pair $(x, y)$ of vertices of $P_{\mathbb{H}^n}$ we associate a pair of connected components of $\mathbb{H}^n$ separated by hyperplanes of $\mathcal{H}$. Choose a point for each of these components in $\mathbb{H}^n$, then consider a geodesic between them. The sequence of hyperplanes of $\mathcal{H}$ crossed by this geodesic gives a path of edges of $P_{\mathbb{H}^n}$ which crosses each hyperplane at most once, i.e. the path is a geodesic of the 1-skeleton of $P_{\mathbb{H}^n}$. □

**Lemma 19.** *Let $\gamma$ be a combinatorial path in $X^0$ with the following sequence of dual hyperplanes of $X$: $ABH_1...H_nA$ such that $\forall i = 1, 2, ..., n$, $H_i \neq A, B$, and $A$, $B$ are not simultaneously in the preimage of $C_1$ and $C_2$ under $p_c$. Then $\gamma$ is fixed-end-point homotopic in $X^0$ to the path associated with the sequence $BAH_1...H_nA$.*

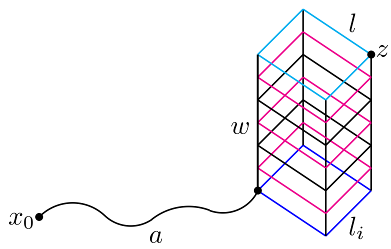
*Proof.* We will show that the first two edges of this path border a square of $X^0$. As defined in Section 2, a vertex of $X$ is a choice of halfspace for every hyperplane of $\mathcal{H}$, such that if a halfspace $\overrightarrow{C}$ is included in another one $\overrightarrow{D}$ and if $\overrightarrow{C}$ belongs to a certain vertex then $\overrightarrow{D}$ belongs to this vertex too. We will say that $\overleftarrow{D}$ and $\overrightarrow{C}$ are *compatible* if they can belong to a same vertex of $X$.

Consider the three first vertices $v_1$, $v_2$ and $v_3$ of the path $\gamma$ and the two walls $\bar{A} = \{\overrightarrow{A}, \overleftarrow{A}\}$ and $\bar{B} = \{\overrightarrow{B}, \overleftarrow{B}\}$ associated with the hyperplanes $A$ and $B$. Suppose that $v_1$ contains the halfspaces $\overrightarrow{A}$ and $\overrightarrow{B}$. As $v_1$ and $v_2$ are separated by an edge dual to $A$ then $v_2$ contains halfspaces $\overleftarrow{A}$ and $\overrightarrow{B}$. Moreover $v_3$ contains the halfspaces $\overleftarrow{A}$ and $\overleftarrow{B}$. Consider the collection $s$ of halfspaces composed with $\overrightarrow{A}$, $\overleftarrow{B}$ and every halfspace which simultaneously belongs to vertices $v_1$, $v_2$ and $v_3$. To prove that $s$ is a vertex of $X$ we will show that every pair of hyperplanes of $s$ is compatible. The last vertex of $\gamma$ contains $\overrightarrow{A}$ and $\overleftarrow{B}$ because $\forall i = 1, 2, ..., n$, $H_i \neq A, B$. The collection of halfspaces associated with $v_1$, $v_2$ and $v_3$ show that halfspaces $\overrightarrow{A}$, $\overrightarrow{B}$, $\overleftarrow{A}$ and $\overleftarrow{B}$ are compatible with all halfspaces shared by $v_1$, $v_2$ and $v_3$, and that the pairs $\overrightarrow{A}$ and $\overrightarrow{B}$, $\overleftarrow{A}$ and $\overrightarrow{B}$, $\overleftarrow{A}$ and $\overleftarrow{B}$ are compatible. Hence $s$ is a vertex of $X$, and the three first vertices of $\gamma$, $s$, and the edges between them describe the boundary of a square of $X$. Since $A$ and $B$ are not simultaneously preimages of $C_1$ or $C_2$ the square is still in $X^0$. □

**Proposition 20.** *The fundamental group $\pi_1(X^0)$ of $X^0$ is an infinite free group generated by $\{\alpha_i\}$.*

**Lemma 21.** *Let $K_i$ be a connected component of a preimage of $C_1 \cap C_2$ in X, let $l$ be an oriented loop obtained as the boundary of a square of X with a non trivial intersection with $K_i$, and let $z$ be a vertex of $l$. Then there exists a loop $\alpha_i' = a' l a'^{-1}$ homotopic to $\alpha_i$ (or $\alpha_i^{-1}$), with $a'$ geodesic in the 1-skeleton of $X^0$ between $x_0$ and $z$.*

*Proof.* The intersection between two hyperplanes has a natural cube complex structure. The neighborhood of $K_i$ in $X$ is a product $W \times \square$ with $\square$ a square and $W$ a cube complex isomorphic to $K_i$. The neighborhood of $K_i$ in $X^0$ will be $W \times \partial(\square)$. There exists a loop $\alpha_i = a l_i a^{-1}$, one of the generators of $\pi_1(P^0_{\mathbb{H}^n})$, such that $l_i$ turns around $K_i$. Then there exists $w_1,\ w_2$ in $W$ such that $w_1 \times \partial(\square) = l$ and $w_2 \times \partial(\square) = l_i$. Denote by $w_2 \times \bullet$ the vertex of $l_i$ which is the last vertex of $a$. Since $W$ is connected, choose a path $w'$ in $W$ from $w_1$ to $w_2$, and consider $w = w' \times \bullet$ a path in $X^0$. Denote by $c$ the concatenation of $a$ and $w$.




We will construct a path $c'$ homotopic to $c$ such that $c'$ is a geodesic in the 1-skeleton of $X$, i.e. such that $c'$ does not cross the same hyperplane of $X$ twice. The path $c$ does not cross any hyperplane that is a preimage of $C_1$ or $C_2$. Indeed, by construction, neither does $a$, nor does $w$ because $C_1$ and $C_2$ do not self-intersect ( because $V_1$ and $V_2$ do not self-intersect). Suppose that $c$ is not geodesic and consider the two nearest edges dual to the same hyperplane $H$. The subpath of $c$ between these edges is not dual to $H$ and we can apply Lemma 19 several times, until the two edges are next to each other. Then one can delete the two edges by homotopy. By recurrence on these pairs of hyperplanes, we obtain a path $c'$ from $x_0$ to $l$ homotopic to $c$ which crosses each hyperplane of $X$ at most once. Then consider $a'$ the path obtained from $c'$ to $z$ by adding one or two edges of $l$. These additional edges are dual to the preimage of $C_1$ or $C_2$, $a'$ is still geodesic in the 1-skeleton of $X$, and $a' l a'^{-1}$ is homotopic to $a l_i a^{-1}$. □

*Proof of Proposition 20.* First we prove that $\pi_1(X^0)$ is generated by $\{\alpha_i\}$. For each combinatorial loop $\gamma$, denote by $|\gamma|$ the length of the loop, then consider

$$L : \pi_1(X^0) \to \mathbb{N}$$

$$L(\gamma) = min\{|\gamma'|,\ \gamma \sim \gamma'\}.$$

We will use a recursive argumentation on the length $L$ of homotopy classes of loops. If $L(\gamma) = 0$ then $\gamma$ is homotopic to $x_0$. Suppose that every loop of length strictly less than $N$ is generated by $\{\alpha_i\}$, and let $\beta$ be a loop of $X^0$, such that $L(\beta) = N$. A loop of $X^0$ can be described as a concatenation of loops of type $blb^{-1}$, with $l$ the boundary of a square of the neighborhood of the preimage of a connected component of $C_1 \cap C_2$. If $\beta$ is not exactly one of these loops then $\beta$ is a concatenation of at least 2 such loops of length less than $N$, and by recurrence $\beta$ is generated by $\{\alpha_i\}$. Applying Lemma 21 to $l$ and $z$, the last vertex of $b$, there exists a loop $\alpha_i' = a' l a'^{-1}$ homotopic to $\alpha_i^{\pm 1}$ such that $a'$ is minimal between $x_0$ and $z$.

$$\beta = blb^{-1} = ba'^{-1} a' l a'^{-1} a' b^{-1} = (ba'^{-1}) a' l a'^{-1} (a' b^{-1}).$$

Fix $j = |b|$, then $N = |\beta| = 2j + 4$. As $a'$ is a geodesic with the same endpoints as $b$ then $|a'| \leq j$. Finally, $L(ba'^{-1}) \leq |ba'^{-1}| \leq 2j < N$, and by recurrence $ba'^{-1}$ is generated by $\alpha_i$. Furthermore, as $a' l a'^{-1}$ is homotopic to $\alpha_i$, $\beta$ is also generated by $\{\alpha_i\}$.

To see that $\pi_1(X^0)$ is a free group, we will construct an injective morphism from $\pi_1(X^0)$ to an infinite free group.

Every hyperplane of $C_2$ in $C$ is divided into several different hyperplanes in $C^0$. Let $E$ be the set of all lifts of these hyperplanes in $X^0$. For every $e \in E$, $H_e$ will be the hyperplane of $X$ that contains $e$ [1]. Consider the infinite free group $\mathbb{F}_\infty$ generated by $E$.

The elements of $E$ are hyperplanes of $X^0$, we choose an orientation of each hyperplane of $X$ which induces an orientation of hyperplanes of $E$. Consider the map $m$ such that if $\gamma$ is a combinatorial loop of $X^0$ then $m(\gamma)$ is the word in $E^{\pm}$ obtained by juxtaposing hyperplanes of $X^0$ crossed by $\gamma$, to the power of $\pm 1$ depending on the orientation. The map $m$ induces the morphism

$$h : \pi_1(X^0) \to \mathbb{F}_\infty.$$

Homotopies between combinatorial loops can be described as a succession of elementary homotopies: going on an edge and coming back is homotopic to the identity, and for every square of $X^0$ an edge of its boundary is homotopic to the path which runs along the three other edges of the boundary. Then $m$ is well defined modulo these elementary homotopies.

The map $h$ is injective. Assume that $\gamma$ is a combinatorial loop of $X^0$ such that $h(\gamma) = 1$. Since $\gamma$ is a loop, every hyperplane of $X$ crosses $\gamma$ an even number of times. If $\gamma$ does not intersect any hyperplanes of $\{H_e, e \in E\}$ then the hyperplanes dual to $\gamma$ are not in $p_c^{-1}(C_2)$ and we use Lemma 19 to gather and eliminate two by two those hyperplanes dual to $\gamma$ by elementary homotopies in the following way. Each hyperplane of $X^0$ dual to $\gamma$ appears an even number of times in the sequence of hyperplanes dual to $\gamma$. Consider two repetitions of the same hyperplane as closed as possible. Then every hyperplane contained between those two repetitions will appear once. If there is not a hyperplane between the two repetitions then $\gamma$ is homotopic to a loop given by the same sequence minus the two occurrences of this hyperplane. If there are some hyperplanes between the repetitions, we can use Lemma 19 several times, since no hyperplane of $p^{-1}(C_2)$ belongs to the sequence, bringing us back to the previous case. Then the homotopy class of $\gamma$ is trivial in $\pi_1(X^0)$.

Now suppose that one of the hyperplanes $K$ of $X$, crossed by $\gamma$, is associated with an element of $E$. Since $h(\gamma) = 1$, the word $m(\gamma)$ contains a subword $ee^{-1}$ with $e \in E$. One can assume that $K = H_e$. Then the enumeration of hyperplanes *of* $X^0$ dual to $\gamma$ contains a sequence $e, h_1, \ldots, h_n, e$, where the $h_i$'s do not belong to $E$. Denote by $c$ the subpath of $\gamma$ associated with this sequence. For every $i \in \{1, \ldots, n\}$ denote $W_i$ the hyperplane of $X$ that contains $h_i$. For every $i \in \{1, \ldots, n\}$, as $h_i$ is not in $E$, then $W_i$ is not contained in $p_c^{-1}(C_2)$. Furthermore if $W_i$ is contained in $p_c^{-1}(C_1)$ then the $h_i$'s do not intersect $e$, because $e$ is a connected component of $p_c^{-1}(C_2)$ cut along $p_c^{-1}(C_1)$. Therefore there exists an even number of $h_j$'s such that $W_i = W_j$. If such a pair of elements $W_i = W_j \subset p_c^{-1}(C_1)$ exists then by applying Lemma 19 several times to the sentence $H_i, H_{i+1}, \ldots, H_j$ which does not contain hyperplanes dual to $p_c^{-1}(C_2)$ we get $j = i + 1$, and $H_i H_i$ is homotopic to a point. Finally, applying Lemma 19 several times to the sentence $H_e, H_1, \ldots H_n, H_e$ with $H_i \not\subset p_c^{-1}(C_1)$, one can reduce the number of $ee^{-1}$ associated with $\gamma$.

□

### 3.3 Ramified covering over a locally $CAT(0)$ cube complex is locally $CAT(0)$.

We want to prove that if $\widehat{C}$ is a ramified covering of a locally $CAT(0)$ cube complex $C$ constructed above the intersection of two unions of hyperplanes (see Definition 13), then $\widehat{C}$ is locally $CAT(0)$. We will prove this with a more general definition of a ramified covering.

**Definition 22.** *Let $C$ be a cube complex and $L$ be a subcomplex of $C$. A cube complex $\widehat{C}$ is a* general ramified covering *of $C$ above $L$ if there exists a combinatorial map $f : \widehat{C} \longrightarrow C$ and a subcomplex $\widehat{L}$ of $\widehat{C}$ such that:*

- $f_{|\widehat{L}} : \widehat{L} \xrightarrow{\simeq} L$
- $f_{|(\widehat{C} \setminus \widehat{L})} : (\widehat{C} \setminus \widehat{L}) \to (C \setminus L)$ *is a cover.*

**Definition 23.** *A subcomplex $L$ of a cube complex $C$ is* locally convex *if for every cube $Q$ of $C$ the subcomplex $L \cap Q$ is either a unique face of $Q$ or the whole cube $Q$.*

We will use the following characterization of being locally $CAT(0)$ in a cube complex.

**Definition 24.** *A cube complex $C$ is locally $CAT(0)$ if and only if for every $v \in C^0$, $link(v, C)$ is a simplicial flag complex.*

[1]The map $e \mapsto H_e$ is not injective

**Proposition 25.** *Let $(\widehat{C}, \widehat{L})$ be a general ramified covering of a cube complex $(C, L)$, with $C$ locally $CAT(0)$ and $L$ locally convex in $C$. Then $\widehat{C}$ is locally $CAT(0)$.*

*Proof.* First, we prove that for every vertex $v$ of $\widehat{C}$, the complex $link(v, \widehat{C})$ is simplicial , and not multi-simplicial, i.e. $link(v, \widehat{C})$ is totally determined by its boundary. Suppose that $v$ is not in $\widehat{L}$, then a small ball around $f(v)$ is homeomorphic to a small ball around $v$, so the $link$ is the same. As $C$ is locally $CAT(0)$, then $link(v, \widehat{C})$ is simplicial. Now suppose that $v$ belongs to $\widehat{L}$. Consider two $(k-1)$-simplices of $link(v, \widehat{C})$ for $k \geq 2$ sharing the same boundary. Denote by $Q_1$ and $Q_2$ the two $k$-cubes of $\widehat{C}$ associated with these $(k-1)$-simplices. As the two simplices have the same boundary, then $Q_1$ and $Q_2$ are glued along subcubes of codimension 1 containing $v$. Let $f$ be the projection associated with the ramified covering $\widehat{C} \to C$. As $f$ is combinatorial then $C_1$ and $C_2$ are projected on $k$-cubes of $C$. As $C$ is locally $CAT(0)$ then $C_1$ and $C_2$ are projected onto the same cube. The map $f$ induces an isomorphism between $\widehat{L}$ and $L$, and $L$ is convex in $C$. If every $P_i$ belongs to $\widehat{L}$ then $Q \subset L$ and $Q_1 = Q_2$. If there exists $i \in \{1, 2, \ldots, k-1\}$ such that $P_i$ does not belong to $\widehat{L}$, the restriction of $f$ to $(Q_1 \cup Q_2) \setminus \widehat{L}$ is a covering on its image of degree 1, because the preimage of $f(P_i)$ is $P_i$. Then $Q_1$ and $Q_2$ are equal.

It remains to see that for every vertex $v \in \widehat{C}$, the simplicial complex $link(v, \widehat{C})$ is flag. Let $v$ be a vertex of $C$ and $e_1, \ldots, e_p$ be two by two connected vertices of $link(v, \widehat{C})$. The function $f$ projects two such vertices on two different vertices of $link(f(v), C)$ because $link(f(v), C)$ is simplicial and not multi-simplicial. As $C$ is locally $CAT(0)$ $link(f(v), C)$ is flag and there exists a $p$-simplex of $link(f(v), C)$ which has $f([e_i, e_j])$ as 1-skeleton. The $(p+1)$-cube associated with this $p$-simplex lifts to a cube with a 0-skeleton of exactly $\{e_1, \ldots, e_p\}$. □

# 4 The cyclic ramified covering of a special cube complex is special.

We will use the following characterization of being special.

**Definition 26.** *(See [HW08]) A cube complex is* special *if it does not contain pathologies of hyperplanes such as self-intersection (see Figure 5), self-osculation (see Figure 6) or inter-osculation (see Figure 7), and furthermore if its hyperplanes are two-sided i.e. a neighborhood is homeomorphic to the product of the hyperplane with an interval.*

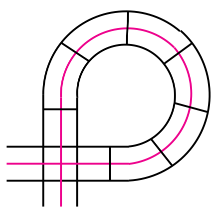


Figure 5: A self-intersecting hyperplane.

**Proposition 27.** *Let $C$ be a special cube complex and let $C_1$ and $C_2$ be two separating unions of hyperplanes of $C$. Then the ramified covering $\widehat{C}$ (in the sense of Definition 13) of the cubical barycentric subdivision $C'$ of $C$ above $C_1 \cap C_2$ is special.*

**Lemma 28.** *Let $C$ be a cube complex and let $C'$ be the cubical barycentric subdivision of $C$. If $C$ is special then $C'$ is special.*

*Proof.* The neighborhood of a hyperplane of $C'$ is the half neighborhood of a certain hyperplane of $C$. Then a pathology of a hyperplane of $C'$ would imply a pathology of the associated hyperplanes of $C$. □

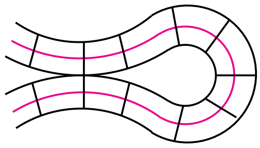


Figure 6: A self-osculating hyperplane.

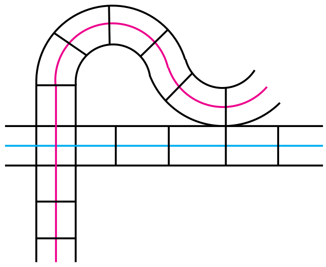


Figure 7: Two inter-osculating hyperplanes.

*Proof of the Proposition 27.* We will use the notations from Definition 13, and denote by $f$ the projection $\widehat{C} \to C'$. We show that if there is a pathology of hyperplanes as defined in 26 in $\widehat{C}$ then there is a pathology in $C'$.

A hyperplane of $\widehat{C}$ is a union of hyperplanes of some $\overline{C'}^i$'s ($k$ copies of the cubical barycentric subdivision of $\overline{C}$ glued together to form $\widehat{C}$) which coincides on the boundary between $\overline{C'}^i$ and $\overline{C'}^{i+1}$ for every $i$. The projection of all different pieces of this hyperplane on $C'$ forms a unique hyperplane of $C'$. If $\widehat{H}$ is not a two-sided hyperplane of $\widehat{C}$, then there is a loop in $\widehat{H}$ such that the parallel transport of a vector orthogonal to $\widehat{H}$ along this loop gives a vector of the opposite direction. Then the projection of the loop has the same property, and the projection of the hyperplane to $C$ is not two-sided.

The map $f$ projects a self-intersecting hyperplane of $\widehat{C}$ on a self-intersecting hyperplane of $C'$. This prevents the first pathology from occurring.

Suppose now that there exists two hyperplanes $\widehat{H}$ and $\widehat{K}$ of $\widehat{C}$ which are inter-osculating. Let $\widehat{e}_1$ and $\widehat{e}_2$ be two edges of $\widehat{C}$ respectively transverse to $\widehat{H}$ and $\widehat{K}$ which share a vertex $\widehat{v}$. Let $e_1$, $e_2$, $H$ and $K$ be the images of $\widehat{e}_1$, $\widehat{e}_2$, $\widehat{H}$ and $\widehat{K}$ under $f$. If $e_1 = e_2$, then $f$ projects $\widehat{H}$ and $\widehat{K}$ on a unique self-intersecting hyperplane, contradicting the fact that $C'$ is special. If $e_1$ and $e_2$ are two osculating edges of $C'$, then $H$ and $K$ are two inter-osculating hyperplanes of $C'$.The case where $e_1$ and $e_2$ form the corner of a square $Q$ remains to be seen. In this case, $Q$ has two different lifts $\widehat{Q}_1$ and $\widehat{Q}_2$ which contain respectively $\widehat{e}_1$ and $\widehat{e}_2$. As there is one lift of the vertex $v = f(\widehat{v})$ in the union of squares $\widehat{Q}_1 \cup \widehat{Q}_2$, the vertex $v$ belongs to the ramification locus. However, as there are two lifts of $e_i$ in $\widehat{Q}_1 \cup \widehat{Q}_2$, the square $Q$ is not included in $K$. As the cubical neighborhood of $C_1 \cap C_2$ is isomorphic to the cube complex $C_1 \cap C_2 \times \square$, where $\square$ is a square, and as the square $Q$ of the cubical barycentric subdivision $C'$ of $C$ is not totally included in $C_1 \cap C_2$, the square $Q$ is a quarter of a square of $C$ given by a point times $\square$ in the neighborhood of $C_1 \cap C_2$. We can assume that $e_i$ belongs to $C_i$. If not, switch the notation of $e_1$ and $e_2$. Consider the path $\widehat{\gamma}$ obtained by the concatenation of a path $\widehat{\gamma}_H$ of $\widehat{H}$ from the center of $\widehat{Q}_1$ to the intersection of $\widehat{H}$ and $\widehat{K}$, and a path $\widehat{\gamma}_K$ of $\widehat{K}$ from this intersection to the center of $\widehat{Q}_2$. Let $\gamma$, $\gamma_H$ and $\gamma_K$ be the image of $\widehat{\gamma}$, $\widehat{\gamma}_H$ and $\widehat{\gamma}_K$ under the projection $f$. The path $\gamma$ is a loop of $C'$. Since $\widehat{\gamma}$ itself is not a loop the (algebraic) intersection number of $\gamma$ with $C_1^+ \cap C_2$ has to be different from zero, otherwise $\gamma$ would lift to a loop in the ramified cover. We will reach a contradiction by showing that in fact $\gamma$ has a trivial intersection with $C_1^+ \cap C_2$. Actually, $H$ and $K$ are disjoint from $C_2$ and $C_1$. For example $e_1$ is an edge of $C'$, contained in an edge of $C$ dual to the hyperplane $C_2$ in $C$. The intersection of every cube $D$ of $C$ with $C_2$ is obtained by setting one coordinate to $\frac{1}{2}$, and the intersection with $H$ is obtained by setting the same coordinate to $\frac{1}{4}$ or $\frac{3}{4}$, so $H$ is a disjoint copy of $C_2$. We can use exactly the same reasoning to prove that $K$ is disjoint from $C_1$. Now, there are two possibilities depending on whether $K$ is totally included in $C_1^-$ or in $C_1^+$. In the first case, as $\gamma_H \subset H$ which is disjoint from $C_2$, and $\gamma_K \subset K \subset C_1^-$, the loop $\gamma$ never crosses $C_1^+ \cap C_2$. Now, if $K$ is totally included in $C_1^+$, as $C_2$ separates $C$, the algebraic intersection number of the loop $\gamma$ with $C_2$ is zero. Since $H$ is disjoint from $C_2$, the path $\gamma_H \subset H$ does not cross $C_2$. Then intersections of $\gamma$ and $C_2$ are in $\gamma_K$ and the intersection number of $\gamma_K$ with $C_2$ is zero. As $\gamma_K$ is included in $C_1^+$, the intersection number of the loop $\gamma$ with $C_1^+ \cap C_2$ is zero, and we have a contradiction.

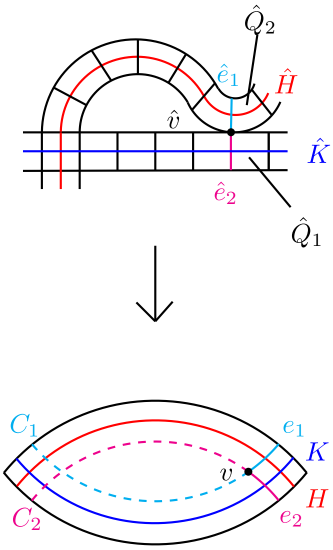



Figure 8: When $e_1$ and $e_2$ are two adjacent edges of a square.

Let $\widehat{H}$ be a self-osculating hyperplane of $\widehat{C}$. Denote by $\widehat{e}_1$ and $\widehat{e}_2$ two distinct edges, transverse to $\widehat{H}$, sharing a vertex $\widehat{v}$. Denote by $e_1$, $e_2$, $v$ and $H$ the image of $\widehat{e}_1$, $\widehat{e}_2$, $\widehat{V}$ and $\widehat{H}$ under $f$. If $e_1 \neq e_2$ then $H$ will be a self-osculating or a self-intersecting hyperplane of $C'$. If $e_1 = e_2$, then $v$ will belong to the ramification locus and $e_1 \subset C_1^+ \cap C_2$. A path $\gamma$ from the center of $\widehat{e}_1$ to the center of $\widehat{e}_2$ is sent onto a loop of $C'$ which crosses $C_1^+ \cap C_2$ non trivially. As $H$ is transverse to $e_1$ $H$ is a copy of a hypeplane of $C_1$, so $H$ is totally included in $C_1^+$, and as $C_2$ separates $C'$, $f(\gamma)$ has to cross $C_1^+ \cap C_2$ trivially.

□

# 5 Gromov-Thurston manifolds.

The main result relies on the following Theorem of [BHW11].

**Theorem 29.** *"Simple type" arithmetic manifolds are cubical and virtually special.*

Let $V$ be a "Simple type" arithmetic manifold, containing many (immersed) compact totally geodesic codimension 1 submanifolds. The fundamental group of $V$ acts cocompactly on the cube complex associated with a finite number of such submanifolds. To show the last Theorem, Gromov and Thurston chose these submanifolds such that the action of $\pi_1(V)$ on the dual cube complex is proper, using the following criteria: let $\mathcal{H}$ be a collection of lifts of a finite number of hyperplanes $W_1, \ldots, W_\ell$ of $V$ described as above. For every pair of point $x, y$, denote by $d_{\mathcal{H}}$ the number of elements of $\mathcal{H}$ witch separate $x$ from $y$. If $d_{\mathcal{H}}$ and the usual distance on $\mathbb{H}^n$ are quasi-isometric then the action of $\pi_1(V)$ on the dual cube complex is proper.

To choose such a collection of hyperplanes in [BHW11], the authors use the fact that hyperplanes of $\mathbb{H}^n$ which project to a compact submanifold of $V$ are dense in the set of hyperplanes of $\mathbb{H}^n$ (see p. 6 of [BHW11]

). Later, they prove that by passing to a finite cover of $V$, the quotient of the cube complex obtained using this construction by the group $\pi_1(V)$ is special.

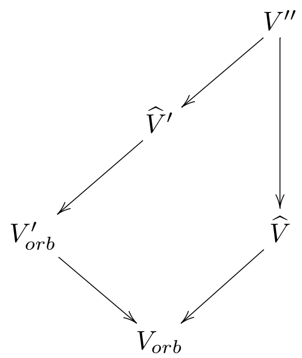
Let $V$ be an arithmetic manifold. By passing to a finite cover, Gromov and Thurston constructed two totally geodesic manifolds, $V_1$ and $V_2$, which separate $V$, and then consider $\widehat{V}$ a ramified covering of $V$ above $V_1 \cap V_2$. Let us prove that $\widehat{V}$ is virtually special.

*Proof of Theorem 1.* In the proof of Proposition 2.1 of [BHW11], the argument of density for choosing $W_1, \ldots, W_\ell$ allows to suppose furthermore that the intersections between hyperplanes of $\mathcal{H}$ are generic. Then let $V'$ be a finite cover of $V$ such that the quotient of the cube complex dual to $\mathcal{H}$ by $\pi_1(V')$ is special. Denote by $\widehat{V'}$ the cyclic ramified cover of degree $k$ of $V'$ above the intersection of the preimage of $V_1$ and $V_2$ in $V'$. By Theorem 5, $\pi_1(\widehat{V}')$ is cubical, and furthermore by Proposition 27, it is special. We will show that there exists a finite cover of $\widehat{V}'$ and $\widehat{V}$. Then the fundamental group of $\widehat{V}$ will be virtually special. To see this we will use covering orbifold theory.

Let $V_{orb}$ be an orbifold. Its underlying space is $V$ and its singular locus is $V_1 \cap V_2$. The orbifold structure is given by the following maps: If $x$ belongs to $V \smallsetminus V_1 \cap V_2$, choose a neighborhood sufficiently small which does not intersect the singular locus and take the identity on $V$. If $x \in V_1 \cap V_2$, a small tubular neighborhood of $x$ is isomorphic to $D^2 \times ]-1,1[^{n-2}$. Then choose the quotient of $D^2 \times ]-1,1[^{n-2}$ by the action of $\mathbb{Z}/k\mathbb{Z}$ on $D^2$.

Let $V'_{orb}$ be an orbifold with $V'$ as the underlying space obtained by pulling back under the covering map $p : V' \to V$ the orbifold structure of $V_{orb}$. For every point $x$ of $V'$, a local map will be the composition of $p$ and of a map of $V'$ at $p(x)$. The projection $V'_{orb} \to V_{orb}$ is a covering orbifold. Ramified covering $\widehat{V}$ is a covering orbifold of $V_{orb}$. Indeed, far from the singular locus, the projection $\widehat{V} \to V$ is the identity, and on a neighborhood of $V_1 \cap V_2$ this projection is a quotient by the cyclic group $\mathbb{Z}/k\mathbb{Z}$. By the same reasoning, the manifold $\widehat{V}'$ is a covering orbifold of $V'$.

Denote by $\tilde{V}_{orb}$ the universal covering orbifold of $V_{orb}$. There exists two groups $G_1$ and $G_2$ such that $\widehat{V}' = \tilde{V}_{orb}/G_1$ and $\widehat{V} = \tilde{V}_{orb}/G_2$. Note that $G_1$ is special. Consider now $V'' = \tilde{V}_{orb}/G_1 \cap G_2$. $V''$ is a manifold and a (classical) cover of $\widehat{V}$ since $\widehat{V}$ is a manifold. Furthermore, as $G_1$ and $G_2$ have a finite index in $G$, the group $G_1 \cap G_2$ has a finite index in $G_2$, and $G_1 \cap G_2$ is special as a subgroup of $G_1$. Then $\pi_1(\widehat{V}) = G_2$ is virtually special.

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