ARAKELOV GEOMETRY

Heights and the Bogomolov conjecture

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Version of September 28, 2012
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CHAPTER 1

HERMITIAN VECTOR BUNDLES ON ARITHMETIC CURVES

§ 1.1
DEFINITIONS; GENERAL CONSTRUCTIONS

A. Definition and comments

Let K be a number field, $\mathfrak{o}_K$ be its ring of integers, $S = \operatorname{Spec} \mathfrak{o}_K$. Let Σ be the set of embeddings of K in $\mathbf{C}$; such an embedding σ is said to be real if $\sigma(K) \subset \mathbf{R}$, and complex otherwise.

DEFINITION 1.1.1. *A Hermitian vector bundle $\overline{E} = (E, h)$ on* $\operatorname{Spec} \mathfrak{o}_K$ *is the data of a projective $\mathfrak{o}_K$-module of finite type E together with a family $(h_\sigma)_{\sigma\in\Sigma}$, where for any $\sigma \in \Sigma$, h_σ is an Hermitian form on the complex vector spaces $\sigma^* E = E \otimes_\sigma \mathbf{C}$. We assume the family (h_σ) to be invariant under complex conjugation.*

For any $\sigma \in \Sigma$ and any $v \in \sigma^* E$, we also write $\| v\|_\sigma = \sqrt{h_\sigma(v, v)}$.

The rank $\operatorname{rank} \overline{E}$ of $\overline{E}$ is defined to be the rank of the (locally free) $\mathfrak{o}_K$-module E; it is also equal to the common dimension of the complex vector spaces $\sigma^* E$.

Remark 1.1.2. Let $\overline{E} = (E, h)$ be a Hermitian vector bundle over S. Let us detail the condition that the family (h_σ) is invariant under complex conjugation.

1) Let $\sigma \in \Sigma$ and let $\overline{\sigma}$ be the conjugate embedding, defined by $a \mapsto \overline{\sigma(a)}$. Recall that for $e \in E$ and $a \in \mathfrak{o}_K$, one has $ae \otimes 1 = e \otimes \sigma(a)$ in $\sigma^* E$. In particular, the map $\overline{\cdot}\colon \sigma^* E \to \overline{\sigma}^* E$ given by $v \otimes z \mapsto v \otimes \overline{z}$ is an anti-linear isomorphism of complex vector spaces. The condition of invariance under complex conjugation means that for any $\sigma \in \Sigma$, any $v_1, v_2 \in \sigma^* E$,

$$h_{\overline{\sigma}}(\overline{v_1}, \overline{v_2}) = \overline{h_\sigma(v_1, v_2)}.$$

The invariance by complex conjugation is also equivalent to the fact that $\| v \otimes 1\|_\sigma = \| v \otimes 1\|_{\overline{\sigma}}$ for any $v \in E$. One implication is obvious. The other follows from the fact that an Hermitian scalar product is characterized by its associated norm.

If σ is a real embedding, $\overline{\sigma} = \sigma$ and the condition implies that h_σ comes from an euclidean scalar product on the real vector subspace $E \otimes_\sigma \mathbf{R}$ of $\sigma^* \mathbf{C}$.

2) Let $(v_1,\dots,v_n)$ be a basis of the K-vector space $E\otimes K$. For any $\sigma\in\Sigma$, the family $(v_i\otimes 1)$ is a basis of σ^*E. Let A_σ be the matrix of h_σ in this basis; in other words, $A_\sigma = (h_\sigma(v_i\otimes 1, v_j\otimes 1))$. Since h_σ is Hermitian, one has $A_\sigma^{\mathrm{T}} = \overline{A_\sigma}$.

The condition of invariance under complex conjugation is equivalent to the additional equalities $A_{\overline{\sigma}} = \overline{A_\sigma}$, for $\sigma\in\Sigma$.

3) The complex space $E_{\mathbf{C}} = E\otimes_{\mathbf{Z}}\mathbf{R}$ is naturally isomorphic to $\bigoplus_{\sigma\in\Sigma}\sigma^*E$, complex conjugation acting by exchanging σ^*E and $\overline{\sigma}^*E$, for $\sigma\in\Sigma$. (See Lemma 1.3.1 below, applied to the field extension $\mathbf{Q}\subset K$.) Let us still write h for the Hermitian scalar product on $E_{\mathbf{C}}$ given by h_σ on σ^*E and such that σ^*E and τ^*E are orthogonal if $\sigma\neq\tau$. The condition of invariance under complex conjugation means that $h(\overline{v},\overline{w}) = \overline{h(v,w)}$ for any two vectors $v,w\in E_{\mathbf{C}}$.

Let us moreover observe that $E_{\mathbf{R}} = E\otimes_{\mathbf{Z}}\mathbf{R}$ is the real part of $E_{\mathbf{C}} = E\otimes_{\mathbf{Z}}\mathbf{C}$, *i.e.*, the set of vectors v such that $\overline{v} = v$. Consequently, h restricts to an Euclidean scalar product on $E_{\mathbf{R}}$. Conversely, an Euclidean scalar product on $E_{\mathbf{R}}$ extends uniquely to a Hermitian scalar product on $E_{\mathbf{C}}$ which is invariant by complex conjugation.

Let $\overline{E} = (E,h)$ and $\overline{F} = (F,h)$ be Hermitian vector bundles over S. We define a morphism of Hermitian vector bundles to be a morphism $\varphi\colon E\to F$ between the underlying $\mathfrak{o}_K$-modules such that $\left\|\varphi(e)\right\|_\sigma \leqslant \|e\|_\sigma$ for any $\sigma\in\Sigma$ and any $e\in\sigma^*E$. This defines a category $\widehat{\mathrm{Vect}}(S)$ of Hermitian vector bundles over S; in that category, a morphism φ is an isomorphism if it is bijective and respects the Hermitian forms.

The ring $\mathfrak{o}_K$ is a Dedekind ring: it is Noetherian, and its local rings are discrete valuation rings. The analogy between number fields and function fields suggests to consider the scheme $S = \operatorname{Spec}\mathfrak{o}_K$ as an affine smooth curve. The *motto* of Arakelov geometry is that these objects are the analogues of vector bundles on the (putative) projective curve which would compactify S.

The analogue of a coherent sheaf is the notion of a *Hermitian coherent sheaf*, defined as a pair (E,h), where E is a $\mathfrak{o}_K$-module of finite type and h is a family of Hermitian forms on the finite dimensional vector spaces σ^*E, for $\sigma\in\Sigma$, invariant under complex conjugation.

B. Constructions of Hermitian vector bundles from linear algebra

In order to develop this analogy, we first observe that some natural constructions carry over to this setting.

B.1. Submodule, quotients. — Let $\overline{E} = (E,h)$ be a Hermitian vector bundle (coherent sheaf) over S.

Let F be a submodule of E and let $G = E/F$.

For any $\sigma\in\Sigma$, the exact sequence $0\to F\to E\to G\to 0$ induces an exact sequence of complex vector spaces

$$0\to\sigma^*F\to\sigma^*E\to\sigma^*G\to 0.$$

We endow $\sigma^* F$ and $\sigma^* G$ with their natural Hermitian forms deduced from h_σ: that on $\sigma^* F$ is actually the restriction of h_σ, while that on $\sigma^* G$ is deduced from the restriction of h_σ to the orthogonal complement of $\sigma^* F$ and the identification of this complement to $\sigma^* G$.

From the point of view of hermitian norms, rather than forms, observe that the hermitian norm $\|\cdot\|_{F,\sigma}$ on $\sigma^* F$ is deduced from the norm $\|\cdot\|_\sigma$ by restriction, while the hermitian norm on $\sigma^* G$ is the quotient norm: for any vector $v \in \sigma^* G$, $\|v\|_{G,\sigma}$ is the least norm of a vector $e \in \sigma^* E$ mapping to v.

We write $\overline{F} = (F, h_F)$ and $\overline{G} = (G, h_G)$ for the so-defined Hermitian vector bundle (coherent sheaf) on S. (If no confusion can arise, we shall allow ourselves to omit the indexes F and G in h_F and h_G.) We say that $\overline{F}$ is a Hermitian submodule of $\overline{E}$ and $\overline{G}$ is a quotient of $\overline{E}$. If, moreover, G is torsion free, then $\overline{G}$ is a Hermitian vector bundle on S; in that case, we say that $\overline{F}$ is a Hermitian subbundle of $\overline{E}$.

B.2. Direct sum. — Let $\overline{E_1} = (E_1, h_1)$ and $\overline{E_2} = (E_2, h_2)$ be hermitian vector bundles (coherent sheaf) over S. The module $E = E_1 \oplus E_2$ then gives rise to a hermitian vector bundle (coherent sheaf) (E, h) by letting h_σ to be the orthogonal direct sum of the forms $h_{1,\sigma}$ and $h_{2,\sigma}$; in other words, for any $\sigma \in \Sigma$, one has

$$h_\sigma(v_1 \oplus v_2, v_1' \oplus v_2') = h_{1,\sigma}(v_1, v_1') + h_{2,\sigma}(v_2, v_2'), \qquad v_1, v_1' \in \sigma^* E_1, \quad v_2, v_2' \in \sigma^* E_2.$$

We write $\overline{E} = \overline{E_1} \overset{\perp}{\oplus} \overline{E_2}$, or even $\overline{E} = \overline{E_1} \oplus \overline{E_2}$ if no confusion can arise. One has $\operatorname{rank}(\overline{E}) = \operatorname{rank}(\overline{E_1}) + \operatorname{rank}(\overline{E_2})$.

The hermitian vector bundles (coherent sheaves) $\overline{E_1}$ and $\overline{E_2}$ can both be seen, either as submodules, or as quotients, of their orthogonal direct sum.

B.3. Tensor products. — Let $\overline{E_1} = (E_1, h_1)$ and $\overline{E_2} = (E_2, h_2)$ be hermitian vector bundles (coherent sheaf) over S. The module $E = E_1 \otimes_{\mathfrak{o}_K} E_2$ then gives rise to a hermitian vector bundle (coherent sheaf) (E, h) by letting, for any $\sigma \in \Sigma$, h_σ to be the natural hermitian form on the tensor product $\sigma^* E = \sigma^* E_1 \otimes_{\mathbf{C}} \sigma^* E_2$.

One writes $\overline{E} = \overline{E_1} \otimes \overline{E_2}$; its rank is given by $\operatorname{rank}(\overline{E}) = \operatorname{rank}(\overline{E_1}) \operatorname{rank}(\overline{E_2})$.

More generally, all classical tensor constructions in linear algebra can be extended in a natural way to the Hermitian framework, but some conventions have to be chosen. We just quote the case of symmetric and alternate products.

Let $\overline{E} = (E, h)$ be a hermitian vector bundle (coherent sheaf) on S. Let p be a non-negative integer. The symmetric and alternate products, $\operatorname{Sym}^p E$ and $\operatorname{Alt}^p E$, are defined by a universal property: they are the target of a p-linear morphism from E^p such that, for any $\mathfrak{o}_K$-module F, $\operatorname{Hom}(\operatorname{Sym}^p E, F)$ and $\operatorname{Hom}(\operatorname{Alt}^p E, F)$ are respectively the sets of symmetric and alternate p-linear morphisms from E^p to F. The p-linear maps $E^p \to \operatorname{Sym}^p E$ and $E^p \to \operatorname{Alt}^p E$ give rise to canonical morphisms $E^{\otimes p} \to \operatorname{Sym}^p E$ and $E^{\otimes p} \to \operatorname{Alt}^p E$. These morphisms are surjective.

The Hermitian norms on $\sigma^*(\operatorname{Sym}^p E) = \operatorname{Sym}^p(\sigma^* E)$ and $\sigma^*(\operatorname{Alt}^p E) = \operatorname{Alt}^p(\sigma^* E)$ are defined as in the appendix. This defines canonical Hermitian vector bundles (coherent sheaves) $\operatorname{Sym}^p \overline{E}$ and $\operatorname{Alt}^p \overline{E}$.

If $\overline{E}$ is a Hermitian vector bundle of rank N, one also writes $\det(\overline{E})$ for $\mathrm{Alt}^N \overline{E}$. [1]

B.4. Homomorphisms. — Let $\overline{E_1} = (E_1, h_1)$ and $\overline{E_2} = (E_2, h_2)$ be hermitian vector bundles (coherent sheaf) over S. The module $E = \mathrm{Hom}(E_1, E_2)$ of $\mathfrak{o}_K$-linear homomorphisms from E_1 to E_2 then gives rise to a hermitian vector bundle (coherent sheaf) (E, h) by letting, for any $\sigma \in \Sigma$, h_σ to be the natural Hermitian form on the tensor product $\sigma^* E = \mathrm{Hom}_{\mathbf{C}}(\sigma^* E_1, \sigma^* E_2)$. One has $\mathrm{rank}(\overline{E}) = \mathrm{rank}(\overline{E_1})\,\mathrm{rank}(\overline{E_2})$.

In particular, for any hermitian vector bundle (coherent sheaf) $\overline{E} = (E, h)$ over S, the module $E^\vee = \mathrm{Hom}_{\mathfrak{o}_K}(E, \mathfrak{o}_K)$ is endowed with a natural structure of hermitian vector bundle $\overline{E}^\vee$. One has $\mathrm{rank}(\overline{E}^\vee) = \mathrm{rank}(\overline{E})$.

C. Canonical isometries

The classical isomorphism

$$\mathrm{Alt}^k(E \oplus F) \simeq \bigoplus_{i=0}^{k} \mathrm{Alt}^i(E) \otimes \mathrm{Alt}^{k-i}(F)$$

gives rise to an isomorphism of hermitian vector bundles

$$\mathrm{Alt}^k(\overline{E} \overset{\perp}{\oplus} \overline{F}) \simeq \bigoplus_{i=0}^{k} \mathrm{Alt}^i(\overline{E}) \otimes \mathrm{Alt}^{k-i}(\overline{F}).$$

When $\overline{E}$ and $\overline{F}$ are hermitian vector bundles, one has a canonical isometry

$$\det(\overline{E} \oplus \overline{F}) \simeq \det(\overline{E}) \otimes \det(\overline{F}).$$

More generally, let $\overline{E}, \overline{F}, \overline{G}$ be hermitian vector bundles, related by an exact sequence $0 \to F \to E \to G \to 0$. We assume that for any embedding $\sigma \in \Sigma$, the hermitian norms on $\overline{F}$ and $\overline{G}$ are the induced and quotient norms respectively. Then, there is a canonical isomorphism of $\mathfrak{o}_K$-modules $\det(E) \simeq \det(F) \otimes \det(G)$ and it induces an isometry of $\det(\overline{E})$ with $\det(\overline{F}) \otimes \det(\overline{G})$.

Let $\overline{E_1} = (E_1, h_1)$ and $\overline{E_2} = (E_2, h_2)$ be hermitian vector bundles (coherent sheaf) over S. Assume that E_1 is locally free. Observe that the natural isomorphism of $\mathfrak{o}_K$-modules:

$$E_1^\vee \otimes E_2 \to \mathrm{Hom}(E_1, E_2)$$

is an isometry.

§ 1.2
ARITHMETIC DEGREE

A. The Arakelov Picard group

Equivalence classes of hermitian line bundles on S form a set $\widehat{\mathrm{Pic}}(S)$. The tensor product induces a group law on this set, the inverse is induced by the dual.

[1] Definir le determinant dans le cas cohérent? Ou carrément le déterminant d'un complexe?

PROPOSITION 1.2.1. *There is an exact sequence of Abelian groups:*

$$1 \to \mu_K \to \mathfrak{o}_K^* \xrightarrow{\log_\Sigma} \left(\mathbf{R}^\Sigma\right)^{F_\infty} \to \widehat{\operatorname{Pic}}(S) \to \operatorname{Pic}(S) \to 0.$$

Proof. — The injection $\mu_K \hookrightarrow \mathfrak{o}_K^*$ is the obvious one.

The map $\log_\Sigma$ associates to any element $a \in \mathfrak{o}_K^*$ the family $(\log|\sigma(a)|)_{\sigma\in\Sigma}$. It is a morphism of Abelian groups.

The kernel of $\log_\Sigma$ contains the roots of unity of K. Let us show the opposite inclusion. An element $a \in \ker(\log_\Sigma)$ is in particular an algebraic integer $a \in \mathfrak{o}_K$ be such that $|\sigma(a)| = 1$ for any $\sigma \in \Sigma$. By the finiteness lemma 1.2.2 below, $\ker(\log_\Sigma)$ is finite. Its elements are torsion, so are roots of unity.

The map $(\mathbf{R}^\Sigma)^{F_\infty} \to \widehat{\operatorname{Pic}}(S)$ associates to a family (u_σ) the hermitian line bundle $\overline{L}(u) = (L, h)$ on S defined as follows: L is the trivial line bundle $\mathfrak{o}_K$ on S, and it is endowed with the hermitian metric (h_σ) characterized by $h_\sigma(1) = \exp(-u_\sigma)$. (In other words, $h_\sigma(a\otimes z) = \exp(-u_\sigma)\,|\sigma(a)|\,|z|$.) If (u_σ) is F_∞-invariant, that is, if $u_{\overline{\sigma}} = u_\sigma$ for any $\sigma \in \Sigma$, then so is (h_σ). The so-defined map is obviously a morphism of Abelian groups.

Let $u = (u_\sigma)$ belong to its kernel. In other words, there is an isomorphism $\overline{L}(0) \simeq \overline{L}(u)$. Such an isomorphism is induced by an automorphism ε of $\mathfrak{o}_K$ such that $|\sigma(\varepsilon(x))| = \exp(-u_\sigma)\,|\sigma(x)|$ for any $x \in L$. There exists $a \in \mathfrak{o}_K^*$ such that $\varepsilon(x) = ax$ for any $x \in L$, hence $|\sigma(a)| = \exp(-u_\sigma)$ for any $\sigma \in \Sigma$, whence $u = -\log_\Sigma(a)$. The kernel of the morphism $u \mapsto \overline{L}(u)$ is therefore contained in the image of $\log_\Sigma$ and the same computation establishes the opposite inclusion.

Finally, the morphism $\widehat{\operatorname{Pic}}(S) \to \widehat{\operatorname{Pic}}(S)$ simply forgets the hermitian metrics. It is surjective because any line bundle can be endowed with an hermitian metric. Its kernel consists of hermitian line bundles whose underlying line bundle is the trivial line bundle. Any such line bundle is of the form $\overline{L}(u)$ for some $u \in (\mathbf{R}^\Sigma)^{F_\infty}$, □

LEMMA 1.2.2 (Finiteness lemma). *Let d be a positive integer and B be a positive real number. The set of algebraic integers of degree $\leqslant d$ and all of whose complex conjugates have absolute value $\leqslant B$ is finite.*

Proof. — If a is an algebraic integer of degree d, let P_a be its minimal polynomial; it is a polynomial with rational coefficients. Since we are in characteristic zero, a is separable and $P_a = (X - a_1)\dots(X - a_d)$, where $a_1, \dots, a_d$ are the complex conjugates of a. Since a is an algebraic integer, the coefficients of P_a are algebraic integers; consequently, they are rational integers. Moreover, the coefficient of X^k in P_a is bounded by $\binom{d}{d-k}B^k$. This shows that the polynomial P_a belongs to a finite list of polynomials in $\mathbf{Z}[X]$, independent of a. As a consequence, the set of such a is finite. □

B. An arithmetic Chow group

By definition, the group $\mathrm{Z}^1(S)$ of (Weil) divisors on S is the free Abelian group generated by the set of integral codimension 1 subschemes of S. Such subschemes are in bijection with maximal ideals of $\mathfrak{o}_K$, the ideal $\mathfrak{p}$ corresponding to the subscheme $[\mathfrak{p}] := \operatorname{Spec}(\mathfrak{o}_K/\mathfrak{p})$.

For any maximal ideal $\mathfrak{p}$, the order function $\mathrm{ord}_{\mathfrak{p}}\colon K^* \to \mathbf{Z}$ is the unique group homomorphism such that

$$\mathrm{ord}_{\mathfrak{p}}(a) = \mathrm{length}(\mathfrak{o}_{K,\mathfrak{p}}/(a))$$

for any non-zero $a \in \mathfrak{o}_K$, where length$(\cdot)$ is the length of module. In fact, $\mathfrak{o}_K$ being a Dedekind ring, $\mathrm{ord}_{\mathfrak{p}}$ is a valuation on K.

For any non-zero $a \in K = \kappa(S)$, we now define its divisor $\mathrm{div}(a)$ by the formula

$$\mathrm{div}(a) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(a)[\mathfrak{p}].$$

It is indeed an element of $\mathrm{Z}^1(S)$ and the induced map $\mathrm{div}\colon K^* \to \mathrm{Z}^1(S)$ is a group homomorphism. We set $\mathrm{Rat}^1(S) = \mathrm{div}(K^*)$ and $\mathrm{CH}^1(S) = \mathrm{Z}^1(S)/\mathrm{Rat}^1(S)$.

By definition, the group $\widehat{\mathrm{Z}}^1(S)$ of arithmetic divisors on S is the direct sum $\mathrm{Z}^1(S) \oplus (\mathbf{R}^{\Sigma})^{F_\infty}$. If $a \in K^*$, we define

$$\widehat{\mathrm{div}}(a) = (\mathrm{div}(a), (-\log|\sigma(a)|)_{\sigma\in\Sigma}) \in \widehat{\mathrm{Z}}^1(S).$$

The map $\widehat{\mathrm{div}}$ is a morphism of Abelian groups from K^* to $\widehat{\mathrm{Z}}^1(S)$; let $\widehat{\mathrm{Rat}}^1(S)$ be its image; the quotient group $\widehat{\mathrm{CH}}^1(S) = \widehat{\mathrm{Z}}^1(S)/\widehat{\mathrm{Rat}}^1(S)$ is called the first arithmetic Chow group of S.

The degree map $\widehat{\deg}\colon \widehat{\mathrm{CH}}^1(S) \to \mathbf{R}$ is defined as follows. First, for $(D, g) \in \widehat{\mathrm{Z}}^1(S)$, one sets

$$\widehat{\deg}(D, g) = \sum_{\mathfrak{p}\in\mathrm{Spm}(\mathfrak{o}_K)} n_{\mathfrak{p}} \log \mathrm{card}(\mathfrak{o}_K/\mathfrak{p}) + \sum_{\sigma\in\Sigma} g_\sigma.$$

Let us prove that for any $a \in K^*$, $\widehat{\deg}(\mathrm{div}(a)) = 0$. It suffices to treat the case of non-zero elements in $\mathfrak{o}_K$. For such an element a, one has

$$\widehat{\deg}(\widehat{\mathrm{div}}(a)) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(a) \log \mathrm{card}(\mathfrak{o}_K/\mathfrak{p}) - \sum_{\sigma\in\Sigma} \log|\sigma(a)|.$$

Moreover,

$$\mathrm{ord}_{\mathfrak{p}}(a) \log \mathrm{card}(\mathfrak{o}_K/\mathfrak{p}) = \mathrm{length}(\mathfrak{o}_{K,\mathfrak{p}}/(a)) \log \mathrm{card}(\mathfrak{o}_K/\mathfrak{p}) = \log \mathrm{card}(\mathfrak{o}_{K,\mathfrak{p}}/(a)).$$

Since the $\mathfrak{o}_K$-module $\mathfrak{o}_K/(a)$ has finite support, one has an isomorphism

$$\mathfrak{o}_K/(a) \simeq \bigoplus_{\mathfrak{p}} \mathfrak{o}_{K,\mathfrak{p}}/(a),$$

from which we deduce that

$$\sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(a) \log \mathrm{card}(\mathfrak{o}_K/\mathfrak{p}) = \log \mathrm{card}(\mathfrak{o}_K/(a)) = \log \mathrm{N}((a))$$

is the logarithm of the norm of the ideal (a).

Let us then observe that $\mathfrak{o}_K/(a)$ is the cokernel of the "multiplication-by-a" map μ_a in $\mathfrak{o}_K$, which is a free $\mathbf{Z}$-module of finite rank. By the theory of elementary divisors, one has $\mathrm{card}(\mathfrak{o}_K/(a)) = \left|\det(\mu_a))\right|$. Moreover, when tensoring with $\mathbf{C}$, one has $\mathfrak{o}_K \otimes_{\mathbf{Z}} \mathbf{C} \simeq \bigoplus_{\sigma\in\Sigma} \mathbf{C}$, and the map μ_a is identified with the linear map given by the multiplication by $\sigma(a)$ on the factor indexed by σ. Consequently, $\det(\mu_a) = \prod_{\sigma\in\Sigma} \sigma(a)$. These equalities imply that $\widehat{\deg}(\widehat{\mathrm{div}}(a)) = 0$, as was to be shown.

It follows that $\widehat{\deg}$ induces a map from $\widehat{\mathrm{CH}}^1(S)$ to $\mathbf{R}$.

Example 1.2.3. If $K = \mathbf{Q}$, then $\widehat{\deg}$ is an isomorphism.

C. Arithmetic Chern class

Let $\overline{L} = (L, h)$ be an hermitian line bundle on S. For any maximal ideal $\mathfrak{p}$, $L_{\mathfrak{p}} = L \otimes \mathfrak{o}_{K,\mathfrak{p}}$ is a free $\mathfrak{o}_{K,\mathfrak{p}}$-module of rank 1. This allows to define $\operatorname{ord}_{\mathfrak{p}}(\ell)$, if ℓ is a non-zero element in $L \otimes K$: it is the unique element of $\mathbf{Z}$ equal to $\operatorname{ord}_{\mathfrak{p}}(a)$, for any basis ε of $L_{\mathfrak{p}}$ and any $a \in K^*$ such that $\ell = a\varepsilon$. The divisor $\operatorname{div}(\ell)$ of a non-zero $\ell \in L \otimes K$ is then defined as the element $\sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(\ell)$ of $\widehat{\mathrm{Z}}^1(S)$. We moreover set $\widehat{\operatorname{div}}(\ell) = (\operatorname{div}(\ell), (-\log \|\ell\|_{\sigma})_{\sigma \in \Sigma})$.

The image of $\widehat{\operatorname{div}}(\ell)$ in $\widehat{\mathrm{CH}}^1(S)$ does not depend on the choice of ℓ; we denote it by $\widehat{c}_1(\overline{L})$ and call it the first arithmetic Chern class of $\overline{L}$. It only depends on the isometry class of $\overline{L}$.

PROPOSITION 1.2.4. *The map $\widehat{c}_1 \colon \widehat{\mathrm{Pic}}(S) \to \widehat{\mathrm{CH}}^1(S)$ is an isomorphism of groups.*

Proof. — Let $\overline{L}$ and $\overline{M}$ be hermitian line bundles on S, let ℓ and m be non-zero elements in $L \otimes K$ and $M \otimes K$. We observe that $\widehat{\operatorname{div}}(\ell \otimes m) = \widehat{\operatorname{div}}(\ell) + \widehat{\operatorname{div}}(m)$; this implies that $\widehat{c}_1$ is a morphism of Abelian groups.

Let $\overline{L} \in \widehat{\mathrm{Pic}}(S)$ be such that $\widehat{c}_1(\overline{L}) = 0$. Let ℓ be any nonzero element in $L \otimes K$; by assumption, $\widehat{\operatorname{div}}(\ell) \in \widehat{\mathrm{Rat}}^1(S)$, so that there exists $a \in K^*$ such that $\widehat{\operatorname{div}}(\ell) = \widehat{\operatorname{div}}(a)$; replacing ℓ by $a^{-1}\ell$, we may assume that $\widehat{\operatorname{div}}(\ell) = 0$. As a consequence, $\operatorname{div}(\ell) = 0$: the element ℓ has neither zeroes, nor poles, so is a basis of L, hence induces an isomorphism $\varphi \colon \mathfrak{o}_K \simeq L$. Moreover, $\|\ell\|_{\sigma} = 1$ for any $\sigma \in \Sigma$, so that φ is an isometry from the trivial hermitian line bundle to $\overline{L}$. This shows that $\widehat{c}_1$ is injective.

Let us now establish its surjectivity. Let (D, g) be any element in $\widehat{\mathrm{Z}}^1(S)$; let $L = \mathscr{O}_S(D)$ be the fractional ideal consisting (besides 0) of elements $a \in K^*$ such that $\operatorname{div}(a) + D \geqslant 0$ is a divisor with nonnegative coefficients. The injection $L \subset K$ extends to a canonical isomorphism $L \otimes K \simeq K$; let ℓ be the element of $L \otimes K$ which maps to 1. Since $\mathfrak{o}_K$ is a Dedekind ring, D is a Cartier divisor and one has $\operatorname{div}(\ell) = D$. For any $\sigma \in \Sigma$, there is a unique hermitian metric h_{σ} on $L \otimes_{\sigma} \mathbf{C}$ such that $\|\ell\|_{\sigma} = \exp(-g_{\sigma})$. Since g belongs to $(\mathbf{R}^{\Sigma})^{F_{\infty}}$, the family $h = (h_{\sigma})$ is invariant under F_{∞} and $\overline{L} = (L, h)$ is a hermitian line bundle on S. Moreover, $\widehat{\operatorname{div}}(\ell) = (D, g)$, hence the desired surjectivity. □

We shall also denote by $\widehat{\deg}(\overline{L})$ the arithmetic degree of $\widehat{c}_1(\overline{L})$.

LEMMA 1.2.5. *For any non zero $\ell \in L$, the $\mathfrak{o}_K$-module $L/\mathfrak{o}_K\ell$ is finite and*

$$\widehat{\deg}(\overline{L}) = \log \operatorname{card}(L/\mathfrak{o}_K\ell) - \sum_{\sigma \in \Sigma} \log \|\ell\|_{\sigma}.$$

Proof. — By definition of the maps $\widehat{c}_1$ and $\widehat{\deg}$,

$$\widehat{\deg}(\overline{L}) = \widehat{\deg}(\widehat{\operatorname{div}}(\ell)) = \sum_{\mathfrak{p}} \operatorname{length}(L_{\mathfrak{p}}/\mathfrak{o}_{K,\mathfrak{p}}\ell) \log \operatorname{card}(\mathfrak{o}_K/\mathfrak{p}) - \sum_{\sigma} \log \|\ell\|_{\sigma}.$$

It thus remains to show that the first part of this sum is equal to $\log \operatorname{card}(L/\mathfrak{o}_K\ell)$. Since $L/\mathfrak{o}_K\ell$ has finite support, one has

$$L/\mathfrak{o}_K\ell \simeq \bigoplus_{\mathfrak{p}} L_{\mathfrak{p}}/\mathfrak{o}_{K,\mathfrak{p}}\ell$$

hence

$$\log\operatorname{card}(L/\mathfrak{o}_K\ell) = \sum_{\mathfrak{p}} \log\operatorname{card}(L_{\mathfrak{p}}/\mathfrak{o}_{K,\mathfrak{p}}\ell) = \sum_{\mathfrak{p}} \operatorname{length}(L_{\mathfrak{p}}/\mathfrak{o}_{K,\mathfrak{p}}\ell)\log\operatorname{card}(\mathfrak{o}_K/\mathfrak{p}).$$

The proposition is proved. □

Example 1.2.6. The arithmetic degree of the hermitian line bundle $\mathfrak{o}_K$ is zero. Let I be an ideal in $\mathfrak{o}_K$, and let us view is a Hermitian line bundle by restriction. One has $\widehat{\deg}(I) = -\log \mathrm{N}(I)$.

D. Degree of hermitian vector bundles

Let $\overline{E} = (E,h)$ be an hermitian coherent sheaf on S. Let $T_E \subset E$ be the torsion subgroup of E; the quotient E/T_E is locally free, hence is a vector bundle on S. Moreover, the projection $E \to E/T_E$ induces an isomorphism of $\sigma^* E$ with $\sigma^*(E/T_E)$, for any $\sigma \in \Sigma$. We endow E/T_E with hermitian forms via these isomorphisms and let $\overline{E/T_E}$ be the corresponding hermitian vector bundle on S.

DEFINITION 1.2.7. *We define the arithmetic degree of $\overline{E}$ by the formula:*

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\det(\overline{E/T_E})) + \log\operatorname{card}(T_E).$$

PROPOSITION 1.2.8. *Let $0 \to \overline{F} \xrightarrow{i} \overline{E} \xrightarrow{p} \overline{G} \to 0$ be an exact sequence of hermitian coherent sheaves on S. Then,*

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{F}) + \widehat{\deg}(\overline{G}).$$

Proof. — Let T_E, T_F, T_G be the torsion subgroups of E,F,G. We first prove a series of particular cases.

(I). *Assume that E, F and G are torsion.* Then $\widehat{\deg}(\overline{E}) = \log\operatorname{card}(E)$, etc., and the result follows from the multiplicativity of cardinalities in exact sequences.

(II). *Assume $\overline{E}$, $\overline{F}$ and $\overline{G}$ are locally free.* Then, the proposition follows from the existence of an isometry $\det(\overline{E}) \simeq \det(\overline{F}) \otimes \det(\overline{G})$, and the behaviour of $\widehat{\deg}$ for a tensor product.

(III). *Assume that $\overline{E}$ is locally free and $\overline{G}$ is a torsion module.* Then, E and F have the same rank, say n, and the $\mathfrak{o}_K$-module E/F is a finite Abelian group. Let $(e_1,\dots,e_n)$ be a family of linearly independant elements in F; let us pose $e = e_1 \wedge \dots \wedge e_n$, viewed as an element of $\det(F) \subset \det(E)$. One has

$$\widehat{\deg}(\overline{F}) = \log\operatorname{card}(\det(F)/e\mathfrak{o}_K) - \sum_{\sigma} \log \|e\|_{\sigma}$$

and

$$\widehat{\deg}(\overline{E}) = \log\operatorname{card}(\det(E)/e\mathfrak{o}_K) - \sum_{\sigma} \log \|e\|_{\sigma}.$$

Consequently,

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{F}) + \log\operatorname{card}(\det(E)/\det(F)).$$

It thus remains to show that $\det(E)/\det(F)$ is an Abelian group of cardinality $\operatorname{card}(G)$. It suffices to prove that for any maximal ideal $\mathfrak{p}$, the localizations $(\det(E)/\det(F))_{\mathfrak{p}}$ and

$G_{\mathfrak{p}}$ have the same cardinality. This follows in turn from the theorem on elementary divisors in the discrete valuation ring $\mathfrak{o}_{K,\mathfrak{p}}$.

(IV). *Assume that G is a torsion module.* From the exact sequence $0 \to \overline{F}/T_F \to \overline{E}/T_E \to T_G/p(T_E) \to 0$ and case (III) above, we obtain

$$\begin{aligned}\widehat{\deg}(\overline{E}) &= \widehat{\deg}(\overline{E}/T_E) + \log\operatorname{card}(T_E)\\ &= \widehat{\deg}(\overline{F}/T_F) + \log\operatorname{card}(T_G/p(T_E)) + \log\operatorname{card}(T_E)\\ &= \widehat{\deg}(\overline{F}) + \log\frac{\operatorname{card}(T_G)\operatorname{card}(T_E)}{\operatorname{card}(T_F)\operatorname{card}p(T_E)}.\end{aligned}$$

Using the exact sequence $0 \to T_F \to T_E \to p(T_E) \to 0$ and applying Case (I), we see that

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{F}) + \log\operatorname{card}(T_G).$$

(V). *Assume that $\overline{G}$ is locally free.* Then, $T_E = T_F$ and we have an exact sequence,

$$0 \to \overline{F}/T_F \to \overline{E}/T_E \to \overline{G} \to 0,$$

so that

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{E}/T_E)+\log\operatorname{card}(T_E) = \widehat{\deg}(\overline{G})+\widehat{\deg}(\overline{F}/T_F)+\log\operatorname{card}(T_F) = \widehat{\deg}(\overline{G})+\widehat{\deg}(\overline{F}).$$

The second equality follows from Case (II), the other two from the definition of the arithmetic degree.

We now prove the general case. Let $E' = p^{-1}(T_G)$, and let us endow it with the induced metric. One has $p(T_E) \subset T_G$, hence $T_E \subset E'$; moreover, $T_E \cap F = T_F$. By case (IV) above, the exact sequence $0 \to \overline{F} \to \overline{E'} \to T_G \to 0$ implies that

$$\widehat{\deg}(\overline{E'}) = \widehat{\deg}(\overline{F}) + \log\operatorname{card}(T_G).$$

Observe that E/E' is torsion free and that $F \subset E'$. Quotienting by E', we thus obtain an isomorphism

$$\overline{E/E'} \simeq \overline{G}/T_G$$

of Hermitian vector bundles over S. Consequently,

$$\widehat{\deg}(\overline{E/E'}) = \widehat{\deg}(\overline{G}/T_G) = \widehat{\deg}(\overline{G}) - \log\operatorname{card}(T_G).$$

Moreover, Case (V) implies that

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{E'}) + \widehat{\deg}(\overline{E/E'}),$$

so that

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{F}) + \log\operatorname{card}(T_G) + \widehat{\deg}(\overline{G}) - \log\operatorname{card}(T_G) = \widehat{\deg}(\overline{F}) + \widehat{\deg}(\overline{G}).$$

This concludes the proof. □

PROPOSITION 1.2.9. *Let $\overline{E}$ and $\overline{F}$ be Hermitian vector bundles over S. One has* $\widehat{\deg}(\overline{E} \otimes \overline{F}) = \operatorname{rank}(\overline{F})\,\widehat{\deg}(\overline{E}) + \operatorname{rank}(\overline{E})\,\widehat{\deg}(\overline{F})$.

Proof. — This follows from the existence of an isometry

$$\det(\overline{E}\otimes\overline{F}) \simeq \widehat{\deg}(\overline{E})^{\otimes\operatorname{rank}(\overline{F})}\otimes\widehat{\deg}(\overline{F})^{\otimes\operatorname{rank}(\overline{E})}.$$

□

Example 1.2.10. Assume $K=\mathbf{Q}$. Let $\overline{E}=(E,h)$ be an hermitian vector bundle over $\operatorname{Spec}\mathbf{Z}$. Let $(e_1,\dots,e_N)$ be a basis of E — it exists since any finitely generated projective $\mathbf{Z}$-module is free. Then, $e=e_1\wedge\dots\wedge e_N$ is a basis of $\operatorname{Alt}^N(E)$. It follows that

$$\widehat{\deg}(\overline{E})=\log\operatorname{card}(\operatorname{Alt}^N(E)/\mathbf{Z}e)-\log\|e\|=-\log\|e\|=-\frac{1}{2}\log\det\left(h(e_i,e_j)\right).$$

Conversely, let $(f_1,\dots,f_N)$ be an orthonormal basis of $E_{\mathbf{R}}$. It induces an isomorphism $E_{\mathbf{R}}\to\mathbf{R}^N$ by which E is identified with a lattice in $\mathbf{R}^N$. We write $\operatorname{covol}(E_{\mathbf{R}}/E)$ for the volume of a fundamental domain of this lattice, for the Euclidean Lebesgue measure on $E_{\mathbf{R}}$, *i.e.*, the Haar measure normalized so that the unit cube $\sum[0,1]f_i$ has volume 1.

For any i, write $e_i=\sum a_{ij}f_j$. One has $h(e_i,e_j)=\sum_k a_{ik}a_{jk}$, hence $\det(h(e_i,e_j))=\det({}^{\mathrm{t}}AA)=\det(A)^2$, where $A=(a_{ij})$. This implies that $\widehat{\deg}(\overline{E})=-\log\operatorname{covol}(E_{\mathbf{R}}/E)$.

§ 1.3
THE RELATIVE RIEMANN–ROCH THEOREM

A. Direct and inverse images for Hermitian vector bundles

Let $K\subset K'$ be an inclusion of number fields, let $S=\operatorname{Spec}\mathfrak{o}_K$, $S'=\operatorname{Spec}\mathfrak{o}_{K'}$ and $\pi\colon S'\to S$ be the morphism induced by the inclusion $\mathfrak{o}_K\subset\mathfrak{o}_{K'}$. It is finite, locally free of degree $[K':K]$.

Let us attach to π natural maps

$$\pi_*\colon\widehat{\operatorname{Vect}}(S')\to\widehat{\operatorname{Vect}}(S),\quad \pi^*\colon\widehat{\operatorname{Vect}}(S)\to\widehat{\operatorname{Vect}}(S').$$

We shall also define the norm $\mathrm{N}_{S'/S}(\overline{L})$ of a hermitian line bundle $\overline{L}$ over S'.

A.1. Inverse images of hermitian vector bundles. — We begin with π^*. Let $\overline{E}=(E,h)$ be an hermitian vector bundle (coherent sheaf) on S; let $N=\operatorname{rank}(\overline{E})$. Then $E'=\pi^*E=E\otimes_{\mathfrak{o}_K}\mathfrak{o}_{K'}$ is a vector bundle (coherent sheaf) on S' of rank $\operatorname{rank}(\overline{E})$. The restriction to K of any embedding $\sigma'\colon K'\hookrightarrow\mathbf{C}$ is an embedding σ of K into $\mathbf{C}$ and $(\sigma')^*E'=E'\otimes_{\mathfrak{o}_{K'},\sigma'}\mathbf{C}$ is canonically isomorphic to $E\times_{\mathfrak{o}_K,\sigma}\mathbf{C}$. We use this isomorphism to define a hermitian $h'_{\sigma'}$ form on $(\sigma')^*E'$. Then, the pair (E',h') is a hermitian vector bundle (coherent sheaf) on S'.

PROPOSITION 1.3.1. *The construction π^* commutes with the constructions of direct sums, tensor products, symmetric and alternate products on $\widehat{\operatorname{Vect}}(S)$ and $\widehat{\operatorname{Vect}}(S')$.*

For any hermitian vector bundle on S, one has $\operatorname{rank}\pi^*\overline{E}=\operatorname{rank}\overline{E}$ and $\widehat{\deg}\pi^*\overline{E}=[K'K]\,\widehat{\deg}\,\overline{E}$.

Proof. — Let us write $d = [K' : K]$. It suffices to prove that $\widehat{\deg}\pi^*\overline{L} = d\,\widehat{\deg}\,\overline{L}$ for any hermitian coherent sheaf of rank one $\overline{L}$ on S. Let T be the torsion subgroup of L. Since $\mathfrak{o}_{K'}$ is locally free over $\mathfrak{o}_K$, $\pi^*T = T\otimes\mathfrak{o}_{K'}$ is the torsion subgroup of $L\otimes\mathfrak{o}_{K'}$ and $(L/T)\otimes\mathfrak{o}_{K'}$ identifies with π^*L/π^*T.

Let ℓ be any nonzero element of L. By definition

$$\widehat{\deg}(\overline{L}) = \log\operatorname{card}(T) + \log\operatorname{card}(L/(T+\mathfrak{o}_K\ell)) - \sum_{\sigma\in\Sigma}\log\|\ell\|_\sigma.$$

Since the support $\operatorname{supp}(T)$ of T is finite, there is on open neighborhood U of $\operatorname{supp}(T)$ in S such that π is free of rank d over U. This implies that as an $\mathfrak{o}_K$-module, $\pi^*T \simeq T^d$. In particular, $\log\operatorname{card}(\pi^*T) = \log\operatorname{card}(T^d) = d\log\operatorname{card}(T)$. Similarly,

$$\log\operatorname{card}(\pi^*L/\pi^*T + \mathfrak{o}_{K'}\ell) = \log\operatorname{card}\pi^*(L/T+\mathfrak{o}_K\ell) = d\log\operatorname{card}(L/T+\mathfrak{o}_K\ell).$$

Finally, any embedding σ of K into $\mathbf{C}$ is the restriction to σ of exactly d embeddings of K' into $\mathbf{C}$. By the definition of h', this implies that

$$\sum_{\sigma'\in\Sigma'}\log h'_{\sigma'}(\ell\otimes 1) = \sum_{\sigma\in\Sigma} d\log h_\sigma(\ell).$$

This gives the desired formula. □

A.2. Direct images of hermitian vector bundles. — Let now $\overline{E}' = (E', h)$ be an hermitian vector bundle (coherent sheaf) on S'. We define a hermitian vector bundle (coherent sheaf) (E, h) on S as follows. First of all, the module $E = \pi_*E'$ is defined to be E', viewed as an $\mathfrak{o}_K$-module. It is of finite type since $\mathfrak{o}_{K'}$ is itself of finite type over $\mathfrak{o}_K$. If E' is locally free of rank N, then E is locally free of rank dN.

LEMMA 1.3.1. *Let $\sigma\in\Sigma$ be an embedding of K into $\mathbf{C}$. As an algebra over $\mathfrak{o}_{K'}$, the tensor product $A = \mathfrak{o}_{K'}\otimes_{\mathfrak{o}_K,\sigma}\mathbf{C}$ is isomorphic to the algebra $\prod_{\sigma'|\sigma}\mathbf{C}$, where the product ranges over all embeddings $\sigma'\in\Sigma'$ which extend σ, the factor indexed by σ' being viewed as an $\mathfrak{o}_{K'}$-algebra via σ'.*

Proof. — It suffices to prove the analogous assertion where $\mathfrak{o}_K$ and $\mathfrak{o}_{K'}$ are replaced by K and K' respectively. Let $a\in K'$ be a primitive element and $P\in K[X]$ be its minimal polynomial. Then, $K' = K[a] \simeq K[X]/(P)$ and $K'\otimes_{K,\sigma}\mathbf{C} = \mathbf{C}[X]/(P^\sigma)$, where P^σ is the polynomial in $\mathbf{C}[X]$ obtained by applying σ to the coefficients of P. If $P^\sigma = \prod(X-a_i)$, then $a_1,\dots,a_d$ are the images of a by the embeddings of K' into $\mathbf{C}$ which extend σ. In other words, $K'\otimes_{K,\sigma}\mathbf{C} \simeq \bigoplus_{\sigma'|\sigma}\mathbf{C}$, where on the factor indexed by σ', $\mathbf{C}$ is viewed as a K'-algebra via σ'. □

Let $\sigma\in\Sigma$ be an embedding of K into $\mathbf{C}$. Then, $\sigma^*\pi_*E'$ is naturally isomorphic to the direct sum

$$\bigoplus_{\sigma'|\sigma}(\sigma')^*E'.$$

Indeed,

$$\sigma^*\pi_*E' = E'\otimes_{\mathfrak{o}_K,\sigma}\mathbf{C} = E'\otimes_{\mathfrak{o}_{K'}}\left(\mathfrak{o}_{K'}\otimes_{\mathfrak{o}_K,\sigma}\mathbf{C}\right)$$

and the isomorphism of Lemma 1.3.1 induces the stated decomposition. For $e = (e'_{\sigma'}) \in \sigma^*\pi_*E'$, we then define

$$\|e\|_\sigma^2 = \left\|(e'_{\sigma'})_{\sigma'|\sigma}\right\|_\sigma^2 = \sum_{\sigma'|\sigma} d_{\sigma'/\sigma}\left\|e'_{\sigma'}\right\|_{\sigma'}^2.$$

A.3. The norm of on Hermitian line bundle. — Let $\overline{L} = (L, h)$ be an hermitian line bundle over S'. We want to define the norm of $\overline{L}$ as an hermitian line bundle $\mathrm{N}_{S'/S}(\overline{L})$ over S, whose underlying line bundle is indeed the norm $\mathrm{N}_{S'/S}(L)$ of the line bundle L.

Let us recall the definition of this line bundle on S. Let (U_i) be an open cover of S such that $L|_{\pi^{-1}(U_i)}$ is trivial for each i; such an open cover exists, since π is finite. For each i, let us choose a section $\varepsilon_i \in \Gamma(\pi^{-1}(U_i), L)$ which generates L everywhere on $\pi^{-1}(U_i)$. Then, the family (f_{ij}) where, for each pair (i,j), $U_{ij} = U_i \cap U_j$ and $f_{ij} \in \Gamma(\pi^{-1}(U_{ij}), \mathscr{O}_{S'}^\times)$ is the unique invertible function on $\pi^{-1}(U_{ij})$ such that $\varepsilon_i = f_{ij}\varepsilon_j$, is a 1-cocycle which represents the line bundle L.

By definition, the line bundle $\mathrm{N}_{S'/S}(L)$ is represented by the cocycle $(\mathrm{N}_{S'/S}(f_{ij}))$ on S, relative to the open cover (U_i). It admits a canonical trivialization over U_i, whose generator is denoted $\mathrm{N}(\varepsilon_i)$. Over $U_i \cap U_j$, one has $\mathrm{N}(\varepsilon_i) = \mathrm{N}(f_{ij})\mathrm{N}(\varepsilon_j)$, whence the notation.

More generally, any section $s \in L$ possesses a norm $\mathrm{N}(s) \in \mathrm{N}_{S'/S}(L)$, defined in such a way that if $s = f\varepsilon_i$ over $\pi^{-1}(U_i)$, then $\mathrm{N}(s) = \mathrm{N}(f)\mathrm{N}(\varepsilon_i)$ over U_i.

Let us now endow $\mathrm{N}(L)$ with an hermitian metric. Let $\sigma \in \Sigma$. Then, $\sigma^*\mathrm{N}(L)$ is the norm of the line bundle $L \otimes_{\mathfrak{o}_K,\sigma} \mathbf{C}$ from $\mathfrak{o}'_K \otimes_{\mathfrak{o}_K,\sigma} \mathbf{C} = K' \otimes_{K,\sigma} \mathbf{C}$ to $\mathfrak{o}_K \otimes_\sigma \mathbf{C} = \mathbf{C}$, viewed as a K-algebra via σ. As we have seen, the $\mathbf{C}$-algebra $\mathfrak{o}_{K'} \otimes_{\mathfrak{o}_K,\sigma} \mathbf{C}$ is isomorphic to the product $\prod_{\sigma'|\sigma} \mathbf{C}$, where σ' ranges over all complex embeddings of K' which extend σ. This furnishes a canonical isomorphism

$$\sigma^*\mathrm{N}(L) = \bigotimes_{\sigma'|\sigma}((\sigma')^*L))$$

and allows to define a canonical hermitian metric on $\sigma^*\mathrm{N}(L)$ so that the previous isomorphism is an isometry.

We have not yet explained the direct image at the level of $\widehat{\mathrm{CH}}^1(S)$ but let us note the formula

$$\pi_*\widehat{c}_1(\overline{L}) = \widehat{c}_1(\mathrm{N}_{S'/S}(\overline{L}))$$

which extends the classical formula $\pi_*c_1(L) = c_1(\mathrm{N}_{S'/S}(L))$.

B. An arithmetic Grothendieck-Riemann-Roch theorem

THEOREM 1.3.1. *Let E be a hermitian vector bundle over S'. Then, there is a canonical isomorphism of hermitian line bundles over S:*

$$\det\pi_*\overline{E} \simeq \mathrm{N}_{S'/S}(\det\overline{E}) \otimes \det\pi_*(\mathfrak{o}_{K'}, 1)^{\operatorname{rank} E}.$$

Proof. — Let us fix n elements of E, say $e_1, \dots, e_n$, which are linearly independent over K, as well as d elements of $\mathfrak{o}_{K'}$, say $a_1, \dots, a_d$, which are linearly independant over K.

Let E' be the submodule of E generated by the e_i, and let E'' be the submodule of π_*E generated by the a_je_i. Then, $\widehat{c}_1(\det\overline{E})$ is represented by the arithmetic cycle

$$\widehat{\operatorname{div}}(e_1\wedge\cdots\wedge e_n)=\sum_{\mathfrak{p}'}\operatorname{length}_{\mathfrak{p}'}(E/E')[\mathfrak{p}']-\sum_{\sigma'\in\Sigma'}\frac{1}{2}\log\det\big(h_{\sigma'}(e_i,e_j)\big)[\sigma']$$

on S', while $\widehat{c}_1(\mathrm{N}_{S'/S}(\det\overline{E}))$ is represented by the arithmetic cycle

$$\begin{aligned}\pi_*\widehat{\operatorname{div}}(e_1\wedge\cdots\wedge e_n)&=\widehat{\operatorname{div}}\mathrm{N}(e_1\wedge\cdots\wedge e_n)\\&=\sum_{\mathfrak{p}}\left(\sum_{\mathfrak{p}'|\mathfrak{p}}\operatorname{length}_{\mathfrak{p}'}(E/E')[\kappa(\mathfrak{p}'):\kappa(\mathfrak{p})]\right)[\mathfrak{p}]\\&\quad-\sum_{\sigma\in\Sigma}\left(\sum_{\sigma'|\sigma}d_{\sigma'/\sigma}\frac{1}{2}\log\det\big(h_{\sigma'}(e_i,e_j)\big)\right)[\sigma]\end{aligned}$$

on S. Let us observe that for each maximal ideal $\mathfrak{p}$ of $\mathfrak{o}_K$, one has

$$\sum_{\mathfrak{p}'|\mathfrak{p}}\operatorname{length}_{\mathfrak{p}'}(E/E')\log\operatorname{card}(\kappa(\mathfrak{p}'))=\sum_{\mathfrak{p}'|\mathfrak{p}}\log\operatorname{card}\big((E/E')\otimes_{\mathfrak{o}_{K'}}\mathfrak{o}_{K',\mathfrak{p}'}\big)=\log\operatorname{card}\big((E/E')\otimes_{\mathfrak{o}_K}\mathfrak{o}_{K,\mathfrak{p}}\big).$$

Consequently, the coefficient of $[\mathfrak{p}]$ in $\pi_*\widehat{\operatorname{div}}(e_1\wedge\cdots\wedge e_n)$ is equal to $\log\operatorname{card}((E/E')_{\mathfrak{p}})/\log\operatorname{card}\kappa(\mathfrak{p})$.

On the other hand, the family (a_ie_j) is a K-basis of π_*E, so that $\widehat{c}_1(\det\pi_*\overline{E})$ is represented by the arithmetic cycle

$$\widehat{\operatorname{div}}(a_1e_1\wedge\cdots\wedge a_de_n)=\sum_{\mathfrak{p}}\operatorname{length}_{\mathfrak{p}}((E/E'')\otimes\mathfrak{o}_{K,\mathfrak{p}})[\mathfrak{p}]-\sum_{\sigma}\frac{1}{2}\log\det\big(h_\sigma(a_ie_j,a_ke_m)\big)_{\substack{(i,k)\\(j,m)}}$$

on S. Let $A\subset\mathfrak{o}_{K'}$ be the sub-$\mathfrak{o}_K$-module generated by $a_1,\dots,a_d$. One has $E''\subset E'\subset E$ and $E''\simeq\mathfrak{A}\otimes E'\simeq A^n$. Consequently,

$$\begin{aligned}\operatorname{length}((E/E'')\otimes\mathfrak{o}_{K,\mathfrak{p}})&=\operatorname{length}((E/E')\otimes\mathfrak{o}_{K,\mathfrak{p}})+\operatorname{length}((\mathfrak{o}_{K'}/A)\otimes E'\otimes\mathfrak{o}_{K,\mathfrak{p}})\\&=\operatorname{length}((E/E')\otimes\mathfrak{o}_{K,\mathfrak{p}})+n\operatorname{length}((\mathfrak{o}_{K'}/A)\otimes\mathfrak{o}_{K,\mathfrak{p}}).\end{aligned}$$

Finally, $a_1\wedge\cdots\wedge a_d$ is a nonzero section of $\pi_*\mathfrak{o}_{K'}$. For $\sigma\in\Sigma$ and $a,b\in o_{K'}$, the scalar product of a and b as elements of $\pi_*\mathfrak{o}_{K'}$ is equal to

$$\sum_{\sigma'|\sigma}\langle a,b\rangle_{\sigma'}=\sum_{\sigma'|\sigma}\overline{\sigma'(a)}\sigma'(b).$$

Consequently, the norm of $a_1\wedge\cdots\wedge a_d$ in $\sigma^*\det\pi_*\mathfrak{o}_{K'}$ is given by

$$\|a_1\wedge\cdots\wedge a_d\|_\sigma^2=\det\big(\langle a_i,a_j\rangle_\sigma\big)_{i,j}=\det\left(\sum_{\sigma'|\sigma}\overline{\sigma'(a_i)}\sigma'(a_j)\right)_{i,j}=|D_\sigma(a_1,\dots,a_d)|^2,$$

where $D_\sigma(a_1,\dots,a_d)=\det\big(\sigma'(a_i)\big)_{i,\sigma'}$. Finally, the Hermitian line bundle $\det(\pi_*\mathfrak{o}_{K'})$ is represented by the arithmetic divisor

$$\widehat{\operatorname{div}}(a_1\wedge\cdots\wedge a_d)=\sum_{\mathfrak{p}}\operatorname{length}(\mathfrak{o}_{K'}/A)_{\mathfrak{p}}[\mathfrak{p}]-\sum_{\sigma}\log|D_\sigma(a_1,\dots,a_d)|[\sigma].$$

These relations imply that

$$\begin{aligned}\widehat{\operatorname{div}}((\bigwedge a_i e_j)) - \mathrm{N}(\bigwedge e_j) - \operatorname{rank}(E)\widehat{\operatorname{div}}(\bigwedge a_i) \\ = \sum_\sigma \Bigg(n\log|D_\sigma(a_1,\dots,a_d)| + \frac{1}{2}\sum_{\sigma'|\sigma} \log\det\big(h_{\sigma'}(e_i,e_j)\big)_{i,j} \\ - \frac{1}{2}\log\det\big(h_\sigma(a_ie_j,a_ke_m)\big)_{\substack{(i,j)\\(j,m)}}\Bigg)[\sigma].\end{aligned}$$

In particular, the finite components of this arithmetic divisor are zero. Let us show that the same holds for its archimedean components.

Actually, the matrix with $((i,j),(k,m))$ coefficient

$$h_\sigma(a_ie_j,a_ke_m) = \sum_{\sigma'|\sigma} \overline{\sigma'(a_i)}\sigma'(a_k)\langle e_j,e_m\rangle_{\sigma'}$$

is equal to the following product of matrices

$$\left(\left[\overline{\tau'(a_i)}\right]_{i,\tau'}\otimes \mathrm{id}\right)\left(\left[\langle e_j,e_m\rangle'_\sigma \delta_{\sigma',\tau'}\right]_{\substack{(j,\tau')\\(m,\sigma')}}\right)\left(\left[\sigma'(a_k)\right]_{\sigma',k}\otimes \mathrm{id}\right).$$

Its determinant thus equals

$$\begin{aligned}\det\big(h_\sigma(a_ie_j,a_ke_m)\big)_{\substack{(i,j)\\(j,m)}} &= \det\Big((\overline{\tau'(a_i)})_{i,\tau'}\Big)^n \prod_{\sigma'|\sigma}\det\big(\langle e_j,e_m\rangle_{\sigma'}\big)\det\big((\sigma'(a_k))_{\sigma',k}\big)^n \\ &= |D_\sigma(a_1,\dots,a_d)|^{2n}\prod_{\sigma'|\sigma}\det\big(\langle e_j,e_m\rangle_{\sigma'}\big)\end{aligned}$$

and

$$\widehat{\operatorname{div}}((\bigwedge a_ie_j)) = \mathrm{N}(\bigwedge e_j) + \operatorname{rank}(E)\widehat{\operatorname{div}}(\bigwedge a_i).$$

Consequently, the rational morphism from $\det(\pi_*\overline{E})$ to $\mathrm{N}(\det\overline{E})\otimes\det(\pi_*\mathfrak{o}_{K'})^n$ which sends $\bigwedge(a_ie_j)$ to $\mathrm{N}(\wedge e_j)\otimes(\bigwedge a_i)^{\otimes n}$ induces an isometry of Hermitian line bundles. □

C. Relative duality

The Grothendieck–Riemann–Roch theorem describes the difference between the operations of norms and of direct-images of hermitian vector bundles. The duality theorem analogously measures the lack of commutativity of the two operations taking a dual and direct-image.

The dualizing module is furnished by the inverse of the different. Let us recall its definition. The trace $\operatorname{tr}_{K'/K}\colon K'\to K$ is a K-linear map and the bilinear map $K'\times K'\to K$ given by $(a,b)\mapsto \operatorname{tr}_{K'/K}(ab)$ is a perfect pairing, since the extension $K\hookrightarrow K'$ is separable. The codifferent $\mathfrak{D}^{-1}_{K'/K}$ is defined as the set of all $a\in K'$ such that $\operatorname{tr}_{K'/K}(a\mathfrak{o}_{K'})\subset\mathfrak{o}_K$. It is a fractional ideal of K' containing $\mathfrak{o}_K$.

In other words, let us identify the K'-vector space $\operatorname{Hom}_K(K',K)$ with K' by the map $u\mapsto \mathrm{a}_u$, where a_u is the unique element of K' such that $\operatorname{tr}(a_ub)=u(b)$ for all $b\in K'$. Under this identification, the trace form maps to 1 and the submodule $\operatorname{Hom}_{\mathfrak{o}_K}(\mathfrak{o}_{K'},\mathfrak{o}_K)$ maps to $\mathfrak{D}^{-1}_{K'/K}$.

For any $\sigma' \in \Sigma'$, there is a canonical isomorphism $(\sigma')^*\mathfrak{D}^{-1}_{K'/K} \simeq (\sigma')^* K' = \mathbf{C}$ by which we endow $\mathfrak{D}^{-1}_{K'/K}$ with the structure of a Hermitian line bundle over S'. This endows $\mathrm{Hom}_{\mathfrak{o}_K}(\mathfrak{o}_{K'}, \mathfrak{o}_K)$ with a structure of Hermitian vector bundle over S' such that $\|\mathrm{tr}\|_{\sigma'} = 1$ for any $\sigma' \in \Sigma'$.

For any Hermitian vector bundle $\overline{F}$ on S, let us pose

$$\pi^! \overline{F} = \mathfrak{D}^{-1}_{K'/K} \otimes_{\mathfrak{o}_{K'}} \pi^* F = \mathfrak{D}^{-1}_{K'/K} \otimes_{\mathfrak{o}_K} F,$$

viewed as an Hermitian vector bundle over S'.

THEOREM 1.3.2. *Let E and F be Hermitian vector bundles over S' and S respectively. There is a canonical isomorphism of Abelian groups*

$$\mathrm{Hom}_{\mathfrak{o}_K}(\pi_* \overline{E}, \overline{F}) \simeq \pi_* \mathrm{Hom}_{\mathfrak{o}_{K'}}(\overline{E}, \pi^! \overline{F})$$

and this isomorphism induces an isometry of Hermitian vector bundles over S.

Proof. — Let us first define this homomorphism on the generic fibre: since $\mathfrak{D}^{-1}_{K'/K} \otimes K' = K'$, we need to define an isomorphism D_K of K-vector spaces

$$\mathrm{Hom}_{K'}(E_{K'}, F_K \otimes K') \xrightarrow{\sim} \mathrm{Hom}_K(E_{K'}, F_K).$$

We define D_K as the morphism given by composition with the morphism $\mathrm{id}_{F_K} \otimes \mathrm{tr}_{K'/K}$ from $F_K \otimes K'$ to F_K. Its surjectivity follows from the fact that $\mathrm{id}_{F_K} \otimes \mathrm{tr}_{K'/K}$ is surjective. Both sides having the same dimension as K-vector spaces, namely $[K' : K] \dim_{K'}(E_{K'}) \dim_K(F_K) = \dim_K(E_{K'}) \dim_K(F_K)$, we see that D_K is an isomorphism.

Let us now show that D_K induces, by restriction, an isomorphism

$$\mathrm{Hom}_{\mathfrak{o}_{K'}}(E, \mathfrak{D}^{-1}_{K'/K} \otimes \pi^* F) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{o}_K}(\pi_* E, F).$$

Since E, resp. F, is a direct factor of a free $\mathfrak{o}_{K'}$-module, resp. of a free $\mathfrak{o}_K$-module, it suffices to treat the case where $E = \mathfrak{o}_{K'}$ and $F = \mathfrak{o}_K$. Then, the left hand side is $\mathfrak{D}^{-1}_{K'/K}$, and the right hand side is $\mathfrak{o}_K$, the map D_K identifies with the "morphism" $\mathfrak{D}^{-1}_{K'/K} \simeq \mathrm{Hom}(\mathfrak{o}_{K'}, \mathfrak{o}_K)$ such that $D_K(b) = (a \mapsto \mathrm{tr}(ab))$. By definition of $\mathfrak{D}^{-1}_{K'/K}$, this is indeed an isomorphism.

It now remains to prove that D_K induces an isometry. Let us fix $\sigma \in \Sigma$. Then,

$$\sigma^* \mathrm{Hom}_{K'}(E_{K'}, F_K \otimes K') = \bigoplus_{\sigma'|\sigma} \mathrm{Hom}_{\mathbf{C}}((\sigma')^* E, \sigma^* F),$$

where σ' ranges over all embeddings of K' into $\mathbf{C}$ which extend σ, while

$$\sigma^* \mathrm{Hom}_K(E_{K'}, F_K) = \mathrm{Hom}_{\mathbf{C}}(\bigoplus_{\sigma'|\sigma} (\sigma')^* E, \sigma^* F) = \bigoplus_{\sigma'|\sigma} \mathrm{Hom}_{\mathbf{C}}((\sigma')^* E, \sigma^* F).$$

These are orthogonal direct sums of Hermitian complex vector spaces. Under these identifications, the isomorphism D_K identifies with the identity morphism. This shows that D_K is an isometry. □

To be really useful, the Grothendieck–Riemann–Roch theorem has to be complemented by the computation of the second term, namely $\det(\pi_* \mathfrak{o}_{K'})$.

PROPOSITION 1.3.3. *Let $\mathfrak{d}_{K'/K} \subset \mathfrak{o}_K$ be the discriminantal ideal end let us consider it as an Hermitian line bundle. Then, there is a canonical isomorphism*

$$\det(\pi_* \mathfrak{o}_{K'})^{\otimes 2} \simeq \mathfrak{d}_{K'/K}$$

of Hermitian line bundles.

Proof. — Let us consider the $\mathfrak{o}_K$-bilinear map on $\mathfrak{o}_{K'}$ given by the trace map. It induces a linear map

$$\tau\colon \det(\pi_*\mathfrak{o}_{K'}) \otimes \det(\pi_*\mathfrak{o}_{K'}) \to \mathfrak{o}_K, \qquad (a_1 \wedge \cdots \wedge a_d, b_1 \wedge \cdots \wedge b_d) \mapsto \det\big(\mathrm{tr}(a_i b_j)\big).$$

Its image is, by definition, the discriminant ideal $\mathfrak{d}_{S'/S}$. (Indeed, this ideal is generated by determinants of the form $\det\big(\mathrm{tr}(\omega_i\omega_j)\big)$; the assertion follows at once if one can choose (ω_i) to be a basis of $\mathfrak{o}_{K'}$ over $\mathfrak{o}_K$; a localization argument then implies the general case.) We now need to compare the hermitian metrics on both sides.

Let $\sigma \in \Sigma$. The isomorphism $\sigma^*\pi_*\mathfrak{o}_{K'} \simeq \bigoplus_{\sigma'|\sigma}(\sigma')^*\mathfrak{o}_{K'}$ shows that the hermitian form on $\sigma^*\pi_*\mathfrak{o}_{K'}$ is given by $\langle a, b\rangle_\sigma = \sum_{\sigma'|\sigma}\overline{\sigma'(a)}\sigma'(b)$. Moreover, the trace form $\sigma^*\mathrm{tr}\colon \sigma^*\pi_*\mathfrak{o}_{K'} \to \sigma^*\mathfrak{o}_K$ identifies with the map $(z_{\sigma'}) \mapsto \sum z_{\sigma'}$. Then, $\sigma^*\tau$ is identified with the map

$$(z_1 \wedge \cdots \wedge z_d) \otimes (w_1 \wedge \cdots \wedge w_d) \mapsto \det\left(\sum_{\sigma'} z_{i,\sigma'} w_{j,\sigma'}\right) = \det\big(z_{i,\sigma'}\big)\det\big(w_{j,\sigma'}\big).$$

so that $\sigma^*\tau$ is an isometry. □

COROLLARY 1.3.4. *One has $\widehat{\deg}(\pi_*\mathfrak{o}_{K'}) = -\frac{1}{2}\log \mathrm{N}(\mathfrak{d}_{K'/K})$.*

Proof. — Indeed, the arithmetic degree of the ideal $\mathfrak{d}_{K'/K}$ in $\mathfrak{o}_K$ is equal to $-\log \mathrm{N}(\mathfrak{d}_{K'/K})$. □

COROLLARY 1.3.5. *For any Hermitian vector bundle $\overline{E}$ on S', one has*

$$\widehat{\deg}\,\pi_*\overline{E} = \widehat{\deg}\,\overline{E} - \frac{1}{2}\mathrm{rank}(\overline{E})\log \mathrm{N}(\mathfrak{d}_{K'/K}).$$

§ 1.4
GLOBAL SECTIONS, AND GEOMETRY OF NUMBERS

A. An arithmetic Riemann inequality

Let K be a number field, let $\mathfrak{o}_K$ be its ring of integers, let Σ be the set of embeddings of K into $\mathbf{C}$. Let us also write $S = \mathrm{Spec}\,\mathfrak{o}_K$.

Let $\overline{E} = (E, h)$ be an hermitian vector bundle over S. Under the function field/number field analogy, global sections of $\overline{E}$ are elements of E satisfying $\|e\|_\sigma \leqslant 1$ for all $\sigma \in \Sigma$. Let us denote this set by $H^0(\overline{E})$ and $h^0(\overline{E}) = \log \mathrm{card}\, H^0(\overline{E})$.

The results below are then analogues of the following theorems for algebraic curves: let X be a projective geometrically integral smooth curve over a field k, let D be a divisor on X, let g be the genus of X; then:

a) if $\deg D < 0$, then $h^0(D) = 0$;

b) if $\deg D = 0$ and $D \neq 0$, then $h^0(D) = 0$.

c) if $\deg D > g-1$, then $h^0(D) \neq 0$; more precisely, $h^0(D) \geqslant \deg(D) + 1 - g$.

LEMMA 1.4.1. *Let $\overline{E}$ be a non-trivial Hermitian line bundle over S such that $\widehat{\deg}\,\overline{E} \leqslant 0$. Then $h^0(\overline{E}) = 0$.*

Proof. — Let us assume that $h^0(\overline{E}) \neq 0$ and let ℓ be any non-zero element in $H^0(\overline{E})$. Then, by definition of the arithmetic degree,

$$\widehat{\deg}(\overline{E}) = \log \operatorname{card}(E/\mathfrak{o}_K \ell) - \sum_{\sigma \in \Sigma} \log \|\ell\|_\sigma ,$$

hence $\widehat{\deg}(\overline{E}) \geqslant 0$, all terms of the sum being nonnegative.

Since $\widehat{\deg}(\overline{E}) \leqslant 0$ by hypothesis, we deduce that $\widehat{\deg}(\overline{E}) = 0$ and that all terms of this sum vanish. In other words, $E = \mathfrak{o}_K \ell$ and $\|\ell\|_\sigma = 1$ for any $\sigma \in \Sigma$. This means precisely that the morphism $\mathfrak{o}_K \to E$ given by $a \mapsto a\ell$ is an isometry. Consequently, $\overline{E}$ is the trivial hermitian line bundle, contradiction. $\square$

For any integer n, let us β_n be the volume of the unit ball in $\mathbf{R}^n$. Let us pose

$$\chi(n,K) = \frac{1}{2} n \log|D_K| + n r_1 \log 2 - r_1 \log \beta_n - r_2 \log \beta_{2n}.$$

By Lemma A.2.1, $\beta_n = \pi^{n/2}/\Gamma(1+n/2)$, so that

$$\chi(n,K) = \frac{1}{2} n \log\left(|D_K| \frac{\Gamma(1+n/2)^{r_1/n}\Gamma(1+2n)^{2r_2/n}}{\pi^d 2^{-2r_1}} \right).$$

PROPOSITION 1.4.2. *Let $\overline{E}$ be an Hermitian vector bundle over S.*

If $\widehat{\deg}\,\overline{E} \geqslant \chi(\operatorname{rank}\overline{E}, K)$, then $h^0(\overline{E}) > 0$. More precisely, $h^0(\overline{E}) > \widehat{\deg}\,\overline{E} - \chi(\operatorname{rank}\overline{E}, K)$.

Proof. — Let $\Sigma_1 \subset \Sigma$ be the set of real embeddings of K, and $\Sigma_2 \subset \Sigma$ be one half of the complex embeddings, chosen in such a way that $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \overline{\Sigma_2}$. Let us pose $r_1 = \operatorname{card}(\Sigma_1)$ and $r_2 = \operatorname{card}(\Sigma_2)$; they satisfy $r_1 + 2r_2 = d$, where $d = [K:\mathbf{Q}]$. Let us pose $E_{\mathbf{R}} = E \otimes_{\mathbf{Z}} \mathbf{R}$; this is a real vector space of dimension nd, with $n = \operatorname{rank}\overline{E}$.

The map $a \mapsto (\sigma(a))_{\sigma \in \Sigma_1 \cup \Sigma_2}$ induces an isomorphism $\mathfrak{o}_K \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$. Consequently, the morphism of real vector spaces

$$E \otimes_{\mathbf{Z}} \mathbf{R} \simeq \bigoplus_{\sigma \in \Sigma_1} \sigma_{\mathbf{R}}^* E \oplus \bigoplus_{\sigma \in \Sigma_2} \sigma^* E$$

is an isomorphism where, for $\sigma \in \Sigma_1$, $\sigma_{\mathbf{R}}$ is the map $K \to \mathbf{R}$ deduced from σ.

Let $\|\cdot\|$ and $\|\cdot\|_2$ be the norms on $E_{\mathbf{R}}$ defined by

$$\|(e_\sigma)\| = \max \|e_\sigma\|_\sigma , \qquad \|(e_\sigma)\|_2 = \left(\sum_{\sigma \in \Sigma_1} \|e_\sigma\|_\sigma^2 + 2 \sum_{\sigma \in \Sigma_2} \|e_\sigma\|_\sigma^2 \right)^{1/2} .$$

Let μ be the the Euclidean Lebesgue measure on $E_{\mathbf{R}}$ corresponding to the norm $\|\cdot\|_2$. The Euclidean lattice $(E, \|\cdot\|_2)$ is equal to $\pi_* \overline{E}$. Its covolume is therefore equal to

$$-\log \mu(E_{\mathbf{R}}/E) = \widehat{\deg} \pi_* \overline{E} = \widehat{\deg} \overline{E} - \frac{1}{2} \operatorname{rank}(\overline{E}) \log |D_K|,$$

wher D_K is the absolute discriminant of K.

The Euclidean norm $\|\cdot\|_2$ defined on $E_{\mathbf{R}}$ induces the Euclidean norm on each factor $\sigma^*_{\mathbf{R}} E$, for $\sigma \in \Sigma_1$, and twice the hermitian norm on the factors $\sigma^* E$, for $\sigma \in \Sigma_2$. Moreover, the unit ball B of $E_{\mathbf{R}}$ for the norm $\|\cdot\|$ is the product of the Euclidean unit balls of these factors. Since r_1 factors have dimension n, and the r_2 other have dimension $2n$, the Euclidean volume of B is given by

$$\mu(B) = \prod_{\sigma \in \Sigma_1} \beta_n \prod_{\sigma \in \Sigma_2} 2^{2n} \beta_{2n} = \beta_n^{r_1} 2^{2nr_2} \beta_{2n}^{r_2}.$$

By Minkowski's theorem (Theorem A.3.1 below), $\operatorname{card}(E \cap B) > 1$ if $\mu(E_{\mathbf{R}}/E)/\mu(B) \leqslant 2^{-nd}$. In other words, $h^0(\overline{E}) > 0$ if

$$\widehat{\deg}(\overline{E}) \geqslant \frac{1}{2} n \log |D_K| + n r_1 \log 2 - r_1 \log \beta_n - r_2 \log \beta_{2n} = \chi(n, K).$$

More generally, Minkowski's Theorem implies that $\operatorname{card}(E \cap B) \geqslant 2^{nd} \mu(E_{\mathbf{R}}/E)/\mu(B)$, so that $h^0(\overline{E}) \geqslant \chi(\operatorname{rank} \overline{E}, K)$. □

COROLLARY 1.4.3. *Let $(\overline{E}_i)$ be a family of Hermitian vector bundles over S such that $\widehat{\deg}(\overline{E}_i)/\operatorname{rank}(\overline{E}_i) \log \operatorname{rank}(\overline{E}_i)$ converges to $+\infty$. Then $h^0(\overline{E}_i) > 0$ for i large enough.*

Proof. — It suffices to show that for $i \gg 0$, $\widehat{\deg}(\overline{E}_i) > \chi(\operatorname{rank} \overline{E}_i, K)$, so that $\overline{E}_i$ satisfies the assumption of Proposition 1.4.2. It suffices to prove this under one of the assumptions (1) $\operatorname{rank}(\overline{E}_i)$ is bounded, or (2) $\operatorname{rank}(\overline{E}_i) \to \infty$. This is clear in case (1) for the hypothesis of the corollary implies that $\widehat{\deg}(\overline{E}_i) \to +\infty$.

Let us thus assume that $\operatorname{rank}(\overline{E}_i) \to \infty$. The Stirling formula

$$\Gamma(1+x) \sim x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi}$$

implies

$$\frac{1}{x} \log \Gamma(1+x) = \log x - 1 + \mathrm{o}(1),$$

hence $\chi(n, K)$ satisfies the following asymptotic expansion when $n \to \infty$:

$$\begin{aligned}
\chi(n, K) &= nd \left(\frac{1}{2} \log |D_K|^{1/[K:\mathbf{Q}]} + \log \frac{2}{\sqrt{\pi}} + \frac{r_1}{d} \frac{1}{2} \left(\log(n/2) - 1 \right) + \frac{r_2}{d} \left(\log n - 1 \right) \right) \\
&= \frac{1}{2} nd \left(\log n + \mathrm{O}(1) \right),
\end{aligned}$$

the field K (hence its degree d) being fixed. Consequently, $\widehat{\deg}(\overline{E}_i) > \chi(\operatorname{rank}(\overline{E}_i), K)$ for i large enough; this proves the Corollary. □

COROLLARY 1.4.4. *Let (D, g) be an arithmetic divisor on S such that $\widehat{\deg}(D, g) \geqslant \chi(1, K)$. There exists $a \in K^*$ such that $\widehat{\operatorname{div}}(a) + (D, g) \geqslant 0$.*

Proof. — We just need to apply Prop. 1.4.2 to the hermitian line bundle $\mathscr{O}(D,g)$ consisting of elements $a \in K^*$ such that $\operatorname{div}(a) + D \geqslant 0$, endowed with the hermitian norm such that $-\log \|1\|_\sigma = g_\sigma$. □

B. An approximate Riemann–Roch equality

If X is an algebraic curve (smooth, proper, geometrically connected) over a field k and D is a divisor on X, the Riemann–Roch equality takes the form $h^0(D) - h^1(D) = \deg(D) + 1 - g$, where g is the genus of X. Moreover, the duality theorem implies $h^1(D) = h^0(\kappa - D)$, where κ is a "canonical divisor". This implies inequalities of the form $h^0(D) \geqslant \deg(D) + 1 - g$, as well as an equality $h^0(D) = \deg(D) + 1 - g$ is $\deg(D) > \deg(\kappa)$.

With our definition of h^0 for arithmetic divisors over S, the spectrum of the ring of integeres of a number field K, there are no such equalities. Indeed, the covolume of a lattice by itself says nothing about the number of lattice points of norm $\leqslant 1$, beyond Minkowski's theorem. Note however that GILLET & SOULÉ (1991) establishes the following inequality which can be seen as an approximate analogue of the Riemann–Roch formula.

THEOREM 1.4.5. *Let $\overline{E}$ be an Hermitian vector bundle over S and let $\overline{E}^* = \mathfrak{D}_K^{-1} \otimes \overline{E}^\vee$. One has*

$$\left| h^0(\overline{E}) - h^0(\overline{E}^*) - \widehat{\deg}(\overline{E}) - \frac{1}{2} \operatorname{rank}(\overline{E}) \log |D_K| \right| \leqslant c(r_1, r_2, n),$$

where $c(r_1, r_2, n)$ is a constant depending only on r_1, r_2 and $n \operatorname{rank} \overline{E}$.

Proof. — As in the proof of Prop. 1.4.2, we view E as a lattice in the real vector space $E_{\mathbf{R}}$, endowed with the norm $\max(\|\cdot\|_\sigma)$. The volume of the unit ball B has been computed in the proof of that proposition and equals $\beta_n^{r_1} 2^{2nr_2} \beta_{2n}^{r_2}$, where β_n is the volume of an n-dimensional euclidean unit ball, r_1 and r_2 are the number of real and complex places of K, and $n = \operatorname{rank} E$. Consequently, $h^0(\overline{E}) = \log \operatorname{card}(E \cap B)$.

On the other hand, let be the Hermitian vector bundle $\overline{E}^\vee = \operatorname{Hom}(\overline{E}, \mathfrak{D}_{K/\mathbf{Q}}^{-1})$. (...) □

LEMMA 1.4.6. *For any interval I in $\mathbf{R}$,* $\operatorname{card}(I \cap \mathbf{Z}) \leqslant \max(2, \frac{3}{2} \operatorname{meas}(I))$.

Proof. — Let $I = [a,b]$; then $I \cap \mathbf{Z} = \{\lceil a \rceil, \dots, \lfloor b \rfloor\}$. We can only increase $I \cap \mathbf{Z}$ by shifting I to the right and replacing a by $\lceil a \rceil$, so that $\operatorname{card}(I \cap \mathbf{Z}) = 1 + \lfloor b - a \rfloor = 1 + \lfloor \operatorname{meas}(I) \rfloor$. When $\operatorname{meas}(I) \geqslant 2$, we can write

$$\operatorname{card}(I \cap \mathbf{Z}) \leqslant 1 + \operatorname{meas}(I) \leqslant \frac{3}{2} \operatorname{meas}(I).$$

When $\operatorname{meas}(I) < 2$, $\lfloor \operatorname{meas}(I) \rfloor \leqslant 1$ so that $\operatorname{card}(I \cap \mathbf{Z}) \leqslant 2$. This proves the lemma. □

PROPOSITION 1.4.7. *Let V be a normed vector space of finite dimension n, let B be its unit ball and let Λ be a lattice in V.*

If $B \cap \Lambda$ generates Λ, then $\operatorname{card}(B \cap \Lambda) \operatorname{vol}(B^*) \operatorname{vol}(V/\Lambda) \leqslant 6^n$.

Proof. — Let us prove the lemma by induction on n. It holds for $n = 0$.

When $n = 1$, $B = [-b, b]$ for some positive real number b. In that case, $B^* = [-\frac{1}{b}, \frac{1}{b}]$, so thatvol$(B^*) = 2/b$. Then

$$\operatorname{card}(B \cap \mathbf{Z}) \operatorname{vol}(B^*) \leqslant \max(2, 3b)\frac{2}{b} = \max(4, 6/b) \leqslant 6.$$

Let us assume that the result holds in dimension $n-1$ Let v be a nonzero primitive element in $B \cap V$, let $V_1 = V/\mathbf{R}v$ and let $p\colon V \to V_1$ be the natural projection; let $\Lambda_1 = p(\Lambda)$ and $B_1 = p(B)$. The compact set B_1 is the unit ball for the norm on V_1 quotient of the norm on V.

Since $p(B \cap \Lambda) \subset B_1 \cap \Lambda_1$, one has

$$\operatorname{card}(B \cap \Lambda) \leqslant \sum_{x \in B_1 \cap \Lambda_1} \operatorname{card}\left(p^{-1}(x) \cap (B \cap \Lambda)\right).$$

For an $x \in V_1$, let us write $C_x = p^{-1}(x) \cap B$, since B is symmetric, one has $C_x = -C_{-x}$. Moreover, $C_x + C_{-x} \subset p^{-1}(0) \cap 2B$. Let $s\colon V_1 \to V$ be a section of the projection p such that $s(\Lambda_1) \subset \Lambda$. For any $x \in V_1$, the map $\varphi_x\colon t \mapsto s(x) + tv$ is an affine bijection from $\mathbf{R}$ to $p^{-1}(x)$ which sends $\mathbf{Z}$ onto $p^{-1}(x) \cap \Lambda$. By Lemma 1.4.6,

$$\operatorname{card}(B \cap \Lambda) \leqslant \sum_{x \in B_1 \cap \Lambda_1} \operatorname{card}(\varphi_x^{-1}(C_x) \cap \mathbf{Z}) \leqslant \sum_{x \in B_1 \cap \Lambda_1} \max(2, \frac{3}{2} \operatorname{vol}(\varphi_x^{-1}(C_x))).$$

According to Corollary A.3.12, $\operatorname{vol}(C_x) \leqslant \operatorname{vol}(p^{-1}(0) \cap B) = 2/\|v\|$. Therefore,

$$\operatorname{card}(B \cap \Lambda) \leqslant \sum_{x \in B_1 \cap \Lambda_1} \max(2, 3/\|v\|) \leqslant \operatorname{card}(B_1 \cap \Lambda_1) \max(2, 3/\|v\|).$$

The lattice Λ_1 is generated by $B_1 \cap \Lambda_1$; applying the induction hypothesis, we thus have

$$\operatorname{card}(B_1 \cap \Lambda_1) \operatorname{vol}(B_1^*) \operatorname{vol}(V_1/\Lambda_1) \leqslant 6^{n-1}.$$

Let φ be the linear map on V that sends a linear form ℓ to $\ell(v)$. For any $t \in \mathbf{R}$, one has $B^* \cap \varphi^{-1}(t) = -B^* \cap \varphi^{-1}(t)$ and $(B^* \cap \varphi^{-1}(t)) + (B^* \cap \varphi^{-1}(-t)) \subset 2(B^* \cap \varphi^{-1}(0))$. Moreover, $B^* \cap \varphi^{-1}(0) = B_1^*$. By Corollary A.3.12, we have

$$\operatorname{vol}(B^*) = \leqslant \operatorname{vol}(B_1^*) \operatorname{vol}(B \cap \mathbf{R}v) = 2\|v\| \operatorname{vol}(B_1^*).$$

Finally,

$$\begin{aligned}
\operatorname{card}(B \cap \Lambda) \operatorname{vol}(B^*) \operatorname{vol}(V/\Lambda) &\leqslant 2\|v\| \max(2, 3/\|v\|) \operatorname{vol}(B_1^*) \operatorname{card}(B_1 \cap \Lambda_1) \operatorname{vol}(V/\Lambda) \\
&\leqslant \max(4\|v\|, 6) 6^{n-1} \frac{\operatorname{vol}(V/\Lambda)}{\operatorname{vol}(V_1/\Lambda_1)} \\
&\leqslant 6^n
\end{aligned}$$

since $\|v\| \leqslant 1$. □

COROLLARY 1.4.8. *Let V be a normed vector space of finite dimension n, let B be its unit ball and let Λ be a lattice in V. Then,*

$$6^{-n} \leqslant \frac{\operatorname{card}(B \cap \Lambda)}{\operatorname{card}(B^* \cap \Lambda^*)} \frac{\operatorname{vol}(V/\Lambda)}{\operatorname{vol}(B)} \leqslant 6^{-n} n^{-n/2} \beta_n^2.$$

Proof. — Let us write $W = V^*$. Let W_0 be the subspace of W generated by $B^* \cap \Lambda^*$, let $B_0^* = B^* \cap W_0$ be the unit ball of W_0 for the norm given by restriction and let $\Lambda_0^* = W_0 \cap \Lambda^*$. By construction, Λ_0^* is a lattice in W_0 which is spanned by $B_0^* \cap \Lambda_0^*$; moreover, $B_0^* \cap \Lambda_0^* = B^* \cap \Lambda$. Consequently,

$$\operatorname{card}(B^* \cap \Lambda^*) \operatorname{vol}(B_0) \operatorname{vol}(W_0/\Lambda_0^*) \leqslant 6^{\dim(W_0)},$$

where B_0 is the unit ball of $(W_0)^* = V/W_0^\perp$ for the quotient norm.

Let B° be the open unit ball in V; for any $x \in B^\circ \cap \Lambda$ and any $y \in B^* \cap \Lambda^*$, the integer $\langle x, y\rangle$ belongs to $(-1, 1)$; consequently, it vanishes. This implies that $B^\circ \cap \Lambda$ is contained in the orthogonal $W_0^\perp$ to W_0 in V. It follows from Theorem A.3.1 that for any positive real number ε,

$$\operatorname{card}(B \cap \Lambda) \geqslant \operatorname{card}(B^\circ \cap \Lambda) \geqslant \operatorname{card}((B^\circ \cap W_0^\perp) \cap \Lambda) \geqslant 2^{-\dim(W_0^\perp)} \frac{\operatorname{vol}(B \cap W_0^\perp) - \varepsilon}{\operatorname{vol}(W_0^\perp / W_0^\perp \cap \Lambda)}.$$

Since $\dim(W_0^\perp) = \dim(W) - \dim(W_0)$, it follows that

$$\operatorname{card}(B \cap \Lambda) \geqslant 6^{\dim(W_0) - \dim(W)} \frac{\operatorname{vol}(B \cap W_0^\perp)}{\operatorname{vol}(W_0^\perp / W_0^\perp \cap \Lambda)}.$$

From these inequalities, we deduce that

$$\frac{\operatorname{card}(B \cap \Lambda) \operatorname{vol}(V/\Lambda)}{\operatorname{card}(B^* \cap \Lambda^*) \operatorname{vol}(B)} \geqslant 6^{-\dim W} \frac{\operatorname{vol}(B \cap W_0^\perp) \operatorname{vol}(B_0)}{\operatorname{vol}(B)} \frac{\operatorname{vol}(W_0/\Lambda_0^*) \operatorname{vol}(V/\Lambda)}{\operatorname{vol}(W_0^\perp / W_0^\perp \cap \Lambda)}.$$

By Corollary A.3.12 to the Brunn-Minkowski inequality,

$$\operatorname{vol}(B) \leqslant \operatorname{vol}(B \cap W_0^\perp) \operatorname{vol}(B_0).$$

Moreover, Λ has a basis containing a basis of $\Lambda \cap W_0^\perp$, so that

$$\operatorname{vol}(V/\Lambda) = \operatorname{vol}(W_0^\perp / W_0^\perp \cap \Lambda) \operatorname{vol}((V/W_0^\perp)/p(\Lambda)) = \operatorname{vol}(W_0^\perp / W_0^\perp \cap \Lambda);$$

by duality,

$$\operatorname{vol}(W_0/\Lambda_0^*) = \operatorname{vol}((V/W_0^\perp)/p(\Lambda))^{-1},$$

so that

$$\operatorname{vol}(V/\Lambda) \operatorname{vol}(W_0/\Lambda_0^*) = \operatorname{vol}(W_0^\perp / W_0^\perp \cap \Lambda).$$

This establishes the lower bound of the corollary.

Exchanging the roles of V and V^*, we have

$$\frac{\operatorname{card}(B \cap \Lambda) \operatorname{vol}(V/\Lambda)}{\operatorname{card}(B^* \cap \Lambda) \operatorname{vol}(B)} \leqslant 6^{-\dim(V)} \frac{\operatorname{vol}(V/\Lambda) \operatorname{vol}(V^*/\Lambda^*)}{\operatorname{vol}(B) \operatorname{vol}(B^*)}.$$

Moreover, $\operatorname{vol}(V/\Lambda) \operatorname{vol}(V^*/\Lambda^*) = 1$, and

$$\operatorname{vol}(B) \operatorname{vol}(B^*) \geqslant \beta_n^2 n^{-n/2}$$

(Bambah's theorem). This implies the required upper bound. $\square$

C. The Riemann–Roch equality via theta-functions

In this Section, we present an alternative definition of h^0, independently due to ROESSLER (1993) and VAN DER GEER & SCHOOF (2000); this definition allows for an exact Riemann–Roch equality. We shall also see, following J.-B. Bost, how this method allows for a straightforward proof of a version of Minkowski's theorem, with a slightly better constant!

DEFINITION 1.4.9. *Let L be a lattice in an Euclidean vector space V. One defines $\hat{h}^0(L) = \log\left(\sum_{v\in L}\exp(-\pi\|v\|^2)\right)$. If $\overline{E}$ is an Hermitian vector bundle over S, one lets $\hat{h}^0(\overline{E}) = \hat{h}^0(\pi_*\overline{E})$, where $\pi\colon S\to \operatorname{Spec}\mathbf{Z}$ is the canonical morphism.*

PROPOSITION 1.4.10. *For any Hermitian vector bundle over S, one has*

$$\hat{h}^0(\overline{E}) - \hat{h}^0(\mathfrak{D}_K^{-1}\otimes\overline{E}^\vee) = \widehat{\deg}(\overline{E}) - \frac{1}{2}\operatorname{rank}(\overline{E})\log|D_K|.$$

Proof. — The function $v\mapsto\exp(-\pi\|v\|^2)$ on a real Euclidean vector space V is equal to its own Fourier transform. Consequently, the Poisson summation formula for a lattice L of V gives

$$\sum_{v\in L}\exp(-\pi\|v\|^2) = \frac{1}{\operatorname{vol}(V/L)}\sum_{v\in L^*}\exp(-\pi\|v\|^2),$$

where L^* is the dual lattice of L. We thus have: $\hat{h}^0(L)-\hat{h}^0(L^*) = -\log\operatorname{vol}(V/L)$.

Let us apply this to the Euclidean lattice $\pi_*\overline{E}$. The duality theorem shows that

$$(\pi_*\overline{E})^\vee = \pi_*(\operatorname{Hom}(\overline{E},\mathfrak{D}_K^{-1})) = \pi_*(\mathfrak{D}_K^{-1}\otimes\overline{E}^\vee).$$

Moreover,

$$-\log\operatorname{covol}(\pi_*\overline{E}) = \widehat{\deg}\,\pi_*\overline{E} = \widehat{\deg}\,\overline{E} - \frac{1}{2}\operatorname{rank}\overline{E}\log|D_K|.$$

Consequently,

$$\hat{h}^0(\overline{E}) - \hat{h}^0(\mathfrak{D}_K^{-1}\otimes\overline{E}^\vee) = \widehat{\deg}\,\overline{E} - \frac{1}{2}\operatorname{rank}\overline{E}\log|D_K|,$$

as was to be shown. □

Since $\hat{h}^0$ is nonnegative, we deduce the following analogue of Riemann's inequality:

COROLLARY 1.4.11. *For any hermitian vector bundle $\overline{E}$ over S, one has*

$$\hat{h}^0(\overline{E}) \geqslant \widehat{\deg}\,\overline{E} - \frac{1}{2}\operatorname{rank}\overline{E}\log|D_K|.$$

Remark 1.4.12. GROENEWEGEN (2001) proves that if $0\to\overline{F}\to\overline{E}\to\overline{G}\to 0$ is an exact sequence of Hermitian vector bundles, then $\hat{h}^0(\overline{E})\leqslant\hat{h}^0(\overline{F})+\hat{h}^0(\overline{G})$, with equality if and only if this exact sequence is split, namely $\overline{F}$ is isomorphic to the orthogonal direct sum $\overline{F}\overset{\perp}{\oplus}\overline{G}$.

In the rest of this Section, whose results were explained to me by J.-B. Bost, we relate the two invariants $\hat{h}^0(\overline{E})$ and $h^0(\overline{E})$. The following Corollary is the announced slight strengthening of Minkowski's theorem.

PROPOSITION 1.4.13. *For any hermitian vector bundle $\overline{E}$ on S, one has*

$$h^0(\overline{E}) \geqslant \widehat{\deg}\,\overline{E} - \frac{1}{2}\operatorname{rank}\overline{E}\log|D_K| - \frac{2+[K:\mathbf{Q}]\operatorname{rank}\overline{E}}{2}\log\frac{2+[K:\mathbf{Q}]\operatorname{rank}\overline{E}}{2\pi} - \frac{1}{2}\log\pi.$$

In order to establish Proposition 1.4.13, we first observe the obvious inequality

$$h^0(\overline{E}) = \log\operatorname{card}\{v\in E;\ \|v\|_\sigma \leqslant 1\ \forall\sigma\in\Sigma\} \geqslant \log\operatorname{card}\{v\in E;\ \sum_\sigma \|v\|_\sigma^2 \leqslant 1\} = h^0(\pi_*\overline{E}),$$

where $\pi\colon S\to\operatorname{Spec}\mathbf{Z}$ is the canonical morphism. We have $\operatorname{rank}\pi_*\overline{E} = [K:\mathbf{Q}]\operatorname{rank}\overline{E}$ and $\widehat{\deg}\,\pi_*\overline{E} = \widehat{\deg}\,\overline{E} - \frac{1}{2}\operatorname{rank}\overline{E}\log|D_K|$ by Corollary 1.3.5. For the rest of the proof, we may thus assume that $F=\mathbf{Q}$ and $S=\operatorname{Spec}\mathbf{Z}$.

For any $\lambda\in\mathbf{R}$, let $\mathscr{O}(\lambda)$ be the Hermitian vector bundle over S associated to the arithmetic divisor $(0,\lambda)$. In other words, $\mathscr{O}(\lambda)=\mathbf{Z}$ with the norm given by $\|1\| = \exp(-\lambda)$. We also write $\overline{E}(\lambda) = \overline{E}\otimes\mathscr{O}(\lambda)$.

LEMMA 1.4.14. a) $\hat{h}^0(\overline{E}(\lambda))$ *is an increasing function of* λ*, and* $\hat{h}^0(\overline{E}(\lambda)) - \lambda\operatorname{rank}E$ *is a decreasing function of* λ*.*

b) *The following inequality holds:*

$$\sum_{v\in E}\|v\|^2\exp(-\pi\|v\|^2) \leqslant \frac{\operatorname{rank}E}{2\pi}\sum_{v\in E}\exp(-\pi\|v\|^2).$$

Proof. — a) Viewed as an element of $\overline{E}(\lambda)$, the squared norm of a vector $v\in E$ is equal to $e^{-2\lambda}\|v\|^2$. Consequently, increasing λ makes the norms decrease, and $\hat{h}^0(\overline{E}(\lambda))$ increases. The other assertion follows from the equalities

$$\widehat{\deg}(\overline{E}(\lambda)) = \widehat{\deg}(\overline{E}) + \lambda\operatorname{rank}E$$

and

$$\hat{h}^0(\overline{E}(\lambda)) - \lambda\operatorname{rank}E = \hat{h}^0(\overline{E}(\lambda)) - \widehat{\deg}\,\overline{E}(\lambda) + \widehat{\deg}\,\overline{E} = \widehat{\deg}\,\overline{E} + \hat{h}^0(\overline{E}^\vee(-\lambda))$$

and the first assertion in a)

b) Let f be the real function given by

$$f(\lambda) = \hat{h}^0(\overline{E}(\lambda)) = \sum_{v\in E}\exp(-\pi e^{-2\lambda}\|v\|^2).$$

It is continuously differentiable and

$$f'(\lambda) = e^{-2\lambda}\sum_{v\in E}2\pi\left\|v^2\right\|\exp(-\pi e^{-2\lambda}\|v\|^2).$$

By *a*), $0\leqslant f'(\lambda)\leqslant \operatorname{rank}E$ for any real number λ. Taking $\lambda=0$ implies the desired inequality. □

Proof of Prop. 1.4.13. — Let λ be a real number. By the Lemma, b), one has

$$\sum_{\substack{v\in\overline{E}\\ \|v\|>e^\lambda}}\exp(-\pi\|v\|^2) < \sum_{v\in\overline{E}}\frac{\|v\|^2}{e^{2\lambda}}\exp(-\pi\|v\|^2) < \frac{\operatorname{rank}E}{2\pi e^{2\lambda}}\sum_{v\in\overline{E}}\exp(-\pi\|v\|^2).$$

Consequently,

$$\sum_{\substack{v\in\overline{E}\\ \|v\|\leqslant e^{\lambda}}} \exp(-\pi\,\|v\|^2) \geqslant \left(1-\frac{\operatorname{rank}E}{2\pi e^{2\lambda}}\right)\exp\left(\hat{h}^0(\overline{E}(\lambda))\right).$$

If, moreover, $2\pi e^{2\lambda} > \operatorname{rank} E$, we get

$$\begin{aligned} h^0(\overline{E}(\lambda)) &= \log\operatorname{card}\{v\in\overline{E};\ \|v\|\leqslant e^{\lambda}\} \\ &\geqslant \log\left(\sum_{\substack{v\in\overline{E}\\ \|v\|\leqslant e^{\lambda}}}\exp(-\pi\,\|v\|^2)\right) \\ &> \left(1-\frac{\operatorname{rank}E}{2\pi e^{2\lambda}}\right)+\hat{h}^0(\overline{E}). \end{aligned}$$

Replacing $\overline{E}$ by $\overline{E}(-\lambda)$ and applying Corollary 1.4.11, we finally obtain the inequality

$$h^0(\overline{E}) > \left(1-\frac{\operatorname{rank}E}{2\pi e^{2\lambda}}\right)+\hat{h}^0(\overline{E}(-\lambda)) > 1-\frac{\operatorname{rank}E}{2\pi e^{2\lambda}} - \lambda\operatorname{rank}\overline{E}+\widehat{\deg}\,\overline{E},$$

which holds for any real number $\lambda > \frac{1}{2}\log(\operatorname{rank}\overline{E}/2\pi)$. The optimal value for λ is given by $e^{2\lambda} = (2+\operatorname{rank}\overline{E})/2\pi$ and the Corollary follows by a straightforward computation. □

Remark 1.4.15. In order to compare this proposition to Minkowski's theorem, let us pose $n = \operatorname{rank}\overline{E}$ and assume, for simplicity that $F = \mathbf{Q}$ and $S = \operatorname{Spec}\mathbf{Z}$. When $n\to\infty$, the asymptotic expansion of the right hand side writes

$$h^0(\overline{E}) - \widehat{\deg}\,\overline{E} \geqslant -\frac{1}{2}n\log n + \frac{1}{2}\log(2\pi)n + \mathrm{O}(1).$$

On the other hand, Minkowski's theorem, in the form of Prop. 1.4.2, together with Stirling's formula $\log\Gamma(1+n/2) \sim \frac{n}{2}\log\frac{n}{2}$, implies that

$$h^0(\overline{E}) - \widehat{\deg}\,\overline{E} \geqslant -\frac{1}{2}n\log n + \frac{1}{2}\log\left(\frac{\pi e}{2}\right)n + \mathrm{O}(\log n).$$

Since $e<4$, the theta-functions approach gives a slightly better result than the original Minkowski theorem. Recall however that the theta-functions approach is specific to Euclidean norms, in the contrary to Minkowski theory.

D. Applications to algebraic number theory

As advocated by SZPIRO (1985), the results of the previous Section allow to give transparent proofs of some important and basic results in Algebraic number theory.

The first one claims that there is no non trivial unramified extension of $\mathbf{Q}$.

THEOREM 1.4.16 (Minkowski). *Let K be a non trivial extension of $\mathbf{Q}$; then, its discriminant satisfies $|D_K| > 1$.*

Proof. — Let $\overline{E}$ be a Hermitian line bundle over S such that $\widehat{\deg}\,\overline{E} = \chi(1,K)$. By Prop. 1.4.2, $h^0(\overline{E}) > 0$; consequently, Lemma 1.4.1 implies that $\chi(1,K) \geqslant 0$. Since $\beta_1 = 2$ and $\beta_2 = \pi$, we obtain the inequality

$$\frac{1}{2}\log|D_K| - r_2 \log \pi \geqslant 0.$$

In particular, $|D_K| \geqslant \pi^{2r_2}$. This is certainly sufficient to prove that K is ramified if $r_2 > 0$, but does not allow to conclude in the case K is totally real.

To prove that $|D_K| > 1$ in the remaining case, it suffices to be able to produce some non-trivial Hermitian line bundle such that $\widehat{\deg}\,\overline{E} = \chi(1,K)$. Since $r_2 = 0$ and $K \neq \mathbf{Q}$, we have $r_1 > 1$. We begin with any Hermitian line bundle $\overline{E}$ such that $\widehat{\deg}\,\overline{E} = \chi(1,K)$. If $\overline{E}$ were trivial, we can rescale the metrics of $\overline{E}$ at two different archimedean places in an uncountable number of ways, while there are only countably many units. Consequently, one of these rescaling will be a non-trivial hermitian line bundle, as desired. □

Remark 1.4.17. The obtained inequality is definitely not the best one available. Techniques of geometry of numbers lead to consider more general convex domains than ellipsoids and give the lower bound

$$\frac{1}{2d}\log|D_K| \geqslant \frac{r_2}{d}\log(\pi/4) + \frac{1}{d}\log d^d d! \gtrsim 1 - \frac{r_2}{d}\log(4/\pi) \gtrsim 1 - \frac{1}{2}\log(4/\pi).$$

Analytic number theory allows for even better bounds.

If X is an algebraic curve over a finite field, the group $\mathrm{Pic}^0(X)$ of isomorphism classes of divisors of degree 0 is a finite group. In the case of number fields, we show below that the analogous group $\widehat{\mathrm{CH}}^1(S)^0$ of isomorphism classes of hermitian line bundles on S whose arithmetic degree is zero is a *compact* Abelian group.

We define a topology on $\widehat{\mathrm{CH}}^1(S)$ to be the quotient topology of the topology of uniform convergence of the coefficients on the group $\widehat{\mathrm{Z}}^1(S)$ of arithmetic divisors. As a topological group, $\widehat{\mathrm{Z}}^1(S)$ is the direct sum of the discrete group $\mathrm{Z}^1(S)$ and of the real vector space $(\mathbf{R}^\Sigma)^{F_\infty}$. The quotient group $\widehat{\mathrm{CH}}^1(S) = \widehat{\mathrm{Z}}^1(S)/\widehat{\mathrm{Rat}}^1(S)$ then possesses the quotient topology. The groups $\widehat{\mathrm{Z}}^1(S)_0$ and $\widehat{\mathrm{CH}}^1(S)_0$ consisting of arithmetic divisors (resp. of classes of arithmetic divisors) of degree 0 are then endowed with the induced topologies.

THEOREM 1.4.18. *The subgroup* $\widehat{\mathrm{Rat}}^1(S)$ *is a discrete subgroup of* $\widehat{\mathrm{Z}}^1(S)_0$. *The quotient group* $\widehat{\mathrm{CH}}^1(S)_0$ *is compact.*

Proof. — a) We have already proved that $\widehat{\mathrm{Rat}}^1(S)$ is contained in $\widehat{\mathrm{Z}}^1(S)_0$. Let us establish its discreteness.

Let t be any positive real number. By definition, the set Ω of arithmetic divisors of the form $(0,g)$, with $|g_\sigma| < t$ for any $\sigma \in \Sigma$ is an open neighborhood of 0 in $\widehat{\mathrm{Z}}^1(S)$. Let $a \in K^*$ such that $\widehat{\mathrm{div}}(a) \in \Omega$. One has $\widehat{\mathrm{div}}(a) = (0, -\log_\Sigma(a))$, hence $a \in \mathfrak{o}_K^*$ and $e^{-t} < |\sigma(a)| < e^t$ for any $\sigma \in \Sigma$. In particular, $a \in \mathfrak{o}_K$ and $|\sigma(a)| < e^t$ for any $\sigma \in \Sigma$. By the finiteness lemma

(Lemma 1.2.2), the set of such a is finite. Consequently, $\Omega \cap \widehat{\mathrm{Rat}}^1(S)$ is finite, hence the discreteness of $\widehat{\mathrm{Rat}}^1(S)$ in $\widehat{\mathrm{Z}}^1(S)$.

b) The quotient group $\widehat{\mathrm{CH}}^1(S)_0$ is Hausdorff. To prove its compactness, we need to show that there exists a compact subset Ω in $\widehat{\mathrm{Z}}^1(S)_0$ which meets every class of arithmetic divisors of degree 0.

Let t be a real number such that $t > \chi(1,K)$ and let (E,h) be a fixed arithmetic divisor of degree t. Let Ω_t be the set of effective arithmetic divisors of degree t; by Lemma 1.4.19 below, it is a compact subset of $\widehat{\mathrm{Z}}^1(S)$.

Let (D,g) be an element in $\widehat{\mathrm{Z}}^1(S)_0$. Since $\widehat{\deg}((D,g)+(E,h)) = t > \chi(1,K)$, there exists $a \in K^*$ such that $\widehat{\mathrm{div}}(a)+(D,g)+(E,h) \geqslant 0$, by Corollary 1.4.4. Consequently, $\widehat{\mathrm{div}}(a)+(D,g)+(E,h) \in \Omega_t$ and (D,g) is equivalent to an element of the translated set $\Omega_t-(E,h)$, which is compact. This concludes the proof. □

LEMMA 1.4.19. *For any real number t, the set of arithmetic divisors of degree t which are effective is a compact subset of $\widehat{\mathrm{Z}}^1(S)$.*

Proof. — Let us denote this set by Ω_t. Let $(D,g) \in \Omega_t$. Writing $D = \sum n_{\mathfrak{p}}[\mathfrak{p}]$, the relation $\widehat{\deg}(D,g) = \sum n_{\mathfrak{p}} \log \mathrm{N}(\mathfrak{p}) + \sum_\sigma g_\sigma$ and the effectivity of (D,g) imply that $0 \leqslant n_{\mathfrak{p}} \leqslant t/\log \mathrm{N}(\mathfrak{p})$ for any maximal ideal $\mathfrak{p}$, and that $g_\sigma \leqslant t$ for any $\sigma \in \Sigma$. Since $n_{\mathfrak{p}}$ is an integer, we see that $n_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ such that $\mathrm{N}(\mathfrak{p}) \geqslant e^t$. It follows that Ω_t is contained in the compact subset $\prod_{\mathrm{N}(\mathfrak{p})<e^t}[0,t] \times \prod_\sigma [0,t]$ of $\widehat{\mathrm{Z}}^1(S)$, in which it is obviously closed. □

COROLLARY 1.4.20. *The class group of $\mathfrak{o}_K$ is finite.*

Proof. — The topological group $\mathrm{Z}^1(S)$ is discrete, hence so is its quotient $\mathrm{CH}^1(S)$. The canonical projection $\widehat{\mathrm{Z}}^1(S)_0 \to \mathrm{Z}^1(S)$ is continuous, and so is the surjective morphism $\widehat{\mathrm{CH}}^1(S)_0 \to \mathrm{CH}^1(S)$ deduced from it by quotienting by $\widehat{\mathrm{Rat}}^1(S)$. Consequently, $\mathrm{CH}^1(S)$ is quasi-compact; since it is discrete, it is also finite. □

COROLLARY 1.4.21 (Dirichlet's unit theorem). *The image of $\mathfrak{o}_K^*$ by the logarithmic map* $\log_\Sigma$ *is a lattice in the hyperplane $(\mathbf{R}^\Sigma)_0^{F_\infty}$ defined by the equation $\sum x_\sigma = 0$ in the real vector space $(\mathbf{R}^\Sigma)^{F_\infty}$. In particular, the Abelian group $\mathfrak{o}_K^*$ is finitely generated, of rank r_1+r_2-1.*

Proof. — To shorten the notation, let A and A_0 be the real vector spaces $A = (\mathbf{R}^\Sigma)^{F_\infty}$ and $A_0 = (\mathbf{R}^\Sigma)_0^{F_\infty}$.

The natural injections $A \hookrightarrow \widehat{\mathrm{Z}}^1(S)$ and $A_0 \hookrightarrow \widehat{\mathrm{Z}}^1(S)_0$ given by $g \mapsto (0,g)$ admit continuous retractions, namely the morphisms $(D,g) \mapsto g$ and $(D,g) \mapsto g + \widehat{\deg}(D,0)h$, where h is any fixed element in A such that $\widehat{\deg}(0,h) = 1$. Consequently, these inclusions are homeomorphisms onto closed subgroups of $\widehat{\mathrm{Z}}^1(S)$ and $\widehat{\mathrm{Z}}^1(S)_0$.

One has $\widehat{\mathrm{Rat}}^1(S) \cap A_0 = \log_\Sigma(\mathfrak{o}_K^*)$; since $\widehat{\mathrm{Rat}}^1(S)$ is discrete in $\widehat{\mathrm{Z}}^1(S)_0$, $\log_\Sigma(\mathfrak{o}_K^*)$ is discrete in A_0 and the quotient group $A_0/\log_\Sigma(\mathfrak{o}_K^*)$ is Hausdorff.

Let Ω be a compact subset of $\widehat{\mathrm{Z}}^1(S)_0$ which meets every class of arithmetic divisors of degree 0. Its projection P to $\mathrm{Z}^1(S)$ is compact and discrete, hence finite. Let us fix, for any principal divisor $p \in P$, some element $a_p \in K^*$ such that $\mathrm{div}(a_p) = p$. Let Ω' be the union of the finitely many translates $\Omega + \widehat{\mathrm{div}}(a_p)$; it is a compact subset in $\widehat{\mathrm{Z}}^1(S)_0$.

Let $g \in A_0$; by definition of Ω, there exists $a \in K^*$ such that $(0, g) - \widehat{\mathrm{div}}(a) \in \Omega$. Let $p = -\mathrm{div}(a)$; by construction, p is a principal divisor in P. Consequently, $(0, g) - \widehat{\mathrm{div}}(a) + \widehat{\mathrm{div}}(a_p)$ belongs to Ω'. Since $\mathrm{div}(a) = \mathrm{div}(a_p)$, $u = a/a_p \in \mathfrak{o}_K^*$ and $\widehat{\mathrm{div}}(u) = \log_\Sigma(u)$. This shows that $g - \log_\Sigma(u)$ belongs to the compact subset $\Omega' \cap A_0$ of A_0. It follows that the quotient group $A_0 / \log_\Sigma(\mathfrak{o}_K^*)$ is compact.

As a discrete cocompact subgroup of the real vector space A_0, $\log_\Sigma(\mathfrak{o}_K^*)$ is a lattice in that vector space. In particular, it is a free Abelian group of finite rank $\dim A_0 = r_1 + r_2 - 1$. Since the kernel of $\log_\Sigma$ is finite, we conclude that $\mathfrak{o}_K^*$ is itself a finitely generated Abelian group of rank $r_1 + r_2 - 1$. □

We observe that the exact sequence

$$0 \to A_0 / \log_\Sigma(\mathfrak{o}_K^*) \to \widehat{\mathrm{CH}}^1(S)_0 \to \mathrm{CH}^1(S) \to 0$$

is an exact sequence of compact Abelian groups.

§ 1.5
SLOPES, THE STUHLER-GRAYSON FILTRATION

A. Sizes of morphisms; successive minima

A.1. Hermitian theory. — Let $\varphi \colon V \to W$ be a morphism of Hermitian vector spaces. Let us introduce various real numbers attached to φ. The first one is its operator norm, defined by

$$\|\varphi\| = \max_{\substack{x \in V \\ \|x\| \leqslant 1}} \|\varphi(x)\| = \max_{x \in V \setminus \{0\}} \frac{\|\varphi(x)\|}{\|x\|}$$

(the maximum is taken in $[0, +\infty[$, so is 0 by convention if $V = 0$).

There exist orthonormal bases $(e_1, \dots, e_n)$ of V and $(f_1, \dots, f_m)$ of W in which the matrix of φ takes the form

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} \lambda_1 & & 0 & 0 & \dots & 0 \\ & \ddots & & \vdots & & \\ & & \lambda_r & 0 & & \\ 0 & \dots & 0 & 0 & \dots & 0 \\ & \dots & & & \ddots & \end{pmatrix}$$

where $\lambda_1, \dots, \lambda_r$ are real numbers such that $\lambda_1 \geqslant \dots \geqslant \lambda_r > 0$.

We let $\mathrm{M}_i(\varphi) = \lambda_1 \dots \lambda_i$, if $i \leqslant r$, and $\mathrm{M}_i(\varphi) = 0$ if $i > r$. In fact, $(\lambda_1, \dots, \lambda_r)$ have the following alternative descriptions:

– for any integer i, λ_i is the ith eigenvalue of the selfadjoint endomorphism $\sqrt{\varphi^* \varphi}$ of V;

– for any integer i, $\mathrm{M}_i(\varphi) = \left\|\bigwedge^i \varphi\right\|$. Moreover, the minimax principle asserts that
$$\mathrm{M}_i(\varphi) = \min_{\operatorname{codim} U = i-1} \left\|\varphi|_U\right\| = \min_{\operatorname{codim} U=i-1} \max_{\substack{x\in U\\ \|x\|\leqslant 1}} \left\|\varphi(x)\right\|;$$
– in particular, the operator norm of φ is equal to $\lambda_1 = \mathrm{M}_1(\varphi)$.

Constructions from linear algebra lead to various inequalities, for example $\mathrm{M}_1(\varphi\otimes\psi) \leqslant \mathrm{M}_1(\varphi)\mathrm{M}_1(\psi)$. Hadamard inequality implies that $\mathrm{M}_i(\varphi) = \mathrm{M}_1(\bigwedge^i \varphi)$.

A.2. p-adic theory. — Let K be a field with a non-trivial absolute value defined by a discrete valuation, let R be its valuation ring. Let us also fix an uniformizing element π of R.

Let E and F be free R-modules of finite rank, let $\varphi\colon V \to W$ be a linear map, where $V = E_K$ and $W = F_K$. We endow the vector space V with the norm $\|\cdot\|$ defined by the formula
$$\|v\| = \inf\{|\pi|^k \,;\, v \in \pi^k E\},$$
and similarly for the vector space W.

According to the structure theorem for modules of finite type over principal ideal rings, there are bases $(e_1,\dots,e_n)$ of E and $(f_1,\dots,f_m)$ of F in which the matrix of φ takes the form
$$\mathrm{Mat}(\varphi) = \begin{pmatrix} \lambda_1 & & 0 & 0 & \dots & 0 \\ & \ddots & & \vdots & & \\ & & \lambda_r & \vdots & & \\ 0 & \dots & 0 & 0 & \dots & 0 \\ & \dots & & & \ddots & \end{pmatrix}$$
where $\lambda_1,\dots,\lambda_r$ are elements of K $|\lambda_1| \geqslant \dots \geqslant |\lambda_r|$. As in the Hermitian case, we let $\mathrm{M}_i(\varphi) = |\lambda_1|\dots|\lambda_i|$ if $i \leqslant r$, and $\mathrm{M}_i(\varphi)$ for $i > r$; these real numbers $\mathrm{M}_i(\varphi)$ have the alternative descriptions:

– for any integer i, $\mathrm{M}_i(\varphi)$ is the operator norm $\left\|\bigwedge^i \varphi\right\|$ of the morphism $\bigwedge^i \varphi\colon \bigwedge^i V \to \bigwedge^i W$, where these spaces are normed as above via the identification $(\bigwedge^i E)_K = \bigwedge^i V$ (similarly for W);
– in particular, $|\lambda_1|$ is the operatorn norm of φ;
– there is a minimax principle:
$$\mathrm{M}_i(\varphi) = \min_{\operatorname{codim} U=i-1} \left\|\varphi|_U\right\| = \min_{\operatorname{codim} U=i-1} \max_{x\in U\cap E} \left\|\varphi(x)\right\|.$$

As in the Hermitian case, constructions from linear algebra lead to various inequalities, for example $\mathrm{M}_1(\varphi\otimes\psi) \leqslant \mathrm{M}_1(\varphi)\mathrm{M}_1(\psi)$, and $\mathrm{M}_i(\varphi) = \mathrm{M}_1(\bigwedge^i \varphi)$.

A.3. Global theory. — We keep the same notation concerning the number field K, Σ, $S = \operatorname{Spec} \mathfrak{o}_K$. Let $\overline{E}$ and $\overline{F}$ be Hermitian vector bundles over S. Let $\varphi\colon E_K \to F_K$ we a linear map.

For any integer i and any complex embedding $\sigma \in \Sigma$, we let $\mathrm{M}_{i,\sigma}(\varphi)$ for the real number attached as above to the morphism of Hermitian vector spaces $\sigma^*\varphi\colon \sigma^*E \to \sigma^*F$.

For any maximal ideal $\mathfrak{p}$ of $\mathfrak{o}_K$, the $\mathfrak{p}$-adic absolute value of $K_{\mathfrak{p}}$ is normalized so that $|a|_{\mathfrak{p}} = 1/\mathrm{N}(\mathfrak{p})$ for any $a \in \mathfrak{p}\setminus\mathfrak{p}^2$. For any integer i, we then write $M_{i,\mathfrak{p}}(\varphi)$ for the real number attached as above to the morphism φ and the $\mathfrak{o}_{K,\mathfrak{p}}$-modules deduced from E and F.

For any integer $i \leqslant \operatorname{rank}(\varphi)$, there are only finitely many maximal ideals $\mathfrak{p}$ such that $\mathrm{M}_{i,\mathfrak{p}}(\varphi) \neq 1$. We thus can define

$$\mathrm{M}_i(\varphi) = \sum_{\mathfrak{p}} \log \mathrm{M}_{i,\mathfrak{p}}(\varphi) + \sum_{\sigma\in\Sigma} \log \mathrm{M}_{i,\sigma}(\varphi).$$

In the particular case where $i = 1$ and $\varphi\colon E_K \to F_K$ is non-zero, we also define the *height of* φ by the formula $h(\varphi) = \log \mathrm{M}_1(\varphi)$, and similarly for $h_{\mathfrak{p}}(\varphi)$ and $h_\sigma(\varphi)$. Observe finally the inequalities $h(\varphi\otimes\psi) \leqslant h(\varphi) + h(\psi)$ and $\mathrm{M}_i(\varphi) = h(\bigwedge^i \varphi) \leqslant i h(\varphi)$.

B. Slopes inequalities

Let $\overline{E}$ be an Hermitian vector bundle. If $\overline{E} \neq 0$, one defines its slope by

$$\widehat{\mu}(\overline{E}) = \frac{\widehat{\deg}(\overline{E})}{\operatorname{rank}(\overline{E})}.$$

The slope of the null vector bundle is undefined.

Let $\overline{E}$ and $\overline{F}$ be two non-zero Hermitian vector bundles on S; one has

$$\widehat{\mu}(\overline{E}\otimes\overline{F}) = \widehat{\mu}(\overline{E}) + \widehat{\mu}(\overline{F}), \qquad \widehat{\mu}(\overline{E}^{\vee}) = -\widehat{\mu}(\overline{E}).$$

Let $0 \to \overline{F} \to \overline{E} \to \overline{G} \to 0$ be an exact sequence of non-zero Hermitian vector bundles on S; one has

$$\widehat{\mu}(\overline{E}) = \frac{\operatorname{rank} F}{\operatorname{rank} E}\widehat{\mu}(\overline{F}) + \frac{\operatorname{rank} G}{\operatorname{rank} E}\widehat{\mu}(\overline{G}).$$

PROPOSITION 1.5.1. *Assume that $\varphi\colon E_K \to F_K$ is injective. Then,* $\widehat{\deg}(\overline{E}) \leqslant \widehat{\deg}(\overline{F}) + \mathrm{M}_n(\varphi)$, *where* $n = \operatorname{rank}(\overline{E})$. *In particular,*

$$\widehat{\mu}(\overline{E}) \leqslant \widehat{\mu}(\overline{F}) + h(\varphi).$$

Proof. — Considering the morphism $\bigwedge^n \varphi\colon \det(E_K) \to \det(F_K)$ and using the inequality $h(\bigwedge^n(\varphi)) \leqslant n h(\varphi)$, we may assume that $\overline{E}$ and $\overline{F}$ have rank 1. In that case, we will even see that the equality $\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{F}) + h(\varphi)$ holds.

Let e be any non-zero element of E such that $\varphi(e) \in F$. By definition, $\widehat{\operatorname{div}}(\varphi(e)) = \widehat{\operatorname{div}}(e) + \widehat{\operatorname{div}}(\varphi)$, where we have set

$$\widehat{\operatorname{div}}(\varphi) = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(\varphi)[\mathfrak{p}] + (-h_\sigma(\varphi)).$$

Since $\widehat{\deg}\,\widehat{\operatorname{div}}(\varphi)) = -h(\varphi)$, we obtain

$$\widehat{\deg}(\overline{F}) = \widehat{\deg}(\widehat{\operatorname{div}}(\varphi(e))) = \widehat{\deg}(\widehat{\operatorname{div}}(e)) + \widehat{\deg}(\widehat{\operatorname{div}}(\varphi)) = \widehat{\deg}(\overline{E}) - h(\varphi).$$

□

COROLLARY 1.5.2. *Let $\overline{E}$ be a non-zero hermitian vector bundle over S.*

a) When $\overline{F}$ runs among the non-zero sub-bundles of $\overline{E}$ (that is, the non-zero submodules such that E/F is torsion free), $\widehat{\deg}\,\overline{F}$ is bounded from above.

b) When $\overline{F}$ runs among the non-zero quotients of $\overline{E}$, $\widehat{\deg}\,\overline{F}$ is bounded from below.

Proof. — a) Observe that a sub-bundle $\overline{F}$ of $\overline{E}$ is determined by its span F_K as a K-vector subspace in E_K, by the formula $F = E \cap F_K$, the hermitian metrics being the ones deduced by restriction.

Let $(e_1, \dots, e_n)$ be a basis of E_K consisting of elements of E. For any $I \subset \{1, \dots, n\}$, let V_I be the K-span of the family $(e_i)_{i\in I}$ and let $\overline{E}_I$ be the hermitian subbundle $\overline{E} \cap V_I$. Any subspace V of E_K has a complement of the form V_I, for some subset $I \subset \{1, \dots, n\}$. Let $\overline{F} = V \cap \overline{E}$ and let us consider the canonical map $\varphi \colon V \oplus V_I \to E_K$. It is bijective and satisfies $\widehat{\operatorname{div}}(\varphi) \geqslant 0$, hence $h(\varphi) \leqslant 0$. It follows from Proposition 1.5.1 that $\widehat{\deg}(\overline{F}) + \widehat{\deg}(\overline{E}_I) \geqslant \widehat{\deg}(\overline{E})$, which gives the asserted upper bound for $\widehat{\deg}(\overline{F})$.

Assertion b) then follows from a) by duality. □

The basic Proposition 1.5.1 allows to define the maximal and minimal slopes of $\overline{E}$, as

$$\widehat{\mu}_{\max}(\overline{E}) = \sup_{0 \neq V \subset E_K} \widehat{\mu}(\overline{E} \cap V), \qquad \widehat{\mu}_{\min}(\overline{E}) = \inf_{V \subsetneq E_K} \widehat{\mu}(\overline{E}/\overline{E} \cap V).$$

These are real numbers such that

$$\widehat{\mu}_{\min}(\overline{E}) \leqslant \widehat{\mu}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{E}).$$

Let moreover $\overline{F}$ be any Hermitian submodule of $\overline{E}$; then, $\overline{E} \cap F_K$ is a Hermitian sub-bundle of $\overline{E}$ containing $\overline{F}$, so that

$$\widehat{\deg}(\overline{E}/\overline{F}) = \widehat{\deg}(\overline{E}/\overline{E} \cap F_K) + \log \operatorname{card}((\overline{E} \cap F_K)/\overline{F}) \geqslant \widehat{\deg}(\overline{E}/\overline{E} \cap F_K).$$

Since $\overline{E} \cap F_K$ and $\overline{F}$ have the same rank, we see that

$$\widehat{\mu}_{\min}(\overline{E}) = \inf_{0 \neq \overline{F} \subset \overline{E}} \widehat{\mu}(\overline{E}/\overline{F}).$$

Moreover, this notation allows for the following Corollary to Proposition 1.5.1.

COROLLARY 1.5.3. *Let $\overline{E}$ and $\overline{F}$ be non-zero hermitian vector bundles over S and let $\varphi \colon E_K \to F_K$ be a linear map.*

a) If φ is injective, then

$$\widehat{\mu}_{\max}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{F}) + h(\varphi).$$

b) If φ is surjective, then

$$\widehat{\mu}_{\min}(\overline{E}) \leqslant \widehat{\mu}_{\min}(\overline{F}) + h(\varphi).$$

c) In any case,

$$\widehat{\mu}_{\min}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{F}) + h(\varphi).$$

Proof. — Assertion a) is proved by applying Proposition 1.5.1 to any hermitian sub-bundle $\overline{E}'$ of $\overline{E}$; part b) follows by duality. Let $\overline{F}'$ be the Hermitian subsheaf of $\overline{F}$ given by $F' = \varphi(E)$. By b),

$$\widehat{\mu}_{\min}(\overline{E}) \leqslant \widehat{\mu}_{\min}(\overline{F}') + h(\varphi) \leqslant \widehat{\mu}_{\max}(\overline{F}') + h(\varphi),$$

hence c). □

The following extension of Proposition 1.5.1 to filtered hermitian vector bundles is very useful in Diophantine approximation. It is usually refered to as the *slope inequality*.

Let $\overline{E}$ be a nonzero hermitian vector bundle over S, let $(\overline{G}_n)_{n\in\mathbf{N}}$ be a sequence of hermitian vector bundles over S. Let V be a K-vector space over K endowed with a decreasing exhausting filtration $(F^nV)_{n\geqslant 0}$ such that, for any $n\in\mathbf{N}$, the graded piece $\mathrm{gr}^n_F V$ is equal to $G_{n,K}$.

Let $\varphi\colon E_K\to V$ be an injective K-linear map. For any $n\in\mathbf{N}$, let $E^n_K=\varphi^{-1}(F^nV)$, let $\overline{E}^n=E^n_K\cap\overline{E}_n$. The morphism φ induces an injective K-linear morphism $\varphi^n\colon E^n_K/E^{n+1}_K\hookrightarrow G_{n,K}$. By Proposition 1.5.1, we have

$$\widehat{\deg}(\overline{E}^n/\overline{E}^{n+1})\leqslant \operatorname{rank}(E^n_K/E^{n+1}_K)\left(\widehat{\mu}_{\max}(\overline{G}_n)+h(\varphi_n)\right),$$

where this inequality has to be interpreted as $0\leqslant 0$ is $\overline{E}^n=\overline{E}^{n+1}$. Since the filtration $(F^nV)_n$ is exhausting and E_K is finite dimensional, one has $\overline{E}^n=0$ for n large enough, hence

$$\widehat{\deg}(\overline{E})\leqslant \sum_{n\geqslant 0}\operatorname{rank}(E^n_K/E^{n+1}_K)\left(\widehat{\mu}_{\max}(\overline{G}_n)+h(\varphi_n)\right). \tag{1.5.4}$$

This is *Bost's slope inequality*.

C. The Stuhler–Grayson filtration

Let $\overline{E}$ be a non-zero Hermitian vector bundle on S.

LEMMA 1.5.5 (Finiteness lemma,II). *For any real number c, there are only finitely many Hermitian submodules $\overline{F}$ of $\overline{E}$ such that $\widehat{\deg}\overline{F}\geqslant c$.*

Proof. — We first reduce to the case where $K=\mathbf{Q}$. Indeed, let $\pi\colon S\to\operatorname{Spec}\mathbf{Z}$ be the natural morphism. Observe that the functor π_* inducing an injection from the set of Hermitian submodules of $\overline{E}$ to those of $\pi_*\overline{F}$. Moreover, the Grothendieck Riemann–Roch theorem (Theorem 1.3.1) compares the variation of arithmetic degrees. We also prove the finiteness result under the supplementary assumption that $\overline{F}$ has given rank, say r.

Let us now treat the case $r=1$. A rank 1 Hermitian subsheaf $\overline{F}$ of $\overline{E}$ has a generator $f\in E$ well defined up to sign, and $\widehat{\deg}(\overline{F})=-\log\|f\|$. In that case, the finiteness assertion is equivalent to the obvious fact that there are only finitely elements $f\in E$ such that $\|f\|\leqslant e^{-c}$ (discretenes of E in $E_{\mathbf{R}}$ plus compactness of the ball).

Let $\overline{F}$ be a rank r Hermitian submodule of $\overline{E}$ such that $\widehat{\deg}\overline{F}\geqslant c$. Let $\overline{F}_1$ be the Hermitian subbundle defined by $F_1=F_K\cap E$. Both Hermitian sheaves $\overline{F}$ and $\overline{F}_1$ have the same rank and satisfy $F\subset F_1$, hence

$$\widehat{\deg}(\overline{F}_1)=\widehat{\deg}(\overline{F})+\log\operatorname{card}(F_1/F).$$

In particular, $\widehat{\deg}(\overline{F}_1)\geqslant c$. By the theory of Grassmann coordinates, the rank r subspace F_K of E_K is determined by the line $\bigwedge^r F_K$ in $\bigwedge^r E_K$. By that rank 1 case, it follows that $\overline{F}_1$ belongs to a finite set of Hermitian subbundles, and $\operatorname{card}(F_1/F) \leqslant \exp(\widehat{\deg}(\overline{F}_1) - c)$.

To conclude, it now suffices to observe that for any positive integer n, the set of submodules F of F_1 such that $\operatorname{card}(F_1/F) \leqslant n$ is finite. By Lagrange's Theorem, such a subbmodule contains nF_1. Finally, the set of submodules of F_1 containing nF_1 is bijective to the set of subbmodules of F_1/nF_1, which is finite since this module is finite. □

PROPOSITION 1.5.6. *There is a largest Hermitian submodule $\overline{F}$ of $\overline{E}$ such that $\widehat{\mu}(\overline{F}) = \widehat{\mu}_{\max}(\overline{E})$. Moreover, $\overline{F}$ is a Hermitian subbundle and is stable under any automorphism of $\overline{E}$.*

Proof. — The existence of such Hermitian submodules follows readily from the Lemma and the definition of $\widehat{\mu}_{\max}(\overline{E})$. Let us prove the existence of a largest submodule. Let $\overline{F}_1$ and $\overline{F}_2$ be Hermitian submodules of $\overline{E}$ such that $\widehat{\mu}(\overline{F}_1) = \widehat{\mu}(\overline{F}_2) = \widehat{\mu}_{\max}(\overline{E})$. Then, the exact sequence

$$0 \to \overline{F_1 \cap F_2} \to \overline{F}_1 \oplus \overline{F}_2 \to \overline{(F_1 + F_2)} \to 0$$

implies that

$$\begin{aligned}\widehat{\deg}(\overline{F_1 + F_2}) &= \widehat{\deg}(\overline{F}_1) + \widehat{\deg}(\overline{F}_2) - \widehat{\deg}(\overline{F_1 \cap F_2}) \\ &\geqslant (\operatorname{rank}(F_1) + \operatorname{rank}(F_2) - \operatorname{rank}(F_1 \cap F_2))\, \widehat{\mu}_{\max}(\overline{E}) \\ &\geqslant \widehat{\mu}_{\max}(\overline{E}) \operatorname{rank}(F_1 + F_2),\end{aligned}$$

so that $\widehat{\mu}(\overline{F_1 + F_2}) = \widehat{\mu}_{\max}(\overline{E})$. In other words, the set of submodules with maximal slope is stable under sum. Since E is Noetherian, there is a maximal such submodule, say $\overline{F}$, as was to be shown.

To prove that $\overline{F}$ is a Hermitian subbundle, we need to prove that $F = F_K \cap E$. However, the Hermitian submodule $\overline{F}_1$ given by $F_1 = F_K \cap E$ satisfies

$$\widehat{\mu}(\overline{F}_1) = \frac{\widehat{\deg}(\overline{F}_1)}{\operatorname{rank}(F_1)} = \frac{\widehat{\deg}(\overline{F}) + \log \operatorname{card}(F_1/F)}{\operatorname{rank}(F)} \geqslant \widehat{\mu}(\overline{F}) + \frac{1}{\operatorname{rank}(F)} \log \operatorname{card}(F_1/F).$$

Since $\overline{F}$ has maximal slope, we obtain $F_1 = F$.

Similarly, for any automorphism g of $\overline{E}$, $\widehat{\mu}(g(\overline{F})) = \widehat{\mu}(\overline{F}) = \widehat{\mu}_{\max}(\overline{E})$. Consequently, $g(\overline{F})$ has maximal slope and so does $\overline{F + g(F)}$. By maximality, we obtain that $g(F) \subset F$. Applying this inclusion to the automorphism g^{-1} then furnishes the equality $F = g(F)$. □

DEFINITION 1.5.7. *A Hermitian vector bundle $\overline{E}$ over S is said to be semistable if $\widehat{\mu}_{\max}(\overline{E}) = \widehat{\mu}(\overline{E})$.*

The largest subbmodule of $\overline{E}$ with slope $\widehat{\mu}_{\max}(\overline{E})$ whose existence is asserted by Prop. 1.5.6 is called the destabilizing *subbbundle of $\overline{E}$ and writen $\overline{E}_{\mathrm{des}}$.*

Remark 1.5.8. Let $\overline{E}$ be a Hermitian vector bundle over S and let $\overline{F}$ be a Hermitian subbundle of $\overline{E}$. From the relations $\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{F}) + \widehat{\deg}(\overline{E/F})$ and $\operatorname{rank}(E) = \operatorname{rank}(F) +$

rank(E/F), we see that the inequalities $\widehat{\mu}(\overline{F}) > \widehat{\mu}(\overline{E})$ and $\widehat{\mu}(\overline{E/F}) < \widehat{\mu}(\overline{E})$ are equivalent. Consequently, $\overline{E}$ is semistable if and only of $\widehat{\mu}_{\min}(\overline{E}) = \widehat{\mu}(\overline{E})$.

Moreover, $\overline{E}$ is semistable if and only if $\overline{E}^{\vee}$ is semistable.

Let $\overline{E}$ be a Hermitian vector bundle over S and let $\overline{E}_{\mathrm{des}}$ be its destabilizing subbundle. By definition, $\widehat{\mu}_{\max}(\overline{E}_{\mathrm{des}}) \leqslant \widehat{\mu}_{\max}(\overline{E}) = \widehat{\mu}(\overline{E}_{\mathrm{des}})$, so that $\overline{E}_{\mathrm{des}}$ is semi-stable. In particular, $\overline{E} = \overline{E}_{\mathrm{des}}$ if and only if $\overline{E}$ is semi-stable.

Let us pose $E_0 = 0$. By induction, one constructs an increasing filtration $(\overline{E}_0, \overline{E}_1, \dots)$ of $\overline{E}$ by Hermitian subbundles such that $\overline{E_{i+1}/E_i} = \overline{E/E_i}_{\mathrm{des}}$. This filtration must be finite and exhaustive since the sequence $(\mathrm{rank}(E_i))$ is increasing. Let d be the smallest integer with $E_d = E$. By construction, the Hermitian subquotients $\overline{E_{i+1}/E_i}$ are semi-stables, their slopes $\mu_1, \dots, \mu_d$ satisfy

$$\widehat{\mu}_{\max}(\overline{E}) = \mu_1 > \mu_2 > \dots > \mu_d = \widehat{\mu}_{\min}(\overline{E}).$$

They are called the successive slopes of $\overline{E}$.

The filtration $0 = \overline{E_0} \subset \overline{E_1} \subset \dots \subset \overline{E_d} = \overline{E}$ is called the Harder–Narasimhan filtration of $\overline{E}$. It was introduced by U. STUHLER STUHLER (1976) and D. GRAYSON GRAYSON (1984), by analogy with the filtration of vector bundles on curves defined by Harder–Narasimhan.

D. Invariance of the Harder–Narasimhan filtration under extension of scalars

We show that this filtration is invariant under extension of scalars.

PROPOSITION 1.5.9. *Let $\overline{E}$ be a Hermitian vector bundle on S. Let K' be a finite extension of K, let $S' = \mathfrak{o}_{K'}$ and let $\pi \colon S' \to S$ be the canonical morphism. Then, $\overline{E}$ is semi-stable if and only if $\pi^*\overline{E}$ is semi-stable.*

Proof. — Let us pose $d = [K' : K]$. Let us observe that $\widehat{\mu}(\pi^*\overline{E}) = d\,\widehat{\mu}(\overline{E})$; considering Hermitian subbmodules of $\pi^*\overline{E}$ of the form $\pi^*\overline{F}$, where $\overline{F}$ is a Hermitian submodule of $\overline{E}$, we see that $\widehat{\mu}_{\max}(\pi^*\overline{E}) \geqslant d\,\widehat{\mu}_{\max}(\overline{E})$. In particular, if $\pi^*\overline{E}$ is semi-stable, then $\widehat{\mu}_{\max}(\pi^*\overline{E}) = \widehat{\mu}(\pi^*\overline{E})$, which implies $\widehat{\mu}(\overline{E}) \geqslant \widehat{\mu}_{\max}(\overline{E})$, so that $\overline{E}$ is semi-stable too.

To prove the converse assertion, let us assume that $\overline{E}$ is semi-stable and let us prove that $\pi^*\overline{E}$ is semi-stable too. In view of the first part of the proof, we may replace the extension K' by its Galois closure. Let us now write $\overline{F}' = (\pi^*\overline{E})_{\mathrm{des}}$. Let us first show that the submodule F' is of the form π^*F. For any element $\tau \in \mathrm{Gal}(K'/K)$, $\tau(F')$ is a submodule of π^*E satisfying

$$\widehat{\mu}(\overline{\tau(F')}) = \widehat{\mu}(\overline{F}') = \widehat{\mu}_{\max}(\pi^*\overline{E}).$$

By the maximality of $(\pi^*\overline{E})_{\mathrm{des}}$, we obtain $\tau(F') \subset F'$. Since this inclusion holds for any $\tau \in \mathrm{Gal}(K'/K)$, we obtain that $\tau(F') = F'$. As a consequence, F' comes from a subbmodule of E, *i.e.*, there exists a submodule F of E such that $\pi^*F = F'$. Then,

$$\widehat{\mu}(\overline{F}) = \frac{1}{[K' : K]}\,\widehat{\mu}(\pi^*\overline{F}) = \frac{1}{[K' : K]}\,\widehat{\mu}_{\max}(\pi^*\overline{E}) \geqslant \widehat{\mu}_{\max}(\overline{E}),$$

so that $\widehat{\mu}(\overline{F}) = \widehat{\mu}_{\max}(\overline{E}) = \widehat{\mu}(\overline{E})$ since $\overline{E}$ is semi-stable.

Let us now argue by contradiction. If $\pi^*\overline{E}$ were not semi-stable, we would have $\widehat{\mu}(\overline{F}') > \widehat{\mu}(\pi^*\overline{E})$, hence $\widehat{\mu}(\overline{F}) > \widehat{\mu}(\overline{E})$, contradiction. $\square$

CHAPTER 2

GEOMETRIC AND ARITHMETIC INTERSECTION THEORY

§ 2.1
CYCLES AND RATIONAL EQUIVALENCE

Our first goal in this Chapter is to provide the intersection theory necessary for our subsequent treatment of heights in Arakelov geometry. Our basic reference is FULTON (1998), supplemented by MATSUMURA (1980); THORUP (1990) and Grothendieck's *Éléments de géométrie algébrique.*

A. Order functions in one-dimensional rings

Let us recall that the dimension of a ring A is the supremum of the lengths of chains $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals in A. The dimension of a Noetherian *local* ring is finite.

Fields, and more generally, Artinian rings, have dimension 0: this means that prime ideals are maximal ideals. The rings of integers of a number field has dimension 1: except for the null ideal, the prime ideals are all maximal.

PROPOSITION 2.1.1. *Let A be a Noetherian domain, let K be its field of fractions. Let us assume that* $\dim(A) = 1$. *Then, for any $a \in A \setminus \{0\}$, the A-module $A/(a)$ has finite length. Moreover, there exists a unique group morphism* $\operatorname{ord}\colon K^* \to \mathbf{Z}$ *such that* $\operatorname{ord}(a) = \ell_A(A/(a))$ *for any $a \in A \setminus \{0\}$.*

Proof. — Let $a \in A \setminus \{0\}$. By the theory of associated primes applied to the A-module $A/(a)$, there exists a chain of ideals $(a) = I_0 \subset I_1 \subset \cdots \subset I_n = A$, as well as prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ in A such that $I_k/I_{k-1} \simeq A/\mathfrak{p}_k$ for any $k \in \{1, \dots, n\}$. Since $a \in I_0$, $a \in \mathfrak{p}_k$ for any k; consequently, the chain $(0) \subset \mathfrak{p}_k$ is a chain of length 1 of prime ideals in A. By hypothesis, $\dim(A) = 1$; these chains are therefore maximal which implies that $\mathfrak{p}_k$ is a maximal ideal for each k. It follows that $A/(a)$ has finite length, equal to n.

Let a and b be nonzero elements in A. Let us consider the sequence of A-modules:

$$0 \to A/(b) \xrightarrow{a} A/(ab) \to A/(a) \to 0,$$

where the first map associates to an element $x \pmod b$ the element $ax \pmod{ab}$. This is an exact sequence: this is obvious in the middle as well as on the right; since A

is a domain, the first map is injective. By additivity of lengths, one deduces the equality

$$\ell_A(A/(b)) - \ell_A(A/(ab)) + \ell_A(A/(a)) = 0.$$

The map ord_A defined on $A \setminus \{0\}$ by $a \mapsto \ell_A(A/(a))$ is a morphism of semigroups; since K^* is the group associated to the semigroup $A \setminus \{0\}$, this map ord_A extends uniquely to a morphism of groups from K^* to $\mathbf{Z}$. □

LEMMA 2.1.2. *Let A be a Noetherian domain of dimension* 1*. For any $a \in K^*$, one has*

$$\mathrm{ord}_A(a) = \sum_{\mathfrak{p}\in\mathrm{Spm}(A)} \mathrm{ord}_{A_\mathfrak{p}}(A_\mathfrak{p}/(a)).$$

Proof. — Since both sides of this equality are additivie, it suffices to check it for $a \in A \setminus \{0\}$. Then, let us consider a filtration

$$(a) = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = A$$

by ideals of A, as in the proof of Prop. 2.1.1. Considering localizations at a maximal ideal $\mathfrak{p}$, we get a similar filtration of ideals of $A_\mathfrak{p}$; however, the only inclusions $I_{k-1}A_\mathfrak{p} \subset I_kA_\mathfrak{p}$ which remain strict are those for which $I_k/I_{k-1} \simeq A/\mathfrak{p}$. Consequently,

$$\ell_A(A/(a)) = n = \sum_{\mathfrak{p}\in\mathrm{Spm}(A)} \sum_{\substack{k=1 \\ I_k/I_{k-1}\simeq A/\mathfrak{p}}}^{n} 1 = \sum_{\mathfrak{p}\in\mathrm{Spm}(A)} \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/(a)),$$

as was to be shown. □

PROPOSITION 2.1.3. *Let A be a one-dimensional Noetherian domain, let K be its field of fractions and let B be the integral closure of A in K. Let us assume that B is a A-module of finite type.*

For any $a \in K^*$, one has

$$\mathrm{ord}_A(a) = \sum_{\mathfrak{p}\in\mathrm{Spm}(A)} \sum_{\substack{\mathfrak{q}\in\mathrm{Spm}(B)\\ \mathfrak{q}\cap A=\mathfrak{p}}} [B/\mathfrak{q} : A/\mathfrak{p}]\, \mathrm{ord}_{B_\mathfrak{q}}(a).$$

Proof. — By assumption, B is generated as a A-module by finitely many elements $b_1, \dots, b_r$. Each of them belonging to K, there exists an element $a \in A \setminus \{0\}$ such that $ab_k \in A$ for $k \in \{1, \dots, r\}$; consequently, $aB \subset A$. It follows that the A-module B/A is a $A/(a)$-module of finite type; the ring $A/(a)$ being Artinian, B/A has finite length. Considering the exact sequence

$$0 \to (B/A)_a \to (B/A) \xrightarrow{a} (B/A) \to (B/A)/(a),$$

where $(B/A)_a$ is the set of elements $b \in (B/A)$ such that $ab = 0$, and using the additivity of lengths in exact sequences, we obtain the equality $\ell_A((B/A)/(a)) = \ell_A((B/A)_a)$.

Let us then consider the diagram of exact sequences

$$0[r]A[r][d]^aB[r][d]^aB/A[r][d]^a00[r]A[r]B[r]B/A[r]0.$$

By the snake's lemma, we obtain an exact sequence

$$0 \to (B/A)_a \to A/(a) \to B/(a) \to (B/A)/(a) \to 0,$$

hence the equality

$$\ell_A(B/(a)) = \ell_A(A/(a)) + \ell_A(B/(A+aB)) - \ell_A((B/A)_a) = \ell_A(A/(a)) = \operatorname{ord}_A(a).$$

Let us now consider a chain of ideals in $B_\mathfrak{q}$ of the form $(a) = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = B_\mathfrak{q}$, with successive quotients I_k/I_{k-1} isomorphic to $B/\mathfrak{q}$. Since $B/\mathfrak{q}$ is a finite extension of degree $[B/\mathfrak{q} : A/\mathfrak{p}]$ of the field $A/\mathfrak{p}$, one has

$$\ell_A(B_\mathfrak{q}/(a)) = [B/\mathfrak{q} : A/\mathfrak{p}]\ell_A(A/(a)).$$

Finally,

$$\operatorname{ord}_A(a) = \ell_A(B/(a)) = \sum_{\mathfrak{q}\in\operatorname{Spm}(B)} [B/\mathfrak{q} : A/\mathfrak{p}] \operatorname{ord}_B(B_\mathfrak{q}/(a)).$$

□

LEMMA 2.1.4. [(1)] *Let A and B be one-dimensional local Noetherian domains such that $A \subset B$; let $K \subset L$ be the corresponding extension of field of fractions. We assume that $\mathfrak{m}_B \cap A = \mathfrak{m}_A$, $\mathfrak{m}_A$ and $\mathfrak{m}_B$ being the maximal ideals of A and B respectively, that B is* flat *over A and that the degree of the field extension $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite, equal to d.*

Then, for any element $a \in K^*$, one has $\operatorname{ord}_B(a) = d \operatorname{ord}_A(a)$.

Proof. — Let $a \in A \setminus \{0\}$ and let $(a) = I_0 \subsetneq \dots I_n = A$ be a chain of ideals such that $I_k/I_{k-1} \simeq A/\mathfrak{m}_A$ for any $k \in \{1,\dots,n\}$; one has $n = \ell_A(A/(a)) = \operatorname{ord}_A(a)$. Let us consider the corresponding chain of ideals in B, namely

$$aB \subset I_0B \subset \cdots \subset I_nB = A.$$

For any $k \in \{1,\dots,n\}$, $I_kB/I_{k-1}B \simeq (I_k/I_{k-1}) \otimes_A B \simeq B/\mathfrak{m}_AB$, since B is flat over A. Consequently,

$$\ell_B(B/(a)) = \ell_A(A/(a))\ell_B(B/\mathfrak{m}_AB).$$

□

B. Definition, basic functoriality

Let X be an excellent Noetherian scheme. We refer to §34 in MATSUMURA (1980) or (GROTHENDIECK, 1965, 7.8) the definition. We will use the following properties of X, namely: X has a finite cover by affine open subsets of the form $\operatorname{Spec} A$, where the ring A satisfies the following properties:

– A is Noetherian;

– A is *universally catenary:* for any A-algebra of finite type B, and any two prime ideals $\mathfrak{p},\mathfrak{q} \in \operatorname{Spec} B$, such that $\mathfrak{p} \subset \mathfrak{q}$, $\dim(B_\mathfrak{q}) - \dim(B_\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}/\mathfrak{p})$. Geometrically, $\dim(B_\mathfrak{q})$ is the codimension of the closed subset $V(\mathfrak{q})$ of $\operatorname{Spec} B$, while $\operatorname{ht}(\mathfrak{q}/\mathfrak{p})$ is the codimension of $V(\mathfrak{q})$ in $V(\mathfrak{p})$. Therefore, the condition means that for any two irreducible closed subsets Z and Z' such that $Z \subset Z'$ in $\operatorname{Spec} B$, all maximal chains of irreducible closed subsets $Z = Z_0 \subsetneq \cdots \subsetneq Z_n = Z'$ have the same length $n = \operatorname{codim}(Z, \operatorname{Spec} B) - \operatorname{codim}(Z', \operatorname{Spec} B)$;

[(1)] Incorrect lemma

– A is a *Nagata* — aka *japanese* — *ring*: for any prime ideal $\mathfrak{p}$ in A, for any finite extension L of the fraction field K of $A/\mathfrak{p}$, the integral closure of $A/\mathfrak{p}$ in L is a finite $(A/\mathfrak{p})$-module.

Any open subset, any closed subset, any scheme of finite type over an excellent scheme is excellent; any localization $S^{-1}A$ of an excellent ring A is again an excellent ring. Fields are excellent rings, as well as rings of integers in number fields and discrete valuation ring of characteristic zero. Therefore, schemes of finite type over such rings are excellent; this will be our main, and essentially only, source of example of excellent schemes; also, the dimension of such schemes is finite.

Let X be an finite dimensional excellent scheme. By definition, the group of cycles on X, $\mathrm{Z}(X)$ is the free abelian group generated by the points of X. A point $x \in X$ can be identified with the irreducible closed subset $\overline{\{x\}}$ of which it is the generic point, or to the integral closed subscheme given by that subset. We shall freely move from one terminology to another, whenever this appears to be convenient. For any subscheme Y of X, we may view $\mathrm{Z}(Y)$ as a subgroup of $\mathrm{Z}(X)$; one has $\mathrm{Z}(X) = \mathrm{Z}(Y) \oplus \mathrm{Z}(X \setminus Y)$; in particular, $\mathrm{Z}(X) = \mathrm{Z}(X_{\mathrm{red}})$.

The group $\mathrm{Z}(X)$ can be endowed with various graduations, defined by functions on X, like the following ones: (y is a point of X, Y is its closure):

– (*dimension*) $y \mapsto \dim(Y)$;

– (*codimension*) $y \mapsto -\operatorname{codim}(Y,X) = -\dim \mathscr{O}_{X,y}$;

– (*modified dimension*) assuming that X is an A-scheme of finite type, where A is a fixed excellent Dedekind (one-dimensional, regular) domain with field of fractions K, define $\dim_A(Y) = \dim_K(Y_K) + 1$ if $Y_K \neq \varnothing$, $\dim_A(Y) = \dim(Y)$ otherwise.

These notions are essentially distinct in general. As an example, the reader may consider the case of $X = \operatorname{Spec} A[T]$, where A is a discrete valuation ring with maximal ideal (a), and the prime ideals (0), (T), $(aT-1)$, (a,T). However, dimension and codimension add to the constant $\dim(X)$ when X is equidimensional (this is the definition). Moreover, the modified dimension coincides with the modified dimension when A is the ring of integers of a number field [(2)], or if X is proper over A.

Since X is catenary, for any integral subschemes V and W of X such that $V \subset W$, $\operatorname{codim}(V,W) = \delta(W) - \delta(V)$ for any such function δ.

In the sequel, we shall mostly consider the dimension function, but sometimes we may have to resort to the modified dimension. Consequently, we let $\mathrm{Z}_p(X)$ be the subgroup of $\mathrm{Z}(X)$ generated by the point $y \in X$ such that $\dim(\overline{\{y\}}) = p$. One has $\mathrm{Z}(X) = \bigoplus_{p\in\mathbf{N}} \mathrm{Z}_p(X)$.

C. The divisor of a rational function

Let X be a finite dimensional integral excellent scheme and let $R(X)$ be its field of fractions. For any integral subscheme V of codimension 1 in X, the local ring $\mathscr{O}_{X,v}$ of X at the generic point v of V is a one-dimensional Noetherian local ring with field of fractions $R(X)$. We also write $\mathscr{O}_{X,V}$ for this ring, and let $\operatorname{ord}_V \colon R(X)^* \to \mathbf{Z}$ be the associated order function.

(2) Is it enough to assume Jacobson?

LEMMA 2.1.5. *For any $a \in R(X)^*$, there are only finitely many integral closed subschemes V of codimension* 1 *such that* $\operatorname{ord}_V(a) \neq 0$.

Proof. — Let $U = \operatorname{Spec} A$ be an affine subset of X on which a and a^{-1} exist as regular functions. In other words, $a \in A^*$. Consequently, for any integral subscheme V of codimension 1 such that $V \cap U \neq \varnothing$, $\operatorname{ord}_V(a) = 0$. Conversely, if the complement $\complement U$ to U in X is closed subset that contains only finitely many integral subschemes of codimension 1, since these are among its irreducible components and X is a Noetherian space. □

This legitimates the following definition:

DEFINITION 2.1.6. *Let X be a finite dimensional excellent integral scheme. For any $a \in R(X)^*$, the* divisor *of a is defined as the following cycle*

$$\sum_{\substack{v \in X \\ \dim \mathscr{O}_{X,v} = 1}} \operatorname{ord}_V(a)[V],$$

where we have identified a point $v \in X$ and its closure $V = \overline{\{v\}}$.
If $\dim(X) = p$, *then* $\operatorname{div}(a) \in Z_{p-1}(X)$.

Let X be an excellent scheme. The subgroup of cycles rationally equivalent to 0 is defined as the subgroup $\operatorname{Rat}(X)$ generated by all $\operatorname{div}(a)$, where $a \in R(V)^*$, V being an integral closed subscheme of X. If $\dim(V) = p+1$, then $\operatorname{div}(a) \in Z_p(X)$; we let $\operatorname{Rat}_p(X)$ be the subgroup of $Z_p(X)$ generated by these cycles. One has $\operatorname{Rat}(X) = \bigoplus_{p \in \mathbf{N}} \operatorname{Rat}_p(X)$.

DEFINITION 2.1.7. *The Chow group of X is defined as the quotient* $\operatorname{CH}(X) = Z(X)/\operatorname{Rat}(X)$. *For any $p \in \mathbf{N}$, let* $\operatorname{CH}_p(X) = Z_p(X)/\operatorname{Rat}_p(X)$; *this gives a graduation on* $\operatorname{CH}(X)$, *namely* $\operatorname{CH}(X) = \bigoplus_{p \in \mathbf{N}} \operatorname{CH}_p(X)$.

D. Direct image by a proper morphism

Let $f\colon X \to Y$ be a proper morphism of finite dimensional excellent schemes. By definition, this means that f is separated, namely the diagonal morphism $X \to X \times_Y X$ is a closed immersion, and that f is universally closed: for any Y-scheme Y', the morphism $f_{Y'}\colon X \times_Y Y' \to Y'$ deduced from f by base change is closed, *i.e.*, the image of closed subset of $X \times_Y Y'$ is closed in Y'.

For any closed irreducible subset V in X, its image $W = f(V)$ is therefore irreducible and closed. There is an induced extension of function fields, $R(W) \hookrightarrow R(V)$.

LEMMA 2.1.8. *One has* $\dim V = \dim W + \operatorname{tr.deg}_{R(W)}(R(V))$. *In particular, the extension $R(W) \hookrightarrow R(V)$ is finite if and only if* $\dim(V) = \dim(W)$.

Proof. — CHECK CHECK CHECK This follows from EGA IV, 5.6.5, at least if dim is replaced by the modified dimension. □

We then define a *push forward* morphism $f_*\colon \mathrm{Z}(X) \to \mathrm{Z}(Y)$ by setting, for any integral closed subscheme $V \subset X$, with $W = f(V)$,

$$f_*([V]) = \begin{cases} [R(V) : R(W)][W] & \text{if the extension } R(W) \subset R(V) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

By linearity, this defines morphisms of abelian groups $f_*\colon \mathrm{Z}(X) \to \mathrm{Z}_p(Y)$, for any $p \in \mathbf{N}$.

Let $g\colon Y \to Z$ be a proper morphism of finite dimensional excellent schemes. One has $(g \circ f)_* = g_* \circ f_*$. Indeed, if V is an integral subscheme of X, $W = f(V)$, $T = g(W)$, one has two extensions $R(T) \subset R(W) \subset R(V)$, and these are finite if and only if $\dim T = \dim W = \dim V$. If this holds, then, $f_*([V)] = [R(V) : R(W)][W]$, $g_*([W]) = [R(W) : R(T)][T]$, hence, by multiplicativity of degrees in finite extensions,

$$(g \circ f)_*([V]) = [R(V) : R(T)][T] = [R(V) : R(W)][R(W) : R(T)][T] = g_*(f_*([V])).$$

THEOREM 2.1.9. *For any $p \in \mathbf{N}$, and any $z \in \mathrm{Rat}_p(X)$, one has $f_*(z) \in \mathrm{Rat}_p(Y)$.*

Consequently, the map f_* induces a morphism of Abelian groups, still written f_*, from $\mathrm{CH}_p(X)$ to $\mathrm{CH}_p(Y)$.

The proof of this theorem depends on two different, more precise, computations which we present as separate propositions.

PROPOSITION 2.1.10. *Let X and Y be integral finite dimensional excellent scheme of the same dimension, and let $f\colon X \to Y$ be a proper surjective morphism. For any $a \in R(X)^*$, one has $f_*(\mathrm{div}(a)) = \mathrm{div}(\mathrm{N}(a))$, where $\mathrm{N}\colon R(X) \to R(Y)$ is the norm of the finite extension $R(Y) \subset R(X)$.*

Proof. — Let X' and Y' be the normaizations of X and Y inside the fields $R(X)$ and $R(Y)$ of rational functions on X and Y respectively; we obtain a diagram of schemes, with proper morphisms,

$$X'[r]^p[d]^{f'} X[d]^f X[r]^q Y.$$

We will show the three equalities:

a) $p_*(\mathrm{div}_{X'}(a)) = \mathrm{div}_X(a)$;

b) $q_*(\mathrm{div}_{Y'}(\mathrm{N}(a))) = \mathrm{div}_Y(\mathrm{N}(a))$;

c) $(f')_*(\mathrm{div}_{Y'}(a) = \mathrm{div}_{X'}(\mathrm{N}(a))$,

from which the Proposition immediately follows since they imply

$$\begin{aligned} f_*(\mathrm{div}_X(a)) &= f_*(p_*(\mathrm{div}_{X'}(a))) = (f \circ p)_*(\mathrm{div}_{X'}(a)) \\ &= (q \circ f')_*(\mathrm{div}_{X'}(a)) = q_*(\mathrm{div}_{X'}(\mathrm{N}(a)) = \mathrm{div}_Y(\mathrm{N}(a)). \end{aligned}$$

Since $R(X) = R(X')$ and $R(Y) = R(Y')$, we are reduced to proving the proposition under one of the two supplementary assumptions:

a) X is the normalization of Y inside $R(Y)$;

b) X and Y are normal.

Let W be an integral subscheme of codimension 1 in Y, let A be the local ring $\mathscr{O}_{Y,W}$. Then, $f_A\colon X\times_Y \operatorname{Spec} A \to \operatorname{Spec} A$ is proper and its fibre at the maximal ideal of A is a scheme of dimension 0. (If this fiber possessed two integral subschemes V and V' such that $V \subsetneq V'$, these would be integral subschemes of X mapping dominantly to W, hence onto W since f is proper; however, this would imply $\dim(V) \leqslant \dim X - 2$, while $\dim(W) = \dim(Y) - 1 = \dim(X) - 1$, which contradicts the fact that a proper morphism decreases dimensions.) Consequently, f_A has finite fibres. By Chevalley's Theorem ((Grothendieck, 1963, 4.4.2)), f_A is a finite morphism, which means there exists an A-algebra B, finite as an A-module, such that $X_A = \operatorname{Spec} B$. Under both assumptions, B is the integral closure of A inside the field of fractions of X.

If $\mathfrak{p}$ is the maximal ideal of A corresponding to W, we have $R(W) = A/\mathfrak{p}$. Let V be any integral closed subscheme of X such that $f(V) = W$; it corresponds to V a maximal ideal $\mathfrak{q} \in \operatorname{Spm}(B)$ and the function field of $R(V)$ is equal to $B/\mathfrak{q}$. For any $b \in R(X)^*$, the multiplicity of V of X in the cycle $\operatorname{div}(b)$ is equal to $\operatorname{ord}_{B_\mathfrak{q}}(b)$, which is in turn equal to $\ell_B(B_\mathfrak{q}/(b))$ if $b \in B$. Consequently, to prove the proposition, it suffices to establish that for any $b \in B \setminus \{0\}$,

$$\sum_{\mathfrak{q}\in\operatorname{Spm}(B)} [B/\mathfrak{q} : A/\mathfrak{p}]\ell_B(B_\mathfrak{q}/(b)) = \ell_A(A/\mathrm{N}(b)).$$

If X is the normalization of Y inside $R(Y)$, this formula is the content of Proposition 2.1.3, since $R(X) = R(Y)$ and $\mathrm{N}(b) = b$ for any $b \in R(X)$.

Let us finally assume that A is integrally closed in $R(X)$. In that case, A is a discrete valuation ring. The A-module B, being of finite type and torsion-free, is free of finite rank, say r, and similarly for its submodule bB, for $b \in B \setminus \{0\}$. In other words, there exists bases $(e_1,\dots,e_r)$, $(f_1,\dots,f_r)$ of B as an A-module, in which the multiplication by b map has a diagonal matrix with entries $(a_1,\dots,a_r)$ in A. In particular, the A-module $B/(b)$ is isomorphic to $A/(a_1)\oplus\cdots\oplus A/(a_r)$.

We go on by observing that $B/\mathfrak{q}$ is a A-module of finite length, equal to the degree $[B/\mathfrak{q} : A/\mathfrak{p}]$ (see the proof of Prop. 2.1.3). Consequently,

$$\begin{aligned}\sum_{\mathfrak{q}\in\operatorname{Spm}(B)} [B/\mathfrak{q} : A/\mathfrak{p}]\ell_B(B_\mathfrak{q}/(b)) &= \sum_{\mathfrak{q}\in\operatorname{Spm}(B)} \ell_A(B_\mathfrak{q}/(b)) \\ &= \ell_A(B/(b)) = \sum_{k=1}^{r} \ell_A(A/(a_i)) = \ell_A(A/(a_1\dots a_r)).\end{aligned}$$

By definition, the norm $\mathrm{N}(b)$ is equal to the determinant of the matrix of the multiplication by b map in the basis $(e_1,\dots,e_r)$, since the family $(e_1,\dots,e_r)$ is also a basis of the field $R(X)$ as a $R(Y)$-vector space. Consequently, there exists a unit $u \in A^*$ such that $\mathrm{N}(b) = ua_1\dots a_r$, hence $\ell_A(A/(a_1\dots a_r)) = \ell_A(A/(\mathrm{N}(b)))$. This concludes the proof of the Proposition. □

Proposition 2.1.11. *Let X and Y be integral finite dimensional excellent schemes such that $\dim X > \dim Y$, and let $f\colon X \to Y$ be a proper surjective morphism. Then $f_*(\operatorname{div}(a)) = 0$ for any $a \in R(X)^*$.*

Proof. — If $\dim(X) > \dim(Y) + 1$, the assertion is obvious: the integral subschemes V appearing in $\operatorname{div}(a)$ have dimension $\dim(X) - 1 > \dim(Y)$, hence $f_*([V]) = 0$ for any of them.

From now on, we thus assume that $\dim(X) = \dim(Y) + 1$ and begin by treating the particular case where Y is the spectrum of a field K and $X = \mathbf{P}^1_K$ is the projective line. We view X as a one-point compactification of the affine line $\operatorname{Spec} K[T]$; in particular, $R(X) = K(T)$. By additivity, we may also assume that $a \in K[T]$ is an irreducible polynomial.

Closed points of X correspond, either to irreducible monic polynomials $P \in K[T]$, or to the point at infinity. The multiplicity of $\operatorname{div}(a)$ at the point corresponding to P is equal to 0 if a is not associated to P, since a is then a unit in the local ring $K[T]_{(P)}$, and to 1 otherwise, because a generates the maximal ideal (P) of this local ring. The multiplicity of $\operatorname{div}(a)$ at the point at infinity is equal to the degree of a; indeed, the local ring $\mathscr{O}_{X,\infty}$ is equal to $K[1/T]_{(1/T)}$ and $a = a_d T^d + \cdots + a_0 = (1/T)^{-d}(a_d + a_{d-1}/T + \cdots + a_0/T^d)$ is equal to $(1/T)^{-d}$ times a unit in that local ring if $d = \deg(a)$ and $a_d \neq 0$.

The degree of the finite extension $K \subset K[T]/(P)$ is equal to the degree of the polynomial P; while the residue field at infinity is equal to K. Consequently, $\operatorname{div}(a) = [P] - d[\infty]$ and $f_*(\operatorname{div}(a)) = d[\operatorname{Spec} K] - d[\operatorname{Spec} K] = 0$.

We now return to the general case, still assuming that $\dim X = \dim Y + 1$. If V is an integral subscheme of codimension 1 in X appearing in $\operatorname{div}(a)$, one has $f_*([V]) = 0$ unless $\dim f(V) = \dim V$, which means $f(V) = Y$. By the same localization argument as the one used in the proof of Proposition 2.1.10, we may assume that Y is the spectrum of a field $\operatorname{Spec} K$.

Let X' be the integral closure of X in its field of functions and let $p\colon X' \to X$ be the canonical map and $f' = f \circ p$. By Prop. 2.1.10, we have $\operatorname{div}_X(a) = p_*(\operatorname{div}_{X'}(a))$. Moreover, since X' is integrally closed, any non-constant rational function in $R(X')$ defines a finite morphism $g\colon X' \to \mathbf{P}^1_K$. We thus may factor f' as the composition of the finite morphism g and the morphism $h\colon \mathbf{P}^1_K \to \operatorname{Spec} K$. By Prop. 2.1.10, $g_*(\operatorname{div}_{X'}(a)) = \operatorname{div}_{\mathbf{P}^1_K}(b)$, where b is the norm of a in the finite extension $R(\mathbf{P}^1_K) \subset R(X')$. Finally,

$$f_*(\operatorname{div}_X(a)) = (f')_*(\operatorname{div}_{X'}(a)) = h_*(\operatorname{div}_{\mathbf{P}^1_K}(b)) = 0$$

in view of the particular case of the projective line we had treated first. This concludes the proof of the proposition. □

E. Flat pull-back of cycles

To be added if needed.

§ 2.2
INTERSECTING WITH DIVISORS

The goal of intersection theory is to define the intersection of two integral subschemes of a scheme X, either as a cycle on X, in which case this requires assigning multiplicities to the components of the scheme-theoretic intersection, or, as a class modulo rational equivalence. More generally, intersection theory defines a structure of a ring on the group $CH(X)$, even compatible with the graduation by codimension.

Many approaches are possible for smooth schemes of finite type over a field...

The goal of this section is to define the part of this ring structure corresponding to intersection with Cartier divisors. Recall that a (Weil) divisor on a scheme X is a cycle all of which components have codimension 1; we let $\mathrm{Z}^1(X) \subset \mathrm{Z}(X)$ be the subgroup of the Weil divisors. However, technical reasons force us to consider only Cartier divisors, that is the divisors which can locally be defined by one regular element. This piece of intersection theory can be defined in more generality; it is anyway a fundamental step in the approach of FULTON (1998).

A. The first Chern class of a line bundle

We first recall some background on line bundles and Cartier divisors.

A.1. Cartier divisors. — Let X be a scheme. The sheaf of regular meromorphic function $\mathscr{M}_X$ on X is the sheaf associated to the presheaf given by $U \mapsto \mathscr{S}(U)^{-1}\Gamma(U, \mathscr{O}_X)$, where $\mathscr{S}(U)$ is the set of regular elements in $\Gamma(U, \mathscr{O}_X)$, that is the elements of that ring which are not zero-divisors. By definition, a regular meromorphic function on X is a global section of the sheaf $\mathscr{M}_X$. The canonical morphism of sheaves $\mathscr{O}_X \to \mathscr{M}_X$ is injective. If X is integral, then the sheaf $\mathscr{M}_X$ is constant, given by the field $R(X)$ of rational functions on X.

By definition, a Cartier divisor on a scheme X is a global section of the sheaf $\mathscr{M}_X^*/\mathscr{O}_X^*$. A Cartier divisor can be thus given on an open cover $(U_i)_{i\in I}$ by prescribing invertible meromorphic functions $f_i \in \mathscr{M}_X(U_i)^*$ such that for any pair (i,j) of elements of I, there exists $g_{ij} \in \mathscr{O}_X(U_i \cap U_j)^*$ such that $f_i|_{U_i\cap U_j} = g_{ij} f_j|_{U_i\cap U_j}$. Two Cartier divisors $[(U_i, f_i)]$ and $[(V_j, g_j)]$ are equal if and only if, for any pair (i,j), f_i/g_j is a unit on restriction to $U_i \cap V_j$. Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X. Principal divisors are the Cartier divisors which belong to the image in $\mathrm{Div}(X)$ of the group $\Gamma(X, \mathscr{M}_X^*)$ of regular meromorphic functions. We write $P(X)$ for the subgroup of principal Cartier divisors; if $f \in \Gamma(X, \mathscr{M}_X^*)$, we write $\mathrm{div}(f)$ for its associated Cartier divisor.

The support $|D|$ of a Cartier divisor $D = [(U_i, f_i)]$ is the set of points $x \in X$ such that, for any i with $x \in U_i$, $f_i \notin \mathscr{O}_{X,x}^*$. It is a closed subset of X.

Let V be an integral subscheme of X of codimension 1. Let $D = [(U_i, f_i)]$ be a Cartier divisor on X. We define $\mathrm{ord}_V(D)$ as the order $\mathrm{ord}_V(f_i)$ of f_i, for any index i such that U_i meets V. This is well defined since it does not depend on the choice of i; indeed, if U_i and U_j are open subsets which meet V, so does their intersection and f_i/f_j, being a unit on $U_i \cap U_j$, is a unit in the local ring $\mathscr{O}_{X,V}$ of X along V.

If X is Noetherian, there are only finitely many integral subschemes V of codimension 1 such that $\mathrm{ord}_V(D) \neq 0$. Indeed, if D is represented by $[(U_i, f_i)]$, where (U_i) is a *finite* cover, one has $\mathrm{ord}_V(f_i) = 0$ for all but finitely many integral subschemes V of codimension 1 which meet U_i. We thus can define a map $\mathrm{Div}(X) \to \mathrm{Z}^1(X)$ by assigning to any Cartier divisor $D = [(U_i, f_i)]$ its associated one-codimensional cycle $[D] = \sum_V \mathrm{ord}_V(D)[V]$. For any regular meromorphic function f on X, the cycle associated to the principal Cartier divisor $\mathrm{div}(f)$ is equal to the cycle $\mathrm{div}(f)$ we had previously defined. This map $\mathrm{Div}(X) \to \mathrm{Z}^1(X)$ is injective; it is bijective if and only if the local rings of X are factorial.

A.2. Line bundles. — Let X be a scheme and let L be a line bundle on X. By definition, there exists an open cover (U_i) of X such that the restriction of L to U_i admits a non-vanishing section s_i. On $U_i \cap U_j$, the sections s_i and s_j are equal up to multiplication by an invertible function $g_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^*)$. The relations $s_i = g_{ij} s_j$, $s_i = g_{ik} s_k$ and $s_j = g_{jk} s_k$ on $U_i \cap U_j \cap U_k$ imply the equality $g_{ik} = g_{ij} g_{jk}$ in $\Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)$. In other words, the family (g_{ij}) is a Čech 1-cocycle for the covering (U_i). Conversely, any 1-cocycle defines a line bundle which is trivial on the open subsets U_i.

Let $D = [(U_i, f_i)]$ be a Cartier divisor on X. One attaches to D a $\mathscr{O}_X$-submodule $\mathscr{I}_D$ of $\mathscr{M}_X$, namely the one which is generated by f_i on U_i. When D is effective, that is $f_i \in \mathscr{O}_X(U_i)$ for each i, then $\mathscr{I}_D$ is an ideal sheaf; in general, $\mathscr{I}_D$ is a fractional Ideal. We write $\mathscr{O}_X(D)$ for the inverse fractional Ideal of $\mathscr{I}_D$; in other words, $\mathscr{O}_X(D)$ is the locally free $\mathscr{O}_X$-submodule of $\mathscr{M}_X$ generated on U_i by f_i^{-1}. For any two Cartier divisors D and E on X, one has

$$\mathscr{O}_X(D+E) = \mathscr{O}_X(D)\mathscr{O}_X(E) \simeq \mathscr{O}_X(D) \otimes \mathscr{O}_X(E).$$

We therefore obtain a morphism of Abelian groups $\mathrm{Div}(X) \to \mathrm{Pic}(X)$, which attaches to any Cartier divisor D the isomorphism class of the line bundle $\mathscr{O}_X(D)$. If $D = \mathrm{div}(f)$ is a principal divisor, then $\mathscr{O}_X(D)$ is free, with basis f^{-1}; this morphism passes to the quotient by $P(X)$ and induces an injection $\mathrm{Div}(X)/P(X) \to \mathrm{Pic}(X)$.

This map is not surjective in general. Indeed, a line bundle L, defined by a 1-cocycle (g_{ij}), belongs to the image of this map if and only if the invertible functions g_{ij} can be written f_i/f_j, for some regular meromorphic functions $f_i \in \Gamma(U_i, \mathscr{M}_X^*)$. Although such functions do not exist in general, they do exist if X is integral. In that case, $\Gamma(U, \mathscr{M}_X^*) = R(X)^*$ for any non-empty open subset U of X; in particular, $\Gamma(U, \mathscr{M}_X^*)$ contains $\Gamma(U \cap V, \mathscr{O}_X^*)$ for any open subset V of X. We thus may fix an index i and set $f_i = 1$ and $f_j = g_{ij}$ for any $j \neq i$. More generally, this map is surjective if the set of associated points of X (assumed to be Noetherian) is contained in an open affine subset of X, see (GROTHENDIECK, 1967, §21).

When it is surjective, one deduces a morphism of Abelian groups $\mathrm{Pic}(X) \to \mathrm{Z}^1(X)$ which assigns to the isomorphism class of a line bundle L the cycle class $c_1(L)$ of any Cartier divisor D which represents L.

B. Intersecting with first Chern classes

Let X be a finite dimensional excellent scheme.

Let L be a line bundle on X.

Let V be an integral subscheme of X; let $p = \dim V$. If $j_V \colon V \hookrightarrow X$ is the canonical closed immersion, the line bundle $j_V^* L$ on V is represented by a Cartier divisor D_V, because V is integral. We thus write $c_1(L) \cap [V]$ for the cycle class of D_V in $\mathrm{CH}_{p-1}(V)$ or for its image in $\mathrm{CH}_{p-1}(X)$. The notation makes sense since the linear equivalence class of the Cartier divisor, *a fortiori* its associated cycle class, does only depend on L and V. By linearity, this defines a morphism of Abelian groups from $\mathrm{Z}_p(X)$ to $\mathrm{CH}_{p-1}(X)$, $\alpha \mapsto c_1(L) \cap \alpha$.

If L and L' are two line bundles on X, one has $c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$. Moreover, if L is trivial, then $c_1(L) \cap \alpha = 0$; indeed, for any integral subscheme V of X as above, the line bundle $j_V^* L$ is trivial, hence one may take $D_V = 0$.

One can refine this construction under the assumption that L is represented by a Cartier divisor $D = [(U_i, f_i)]$ on X.

Let V be an integral subscheme of X such that $V \not\subset |D|$. Let us fix an index i. One may find a non-zero divisor $g \in \mathscr{O}_X(U_i)$ such that $g f_i \in \mathscr{O}_X(U_i)$; then, both $g f_i$ and g map to a non zero element in $O_V(U_i)$ under the surjective map $\mathscr{O}_X \to \mathscr{O}_V$; their quotient gives us a well-defined element $f_i \mid_V \in R(V)^*$. The family $[(U_i \cap V, f_i \mid_V)]$ then furnishes a well-defined Cartier divisor on $|D| \cap V$ which we write $D \cap V$; we also write $D \cdot [V]$ for its associated cycle in $\mathrm{Z}_{p-1}(|D| \cap V)$, as well as for its classes in $\mathrm{CH}_{p_1}(|D| \cap V)$, $\mathrm{CH}_{p-1}(|D|)$ or $\mathrm{CH}_{p-1}(X)$.

This construction does not work if $V \subset |D|$. However, since V is integral, there exists a Cartier divisor D_V on V, well-defined up to a principal divisor, such that $\mathscr{O}_V(D_V) \simeq j_V^* \mathscr{O}_X(D)$. Again, we write $D \cap [V]$ for the cycle class in $\mathrm{CH}_{p-1}(|V|)$, $\mathrm{CH}_{p-1}(|D|)$ or $\mathrm{CH}_{p_1}(X)$ of the Cartier divisor D_V. In other words, although we can't define a Cartier divisor $D \cap V$, its linear equivalence class is well-defined.

By linearity again, we obtain a morphism of Abelian groups $\mathrm{Z}_p(X) \to \mathrm{CH}_{p-1}(|D|)$, $\alpha \mapsto D \cdot \alpha$.

PROPOSITION 2.2.1 (Projection formula). *Let $f \colon X \to Y$ be a proper morphism of excellent schemes of finite dimension. Let L be a line bundle on Y and $\alpha \in \mathrm{Z}_p(X)$. The following formula holds in* $\mathrm{CH}_{p-1}(Y)$*:*

$$f_*(c_1(f^* L) \cap \alpha) = c_1(L) \cap f_*(\alpha).$$

Proof. — We may assume that $\alpha = [V]$, for some integral closed subscheme of dimension p of X; let $W = f(V)$, so that $f_*([V]) = d[W]$ where $d = [R(V) : R(W)]$ if $\dim(V) = \dim(W)$, and $d = 0$ otherwise. By definition of $c_1(L) \cap [W]$ and $c_1(f^* L) \cap [V]$, we may assume that $V = X$ and f is surjective.

Let $D = [(U_i, u_i)]$ be a Cartier divisor on Y such that $L \simeq \mathscr{O}_Y(D)$; we assume that U_i is affine for each i. Consequenrly, for any i, there is an element $u \in \Gamma(U_i, \mathscr{O}_Y)$ which is not a zero divisor such that $uu_i \in \Gamma(U_i, \mathscr{O}_Y)$. Since f is a dominant morphism of integral schemes, the quotient $f^*(uu_i)/f^*(u)$ gives a well-defined element in $\Gamma(f^{-1}U_i, \mathscr{M}_X^*)$ which we write $f^*(u_i)$. Let $f^* D$ be the Cartier divisor $[(f^{-1}(U_i), f^*(u_i))]$ on X; by construction, one has $\mathscr{O}_X(f^* D) \simeq f^*(L)$. By definition, $c_1(L) \cap [X]$ is the cycle class of $f^* D$, hence $f_*(c_1(L) \cap [X]) = f_*([f * D])$.

To compute $f_*([f^*D])$, we use the fact that, on each U_i, D is the divisor of a rational function on U_i. One has indeed $D\cap U_i = \operatorname{div}_{U_i}(u_i)$ and $f^*D\cap f^{-1}(U_i) = \operatorname{div}_{f^{-1}(U_i)}(f^*(u_i))$; The morphism $f_{U_i}\colon f^{-1}(U_i)\to U_i$ is proper. By Proposition 2.1.10 one has $(f_{U_i})_*(\operatorname{div}_{f^{-1}(U_i)}(f^*u_i)) = 0$ if $\dim f^{-1}(U_i) > \dim U_i$. On the contrary, if $\dim(f^{-1}(U_i)) = \dim(U_i)$, then Prop. 2.1.10 implies that $(f_{U_i})_*(\operatorname{div}(f^*u_i)) = \operatorname{div}(\mathrm{N}(u_i)) = d\operatorname{div}(u_i)$, where $d = [R(X):R(Y)]$.

If $\dim(X) > \dim(Y)$, we thus conclude that $f_*([f^*D]) = 0$, while $f_*([f^*D]) = d[D]$ if $\dim(X) = \dim(Y)$. The proof of the Proposition is finished. □

C. Passing to rational equivalence

THEOREM 2.2.2. *Let X be an integral excellent scheme of dimension p, let D and D' be two Cartier divisors on X. One has $D\cdot[D'] = D'\cdot[D]$ in* $\mathrm{CH}_{p-2}(|D|\cap|D'|)$.

Proof. — 1) *Assume that D and D' are effective and do not have any component in common.* In that case, $D\cdot[D']$ and $D'\cdot[D]$ are defined as *cycles* supported by $|D|\cap|D'|$, and not only classes of cycles modulo rational equivalence. We shall show that these cycles are actually equal.

Let $W\subset X$ be any integral closed subscheme of dimension $p-2$. Let A be the local ring $\mathscr{O}_{X,W}$, let $\mathfrak{m}$ be its maximal ideal, and let a, a' be elements of A defining D and D' in a neighborhood of the generic point of W (such elements exist since D and D' are assumed to be effective). One has $\dim(A) = 2$; except for its maximal ideal $\mathfrak{m}$ and the zero ideal, prime ideals of A have height 1 and correspond to integral closed subschemes V of dimension $p-1$ of X containing W. Then, by definition of $[D']$, the multiplicity of $[V]$ in $[D']$ is equal to $\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/(a'))$. Consequently, the multiplicity m_W of $[W]$ in $D\cdot[D']$ is given by

$$\sum_{\mathrm{ht}(\mathfrak{p})=1}\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/(a'))\operatorname{ord}_{A/\mathfrak{p}}(a) = \sum_{\mathrm{ht}(\mathfrak{p})=1}\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/(a'))\ell_{A/\mathfrak{p}}(A/\mathfrak{p}+(a)).$$

To go on with that computation, we need a further definition. For any A-module M, let us pose

$$\chi(M) = \sum_{\mathrm{ht}(\mathfrak{p})=1}\ell_{A_\mathfrak{p}}(M_\mathfrak{p})\operatorname{ord}_{A/\mathfrak{p}}(a),$$

provided the relevant modules $M_\mathfrak{p}$ have finite length. This expression is additive in exact sequences and tailored so that $m_W = \chi(A/(a'))$.

We have $\chi(A/\mathfrak{m}) = 0$ for all the localizations $(A/\mathfrak{m})_\mathfrak{p}$ are then zero. On the other hand, if $\mathfrak{q}\neq(0)$ is a prime ideal in A, $\mathrm{ht}(\mathfrak{q}) = 1$ and

$$\chi(A/\mathfrak{q}) = \sum_{\mathrm{ht}(\mathfrak{p})=1}\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{q}_\mathfrak{p})\operatorname{ord}_{A/\mathfrak{p}}(a) = \operatorname{ord}_{A/\mathfrak{q}}(a) = \ell(A/\mathfrak{q}+(a)).$$

Consequently, if $M = A/\mathfrak{q}$, we see that $\chi(M) = \ell(M/aM)-\ell(M_a)$, where $M_a = \{m\in M\,;\ am = 0\}$. The right-hand side of this formula being also additive in exact sequences, it follows that this equality holds for any A-module M such that M/aM and M_a have finite length (in fact, this holds if and only is a torsion A-module of finite type). Indeed, by the theory of associated prime ideals, there exists a chain a submodules $M = M_n\subset$

$M_{n-1} \subset \cdots \subset M_0 = 0$, where, for each $k \in \{1, \ldots, n\}$, M_k / M_{k-1} is isomorphic to $A/\mathfrak{p}_i$, for some prime ideal $\mathfrak{p}_i$ of A.

Finally, we have $m_W = \ell_A(A/(a, a')) - \ell_A((A/(a'))_a)$. However, the assumption that F and D' have no common component implies that the multiplication by a is injective in $A/(a')$, hence $A/(a')_a = 0$ and $m_W = \ell_A(A/(a, a'))$.

This formula being symmetric in a and a', we conclude that the multiplicity of W in $D \cdot [D']$ equals that of W in $D' \cdot [D]$. Finally, $D \cdot [D'] = D' \cdot [D]$.

2) *Assume that D and D' are effective.* The excess of intersection is then defined by

$$\varepsilon(D, D') = \max_V \operatorname{ord}_V(D) \operatorname{ord}_V(D'),$$

where V runs among the integral closed subschemes of dimension $p-1$ in X. The proof goes by induction on $\varepsilon(D, D')$; the case $\varepsilon(D, D') = 0$, meaning that $[D]$ and $[D']$ have no common component, having being treated by Case 1).

Let $\pi \colon Y \to X$ be the blow-up of the ideal sheaf $\mathscr{I}_D + \mathscr{I}_{D'}$ in X. The morphism π is proper, surjective and birational. Moreover, the Cartier divisors $\pi^* D$ and $\pi^* D'$ on Y decompose as sums $\pi^* D = E + C$, $\pi^* D' = E + C'$, where E, C, C' are Cartier divisors such that $|E| = \pi^{-1}(|D| \cap |D|')$ and $|C| \cap \left|C'\right| = \varnothing$. We claim that if $\varepsilon(D, D') > 0$, then $\varepsilon(C, E) < \varepsilon(D, D')$ and $\varepsilon(C, E') < \varepsilon(D, D')$.

We postpone the proof of this claim and go on with the proof of Case 2). On Y, $C \cdot [E] = E \cdot [C]$, $C' \cdot [E] = E \cdot [C']$ (by induction), while $C \cdot [C'] = C' \cdot [C] = 0$ since $|C|$ and $\left|C'\right|$ are disjoint. Consequently, $\pi^* D \cdot [\pi^* D'] = \pi^* D' \cdot [\pi^* D]$ in $\mathrm{CH}_{p-2}(|\pi^* D| \cap \left|\pi^* D'\right|)$. Then, using that $D = \pi_* \pi^* D$, $D' = \pi_* \pi^* D'$, and applying the projection formula in the form established during the proof of Prop. 2.2.1, we obtain

$$\begin{aligned}
D \cdot [D'] &= \pi_*(\pi * D \cdot ([E] + [C'])) = \pi_*((E + C) \cdot ([E] + [C'])) = \pi_*(E \cdot [E] + E \cdot [C'] + C \cdot [E]) \\
&= \pi_*(E \cdot [E] + C' \cdot [E] + E \cdot [C] \\
&= D' \cdot [D]
\end{aligned}$$

by symmetry.

3) *The Cartier divisor D' is effective.* Let $\mathscr{I} = \mathscr{O}_X(D) \cap \mathscr{O}_X$ be the denominator ideal of D; in other words, $\mathscr{I}(U)$ is the set of $a \in \mathscr{O}_X(U)$ such that $\operatorname{div}(a) + D$ is effective. Let $\pi \colon Y \to X$ be the blow-up of the Ideal sheaf $\mathscr{I}$. By definition of a blow-up, $\pi^* \mathscr{I} = \mathscr{O}_Y(-E)$, for some effective Cartier divisor E on Y; moreover, the Cartier divisor $C = \pi^* D + E$ on Y is effective. Then, Case 2) and the projection formula imply that

$$\begin{aligned}
D \cdot [D'] &= \pi_*(\pi^* D \cdot [\pi^* D']) = \pi_*((C - E) \cdot [\pi^* D']) = \pi_*(C \cdot [\pi^* D']) - \pi_*(E \cdot [\pi^* D']) \\
&= \pi_*(\pi^* D' \cdot [C]) - \pi_*(\pi^* D' \cdot [E]) = \pi_*(\pi^* D' \cdot ([C] - [E])) = \pi_*(\pi^* D' \cdot [\pi^* D]) \\
&= D' \cdot [D].
\end{aligned}$$

4) The *general case* is proved in a similar manner, introducing the denominator ideal of D' and resorting to Case 3).

It remains to prove the claim above that we had temporarily assumed. Let $U = \operatorname{Spec} A$ be an affine open subset of X on which D and D' are defined by elements a

and $a' \in A$. Then $Y_A = \operatorname{Proj} R(I)$, where $I = (a, a')$ and $R(I) = \bigoplus_{n=0}^{\infty} I^n$. The morphism of graded algebras $A[T, U] \mapsto R(I)$ such that $P \mapsto P(a, a')$ is surjective and induces a closed immersion of Y_A in the projective line $\mathbf{P}^1_A$ over $\operatorname{Spec} A$. In fact, this immersion factors through the closed subscheme $V(a'T - aU)$ of $\mathbf{P}^1_A$. Let t and u be the images of T and U in $R(I)$. By definition of a blow-up, $\pi^* I$ is the ideal of an effective Cartier divisor E on Y.

On the open set $D^+(T)$ of $\operatorname{Proj} R(I)$ where $t \neq 0$, one may write $a' = au/t$, hence an equality of ideals $(a, a') = (1, u/t)(a)$ showing that $E = \pi^* D$ on $D^+(T)$, while $\pi^* D' = E + \operatorname{div}(u/t) = E + C'$. On the other hand, on the open set $D^+(U)$, one finds $\pi^* D' = E$ and $\pi^* D = E + C$, with $C = \operatorname{div}(t/u)$. This shows that $|C|$ and $\left|C'\right|$ are the traces on Y of the zero and infinity sections of $\mathbf{P}^1_A$, hence they are disjoint.

Let W be an integral closed subscheme of codimension 1 in Y, let $V = \pi(W)$. We have $\operatorname{ord}_V(D) \geqslant \operatorname{ord}_W(E+C) = \operatorname{ord}_W(E) + \operatorname{ord}_W(C)$, and similarly $\operatorname{ord}_V(D') \geqslant \operatorname{ord}_W(E) + \operatorname{ord}_W(C')$. Consequently,

$$\operatorname{ord}_V(D) \operatorname{ord}_V(D') \geqslant \operatorname{ord}_W(E)^2 + \operatorname{ord}_W(E) \operatorname{ord}_W(C).$$

Assume $\varepsilon(E, C) > 0$ and let W be the component of $|E| \cap |C|$ such that $\varepsilon(E, C) = \operatorname{ord}_W(E) \operatorname{ord}_W(C)$. Necessarily, W is a component of E, and $\operatorname{ord}_V(D) \operatorname{ord}_V(D') \geqslant 1 + \operatorname{ord}_W(E) \operatorname{ord}_W(C)$; therefore, $\varepsilon(D, D') \geqslant 1 + \varepsilon(E, C)$. If $\varepsilon(E, C) = 0$, this inequality also holds provided $\varepsilon(D, D') > 0$, which concludes the prof of the asserted Claim. □

This Theorem has a number of important consequences, the first of which being that intersecting with a divisor passes through rational equivalence.

COROLLARY 2.2.3. *Let X be an excellent scheme of finite dimension, let L be a line bundle on X and let $\alpha \in \operatorname{Rat}_p(X)$. Then, $c_1(L) \cap \alpha = 0$ in $\operatorname{CH}_{p-1}(X)$.*

Proof. — We reduce to proving that $c_1(L) \cap [\operatorname{div}(u)] = 0$ if X is integral and $u \in R(X)^*$. Let D be a Cartier divisor representing L. By the Theorem,

$$c_1(L) \cap [\operatorname{div}(u)] = D \cdot [\operatorname{div}(u)] = \operatorname{div}(u) \cdot [D] = 0$$

in $\operatorname{CH}_{p-1}(X)$ since the Cartier divisor $\operatorname{div}(u)$ is linearly equivalent to zero. □

Let X be an excellent scheme of finite dimension. It follows that one can define, for any line bundle L on X, morphisms of Abelian groups $\operatorname{CH}_p(X) \to \operatorname{CH}_{p-1}(X)$, written $\alpha \mapsto c_1(L) \cap \alpha$. If, moreover L is represented by a Cartier divisor D, one even obtains morphisms $\operatorname{CH}_p(X) \to \operatorname{CH}_{p-1}(|D|)$.

COROLLARY 2.2.4. *Let X be an excellent scheme of finite dimension, let L and L' be two line bundles on X and let $\alpha \in \operatorname{CH}_p(X)$. Then, one has*

$$c_1(L) \cap c_1(L') \cap \alpha = c_1(L') \cap c_1(L) \cap \alpha$$

in $\operatorname{CH}_{p-2}(X)$.

If L and L' are represented by Cartier divisors D and D', the proof below shows that this equality even holds on $\operatorname{CH}_{p-2}(|D| \cap \left|D'\right|)$.

Proof. — It suffices to show the asserted formula when $\alpha = [V]$, for some integral closed subscheme V of X of dimension p. We may then reduce to the case where $V = X$; in particular, L and L' are represented by Cartier divisors D and D' on X. By definition, $c_1(L') \cap [X] = [D']$, hence $c_1(L) \cap c_1(L') \cap [X] = D \cdot [D']$. By symmetry, $c_1(L') \cap c_1(L) \cap [X] = D' \cdot [D]$, hence the desired equality. □

§ 2.3
INTERSECTION THEORY (FORMULAIRE)

§ 2.4
GREEN CURRENTS ON COMPLEX VARIETIES

Our references for this section are DE RHAM (1973), GRIFFITHS & HARRIS (1978), DEMAILLY (1997) and WELLS (2008).

A. Differential forms on a complex manifold, currents

Let X be a complex analytic manifold; for simplicity, we assume that all connected components of X have the same dimension, say n. By definition of a manifold, X can be covered by open charts (U_i, ψ_i) where $\psi_i \colon U_i \to \mathbf{C}^n$ is a biholomorphic morphism of U_i onto an open subset of $\mathbf{C}^n$. In other words, for any two indices i and j, $\psi_i(U_i \cap U_j)$ and $\psi_j(U_i \cap U_j)$ are open subset of $\mathbf{C}^n$ and $\psi_j \circ \psi_i^{-1}$ defines a biholomorphic isomorphism of the former onto the latter.

To properly describe differential forms on X, we need a small digression about $\mathbf{C}^n$. Let V be a complex vector space of dimension n; let $(z_1, \dots, z_n)$ be a basis of V^*. Their conjugates $\overline{z}_1, \dots, \overline{z}_n$ are defined by $\overline{z}_j(v) = \overline{z_j(v)}$ for $v \in V$; these are complex valued linear forms on the real space $V_{\mathbf{R}}$ associated to V. We can also write $z_j = x_j + \mathrm{i} y_j$, where $x_j, y_j \colon V \to \mathbf{R}$ are real linear forms on $V_{\mathbf{R}}$. We thus observe that $(z_1, \overline{z}_1, \dots, z_n, \overline{z}_n)$ and $(x_1, y_1, \dots, x_n, y_n)$ are two bases of the complex vector space $\mathrm{Hom}_{\mathbf{R}}(V_{\mathbf{R}}, \mathbf{C})$. In fact, $(x_1, \dots, y_n)$ is also a basis of the real vector space $V_{\mathbf{R}}^*$. They are related by

$$
\begin{aligned}
z_j &= x_j + \mathrm{i} y_j, & x_j &= \frac{1}{2}(z_j + \overline{z}_j) \\
\overline{z}_j &= x_j - \mathrm{i} y_j, & y_j &= \frac{1}{2\mathrm{i}}(z_j - \overline{z}_j).
\end{aligned}
$$

For any basis $u_1, \dots, u_{2n}$ of $V_{\mathbf{R}}^*$, and any integer m, the space $\mathrm{Hom}_{\mathbf{R}}(\bigwedge^m V_{\mathbf{R}}, \mathbf{C})$ of complex valued degree m covectors on $V_{\mathbf{R}}$ admits the family $(u_{i_1} \wedge \dots \wedge u_{i_m})_{i_1 < \dots < i_m}$ as a basis. If we consider the basis $(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n)$ introduced above, we say that a covector has bidgree (p, q) if it a linear combination of the covectors $(z_{i_1} \wedge \dots \wedge z_{i_p} \wedge \overline{z}_{j_1} \wedge \dots \overline{z}_{j_q}$, for $1 \leqslant i_1 < \dots < i_p \leqslant n$ and $1 \leqslant j_1 < \dots < j_q \leqslant n$. In other words, $\mathrm{Hom}_{\mathbf{R}}(V_{\mathbf{R}}, \mathbf{C}) = V \oplus \overline{V}$

and we have decomposed $\mathrm{Hom}(\bigwedge^m V_{\mathbf{R}},\mathbf{C})$ as the direct sum

$$\bigoplus_{p+q=m}\mathrm{Hom}(\bigwedge^p V,\mathbf{C})\otimes\mathrm{Hom}(\bigwedge^q \overline{V},\mathbf{C})=\bigoplus_{p+q=m}\bigwedge^p V^*\otimes\bigwedge^q \overline{V}^*.$$

We write $V^*_{(p,q)}$ for the space of covectors of bidgree (p,q) on V. We thus have $V^*_{(1,0)}=V$, $V^*_{(0,1)}=\overline{V}$ and $V^*_{(p,q)}=\bigwedge^p V\otimes\bigwedge^q\overline{V}$.

We observe that for any $\lambda\in\mathbf{C}$ and covector F of bidegree (p,q) on V, $F(\lambda v_1,\dots,\lambda v_d)=\lambda^p\overline{\lambda}^q F(v_1,\dots,v_d)$. Since the maps $\lambda\mapsto\lambda^p\overline{\lambda}^q$, for varying p and q, are linearly independant, this implies that the spaces $V^*_{(p,q)}$ do not depend on the choice of a basis of V^*.

If we apply these considerations to the tangent bundle of a complex manifold X, we obtain a decomposition of the bundle of complex valued degree m differential forms on $X_{\mathbf{R}}$ as a direct sum

$$\mathbf{C}\otimes\bigwedge^m T^*X_{\mathbf{R}}=\mathrm{Hom}(\bigwedge^d TX,\mathbf{C})=\bigoplus_{p+q=m}T^*_{(p,q)}X,$$

where $T^*_{(p,q)}X=\bigwedge^p T^*X\otimes\bigwedge^q\overline{T^*X}$. In other words, a differential form of bidegree (p,q) on X can be written an a chart U_i as a linear combination

$$\sum_{\substack{I=\{i_1<\cdots<i_p\}\\ J=\{j_1<\cdots<j_q\}}}\omega_{IJ}\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_p}\wedge\mathrm{d}\overline{z}_{j_1}\wedge\cdots\wedge\mathrm{d}\overline{z}_{j_q}.$$

We define $\mathscr{A}^{(p,q)}(X)$ to be the space of $\mathscr{C}^\infty$-differential forms of bidgree (p,q) in X, and $\mathscr{A}^m(X)=\bigoplus_{p+q=m}\mathscr{A}^{(p,q)}(X)$ the space of degree m differential forms in X. The support of a form α on X is the smallest closed subset S of X such that $\alpha\mid_{X\setminus S}\equiv 0$. We then let $\mathscr{A}_{\mathrm{c}}^{(p,q)}(X)$ and $\mathscr{A}_{\mathrm{c}}^m(X)=\bigoplus_{p+q=m}\mathscr{A}_{\mathrm{c}}^{(p,q)}(X)$ to be the subspaces consisting of differential forms with compact support.

These are infinite dimensional complex vector spaces; we define a structure of locally convex topological vector space on them by considering the family of seminorms

$$\omega\mapsto\sup_{x\in K}\sup_{I,J}\left\|\omega_{IJ}\right\|_{r,K},$$

where K is a compact subset of an open set of definition of a chart U_i, $\left\|\omega_{IJ}\right\|_{r,K}$ is the $\mathscr{C}^r$-norm of the $\mathscr{C}^\infty$-function $\omega_{IJ}\circ\psi_i^{-1}$ defined on a the compact set $\psi_i(K)$ in $\mathbf{C}^n$.

The spaces of *currents* are the topological duals of these spaces. Namely, we define

$$\mathscr{D}_{(p,q)}(X)=(\mathscr{A}_{\mathrm{c}}^{(p,q)}(X))'\qquad\mathscr{D}_m(X)=(\mathscr{A}_{\mathrm{c}}^{2n-m}(X))'=\bigoplus_{p+q=m}\mathscr{D}_{(p,q)}(X)\tag{2.4.1}$$

$$\mathscr{D}^{\mathrm{c}}_{(p,q)}(X)=(\mathscr{A}^{(p,q)}(X))'.\qquad\mathscr{D}^{\mathrm{c}}_m(X)=(\mathscr{A}^m(X))'=\bigoplus_{p+q=m}\mathscr{D}^{\mathrm{c}}_{(p,q)}(X).\tag{2.4.2}$$

Theses are the space of currents of bidimension (p,q), the space of currents of dimension m, and the analogous spaces of currents with compact support. A current of bidimension (p,q) is also said to have bidegree $(n-p,n-q)$, where n is the complex dimension of X (which is assumed to be equidimensional); a current of dimension m is also said to have degree $n-m$. We write $\mathscr{D}^{(p,q)}(X)$, $\mathscr{D}^m(X)$ for the spaces of currents respectively of bidegree (p,q) and of degree m, and similarly for currents with compact support.

By definition, an element of $\mathscr{D}_{(p,q)}(X)$ is a linear form T on $\mathscr{A}_{\mathrm{c}}^{(p,q)}(X)$ such that for any compact subset $K\subset U_i$ there exists a real number C_K and an integer r such that

$$|T(\alpha)|\leqslant C_K\,\|\alpha\|_{K,r}$$

for any differential form α of bidgree (p,q) with support in K.

The support of a current T is the smallest closed subset S of X such that $T(\alpha)=0$ for any form α whose support is disjoint from S. By duality, the obvious injections $\mathscr{A}_{\mathrm{c}}^{(p,q)}(X)\to\mathscr{A}^{(p,q)}(X)$ induce morphisms $\mathscr{D}^{\mathrm{c}}_{(p,q)}(X)\to\mathscr{D}_{(p,q)}(X)$ at the level of currents. These morphisms are injective too and identify the space of currents with compact supports with the subspaces of currents whose support is compact.

Let us recall that a differential form on X of bidgree (n,n) can be integrated on X, at least if it has compact support. We deduce from that canonical maps $\mathscr{A}^{(n-p,n-q)}(X)\to\mathscr{D}_{(p,q)}(X)$, $\mathscr{A}_{\mathrm{c}}^{(n-p,n-q)}(X)\to\mathscr{D}^{\mathrm{c}}_{(p,q)}(X)$, defined, for $\alpha\in\mathscr{A}^{(p,q)}(X)$, as the linear form $\omega\mapsto\int_X\omega\wedge\alpha$, and similarly for $\alpha\in\mathscr{A}_{\mathrm{c}}^{(p,q)}(X)$. In fact, these maps even extend to the set of differential forms with locally integrable coefficients, because the relevant integrals then converge absolutely. These maps are injective: using a partition of unity and local coordinates, this amounts to the classical fact that if α is any locally integrable function on $\mathbf{C}^n$ such that $\int_{\mathbf{C}^n}\alpha\varphi=0$ for any $\varphi\in\mathscr{C}_{\mathrm{c}}^\infty(\mathbf{C}^n)$, then $\alpha=0$.

We shall write $[\alpha]$ for the current associated to a form α.

B. Differential calculus

Let X be a complex analytic manifold. Differential forms on X can be differentiated: there are differential operators $\mathrm{d}=\partial+\overline{\partial}$, where $\partial:\mathscr{A}^{(p,q)}(X)\to\mathscr{A}^{(p+1,q)}(X)$ and $\overline{\partial}:\mathscr{A}^{(p,q)}(X)\to\mathscr{A}^{(p,q+1)}(X)$ are given in local coordinates by the formulae

$$\partial\omega=\sum_{I,J}\frac{\omega_{IJ}}{\partial z_k}\mathrm{d}z_k\wedge\mathrm{d}z_I\wedge\mathrm{d}\overline{z}_J$$
$$\overline{\partial}\omega=\sum_{I,J}\frac{\omega_{IJ}}{\partial\overline{z}_k}\mathrm{d}\overline{z}_k\wedge\mathrm{d}z_I\wedge\mathrm{d}\overline{z}_J.$$

In this formula, ω is a differential form of bidegree (p,q) given locally as $\omega=\sum\omega_{IJ}\mathrm{d}z_I\wedge\mathrm{d}\overline{z}_J$ while, for multiindices $I=(i_1<i_2<\cdots<i_p)$ and $J=(j_1<\cdots<j_q)$, we have set $\mathrm{d}z_I=\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_p}$ and $\mathrm{d}\overline{z}_J=\mathrm{d}\overline{z}_{j_1}\wedge\cdots\wedge\mathrm{d}\overline{z}_{j_q}$.

Since $\mathrm{d}\circ\mathrm{d}=0$, one obtains a complex,

$$0\to\mathscr{A}^0(X)\xrightarrow{\mathrm{d}}\mathscr{A}^1(X)\xrightarrow{\mathrm{d}}\dots,$$

called the De Rham complex of X, and whose cohomology groups $H^m_{\mathrm{dR}}(X)$ are called the *De Rham cohomology groups* of X. Explicitly:

$$H^m_{\mathrm{dR}}(X)=\frac{\ker\big(\mathrm{d}\colon\mathscr{A}^m(X)\to\mathscr{A}^{m+1}(X)\big)}{\operatorname{im}\big(\mathrm{d}\colon\mathscr{A}^{m-1}(X)\to\mathscr{A}^m(X)\big)}.$$

Forms α such that $\mathrm{d}\alpha = 0$ are called closed; those of the form $\mathrm{d}\beta$ are called exact. With this terminology, $H^m_{\mathrm{dR}}(X)$ is the vector space of closed m-forms modulo its subspace of exact forms. For $m = 0$, $H^0_{\mathrm{dR}}(X)$ is the space of locally constant functions; if X is connected and non-empty, one thus has $H^0_{\mathrm{dR}}(X) = \mathbf{C}$. According to the Poincaré lemma, the remaining spaces are zero if $X = \mathbf{C}^n$ or, more generally, if X is starshaped with respect to one of its points. As a consequence, the complex of *sheaves* given on an open subset U of X by the De Rham complex of U is a resolution of the constant sheaf $\mathbf{C}_X$. Since there exists $\mathscr{C}^\infty$-partitions of unity, these sheaves are fine, from which general cohomological techniques imply that the De Rham cohomology groups of X coincide with the cohomology groups $H^m(X,\mathbf{C}_X)$ of the constant sheaf $\mathbf{C}_X$.

Since $\mathrm{d}^2 = (\partial+\overline{\partial})^2 = \partial^2+\partial\overline{\partial}+\overline{\partial}\partial+\overline{\partial}^2$, consideration of degrees imply that $\partial^2 = \overline{\partial}^2 = 0$. Consequently, one also obtains, for each integer p, a complex

$$0 \to \mathscr{A}^{(p,0)}(X) \xrightarrow{\overline{\partial}} \mathscr{A}^{(p,1)}(X) \xrightarrow{\overline{\partial}} \dots,$$

called the Dolbeault complex of Ω^p_X, and whose cohomology groups $H^q_{\mathrm{Dol}}(\Omega^p_X)$ are called the *Dolbeault cohomology groups* of Ω^q_X. Explicitly:

$$H^q_{\mathrm{Dol}}(\Omega^p_X) = \frac{\ker\left(\overline{\partial}\colon \mathscr{A}^{(p,q)}(X) \to \mathscr{A}^{(p,q+1)}(X)\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathscr{A}^{(p,q-1)}(X) \to \mathscr{A}^{(p,q)}(X)\right)}.$$

According to the complex Poincaré lemma, the spaces $H^q_{\mathrm{Dol}}(\Omega^p_X)$ are zero for $q > 0$ if $X = \mathbf{C}^n$ or, more generally, if X is starshaped with respect to one of its points. Moreover, one knows that for a $\mathscr{C}^\infty$-functions u on an open subset U of $\mathbf{C}^n$, u is holomorphic if and only if $\overline{\partial}u = 0$. In local coordinates, this implies that $H^0_{\mathrm{Dol}}(\Omega^p_X)$ coincides with the space holomorphic differential forms of degree p on X. As a consequence, the complex of *sheaves* given on an open subset U of X by the Dolbeault complex of U is a resolution of the sheaf Ω^p_X of holomorphic differential forms of degree p. These sheaves being fine, one concludes in a similar manner than for De Rham cohomology that $H^q_{\mathrm{Dol}}(\Omega^p_X) = H^q(X,\Omega^p_X)$.

There are similar De Rham and Dolbeault complexes with compact support, obtained by replacing the spaces $\mathscr{A}^{(p,q)}(X)$ and $\mathscr{A}^m(X)$ by their subspaces $\mathscr{A}^{(p,q)}_{\mathrm{c}}(X)$ and $\mathscr{A}^m_{\mathrm{c}}(X)$. The local formulae for d, ∂ and $\overline{\partial}$ show that one indeed gets complexes whose cohomology groups are called (surprise!) the De Rham cohomology groups with compact support of X, and the Dolbeault cohomology groups with compact support of Ω^p_X. Again, these spaces are isomorphic to the cohomology groups with compact support of the sheaves $\mathbf{C}_X$ and Ω^p_X respectively.

Since currents are dual to forms, the operators d, ∂ and $\overline{\partial}$ acts by transposition on the spaces of currents. However, if ω and α are $\mathscr{C}^1$-differential forms, one of them having compact support, then $\mathrm{d}(\omega\wedge\alpha) = \mathrm{d}\omega\wedge\alpha + (-1)^m\omega\wedge\mathrm{d}\alpha$, where α is of degree m. By Stokes's formula,

$$\int_X \mathrm{d}\omega\wedge\alpha = (-1)^{m+1}\int_X \omega\wedge\mathrm{d}\alpha;$$

in other words,

$$\langle [\mathrm{d}\omega], \alpha\rangle = (-1)^{m+1} \langle [\omega], \mathrm{d}\alpha\rangle,$$

$m+1$ being the degree of $\mathrm{d}\alpha$. In view of this formula, we define, for any current $T \in \mathscr{D}_m(X)$, the current $\mathrm{d}T$ by the formula

$$\langle \mathrm{d}T, \alpha\rangle = (-1)^{m+1} \langle T, \mathrm{d}\alpha\rangle,$$

for any $\alpha \in \mathscr{A}_\mathrm{c}^m(X)$. We give a similar definition for ∂ and $\overline{\partial}$, so that the equality $\mathrm{d} = \partial + \overline{\partial}$ holds for currents.

Currents furnishes new De Rham and Dolbeault complexes whose cohomology, in fact, coincide with that of the original complexes. This follows from the fact that the classical and complex Poincaré lemmas still hold at the level of currents so that the De Rham, resp. the Dolbeault complexes, with currents are also resolutions of the constant sheaf $\mathbf{C}_X$, resp. of the sheaf Ω^p_X. (Technically, the inclusion of the sheafified De Rham complex into the sheafified De Rham complexes with currents is a quasi-isomorphism.)

C. Hodge Theory

Let X be a compact complex analytic manifold, together with a hermitian metric h on its cotangent bundle; this endows the spaces $T^*_{(p,q)}$ by hermitian metrics, and in particular the space $T^*_{(n,n)}X$ of volume forms. Let $\mathrm{d}V$ be the volume form which is of norm 1 everywhere; at any point of X, it is given by $\mathrm{d}V = (\mathrm{i}/2)^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \dots$ if $(\mathrm{d}z_1, \dots, \mathrm{d}z_n)$ is a unitary basis of $T^*_{(1,0)}$.

We then can use the volume form $\mathrm{d}V$ to define at each point a Hodge operator $\star\colon T^*_{(p,q)}X \to T^*_{(n-q,n-p)}X$, where $n = \dim X$, defined by the formulae

$$\alpha \wedge \star\overline{\beta} = h(\alpha, \beta)\,\mathrm{d}V$$

for any two covectors α and $\beta \in T^*_{(p,q)}X$ at a point x. Explicitly, $\star$ is antilinear and

$$\star(\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J) = \dots \mathrm{d}z_{J'} \wedge \mathrm{d}\overline{z}_{I'},$$

where I and I' are complementary multiindices, as well as J and J'. Indeed,

$$\begin{aligned}
\delta_{I,I'}\delta_{J,J'} &= h(\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J, \mathrm{d}z_{I'} \wedge \mathrm{d}\overline{z}_{J'}) \\
&= \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J \wedge \mathrm{d}z_{I'} \wedge \mathrm{d}\overline{z}_{J'} \\
&= (-1)^{pq} \mathrm{d}z_I \wedge \mathrm{d}z_{I'} \wedge \mathrm{d}\overline{z}_J \wedge \mathrm{d}\overline{z}_{J'} \\
&= (-1)^{pq} \varepsilon_{I,I'} \varepsilon_{J,J'}
\end{aligned}$$

where $\varepsilon_{I,I'}$ is the signature of the permutation (I, I').

This allows to define the (so called, formal) adjoint d^* to the operator $\mathrm{d}\colon \mathscr{A}^m(X) \to \mathscr{A}^{m+1}(X)$ by the formula

$$\mathrm{d}^* = (-1)^{m+1} \star^{-1} \mathrm{d}\star\colon \mathscr{A}^m(X) \to \mathscr{A}^{m-1}(X).$$

Indeed, for $\alpha \in \mathscr{A}^m(X)$ and $\beta \in \mathscr{A}^{m+1}(X)$, one has

$$\begin{aligned}\int_X h(\mathrm{d}\alpha, \beta)\,\mathrm{d}V &= \int_X \mathrm{d}\alpha \wedge \star\overline{\beta} \\ &= \int_X \mathrm{d}(\alpha \wedge \star\overline{\beta}) - (-1)^m \alpha \wedge \mathrm{d}\star\overline{\beta} \\ &= (1)^{m+1} h(\alpha, \star^{-1}\overline{\mathrm{d}\star\overline{\beta}})\,\mathrm{d}V.\end{aligned}$$

A computation shows that $\star^{-1}\overline{d\star\overline{\beta}} = -\star\mathrm{d}\star\beta = \mathrm{d}^*\beta$. The *laplacian* is the differential operator of order 2 given by $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. One checks in local coordinates that it is an elliptic operator.

Harmonic forms are the differential forms annihilated by Δ. Moreover, for any $\alpha \in \mathscr{A}^m(X)$, the equation

$$\int_X h(\Delta\alpha, \alpha)\,\mathrm{d}V = \int_X h(\mathrm{d}\alpha, \mathrm{d}\alpha)\,\mathrm{d}V + \int_X h(\mathrm{d}^*\alpha, \mathrm{d}^*\alpha)\,\mathrm{d}V$$

implies that a form α is harmonic if and only if $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$. The De Rham complex is *elliptic*: the associated complex of vector bundles on the cotangent space T^*X given by the principal symbols of the differential operators $\mathrm{d}\colon \mathscr{A}^m \to \mathscr{A}^{m+1}$ is exact. (The fibre a point (x,ξ) of T^*X of this complex is the complex of cotangent spaces $\bigwedge^m T^*_x X$ related by the morphism "taking exterior product with ξ". For a covector v, the relation $\xi \wedge v = 0$ is equivalent to the fact that v can be written $\xi \wedge w$. Consequently, this complex is exact.)

The theory of elliptic differential operators then implies that the space $\mathscr{H}^m(X)$ is finite dimensional, as well as the existence of *Green operators* $G\colon \mathscr{A}^m(X) \to \mathscr{A}^m(X)$ commuting with d and d^* such that $\mathrm{id} - \Delta G$ is a projector onto $\mathscr{H}^m(X)$. Consequently, for any $\alpha \in \mathscr{A}^m(X)$, one can write

$$\alpha = \mathscr{H}\alpha + \mathrm{d}(\mathrm{d}^*G\alpha) + \mathrm{d}^*(\mathrm{d}G\alpha),$$

where $\mathscr{H}\alpha \in \mathscr{H}^m(X)$ is harmonic. From this, one deduces that there is an orthogonal decomposition

$$\mathscr{A}^m(X) = \mathscr{H}^m(X) \overset{\perp}{\oplus} \mathrm{d}\mathscr{A}^{m-1}(X) \overset{\perp}{\oplus} \mathrm{d}^*\mathscr{A}^{m+1}(X).$$

Observe moreover that the kernel of d consists of the first two terms, while its image is the second one. Consequently, the spaces $\mathscr{H}^m(X)$ is naturally isomorphic to the De Rham cohomology group $H^m_{\mathrm{dR}}(X)$.

Similarly, one defines the (formal) adjoints $\partial^* = -\star\overline{\partial}\star$ and $\overline{\partial}^* = -\star\partial\star$ to ∂ and $\overline{\partial}$ which allow to define laplacians Δ_∂ and $\Delta_{\overline{\partial}}$. The ∂ and $\overline{\partial}$-complexes are elliptic too, leading to Green operators G_∂, $G_{\overline{\partial}}$, as well as to orthogonal decompositions

$$\begin{aligned}\mathscr{A}^{(p,q)}(X) &= \mathscr{H}_\partial^{(p,q)} \overset{\perp}{\oplus} \partial\mathscr{A}^{(p-1,q)}(X) \overset{\perp}{\oplus} \partial^*\mathscr{A}^{(p+1,q)}(X) \\ &= \mathscr{H}_{\overline{\partial}}^{(p,q)} \overset{\perp}{\oplus} \overline{\partial}\mathscr{A}^{(p,q-1)}(X) \overset{\perp}{\oplus} \overline{\partial}^*\mathscr{A}^{(p,q+1)}(X),\end{aligned}$$

where $\mathscr{H}_{\partial}^{(p,q)}$ and $\mathscr{H}_{\overline{\partial}}^{(p,q)}$ are finite dimensional vector spaces, equal respectively to the kernels of Δ_∂ and $\Delta_{\overline{\partial}}$ acting on $\mathscr{A}^{(p,q)}(X)$. Again, a form belongs to $\mathscr{H}_{\overline{\partial}}^{(p,q)}(X)$ if and only if it is $\overline{\partial}$ and $\overline{\partial}^*$-closed. Consequently, the space $\mathscr{H}_{\overline{\partial}}^{(p,q)}(X)$ is in natural bijection with the Dolbeault cohomology group $H^q_{\mathrm{Dol}}(\Omega^p_X)$.

Let us define a differential form of bidegree $(1,1)$ on X by the formula

$$\omega = -\Im h = \frac{\mathrm{i}}{2}(h-\overline{h}).$$

In local coordinates $(z_1,\dots,z_n)$, let $h_{jk} = h(\mathrm{d}z_j, \mathrm{d}z_k)$; one has

$$\omega = \frac{\mathrm{i}}{2}\sum_{j,k} h_{j,k}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

One says that X is *Kähler* if this differential form is closed. Under this assumption, there is a fundamental coincidence of all spaces of harmonic forms, due to the equalities

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$$

between the three laplacians, themselves a consequence of the commutation relations

$$\partial\overline{\partial}^* = -\overline{\partial}^*\partial, \qquad \overline{\partial}\partial^* = -\partial^*\overline{\partial}.$$

Moreover, the Green operators G, G_∂ and $G_{\overline{\partial}}$ are also related by the formula $G_\partial = G_{\overline{\partial}} = 2G$. The first consequence of these equalities is a decomposition the De Rham cohomology groups :

$$H^m_{\mathrm{dR}}(X) = \bigoplus_{p+q=m} H^q_{\mathrm{Dol}}(\Omega^p_X).$$

We shall also need the following proposition.

PROPOSITION 2.4.3 ($\partial\overline{\partial}$-lemma). *Let X be a compact Kähler complex manifold. Let $\alpha \in \mathscr{A}^{(p,q)}(X)$ be a differential form of bidegree (p,q) such that* $\mathrm{d}\alpha = 0$. *Then, the following properties are equivalent:*

a) *There exists a differential form β such that $\alpha = \partial\overline{\partial}\beta$;*

b) *α is* d*-exact;*

c) *α is ∂-exact;*

d) *α is $\overline{\partial}$-exact;*

e) *α is orthogonal to the space $\mathscr{H}^{(p,q)}(X)$ of harmonic forms of bidegree (p,q).*

Proof. — We first remark that for a form α of bidegree (p,q), the equality $\mathrm{d}\alpha = 0$ implies that $\partial\alpha = \overline{\partial}\alpha = 0$. Indeed, $\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$, while $\partial\alpha$ and $\overline{\partial}\alpha$ do not have the same bidegrees.

Assume now that $\alpha = \partial\overline{\partial}\beta$. Then $\alpha = \partial(\overline{\partial}\beta) = \overline{\partial}(-\partial\beta) = \mathrm{d}(\overline{\partial}\beta - \partial\beta)$ is d, ∂ and $\overline{\partial}$-exact. Moreover, if any of these exactness properties holds, then the Hodge decomposition of $\mathscr{A}^{(p,q)}(X)$ for the corresponding laplacian implies that α is orthogonal to the space of harmonic forms.

Let us finally assume that α is orthogonal to $\mathscr{H}^{(p,q)}(X)$ and let us prove the existence of a form β such that $\alpha = \partial\overline{\partial}\beta$. The decomposition of $\mathscr{A}^{(p,q)}(X)$ for the $\overline{\partial}$ operator implies that there exist differential forms u and v such that $\alpha = \partial u + \partial^* v$. Since $\partial\alpha = 0$, one obtains

$$0 = \partial\partial u + \partial\partial^* v = \partial\partial^* v.$$

Integrating this relation on X and using the adjointness property, one gets $\int_X h(\partial^* v, \partial^* v)\,\mathrm{d}V = 0$ hence $\partial^* v = 0$ and $\alpha = \partial u$. Decomposing u as a sum $h + \overline{\partial}\beta + \overline{\partial}^*\gamma$, with $h \in \mathscr{H}^{(p-1,q)}(X)$, $\beta \in \mathscr{A}^{(p-1,q-1)}(X)$ and $\gamma \in \mathscr{A}^{(p-1,q+1)}(X)$, we finally obtain

$$\alpha = \partial u = \partial h + \partial\overline{\partial}\beta + \partial\overline{\partial}^*\gamma = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\gamma,$$

hence

$$0 = \overline{\partial}\alpha = \overline{\partial}\partial\beta + \overline{\partial}\partial\overline{\partial}^*\gamma = -\overline{\partial}\overline{\partial}^*\partial\gamma.$$

This implies that $\overline{\partial}^*\partial\gamma = 0$, and finally $\alpha = \partial\overline{\partial}\beta$, as we needed to show. □

PROPOSITION 2.4.4 ($\partial\overline{\partial}$-lemma). *Let X be a Kähler compact complex manifold. Let $\alpha \in \mathscr{A}^{(p,q)}(X)$ be a differential form of bidegree (p,q). If α is* d*, ∂ or $\overline{\partial}$-exact, then there exists a form η of bidegree $(p-1,q-1)$ such that $\alpha = \partial\overline{\partial}\eta$.*

Proof. — Following (GRIFFITHS & HARRIS, 1978, p. 115), we shall actually give an explicit solution η for we shall need it for the next proposition. We first observe that for a form α of bidegree (p,q), being d, ∂ or $\overline{\partial}$-closed are equivalent. Indeed, $\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$ is a sum of forms of distinct bidegrees.

Let α_0 be the harmonic projection of α. The Hodge decomposition for ∂ writes

$$\alpha = \alpha_0 + \partial(\partial^* G\alpha) + \partial^*(\partial G\alpha) = \alpha_0 + \partial(\partial^* G\alpha)$$

since $\partial G\alpha = \frac{1}{2}\partial G_\partial = \frac{1}{2}G_\partial\partial\alpha = 0$. Let us observe that the form $\partial^* G\alpha$, of bidegree $(p-1,q)$, is $\overline{\partial}$-closed; indeed,

$$\overline{\partial}\partial^* G\alpha = -\partial^*\overline{\partial}G\alpha = -\frac{1}{2}\partial^*\overline{\partial}G_{\overline{\partial}}\alpha = -\frac{1}{2}\partial^* G_{\overline{\partial}}\overline{\partial}\alpha = 0.$$

As any form in the image of ∂^*, it is also orthogonal to the space of harmonic forms, so that its Hodge decomposition writes

$$\begin{aligned}\partial^* G\alpha &= \overline{\partial}(\overline{\partial}^* G_{\overline{\partial}}\partial^* G\alpha) + \overline{\partial}^*(\overline{\partial}G_{\overline{\partial}}\partial^* G\alpha)\\ &= \overline{\partial}(2\overline{\partial}^*\partial^* G_{\overline{\partial}}^2\alpha).\end{aligned}$$

This implies $\alpha = \alpha_0 + \partial\overline{\partial}\eta$, with $\eta = 2\overline{\partial}^*\partial^* G_{\overline{\partial}}^2\alpha$.

If α is d-exact, the orthogonal decompositions of $\mathscr{A}^m(X)$ induced by the Laplacian Δ shows that α is orthogonal to the space of harmonic forms, so that $\alpha_0 = 0$. By a similar argument involving the other two Laplacians, we conclude that $\alpha_0 = 0$ if α is d, ∂ or $\overline{\partial}$-exact, which concludes the proof of the Proposition. □

COROLLARY 2.4.5 ($\partial\overline{\partial}$-lemma for currents). *Let X be a Kähler compact complex manifold. Let T be a closed current of bidegree (p,q) on X. If T is* d*, ∂ or $\overline{\partial}$-exact, then there exists a current U of bidegree $(p-1,q-1)$ such that $T = \partial\overline{\partial}U$.*

Proof. — In fact, the preceding proof holds (almost) verbatim by replacing forms by currents. One should observe however that for a form α, orthogonality with a harmonic form ω means $\int_X h(\alpha,\omega)\,\mathrm{d}V = 0$. But this is equivalent to the relation $\langle[\alpha],\star\overline{\omega}\rangle = 0$. Since complex conjugaison and the $\star$-operator preserve harmonic forms, orthogonality with $\mathscr{H}^{(p,q)}$ is equivalent to vanishing on $\mathscr{H}^{(n-p,n-q)}$, which gives an adequate formulation for currents. □

In the sequel, we shall use the operator dd^c which is a multiple of $\partial\overline{\partial}$. Namely, we define $\mathrm{d}^c = (\partial - \overline{\partial})/2\mathrm{i}\pi$. This is a real differential operator, and

$$\mathrm{dd}^c = \frac{\mathrm{i}}{\pi}\partial\overline{\partial}.$$

Note that while we follow the convention of DEMAILLY (1997), other authors, notably GILLET & SOULÉ (1990) and GRIFFITHS & HARRIS (1978), use half of it. Our choice is motivated by the will to eliminate a factor 2 in the definition of the arithmetic degree, as well as to be closer to the convention of potential theory.

D. Push-forward and pull-backs of forms and currents

Let $f\colon X \to Y$ be a morphism of complex analytic manifolds; for simplicity we assume that X and Y are everywhere of dimensions d and e respectively. By classical differential calculus, f induces maps $f^*\colon \mathscr{A}^{(p,q)}(X) \to \mathscr{A}^{(p,q)}(Y)$. By duality, one obtains a map $f_*\colon \mathscr{D}^{\mathrm{c}}_{(p,q)}(X) \to \mathscr{D}^{\mathrm{c}}_{(p,q)}(Y)$, given by

$$\langle f_* T, \alpha\rangle = \langle T, f^*\alpha\rangle$$

for $T \in \mathscr{D}^{\mathrm{c}}_{(p,q)}(X)$ and $\alpha \in \mathscr{A}^{(p,q)}(Y)$. If, moreover, f is proper, then f^* maps $\mathscr{A}^{(p,q)}_{\mathrm{c}}(X)$ into $\mathscr{A}^{(p,q)}_{\mathrm{c}}(Y)$. Consequently, the map f_* at the level of currents with compact support extends to a map

$$f_*\colon \mathscr{D}_{(p,q)}(X) \to \mathscr{D}_{(p,q)}(Y)$$

on currents of bidimension (p,q).

If f is a locally trivial fibration, then one also can integrate differential forms on X along the fibres of f, at least if these integrals converge, which happens for the differential forms with compact support. (Locally on X, $X = Y \times Z$, for some complex manifold Z, integrate on Z with respect to the coordinates on Z, making use of theorems concerning regularity of integrals with parameters.) According to a famous lemma of Ehresmann, this is in particular the case if f is a proper submersion. In those cases, one obtains a map

$$f_*\colon \mathscr{A}^{(p,q)}_{\mathrm{c}}(X) \to \mathscr{A}^{(p-d+e,q-d+e)}_{\mathrm{c}}(Y),$$

hence, by duality, a morphism

$$f^*\colon \mathscr{D}_{(p,q)}(Y) \to \mathscr{D}_{(p+d-e,q+d-e)}(X)$$

which preserves currents with compact support if f is proper.

If $\alpha \in \mathscr{A}^{(d-p,d-q)}(Y)$ is a form and $T = [\alpha]$ is the current on Y associated to α, then $f^* T$ is the current associated to the form $f^*\alpha$. Similarly, if $\alpha \in \mathscr{A}^{(p,q)}(X)$ and $T = [\alpha]$ is the current on X associated to α, then f_*T is the form associated to the form $f_*\alpha$ given by integration on the fibres of f.

The maps f_* and f^* commute with the operators d, ∂ and $\overline{\partial}$. One has $f^*(T\wedge\alpha) = f^*T\wedge f^*\alpha$ if T is a current and α a form on Y; one also has a projection formula, namely $f_*(T\wedge f^*\alpha) = f_*T\wedge\alpha$ if α is a form on Y and T is a current on X.

E. The current associated to an analytic subvariety

Let X be a complex analytic manifold and let $A\subset X$ be an analytic subset everywhere of dimension d. Let A_{reg} be the regular locus in A, that is the set of points of A possessing a neighborhood Ω in X such that $A\cap\Omega$ is a submanifold of dimension d of X. The singular locus $A_{\mathrm{sing}} = A\setminus A_{\mathrm{sing}}$ is the complement to the regular locus. By a fundamental theorem [(3)], it is an analytic subset of X.

The integration current on A is the current of bidimension (d,d), equivalently of bidegree $(n-d,n-d)$, defined by the formula

$$\langle\delta_A,\omega\rangle = \int_{A_{\mathrm{reg}}} j_A^*\omega$$

where $j_A\colon A_{\mathrm{reg}}\hookrightarrow X$ is the canonical immersion and $\omega\in\mathscr{A}_{\mathrm{c}}^{(d,d)}(X)$ is a differential form of bidegree (d,d) with compact support on X. When $A_{\mathrm{sing}}\neq\varnothing$, the support of $j_A^*\omega$ is not necessarily compact, and one first has to prove the convergence of the written integral. This can be done using the Weierstrass preparation theorem which allows, locally, to describe A as a finite cover of $\mathbf{C}^d$ defined in $\mathbf{C}^{d+1}=\mathbf{C}^d\times\mathbf{C}$ by a monic polynomial in the last variable whose coefficients are analytic functions in the first d variables.

Another way of defining δ_A is to use resolution of singularities. Namely, there exists a complex analytic manifold Y, a proper birational morphism $\pi\colon Y\to X$ which is an isomorphism outside A_{sing} such that the closure B of $\pi^{-1}(A_{\mathrm{reg}})$ in Y is a closed submanifold. Then δ_B is well-defined, as a current of bidegree $(n-d,n-d)$ on Y (if $j_B\colon B\hookrightarrow Y$ is the canonical closed immersion, one has $\delta_B(\omega)=\int_B j_B^*\omega$ for any $\omega\in\mathscr{A}^{(d,d)}(Y)$), and one may set $\delta_A=\pi_*\delta_B$. In other words, for any $\omega\in\mathscr{A}_{\mathrm{c}}^{(d,d)}(X)$,

$$\langle\delta_A,\omega\rangle = \int_B (j_B^*\pi^*)\omega.$$

That both definitions agree follows from the fact that the complement in B to $\pi^{-1}(A_{\mathrm{reg}})$, being an analytic subset of dimension $<d$, has measure 0.

THEOREM 2.4.6 (Lelong). *For any analytic subset A of X, the current δ_A is closed:* $\mathrm{d}\delta_A = 0$.

Proof. — This is easy if one uses the second definition. Indeed, using the above notation, one has

$$\langle\mathrm{d}\delta_A,\omega\rangle = \langle\mathrm{d}\delta_B,\pi^*\omega\rangle = -\langle\delta_B,\mathrm{d}\pi^*\omega\rangle = 0$$

for any differential form $\omega\in\mathscr{A}_{\mathrm{c}}^{(d,d)}(X)$ since $j_B^*\omega$ is automatically closed, being a form of bidegree (d,d) on the complex d-dimensional manifold B. □

[(3)] Reference

Let us now assume that X is a smooth complex algebraic variety. By linearity, any cycle $A \in \mathrm{Z}_p(X)$ defines a current δ_A of bidimension (p,p).

PROPOSITION 2.4.7 (Poincaré–Lelong Formula). *Let X be a smooth complex algebraic variety, everywhere of dimension n, and let f a regular meromorphic function on X. Then,* $\log\left|f\right|$ *is locally integrable on X and*

$$\mathrm{dd}^c \log\left|f\right|^{-1} + \delta_{\mathrm{div}(f)} = 0.$$

Proof. — This formula can be checked locally for the complex topology, around any point $x \in X$. By the very definition of the currents involved [(4)], we may also except a set of measure 0 in $\left|\mathrm{div}(f)\right|$. We shall therefore assume, either that $x \not\in \left|\mathrm{div}(f)\right|$, or that x is a smooth point of $\left|\mathrm{div}(f)\right|$.

First assume that x does not belong to $\left|\mathrm{div}(f)\right|$; then f is holomorphic and non zero around x. Consequently, there exists a neighborhood Ω of x in X and a holomorphic function g on Ω such that $f = e^{-g}$. On Ω, the current $\log\left|f\right|^{-1}$ is defined by the $\mathscr{C}^\infty$ function $\frac{1}{2}(g+\overline{g})$. We observe that $\overline{\partial} g = 0$ since g is holomorphic on Ω; consequently, $\partial\overline{g} = \overline{\overline{\partial} g} = 0$ too. Then, the current $\mathrm{dd}^c \log\left|f\right|^{-1}$ is the one associated to the form

$$\mathrm{dd}^c \log\left|f\right|^{-1} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}(g+\overline{g}) = \frac{\mathrm{i}}{2\pi}\partial(\overline{\partial} g) - \frac{\mathrm{i}}{2\pi}\overline{\partial}(\partial\overline{g}) = 0.$$

We now assume that x is a smooth point of $\left|\mathrm{div}(f)\right|$. This means that x belongs to exactly one of the components of $\left|\mathrm{div}(f)\right|$, say Z, and is a regular point of Z. In that case, there exists an open neighborhood Ω of x in X, a holomorphic function z_1 on Ω which defines $Z \cap \Omega$ and a non-vanishing holomorphic function u on Ω such that $f = uz^m$. Moreover, up to shrinking Ω, one may assume that there exist holomorphic function $z_2, \dots, z_n$ on Ω such that $(z_1, \dots, z_n)$ is an biholomorphic isomorphism from Ω to an open neighborhood U of the origin in $\mathbf{C}^n$. We thus may assume that Ω is an open neighborhood of the origin $z = 0$ in $\mathbf{C}^n$.

One has $\log\left|f\right| = \log|u| + \log|z_1|$, hence

$$\mathrm{dd}^c \log\left|f\right| = \mathrm{dd}^c \log|u| + m\,\mathrm{dd}^c \log|z_1| = m\,\mathrm{dd}^c \log|z_1|$$

by the first case we have treated. Moreover, $\log|z_1| = (z_1)^* \log|z|$, while $\delta_{z_1=0} = (z_1)^*\delta_0$ where $\log|z|$ and δ_0 are currents of bidegrees respectively $(0,0)$ and $(1,1)$ in $\mathbf{C}$. Since,

$$\mathrm{dd}^c \log|z_1|\,\mathrm{dd}^c (z_1)^* \log|z|,$$

the Proposition follows from the following equality

$$\mathrm{dd}^c \partial\overline{\partial} \log|z| = \delta_0$$

of currents on $\mathbf{C}$, which is proved as a separate Lemma. □

LEMMA 2.4.8. *Let α be a $\mathscr{C}^\infty$-function with compact support on* $\mathbf{C}$*. One has*

$$\int_{\mathbf{C}} \log|z|\,\partial\overline{\partial}\alpha = -\mathrm{i}\pi\alpha(0).$$

(4) Détailler...

Proof. — One has $\langle \delta_0, \alpha\rangle = \alpha(0)$, while

$$\begin{aligned}\langle \frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log|z|, \alpha\rangle &= \frac{\mathrm{i}}{\pi}\langle \log|z|, \overline{\partial}\partial\alpha\rangle \\ &= \frac{\mathrm{i}}{\pi}\int_{\mathbf{C}} \log|z|\,\overline{\partial}\partial\alpha \\ &= \frac{\mathrm{i}}{\pi}\lim_{\varepsilon\to 0^+}\int_{|z|\geqslant\varepsilon} \log|z|\,\overline{\partial}\partial\alpha.\end{aligned}$$

Since $\mathrm{d}\partial\alpha = \overline{\partial}\partial\alpha + \partial^2\alpha = \overline{\partial}\partial\alpha$, one has the following equality of differential forms on $\mathbf{C}^*$:

$$\log|z|\,\overline{\partial}\partial\alpha = \mathrm{d}(\log|z|\,\partial\alpha) - \mathrm{d}\log|z| \wedge \partial\alpha.$$

Moreover, choosing a local determination of $\log z$, we see that

$$\mathrm{d}\log|z| = \frac{1}{2}\mathrm{d}\log|z|^2 = \frac{1}{2}\mathrm{d}\log z + \frac{1}{2}\mathrm{d}\log\overline{z} = \frac{\mathrm{d}z}{2z} + \frac{\mathrm{d}\overline{z}}{2\overline{z}}.$$

For degree reasons, $\mathrm{d}z \wedge \partial\alpha = 0$; we thus obtain that

$$\log|z|\,\overline{\partial}\partial\alpha = \mathrm{d}(\log|z|\,\partial\alpha) - \frac{\mathrm{d}\overline{z}}{2\overline{z}} \wedge \partial\alpha.$$

Then, Stokes formula implies, for any positive real number ε, that

$$\int_{|z|\geqslant\varepsilon} \log|z|\,\overline{\partial}\partial\alpha = -\int_{|z|=\varepsilon} \log|z|\,\partial\alpha + \int_{|z|>\varepsilon} \frac{\mathrm{d}\overline{z}}{2\overline{z}} \wedge \partial\alpha.$$

Moreover, still on $\mathbf{C}^*$, we have

$$\frac{\mathrm{d}\overline{z}}{\overline{z}} \wedge \partial\alpha = \frac{\mathrm{d}\overline{z}}{\overline{z}} \wedge \mathrm{d}\alpha = -\mathrm{d}\left(\alpha\frac{\mathrm{d}\overline{z}}{\overline{z}}\right),$$

so that

$$\int_{|z|>\varepsilon} \frac{\mathrm{d}\overline{z}}{\overline{z}} \wedge \partial\alpha = \int_{|z|=\varepsilon} \alpha\frac{\mathrm{d}\overline{z}}{\overline{z}} = \int_0^{2\pi} \alpha(\varepsilon e^{\mathrm{i}\theta})(-\mathrm{i})\,\mathrm{d}\theta = -\mathrm{i}\int_0^{2\pi} \alpha(\varepsilon e^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Finally,

$$\int_{|z|\geqslant\varepsilon} \log|z|\,\overline{\partial}\partial\alpha = -\int_{|z|=\varepsilon} \log|z|\,\partial\alpha - \frac{1}{2}\mathrm{i}\int_0^{2\pi} \alpha(\varepsilon e^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

When $\varepsilon \to 0$, the absolute value of the first term of the right hand side is majorized by a constant time $\varepsilon\log\varepsilon^{-1}$, hence goes to 0. The second term however converges to $-\mathrm{i}\pi\alpha(0)$. This shows that

$$\int_{\mathbf{C}} \log|z|\,\overline{\partial}\partial\alpha = -\mathrm{i}\pi\alpha(0),$$

as claimed. □

F. Green currents

DEFINITION 2.4.9. *Let X be a complex analytic manifold everywhere of dimension n and let Z be a cycle of dimension p on X. Assume that X is Kähler. A* Green current *for Z is a current g_Z of bidimension $(p+1,p+1)$ on X such that*

$$\omega_Z = \frac{\mathrm{i}}{\pi}\partial\bar\partial g_Z + \delta_Z$$

is (the current associated to) a $\mathscr{C}^\infty$-differential form of bidegree (p,p) on X.

With the above notation, we remark that δ_Z and ω_Z define the same cohomology class in $H^{(n-p,n-p)}_{\mathrm{Dol}}(X)$. Conversely, representing the cohomology class of the current δ_Z by a form paves the way to a proof of the existence of Green currents.

THEOREM 2.4.10. *a) Any cycle has a Green current.*

b) If g and g' are two Green currents for a cycle Z, then there are currents $u \in \mathscr{D}_{(p+1,p+2)}(X)$ and $v \in \mathscr{D}_{(p+2,p+1)}(X)$, and a form $\alpha \in \mathscr{A}^{(n-p-1,n-q-1)}(X)$ such that $g' - g = \alpha + \partial u + \bar\partial v$.

Proof. — a) The current δ_Z is closed; since the De Rham cohomology of currents coincides with that of forms, and is represented by harmonic forms, there exists a harmonic $\mathscr{C}^\infty$-differential form ω such that $\delta_Z - \omega$ is an exact current. By the $\partial\bar\partial$-lemma for currents, there exists a current g such that $\delta_Z - \omega = \partial\bar\partial g$. This implies that $-2\mathrm{i}\pi g$ is a Green current for Z.

b) Let $h = g' - g$. By assumption, $\partial\bar\partial h$ is (the current associated to) a $\mathscr{C}^\infty$-differential form on X. The assertion thus follows from the following regularity lemma. □

PROPOSITION 2.4.11 (Regularity Lemma). *Let X be a complex Kähler manifold and let T be a current of bidegree (p,q) n X.*

a) Assume that T is smooth and can be written $T = \partial U + \bar\partial V$ for some currents U and V of bidegrees $(p-1,q)$ and $(p,q-1)$. Then there exists forms u and v such that $T = \partial u + \bar\partial v$.

b) If $\partial\bar\partial T$ is smooth, then there exist currents U, V of bidegrees $(p-1,q)$ and $(p,q-1)$, and a form α of bidegree (p,q) such that $T = \alpha + \partial U + \bar\partial V$.

c) If $\partial\bar\partial T = 0$, there one can take α to be harmonic. If moreover T is smooth, then U and V can be chosen to be smooth.

Proof. — b) The differential form $\partial(\bar\partial T)$ of bidegree $(p+1,q+1)$ is ∂-exact as a current; since the ∂-cohomology of forms and currents coincide, it is ∂-exact as a form and there exists a differential form $\alpha_1 \in \mathscr{A}^{(p,q+1)}(X)$ such that $\partial(\bar\partial T) = \bar\partial\alpha_1$. Now, $\partial T + \alpha_1$ is a $\bar\partial$-closed current of bidegree $(p,q+1)$; by the same cohomological argument, it is equal to a $\bar\partial$-closed differential form up to a $\bar\partial$-exact current. This means that there exists a form $\beta_1 \in \mathscr{A}^{(p,q+1)}$ and a current T_1 of bidegree $(p-1,q)$ such that $\bar\partial\beta_1 = 0$ and $\partial T + \alpha_1 = \beta_1 - \bar\partial T_1$. We have shown the existence of a current T_1 of bidegree $(p+1,q-1)$ and of a differential form $u_0 \in \mathscr{A}^{(p+1,q)}$ such that $\partial T = u_0 - \bar\partial T_1$. Moreover, $\partial\bar\partial T = -\bar\partial u_0$ is smooth. We thus may iterate this argument and construct a sequence (u_k) of differential forms, with $u_k \in \mathscr{A}^{(p+k+1,q-k)}(X)$, and (T_k) of currents, with $T_k \in$

$\mathscr{D}^{(p+k,q-k)}$, satisfying $T_0 = T$ and $\partial T_k = u_k - \bar{\partial} T_{k+1}$ for $k \geqslant 0$. Considering the degrees of these currents, we see that $T_k = 0$ for $k > q$.

Let us now show by decreasing induction on k that there exists a form $\alpha_k \in \mathscr{A}^{(p+k,q-k)}$ and currents U_k, V_k such that $T_k = \alpha_k + \partial U_k + \bar{\partial} V_k$. This holds obviously for $k > q$. Assume it holds for $k+1$. Then, $\partial T_k = u_k - \bar{\partial}\alpha_{k+1} - \bar{\partial}\partial U_{k+1}$, so that $u_k - \bar{\partial}\alpha_{k+1} = \partial(T_k - \bar{\partial} U_{k+1})$ is a differential form of bidegree $(p+k+1, q-k)$ which is ∂-exact as a current. It follows that it is ∂-exact, hence there exists a form $v \in \mathscr{A}^{(p+k,q-k)}(X)$ such that $u_k - \bar{\partial}\alpha_{k+1} = \partial v$. Then, $T_k - \bar{\partial} U_{k+1} - v$ is a ∂-closed current so is equal to a ∂-closed form w up to a ∂-exact current ∂U_k. Finally, $T_k = v + w + \partial U_k + \bar{\partial} U_{k+1}$; it remains to set $\alpha_k = v + w$ and $V_k = U_{k+1}$. By induction, the desired assertion holds for all k, in particular for $k = 0$. This shows b).

We remark for further use that we have $\partial\bar{\partial} T = \partial\bar{\partial}\alpha$.

a) By hypothesis, $\partial\bar{\partial} U = -\bar{\partial} T$ is smooth. Applying assertion *b*) to U, it follows that there exist currents g_1, g_2, and a form α such that $U = \alpha + \partial g_1 + \bar{\partial} g_2$. Then,

$$T = \partial U + \bar{\partial} V = \partial\alpha + \partial\bar{\partial} g_2 + \bar{\partial} V.$$

Applying also *b*) to V which satisfies $\partial\bar{\partial} V = \partial T$, we see that there exist currents h_1, h_2 and a form β such that $V = \beta + \partial h_1 + \bar{\partial} h_2$. As a consequence,

$$T = \partial\alpha + \bar{\partial}\beta + \partial\bar{\partial}(g_2 - h_1).$$

Applying *b*) to $g_2 - h_1$, we get a form γ such that $\partial\bar{\partial}(g_2 - h_1) = \partial\bar{\partial}\gamma$. Finally, the current

$$T = \partial(\alpha + \bar{\partial}\gamma) + \bar{\partial}\beta$$

is of the required form.

c) With the notation of *b*), one has $\partial\bar{\partial}\alpha = 0$. In view of the decomposition given by *b*), it suffices to treat the smooth case. Observe then that $\partial\alpha$ is a ∂-exact form of bidegree $(p+1, q)$. By the $\partial\bar{\partial}$-lemma there exists a form $u \in \mathscr{A}^{(p,q-1)}(X)$ such that $\partial\alpha = \partial\bar{\partial} u$. Similarly, there exist a form v such that $\bar{\partial}\alpha = \partial\bar{\partial} v$. Then, $\alpha - \partial v - \bar{\partial} u$ is both ∂- and $\bar{\partial}$-closed, so is d-closed. It follows that there exists a harmonic form ω and a form w such that $\alpha = \omega + \mathrm{d}w = \omega + \partial w + \bar{\partial} w$. □

G. Hermitian line bundles and Green currents for divisors

Green currents for divisors are closely related to hermitian metrics on the associated line bundle.

Let X be a complex analytic manifold and let L be a line bundle on X. A Hermitian metric on L is the datum, for any section s of L over an open set U of a function $\|s\| : U \to \mathbf{R}_+$ satisfying the following properties:

– for any open sets $V \subset U$ and any section $s \in \Gamma(U, L)$, $\|s\|(x) = \|s|_V\|(x)$ for any $x \in V$;

– for any open set U, any section $s \in \Gamma(U, L)$ and any holomorphic function $f \in \Gamma(U, \mathscr{O}_X)$, $\|fs\| = |f| \|s\|$.

One says that a hermitian metric is smooth (continuous,...) if for any non-vanishing local section $s \in \Gamma(U, L)$, the associated function $\log\|s\|$ is smooth (continuous,...) on U. We write $\overline{L} = (L, \|\cdot\|)$ to indicate that L is a hermitian bundle, with hermitian metric $\|\cdot\|$.

PROPOSITION 2.4.12. *Let X be a complex analytic manifold and $\overline{L}$ be a hermitian line bundle on X endowed with a smooth hermitian metric. There exists a the differential form $c_1(\overline{L})$ of of bidegree $(1,1)$ on X such that for any open set $U \subset X$, any non-vanishing section $s \in \Gamma(U, L)$,*

$$\mathrm{dd}^c \log\|s\|^{-1} = c_1(\overline{L})\,|_U .$$

This differential form is called the curvature form of the Hermitian line bundle $\overline{L}$. It is closed and real; its cohomology class in $H^{1,1}_{\mathrm{Dol}}(X)$ or in $H^2_{\mathrm{dR}}(X)$ depends only on the underlying line bundle L, but not on the choice of an hermitian metric. This is the *first Chern class* of L.

PROPOSITION 2.4.13. *Let $\omega \in \mathscr{A}^{(1,1)}(X)$ be a real closed differential form of bidegree $(1,1)$ whose cohomomology class is that of L. Then there exists a Hermitian metric on L, unique up to multiplication by the exponential of a harmonic function, whose curvature form equals ω.*

Proof. — Let us fix an arbitrary hermitian metric on L. Then the cohomology class of $c_1(\overline{L}) - \omega$ is zero. Consequently, this differential form is exact and, by the $\partial\overline{\partial}$-lemma, can be written $\mathrm{dd}^c h$, where h is a $\mathscr{C}^\infty$-function on X. A priori, h is only complex valued, but since $c_1(\overline{L})$ and ω are real, and dd^c is a real operator, we may replace h by its imaginary part. Then, multiplying the metric on L by e^h gives a new hermitian metric whose curvature form is $c_1(\overline{L}) - \mathrm{dd}^c h = \omega$. □

PROPOSITION 2.4.14. *Let X be a complex analytic manifold and $\overline{L}$ be a hermitian line bundle on X endowed with a smooth hermitian metric.*

Let s be a regular meromorphic section of L on X. Then the function $\log\|s\|^{-1}$ *on* $X \setminus |\mathrm{div}(s)|$ *is locally integrable on X and is a Green current for the cycle* $\mathrm{div}(s)$. *More precisely,*

$$\mathrm{dd}^c \log\|s\|^{-1} + \delta_{\mathrm{div}(s)} = c_1(\overline{L}).$$

Conversely, for any real Green current g for $\mathrm{div}(s)$, *there is a unique smooth hermitian metric on L such that $g = \log\|s\|^{-1}$.*

Proof. — We may reason locally and assume that L admits a trivialization ε; by assumption, the function $h = \log\|\varepsilon\|^{-1}$ is smooth on X. Moreover, there exists a unique meromorphic function f on X such that $s = f\varepsilon$; then $\mathrm{div}(s) = \mathrm{div}(f)$ and $\log\|s\|^{-1} = -\log|f| + h$. Since the logarithm of a meromorphic function is locally integrable, this implies that $\log\|s\|$ is $\mathrm{L}^1_{\mathrm{loc}}$. By the Poincaré–Lelong formula,

$$\mathrm{dd}^c \log\|s\|^{-1} + \delta_{\mathrm{div}(s)} = -\,\mathrm{dd}^c \log|f| + \delta_{\mathrm{div}(f)} + \mathrm{dd}^c h = \mathrm{dd}^c h,$$

which proves that $\log\|s\|^{-1}$ is indeed a Green function for the cycle $\mathrm{div}(s)$.

For the converse assertion, let ω be the smooth form $\mathrm{dd}^c g + \delta_{\mathrm{div}(f)}$. To prove the existence of a unique smooth hermitian metric on L satisfying $\log\|s\|^{-1} = g$, we may reason locally and assume, as for the first part, that L admits a trivialization ε. Smooth hermitian metric on L are then in bijection with smooth functions on X, the bijection associating to a metric $\|\cdot\|$ the function $h = \log\|\varepsilon\|^{-1}$. Let f be the unique meromorphic function such that $s = f\varepsilon$. We now observe that

$$\mathrm{dd}^c(g+\log\left|f\right|) = -\delta_{\mathrm{div}(f)} + \omega + \delta_{\mathrm{div}(f)} = \omega$$

is a smooth form on X. By the regularity lemma, $g+\log\left|f\right|$ is smooth. Consequently, there exists a smooth hermitian metric $\|\cdot\|$ on L such that $\log\|\varepsilon\|^{-1} = g+\log\left|f\right|$. For this metric, one has

$$\log\|s\|^{-1} = \log\left\|f\varepsilon\right\|^{-1} = -\log\left|f\right| + \log\|\varepsilon\|^{-1} = g,$$

as claimed. □

§ 2.5
ARITHMETIC CHOW GROUPS

A. Definition

Let K be a number field, let $\mathfrak{o}_K$ be its ring of integers, let Σ be the set of embeddings of K into $\mathbf{C}$.

Let X be a quasi-projective scheme over $\mathfrak{o}_K$, whose generic fiber X_K is smooth and equidimensional; we will say that X is an *arithmetic scheme* over $\mathfrak{o}_K$. Such schemes are excellent and finite dimensional. They also give rise, for any embedding $\sigma\colon K \hookrightarrow \mathbf{C}$, to a complex analytic manifold $\sigma^*X(\mathbf{C})$; moreover, if $\overline{\sigma}$ is the complex embedding conjugate to σ, the complex conjugation induces an antiholomorphic isomorphism $F_\infty\colon \sigma^*X(\mathbf{C}) \to \overline{\sigma}^*X(\mathbf{C})$.

For any integer p, F_∞ acts on the vector space $\bigoplus_\sigma \mathscr{A}^{(p,p)}(\sigma^*X(\mathbf{C}))$; we let $\mathscr{A}^{(p,p)}(X_{\mathbf{R}})$ to be the set of families (ω_σ) such that $F_\infty^*\omega_\sigma = (-1)^p\omega_{\overline{\sigma}}$ for all σ. We define analogously $\mathscr{D}_{(p,p)}(X_{\mathbf{R}})$. The operator dd^c, acting componentwise, maps $\mathscr{A}^{(p,p)}(X_{\mathbf{R}})$ into $\mathscr{A}^{(p+1,p+1)}(X_{\mathbf{R}})$, and similarly for currents. For $\omega \in \mathscr{A}^{(p,p)}(X_{\mathbf{R}})$, we also write $[\omega]$ for the element $([\omega_\sigma])$ of $\mathscr{D}^{(p,p)}(X_{\mathbf{R}})$.

Let p be an integer. An arithmetic cycle of dimension p on X is a pair $(Z,(g_\sigma))$ where Z is a cycle of dimension p on X and where, for any $\sigma\in\Sigma$, g_σ is a Green current for the cycle σ^*Z on the complex analytic manifold $\sigma^*X(\mathbf{C})$. We also impose that for any embedding σ of K into $\mathbf{C}$, one has the relation $g_{\overline{\sigma}} = F_\infty^* g_\sigma$. For any arithmetic cycle $\hat{Z} = (Z,(g_\sigma))$, we write g_Z, δ_Z and ω_Z for the elements (g_σ), (δ_{σ^*Z}) and (ω_σ), where $\omega_\sigma = \mathrm{dd}^c g_\sigma + \delta_{\sigma^*Z})$; these are elements of $\mathscr{D}_{(p+1,p+1)}(X_{\mathbf{R}})$, $\mathscr{D}_{(p,p)}(X_{\mathbf{R}})$ and $\mathscr{A}^{(n-p,n-p)}(X_{\mathbf{R}})$ respectively which satisfy $\mathrm{dd}^c g_Z + \delta_Z = [\omega_Z]$. (The integer n is the dimension of the generic fibre X_K of X.) We also write $\zeta(\hat{Z}) = Z$.

We let $\hat{Z}_p(X)$ be the group of arithmetic cycles of dimension p on X.

For any $\omega\in\mathscr{A}^{(n-p-1,n-p-1)}(X_{\mathbf{R}})$, then $a(\omega) = (0,\omega)$ is an arithmetic cycle in $\hat{Z}_p(X)$.

Let W be a closed integral subscheme of dimension p on X and let $f \in K(W)^*$ be a regular rational function on X. Then for any $\sigma \in \Sigma$, the function $\log\left|f\right|_\sigma^{-1}$ on $\sigma^* W(\mathbf{C})$, extended by 0 outside of $\sigma^* W(\mathbf{C})$ in $\sigma^* X(\mathbf{C})$, is a Green current for the cycle $\operatorname{div}(f)$. This follows from the Poincaré–Lelong formula: if $\pi\colon V \to W_K$ is a resolution of singularities of W_K $i\colon W_K \hookrightarrow X_K$ the canonical closed immersion and $j = i\circ\pi$ thir composition, then $\log\left|f\right|_\sigma^{-1} = (j_\sigma)_* \log\left|\pi^* f\right|^{-1}$. The pair $\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), (\log\left|f\right|_\sigma^{-1}))$ is then an arithmetic cycle on X.

We define the group $\widehat{\operatorname{Rat}}_p(X)$ of arithmetical cycles which are rationally equivalent to zero to be the subgroup of $\hat{Z}_p(X)$ generated by the arithmetic cycles of the form $\widehat{\operatorname{div}}(f)$, as well as the arithmetic cycles of the form $(0, (\partial u_\sigma + \bar\partial v_\sigma))$, where u_σ and v_σ are currents of bidimension $(p-2, p-1)$ and $(p-1, p-2)$ on $\sigma^* X(\mathbf{C})$.

Finally, we define $\widehat{\mathrm{CH}}_p(X) = \widehat{\mathrm{Z}}_p(X)/\widehat{\operatorname{Rat}}_p(X)$. This is the *arithmetic Chow group of dimension p of X*. Graduating the cycles by codimension insted of dimension gives rise to analogous groups $\widehat{\mathrm{Z}}^p(X)$, $\widehat{\operatorname{Rat}}^p(X)$ and $\widehat{\mathrm{CH}}^p(X) = \widehat{\mathrm{Z}}^p(X)/\widehat{\operatorname{Rat}}^p(X)$ of arithmetic cycles of codimension p on X.

PROPOSITION 2.5.1. *The maps $a\colon \mathscr{A}^{(n-p-1,n-p-1)}(X_{\mathbf{R}}) \to \widehat{\mathrm{Z}}_p(X)$ and $\zeta\colon \widehat{\mathrm{Z}}_p(X) \to \mathrm{Z}_p(X)$ induce an exact sequence*

$$\mathscr{A}^{(n-p-1,n-p-1)}(X_{\mathbf{R}}) \xrightarrow{a} \widehat{\mathrm{CH}}_p(X) \xrightarrow{\zeta} \mathrm{CH}_p(X) \to 0.$$

B. Push forward of arithmetic cycles

Let $f\colon Y \to X$ be a proper morphism of arithmetic varieties over $\mathfrak{o}_K$ such that f_K is smooth.

For any arithmetic cycle $\hat{Z} = (Z, g)$, let us define $f_* \hat{Z}$ to be the pair $(f_* Z, f_* g)$; there, $f_* Z$ is a cycle of dimension p on X while $f_* g = (f_* g_\sigma)$, where, for any σ, $f_* g_\sigma$ is a current of bidimensions $(n-p+1, n-p+1)$ on $\sigma^* X(\mathbf{C})$. This pair is an arithmetic cycle on X because $\mathrm{dd}^c f_* g = f_* \mathrm{dd}^c g = f_*(-\delta_Z + \omega_Z)$, $f_* \delta_Z = \delta_{f_* Z}$ and $f_* \omega_Z$ is smooth, since f is assumed to be smooth on the generic fibre. Consequently, there is a morphism $f_*\colon \widehat{\mathrm{Z}}_p(Y) \to \widehat{\mathrm{Z}}_p(X)$.

Since direct image of currents commutes with derivation, The direct image of an arithmetic cycle of the form $(0, \partial u + \bar\partial v)$ still is of this form. Let W be a closed integral subscheme of dimension p in Y and let u be a non zero rational function on W. We want to show that $f_* \widehat{\operatorname{div}}(u)$ is zero in $\widehat{\mathrm{CH}}_p(X)$. Let us first assume that $\dim f(W) < \dim(W)$. Then, $f_* \operatorname{div}(u) = 0$. Moreover,$f_* \log|u|^{-1}$ is zero too. Indeed, for any form $\alpha \in \mathscr{A}^{(p,p)}(X)$,

$$\langle f_* \log|u|^{-1}, \alpha\rangle = \int_W \log|u|^{-1} f^* \alpha = 0$$

since $f^* \alpha$ vanishes on the fibers of $f\colon W \to f(W)$ which have positive dimension. Consequently, $f_* \widehat{\operatorname{div}}(u) = 0$.

Let us now assume that $\dim f(W) = \dim W$. In that case, the field extension $R(f(W)) \subset R(W)$ has finite degree, say d and $f_*(W) = d[f(W)]$. Then, $f_* \operatorname{div}(u) = \operatorname{div}(\mathrm{N}(u))$, where N is the norm map from $R(W)$ to $R(f(W))$. Moreover, there exists a Zariski open subset W' of W such that $f\colon W' \to f(W')$ is a finite étale cover of constant degree d. Since the complementary subsets to $f(W')$ in $f(W)$ and to W' in W are null sets, we may compute $\langle f_* \log|u|^{-1}$ as follows: for any form $\alpha \in \mathscr{A}^{(p,p)}(X)$,

$$\begin{aligned}\langle f_* \log|u|^{-1}, \alpha\rangle &= \int_W \log|u|^{-1} f^*\alpha = \int_{W'} \log|u|^{-1} f^*\alpha \\ &= \int_{f(W')} \sum_{w \in f^{-1}(z) \cap W} \log|u(w)|^{-1} \alpha(z) \\ &= \int_{f(W')} \log|\mathrm{N}(u)|^{-1} \alpha \\ &= \langle \log|\mathrm{N}(u)|^{-1}, \alpha\rangle,\end{aligned}$$

so that $f_* \log|u|^{-1} = \log|\mathrm{N}(u)|^{-1}$.

These calculations show that the morphism $f_* \colon \widehat{\mathrm{Z}}_p(Y) \to \widehat{\mathrm{Z}}_p(X)$ passes through rational equivalence and it induces a morphism $f_* \colon \widehat{\mathrm{CH}}_p(Y) \to \widehat{\mathrm{CH}}_p(X)$.

C. The first arithmetic Chern class of a metrized line bundle

Let K be a number field, $\mathfrak{o}_K$ be its ring of integers and Σ the set of embeddings of K into $\mathbf{C}$. Let X be an arithmetic scheme over $\mathfrak{o}_K$.

Generalizing the definition already given when $X = \operatorname{Spec} \mathfrak{o}_K$, a Hermitian vector bundle $\overline{E} = (E, h)$ on X is the data of a vector bundle E on X and, for any $\sigma \colon K \hookrightarrow \mathbf{C}$, of a Hermitian metric h_σ on the complex vector bundle $\sigma^* E$ on $\sigma^* X(\mathbf{C})$, compatible with complex conjugation: for any $\sigma \in \Sigma$, any open set $U \subset \sigma^* X(\mathbf{C})$, any two sections $v_1, v_2 \in \Gamma(U, \sigma^* E)$, one has $h_{\overline{\sigma}}(F_\infty^* v_1, F_\infty^* v_2) = \overline{h_\sigma(v_1, v_2)}$. We also write $\|v\|_{\sigma,\overline{E}}$, or simply $\|v\|_\sigma$, for the Hermitian norm $\sqrt{h_\sigma(v,v)}$ of a local section v. The compatibility with complex conjugation can then be shown to be equivalent to the equalities $\left\|F_\infty^* v\right\|_{\overline{\sigma}} = \|v\|_\sigma$, for any $\sigma \in \Sigma$, any open set $U \subset \sigma^* X(\mathbf{C})$ and any section $v \in \Gamma(U, \sigma^* E)$.

Let $\overline{E} = (E, h_E)$ and $\overline{F} = (F, h_F)$ be hermitian vector bundles on X. A morphism from $\overline{E}$ to $\overline{F}$ is a morphism $\varphi \colon E \to F$ of the underlying vector bundles which is norm decreasing, in the sense that for any $\sigma \in \Sigma$, any open set $U \subset \sigma^* X(\mathbf{C})$ and any section $v \in \Gamma(U, \sigma^* E)$, $\left\|\varphi(v)\right\|_\sigma \leqslant \|v\|_\sigma$.

All constructions of Hermitian vector bundles from linear algebra explained in Chapter 1 apply in that more general setting and we do not repeat them.

A Hermitian vector bundle of rank 1 is called a Hermitian line bundle. The tensor product of Hermitian line bundles induces a group structure on the set $\widehat{\mathrm{Pic}}(X)$ of isomorphism classes of Hermitian line bundles on X.

There is an exact sequence of Abelian groups

$$\Gamma(X, \mathscr{O}_X^*) \xrightarrow{\log_\Sigma} A^0(X_{\mathbf{R}}) \to \widehat{\mathrm{Pic}}(X) \to \mathrm{Pic}(X) \to 0.$$

THEOREM 2.5.2. *Let $\overline{L}$ be a Hermitian line bundle on X and let s be a regular meromorphic section of L. Then, the pair* $\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), (\log\|s\|_\sigma^{-1}))$ *is an arithmetic cycle of codimension* 1 *on X whose class in* $\widehat{\mathrm{CH}}^1(X)$ *only depends on $\overline{L}$.*

We write $\widehat{c}_1(\overline{L})$ for the class of this cycle; this is the first arithmetic Chern class of $\overline{L}$.

If s and s' are regular meromorphic sections of Hermitian line bundles $\overline{L}$ and $\overline{L}'$, then $s \otimes s'$ is a regular meromorphic section of the Hermitian line bundle $\overline{L} \otimes \overline{L}'$ one has $\widehat{\operatorname{div}}(s \otimes s') = \widehat{\operatorname{div}}(s) + \widehat{\operatorname{div}}(s')$. Consequently, the map $\widehat{c}_1 \colon \widehat{\operatorname{Pic}}(X) \to \widehat{\operatorname{CH}}^1(X)$ is a morphism of Abelian groups; it is injective if X is normal, and an isomorphism if X is locally factorial.

Proof. — The fact that $\widehat{\operatorname{div}}(s)$ is an arithmetic cycle of codimension 1 is a restatement of the fact that $\operatorname{div}(s)$ is a codimension 1 cycle on X and that $\log \|s\|_\sigma^{-1}$ is a Green current on $\sigma^* X(\mathbf{C})$ for this cycle. The compatibility with complex conjugation follows from that of the Hermitian metric of $\overline{L}$.

Let s and s' be two regular meromorphic section of L. Then there exists a regular meromorphic function f on X such that $s' = fs$. Then, $\widehat{\operatorname{div}}(s') = \widehat{\operatorname{div}}(s) + \widehat{\operatorname{div}}(f)$, which shows that the classes of $\widehat{\operatorname{div}}(s')$ and of $\widehat{\operatorname{div}}(s)$ in $\widehat{\operatorname{CH}}^1(X)$ are equal. □

D. Intersecting with first arithmetic Chern classes

Let X be an arithmetic variety over $\mathfrak{o}_K$. Let $\overline{L}$ be a Hermitian line bundle on X. The aim of this section is to let the first arithmetic Chern class $\widehat{c}_1(\overline{L})$ act on the arithmetic Chow groups, as morphisms of Abelian groups $\widehat{\operatorname{CH}}_p(X) \to \widehat{\operatorname{CH}}_{p-1}(X)$, for $p \geqslant 1$ which are compatible with the morphisms $\operatorname{CH}_p(X) \to \operatorname{CH}_{p-1}(X)$ defined by intersection theory.

Let (Z, g_Z) be a arithmetic cycle of dimension p such that Z is integral; let $j \colon Z \hookrightarrow X$ be the canonical closed immersion. Let s be a regular meromorphic section of $j^*\overline{L} = \overline{L}|_Z$.

Let us first assume that Z_K is smooth. Then, for any $\sigma \in \Sigma$, the function $\log \|s\|_\sigma^{-1}$ on $\sigma^* Z(\mathbf{C})$ is a Green current for the divisor $\operatorname{div}(s)$. Consequently, $j_* \log \|s\|_\sigma^{-1}$ is a current of bidegree $(n-p-2, n-p-2)$ and one has the equality

$$\mathrm{dd}^c j_* \log \|s\|_\sigma^{-1} = j_* \,\mathrm{dd}^c \log \|s\|_\sigma^{-1} = j_* \left(-\delta_{\operatorname{div}(s)} + c_1(j^*\overline{L})\right) = -\delta_{j_* \operatorname{div}(s)} + j_* c_1(j^*\overline{L})$$

of currents on $\sigma^* X(\mathbf{C})$. To compensate for the term $j_* c_1(j^*\overline{L})$ on the second hand, we observe that $c_1(j^*\overline{L}) = j^* c_1(\overline{L})$ a closed differential form of bidegree $(1,1)$ on $\sigma^* X(\mathbf{C})$. Consequently, the current $c_1(\overline{L}) \wedge g_Z$ on $\sigma^* X(\mathbf{C})$ has bidimension $(p+2, p+2)$ and satisfies

$$\mathrm{dd}^c \left(c_1(\overline{L}) \wedge g_Z\right) = c_1(\overline{L}) \wedge \mathrm{dd}^c g_Z = c_1(\overline{L}) \wedge (-\delta_Z + \omega_Z) = -j_* c_1(\overline{L}) + c_1(\overline{L}) \wedge \omega_Z.$$

All in all, these computations show that

$$j_* \log \|s\|^{-1} + c_1(\overline{L}) \wedge g_Z$$

is a Green current on $\sigma^* X(\mathbf{C})$ for the cycle $j_* \operatorname{div}(s)$.

Consequently, we define an arithmetic cycle of dimension $p-1$ on X as

$$\widehat{\operatorname{div}}(s) \cap (Z, g_Z) = (\operatorname{div}(s), (\log \|s\|_\sigma^{-1} + c_1(\overline{L}) \wedge g_Z)).$$

It has support in $|\operatorname{div}(s)| \cap |Z|$.

In the general case, we let $\pi^* \tilde{Z} \to Z$ be a proper birational morphism such that $\tilde{Z}_K$ is smooth and define

$$\widehat{\operatorname{div}}(s) \cap (Z, g_Z) = (\operatorname{div}(s), \left(j_* \log \|\pi^* s\| + c_1(\overline{L}) \wedge g_Z\right)_\sigma,$$

where $j\colon \tilde{Z} \to X$ is the composition of π and the closed immersion of Z into X.

Let s' be another regular meromorphic section of L on Z and let f be the regular meromorphic function such that $s' = fs$. One has $\operatorname{div}(s') = \operatorname{div}(f) + \operatorname{div}(s)$ and

$$j_* \log \|\pi^* s'\| + c_1(\overline{L}) \wedge g_Z = j_* \log |f| + j_* \log \|\pi^* s\| + c_1(\overline{L}) \wedge g_Z,$$

so that

$$\widehat{\operatorname{div}}(s') \cap (Z, g_Z) = j_* \widehat{\operatorname{div}}(f) + \widehat{\operatorname{div}}(s) \cap (Z, g_Z).$$

In particular, the class of $\widehat{\operatorname{div}}(s) \cap (Z, g_Z)$ in the arithmetic Chow group $\widehat{\mathrm{CH}}_{p-1}(X)$ depends only on $\overline{L}$. We shall write $\widehat{c}_1(\overline{L}) \cap (Z, g_Z)$ for this class. Since any line bundle over an integral scheme possesses a regular meromorphic section, this allows to define, by linearity, a morphism of Abelian groups $\widehat{\mathrm{Z}}_p(X) \to \widehat{\mathrm{CH}}_{p-1}(X)$, $\alpha \mapsto \widehat{c}_1(\overline{L}) \cap \alpha$. This morphisms commute with the morphisms $\zeta\colon \widehat{\mathrm{Z}}_p(X) \to \mathrm{Z}_p(X)$ and $\widehat{\mathrm{CH}}_p(X) \to \mathrm{CH}_p(X)$, in the sense that

$$\zeta(\widehat{c}_1(\overline{L}) \cap \alpha) = c_1(L) \cap \zeta(\alpha).$$

PROPOSITION 2.5.3 (Projection formula). *Let $f\colon X \to Y$ be a proper morphism of arithmetic varieties over $\mathfrak{o}_K$ such that $f_K\colon X_K \to Y_K$ is smooth. Let $\overline{L}$ be an Hermitian line bundle on Y and let $\alpha \in \widehat{\mathrm{Z}}_p(X)$. Then, the following formula holds in $\widehat{\mathrm{CH}}_{p-1}(Y)$:*

$$f_*(\widehat{c}_1(f^*\overline{L}) \cap \alpha) = \widehat{c}_1(\overline{L}) \cap f_*(\alpha).$$

Proof. — Let us assume that α is an arithmetic cycle (Z, g_Z), where Z is integral and let $W = f(Z)$. Let us recall that, according to Prop. **??**, the formula holds at the level of cycles; basically, it remains to check that the Green currents agree.

Let s be a regular meromorphic section of L on W. Consequently, f^*s is a regular meromorphic section of f^*L on Z. Then, $f^*(\overline{L}) \cap \alpha$ is the class in $\widehat{\mathrm{CH}}_{p-1}(X)$ of the arithmetic cycle

$$\widehat{\operatorname{div}}(f^*s) \cap (Z, g_Z) = (\operatorname{div}(f^*s), j_* \log \|f^*s\|^{-1} + c_1(f^*\overline{L}) \wedge g_Z),$$

where $j\colon \tilde{Z} \to X$ is the composition of a generic resolution of singularities of Z and of the immersion of Z into X. Then,

$$f_*(\widehat{\operatorname{div}}(f^*s) \cap (Z, g_Z)) = (f_* \operatorname{div}(f^*s), f_* j_* \log \|f^*s\|^{-1} + f_*(c_1(f^*\overline{L}) \wedge g_Z)).$$

By Prop. **??**, $f_* \operatorname{div}(f^*s) = \operatorname{div}(s) \cap f_*[Z]$. Let us pose $d = [R(Z) : R(W)]$ if $\dim(Z) = \dim(W)$ and $d = 0$ otherwise. By definition, $f_*(Z, g_Z) = (f_*Z, f_*g_Z) = (dZ, f_*g_Z)$.

On the other hand, the projection formula for currents implies that

$$f_* j_* \log \|f^*s\|^{-1} = dk_* \log \|s\|^{-1} \quad \text{and} \quad f_*(c_1(f^*\overline{L}) \wedge g_Z) = c_1(\overline{L}) \wedge f_*g_Z,$$

hence

$$f_* j_* \log \|f^*s\|^{-1} + f_*(c_1(f^*\overline{L}) \wedge g_Z) = dk_* \log \|s\|^{-1} + c_1(\overline{L}) \wedge f_*g_Z.$$

The proposition follows from that. □

THEOREM 2.5.4. *Let* $\overline{L}$ *and* $\overline{M}$ *be Hermitian line bundles on* X. *Let* (Z, g_Z) *be an arithmetic cycle on* X. *Let* s *and* t *be regular meromorphic sections of* L *and* M *on* Z. *Assume that the divisors* $\operatorname{div}(s)$ *and* $\operatorname{div}(t)$ *on* Z *which have no common component meeting the generic fibre* Z_K. *Then,* $\widehat{\operatorname{div}}(s) \cap \left(\widehat{\operatorname{div}}(t) \cap (Z, g_Z)\right) = \widehat{\operatorname{div}}(t) \cap \left(\widehat{\operatorname{div}}(s) \cap (Z, g_Z)\right)$ *in* $\widehat{\mathrm{CH}}_{p-2}(X)$.

Proof. — By definition,

$$\widehat{\operatorname{div}}(t) \cap (Z, g_Z) = (\operatorname{div}(t) \cap Z, j_* \log \|t\|^{-1} + c_1(\overline{M}) \wedge g_Z)$$

and

$$\begin{aligned}\widehat{\operatorname{div}}(s) \cap \left(\widehat{\operatorname{div}}(t) \cap (Z, g_Z)\right) &= \Big(\operatorname{div}(s) \cap \operatorname{div}(t), \\ &\qquad k_* \log \|s\|^{-1} + c_1(\overline{L}) \wedge (j_* \log \|t\|^{-1} + c_1(\overline{M}) \wedge g_Z)\Big) \\ &= \Big(\operatorname{div}(s) \cap \operatorname{div}(t) \cap Z, \\ &\qquad \log \|s\|^{-1} \delta_{\operatorname{div}(t) \cap [Z]} + c_1(\overline{L}) \wedge \log \|t\|^{-1} \delta_Z + c_1(\overline{L}) \wedge c_1(\overline{M}) \wedge g_Z\Big).\end{aligned}$$

Similarly,

$$\begin{aligned}\widehat{\operatorname{div}}(t) \cap \left(\widehat{\operatorname{div}}(s) \cap (Z, g_Z)\right) &= \big(\operatorname{div}(t) \cap \operatorname{div}(s) \cap Z, \log \|t\|^{-1} \delta_{\operatorname{div}(s) \cap [Z]} \\ &\qquad + c_1(\overline{M}) \wedge \log \|s\|^{-1} \delta_Z + c_1(\overline{M}) \wedge c_1(\overline{L}) \wedge g_Z\Big).\end{aligned}$$

To prove the Theorem in that case, it thus suffices to prove the equality of currents

$$\log \|s\|^{-1} \delta_{\operatorname{div}(t) \cap [Z]} + c_1(\overline{L}) \wedge \log \|t\|^{-1} \delta_Z = \log \|t\|^{-1} \delta_{\operatorname{div}(s) \cap [Z]} + c_1(\overline{M}) \wedge \log \|s\|^{-1} \delta_Z$$

modulo ∂ and $\overline{\partial}$-exact currents.

Let us observe that for any two $\mathscr{C}^\infty$-functions u and v,

$$\begin{aligned}u \,\mathrm{dd}^c v - v \,\mathrm{dd}^c u &= \frac{\mathrm{i}}{\pi} \left(u \partial \overline{\partial} v - v \partial \overline{\partial} u\right) \\ &= \frac{\mathrm{i}}{\pi} \left(\partial (u \overline{\partial} v) + \overline{\partial} v \wedge \partial u + v \overline{\partial} \partial u\right) \\ &= \frac{\mathrm{i}}{\pi} \left(\partial (u \overline{\partial} v) + \overline{\partial} (v \wedge \partial u)\right).\end{aligned}$$

We may multiply this relation by any current T. If T is ∂ and $\overline{\partial}$-closed, we obtain

$$u \,\mathrm{dd}^c v \wedge T - v \,\mathrm{dd}^c u \wedge T = \frac{\mathrm{i}}{\pi} \left(\partial (u \overline{\partial} v) \wedge T + \overline{\partial} (v \wedge \partial u) \wedge T\right) = \frac{\mathrm{i}}{\pi} \left(\partial (u \overline{\partial} v \wedge T) + \overline{\partial} (v \wedge \partial u \wedge T)\right).$$

At least at a formal level, the desired equality follows from that by setting $u = \log \|s\|^{-1}$, $v = \log \|t\|^{-1}$ and $T = \delta_Z$. However, u and v are not $\mathscr{C}^\infty$ so an additional analytic investigation is necessary. (...) □

COROLLARY 2.5.5. *Let* $\overline{L}$ *be an hermitian line bundle on* X. *Let* W *be an integral closed subscheme of dimension* p *in* X, *let* g_W *be a Green current for* W *and let* $u \in R(W)^*$ *be a nonzero rational function. Then, for any regular meromorphic section* s *of* $\overline{L}\,|_W$ *which* *meets properly* $\operatorname{div}(u)$ *in the generic fibre, the arithmetic cycle* $\widehat{\operatorname{div}}(s) \cap \widehat{\operatorname{div}}(u) \cap (W, g_W)$ *belongs to* $\widehat{\operatorname{Rat}}_{p-2}(X)$.

Proof. — By the Theorem, $\widehat{\operatorname{div}}(s) \cap \widehat{\operatorname{div}}(u) \cap (W, g_W) = \widehat{\operatorname{div}}(u) \cap \widehat{\operatorname{div}}(s) \cap (W, g_W)$. However, for any prime arithmetic cycle (Z, g_Z) which meets properly $\operatorname{div}(u)$ on the generic fibre, the very definition of the arithmetic cycle $\widehat{\operatorname{div}}(u) \cap (Z, g_Z)$ implies that it belongs to $\widehat{\operatorname{Rat}}(X)$. The corollary follows from that. □

COROLLARY 2.5.6. *There exist well-defined morphisms of Abelian groups* $\widehat{\operatorname{CH}}_p(X) \to \widehat{\operatorname{CH}}_{p-1}(X)$, *written* $\alpha \mapsto \widehat{c}_1(\overline{L}) \cap \alpha$ *such that* $\widehat{c}_1(\overline{L}) \cap [(Z, g_Z)] = \widehat{\operatorname{div}}(s) \cap (Z, g_Z)$ *for any arithmetic cycle* (Z, g_Z) *and any regular meromorphic section* s *of* $\overline{L}$ *on* Z.

E. Arithmetic intersection theory (formulaire)

CHAPTER 3

HEIGHTS

§ 3.1
HERMITIAN LINE BUNDLES AND THE HEIGHT MACHINE

A. The height of an algebraic point

Let K be a number field, $\mathfrak{o}_K$ its ring of integers, Σ the set of embeddings of K into $\mathbf{C}$. Let X be a proper arithmetic variety over $\mathfrak{o}_K$ and $\overline{L}$ be an continuous Hermitian line bundle on X.

The notion of a height function goes back to NORTHCOTT (1950) and **?**. We owe the introduction of Hermitian line bundles to ARAKELOV (1974), the paper that really gave its birth to Arakelov geometry.

We first observe that for any point $P \in X(\overline{K})$, let K' be a finite extension of K such that P belongs to $X(K')$. Since X is proper over $\mathfrak{o}_K$ and $\mathfrak{o}_{K'}$ is a Dedekind ring, there exists a unique morphism $\varepsilon_P \colon \operatorname{Spec}\mathfrak{o}_{K'} \to X$ extending the point P, viewed as a morphism from $\operatorname{Spec}\mathfrak{o}_{K'}$ to X. We can then consider the Hermitian line bundle $\varepsilon_P^*\overline{L}$ over $\mathfrak{o}_{K'}$ and its arithmetic degree $\widehat{\deg}\,\varepsilon_P^*\overline{L}$. In fact, the quantity $\frac{1}{[K':K]}\widehat{\deg}\,\varepsilon_P^*\overline{L}$ does not depend on the choice of the field extension K'. We write it $h_{\overline{L}}(P)$.

If $f\colon Y \to X$ is a morphism of proper arithmetic varieties, and $P \in Y(\overline{K})$, then $h_{f^*\overline{L}}(P) = h(f(P))$. Indeed, if K' is a finite extension of K such that P belongs to $Y(K')$ and $\varepsilon_P \colon \operatorname{Spec}\mathfrak{o}_{K'} \to Y$ maps $\operatorname{Spec}K'$ to P, then $f\circ\varepsilon_P \colon \operatorname{Spec}\mathfrak{o}_{K'} \to X$ maps $\operatorname{Spec}K'$ to the point $f(P)$. Moreover, $(f\circ\varepsilon_P)^*\overline{L} = \varepsilon_P^*(f^*\overline{L})$, hence the desired formula.

Example 3.1.1. Assume that X is the projective space $\mathbf{P}^n_{\mathbf{Z}}$. Let us define a continuous Hermitian metric on the line bundle $\mathscr{O}(1)$ by the following formula: for any $\sigma \in \Sigma$, any $j \in \{0,\dots,n\}$, the norm of the global section T_j of $\mathscr{O}(1)$ at a point x of $X_\sigma(\mathbf{C})$ with homogeneous coordinates $[x_0 : \dots : x_n]$ is given by

$$\|T_0\|_\sigma(x) = \frac{|x_j|}{\max(|x_0|,\dots,|x_n|)}.$$

With this definition, I claim that for any point $P \in \mathbf{P}^n(\overline{K})$, $h_{\overline{\mathscr{O}(1)}}(P)$ is the “Weil height” of the point P. Let K' be a finite extension of K, $P \in \mathbf{P}^n(K')$ be a point of homogeneous coordinates $[x_0:\ldots:x_n]$. Northcott and Weil's definition of the height of P is given by

$$h_{\mathrm{W}}(P)=\frac{1}{[K':K]}\sum_{v\in M_{K'}}\log\max(|x_0|_v,\ldots,|x_n|_v),$$

where $M_{K'}$ is the set of (inequivalent) normalized absolute values of K'. Let us fix $m\in\{0,\ldots,n\}$ such that $x_m\neq 0$. Then, the product formula $\prod_v |x_m|_v=1$ allows us to rewrite this equality as

$$h_{\mathrm{W}}(P)=\frac{1}{[K':K]}\sum_{\substack{v\in M_{K'}\\ v\text{ non-archimedean}}}\log\frac{\max(|x_0|_v,\ldots,|x_n|_v)}{|x_m|_v}+\frac{1}{[K':K]}\sum_{\sigma\in\Sigma}\log\|X_m\|_\sigma^{-1}(P).$$

Now, $\varepsilon_P^*\mathscr{O}(1)$ identifies with the sub $\mathfrak{o}_{K'}$-module $(x_0,\ldots,x_n)$ of $\mathfrak{o}_{K'}$, $\varepsilon_P^*X_j$ corresponding to x_j. Consequently,

$$\varepsilon_P^*\mathscr{O}(1)/_{K'}\varepsilon_P^*X_m=(x_0,\ldots,x_n)/(x_m)$$

whose cardinality is equal to

$$\prod_{v\text{ finite}}\left(\max(|x_0|_v,\ldots,|x_n|_v)/|x_m|_v\right).$$

This shows that

$$h_{\mathrm{W}}(P)=\frac{1}{[K':K]}\widehat{\deg}\,\varepsilon_P^*\overline{\mathscr{O}(1)}=h_{\overline{\mathscr{O}(1)}}(P),$$

as claimed.

More generally, we shall prove that for any continuous Hermitian line bundle $\overline{L}$ on an arithmetic variety X over $\mathfrak{o}_K$, the function $h_{\overline{L}}$ is a height function for the line bundle L_K on X_K. The proof uses some properties of the function $h_{\overline{L}}$ which are of independent interest.

PROPOSITION 3.1.2. *Let s be a global non-zero section of L_K. Then, there exists a real number c such that $h_{\overline{L}}(P)\geqslant c$ for any point $P\in X(\overline{K})$ such that $P\not\in|\mathrm{div}(s)|$.*

Proof. — Let s be a global non-vanishing section of L_K. Since the field K is a flat $\mathfrak{o}_K$-module, one has $\Gamma(X,L)\otimes_{\mathfrak{o}_K}K=\Gamma(X_K,L_K)$. Consequently, there exists a positive integer N such that Ns belongs to $\Gamma(X,L)$. Moreover, for any $\sigma\in\Sigma$, then $\|s\|_\sigma$ is a continuous function on $X_\sigma(\mathbf{C})$; since $X_\sigma(\mathbf{C})$ is compact, there exists a real number c_σ such that $\|s\|_\sigma(x)\leqslant c_\sigma$ for any $x\in X_\sigma(\mathbf{C})$.

Now, for any point $P\in X(K')$, the definition of $h_{\overline{L}}(P)$ is

$$h_{\overline{L}}(P)=\frac{1}{[K':K]}\widehat{\deg}\,\varepsilon_P^*\overline{L}.$$

If $P\not\in|\mathrm{div}(s)|$, we may consider the section $\varepsilon_P^*(Ns)$ of ε_P^*L and obtain that

$$h_{\overline{L}}(P)\geqslant\frac{1}{[K':K]}\sum_{\sigma\in\Sigma}\sum_{\sigma'|\sigma}\log\|Ns\|_{\sigma'}^{-1}(P)\geqslant -N-\sum_{\sigma\in\Sigma}c_\sigma.$$

This concludes the proof of the proposition. □

COROLLARY 3.1.3. *Let $\mathrm{B}(L)$ be the intersection of the zero-sets in X_K of the global sections of all positive powers of L_K. Then, $h_{\overline{L}}$ is bounded from below in $X(\overline{K})\setminus\mathrm{B}(L)$.*

COROLLARY 3.1.4. *If some positive power of L_K is generated by its global sections, then $h_{\overline{L}}$ is bounded from below on $X(\overline{K})$.*

Proof. — Since X_K is a Noetherian space, there exists an integer $n \geqslant 1$ and a finite set of sections $\{s_1,\dots,s_r\}$, of $L_K^{\otimes n}$ whose intersection of divisors $\left|\operatorname{div}(s_j)\right|$ is empty. For any $j \in \{1,\dots,r\}$, let c_j be a real number such that $h_{\overline{L}^n}(P) \geqslant c_j$ for $P \in X(\overline{K}) \setminus \left|\operatorname{div}(s_j)\right|$. Setting $c = \min(c_j/n)$, we obtain $h_{\overline{L}}(P) \geqslant c$ for any $P \in X(\overline{K})$. □

COROLLARY 3.1.5. *If L_K is the trivial line bundle, then $h_{\overline{L}}$ is a bounded function on $X(\overline{K})$.*

Proof. — Indeed, both L_K and its inverse are generated by their global sections. □

THEOREM 3.1.6. *Let X be a projective arithmetic variety over $\mathfrak{o}_K$ and let $\overline{L}$ be a continuous Hermitian line bundle on X. The above-defined function $h_{\overline{L}}\colon X(\overline{K}) \to \mathbf{R}$ is a height function for the line bundle L on X_K.*

Before we prove this theorem, let us first observe that the function $h_{\overline{L}}$ gives rise to a *pairing*

$$\widehat{\operatorname{Pic}}(X) \times X(\overline{K}) \to \mathbf{R}, \quad (\overline{L}, P) \mapsto h_{\overline{L}}(P),$$

which is linear on $\overline{L}$ and induces a morphism of Abelian groups from $\widehat{\operatorname{Pic}}(X)$ to the space $\mathscr{F}(X(\overline{K}))$ of real-valued functions on $X(\overline{K})$. Let $\mathscr{F}_{\mathrm{b}}(X(\overline{K}))$ be the subspace of bounded functions. The classical "height machine" in Diophantine geometry defines a morphism $\operatorname{Pic}(X_K) \to \mathscr{F}(X(\overline{K}))/\mathscr{F}_{\mathrm{b}}(X(\overline{K}))$, where, for any line bundle $M \in \operatorname{Pic}(X)$, h_M is a function on $X(\overline{K})$, well-defined up to the addition of a bounded function. Moreover, h_M is characterized by its functoriality property $h_M \circ f = h_{f^*M}$ for any morphism $f\colon Y \to X$ of proper algebraic varieties, as well as its normalization when $X = \mathbf{P}^n$ and $M = \mathscr{O}(1)$.

In other words, what the theory of Hermitian line bundles allows to do is to define specific representatives of the height function h_M, attached to a specific choices of a Hermitian line bundle $\overline{M} \in \widehat{\operatorname{Pic}}(X)$ mapping to M by the canonical map $\widehat{\operatorname{Pic}}(X) \to \operatorname{Pic}(X_K)$.

Proof. — By the theory of ample line bundles, there exist two very ample line bundles L_1 and L_2 on X_K such that, on X_K, $L \simeq L_1 \otimes L_2^{-1}$. Let j_1 be a closed embedding of X_K into a projective space $\mathbf{P}^{n_1}$ such that $L_1 \simeq j_1^*\mathscr{O}(1)$. Let X_1 be the Zariski closure of $j_1(X_K)$ in $\mathbf{P}^{n_1}_{\mathfrak{o}_K}$, and let us still write j_1 for the closed embedding of X_1 into $\mathbf{P}^{n_1}$. Let $\overline{L_1}$ be the metrized line bundle $j_1^*\overline{\mathscr{O}(1)}_W$. Let X_2, j_2 and $\overline{L_2}$ be defined similarly. Finally, let Y be the Zariski closure of X_K, diagonally embedded into $X \times X_1 \times X_2$; let p, p_1, p_2 be the three projections from Y to X, X_1, X_2 respectively and let $\overline{M} = p^*\overline{L}$, $\overline{M}_1 = p_1^*\overline{L}_1$, $\overline{M}_2 = p_2^*\overline{L}_2$.

By Example 3.1.1, one has, for any point $P \in Y(\overline{K}) = X(\overline{K})$, the formula

$$h_{\overline{M_1}}(P) = h_W(j_1(P)), \qquad h_{\overline{M_2}}(P) = h_W(j_2(P)).$$

Since the restrictions to X_K of the line bundles L and $M_1 \otimes M_2^{-1}$ are isomorphic, the very definition of a height function for the line bundle L implies that the difference $h_{\overline{M_1}} - h_{\overline{M_2}}$ is a height function for L. Moreover, applying Corollary 3.1.5, we see that

$$h_{\overline{M}} - \left(h_{\overline{M}_1} - h_{\overline{M}_2}\right)$$

is bounded on $X(\overline{K})$, and $h_{\overline{M}} = h_{\overline{L}}$. Finally, $h_{\overline{L}}$ differs by a bounded function of a height function for L_K, hence is a height function for L_K. □

One of the reasons for the usefulness of the concept of height is the following finiteness theorem, originally due to NORTHCOTT (1950).

THEOREM 3.1.7. *Assume that L_K is ample. Then, for any real numbers d and B, there are only finitely many points $P \in X(\overline{K})$ such that $[K(P):K] \leqslant d$ and $h_{\overline{L}}(P) \leqslant B$.*

Proof. — To be written. □

§ 3.2
HEIGHTS FOR SUBVARIETIES

A. Degrees of projective varieties

Let K be a field and let X be a proper algebraic variety over K. Let $\pi\colon X \to \operatorname{Spec} K$ be the structural morphism. It defines a map $\pi_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} K)$. Composed with the isomorphism $\deg\colon \mathrm{CH}_0(\operatorname{Spec} K) \to \mathbf{Z}$, we obtain a map $\deg\colon \mathrm{CH}_0(X) \to \mathbf{Z}$, called the degree.

DEFINITION 3.2.1. *Let L be a line bundle on X and let Y be a closed integral subscheme of X. The degree of Y with respect to L is defined by*

$$\deg_L(Y) = \deg(\underbrace{c_1(L) \cap \cdots \cap c_1(L)}_{\dim(Y)\ \textit{factors}} \cap Y).$$

We extend $\deg_L$ by linearity on $\mathrm{Z}_p(X)$. Observe that $\deg_L$ defines morphisms of Abelian groups $\deg_L\colon \mathrm{CH}_p(X) \to \mathbf{Z}$, for any nonnegative integer p.

In view of the definition of the intersection $c_1(L) \cap Z$, when Z is a cycle on X, the degree of integral subschemes can be defined by the following inductive formula:

$$\deg_L(Z) = \deg_L(\operatorname{div}(s)),$$

where s is a regular meromorphic section of L on Z. Recast in that context, the projection formula writes

$$\deg_L(f_*Z) = \deg_{f^*L}(Z),$$

where $f\colon Y \to X$ is a morphism of proper algebraic varieties over K, L is a line bundle on X and Z a cycle in Y.

B. Definition of the height

FALTINGS (1991) used arithmetic intersection theory to extend the height function, relative to a Hermitian line bundles, from algebraic points to subvarieties.

If the idea is to replace intersection of $c_1(L)$ with intersection of $\widehat{c}_1(\overline{L})$, we need to pay attention to the fact that subvarieties do not define arithmetic cycles in general. Faltings's definition, restricted to the (special, but essentially universal) case $X = \mathbf{P}^n$, used "Arakelov Chow groups", the image of a canonical section of the morphism $\widehat{\mathrm{CH}}_p(X) \to \mathrm{CH}_p(X)$ associated to a Kähler form on X. A more general definition using arithmetic intersection theory runs as follows. Let $Y \subset X$ be a closed integral subscheme. Two possibilities arise:

– If Y is contained in a vertical fibre of X, namely $Y \subset X \otimes (\mathfrak{o}_K/\mathfrak{p})$, we let

$$h_{\overline{L}}(Y) = \deg_L(Y) \log \operatorname{card}(\mathfrak{o}_K/\mathfrak{p}).$$

In particular, if Y is a closed point y of Y,

$$h_{\overline{L}}(Y) = \log \operatorname{card} \kappa(y).$$

– Otherwise, Y is flat and proper over $\operatorname{Spec} \mathfrak{o}_K$. Let $\tilde{f} \colon \tilde{Y} \to Y$ be a generic resolution of singularities of Y and $f \colon \tilde{Y} \to X$ be the composition of $\tilde{f}$ with the immersion of Y in X. On the arithmetic variety $\tilde{Y}$, $[\tilde{Y}] = (\tilde{Y}, 0)$ is a well-defined arithmetic cycle; if we let p be its dimension, then we define

$$h_{\overline{L}}(Y) = \widehat{\deg} \pi_* (\underbrace{\widehat{c}_1(f^*\overline{L}) \cap \cdots \cap \widehat{c}_1(f^*\overline{L})}_{p \text{ factors}} \cap [\tilde{Y}]),$$

where $\pi \colon \tilde{Y} \to \operatorname{Spec} \mathfrak{o}_K$ is the structural morphism.

In the latter case, we still need to prove that the given definition is independent of the choice of a generic resolution. Anyway, if $\tilde{f}_1 \colon \tilde{Y}_1 \to Y$ and $\tilde{f}_2 \colon \tilde{Y}_2 \to Y$ are two such resolutions, we may consider the fiber product $\tilde{Y}_1 \times_Y \tilde{Y}_2$; it maps birationally to Y but might be singular again. Let thus consider a generic resolution of its singularities $\tilde{f} \colon \tilde{Y} \to \tilde{Y}_1 \times_Y \times Y_2$. For $i = 1$ or 2, let p_i be the natural projection to $\tilde{Y}_1$ and $\tilde{Y}_2$; one has $(p_i)_*[\tilde{Y}] = [\tilde{Y}_i]$, hence

$$(p_i)_*(\widehat{c}_1(f^*\overline{L}) \cap \cdots \cap \widehat{c}_1(f^*\overline{L}) \cap [\tilde{Y}]) = \widehat{c}_1(f_1^*\overline{L}) \cap \cdots \cap hc_1(f_i^*\overline{L}) \cap [\tilde{Y}_i],$$

so that

$$(f_1)_*(\widehat{c}_1(f_1^*\overline{L}) \cap \cdots \cap \widehat{c}_1(f_1^*\overline{L}) \cap [\tilde{Y}_1]) = (f_2)_*(\widehat{c}_1(f_2^*\overline{L}) \cap \cdots \cap \widehat{c}_1(f_2^*\overline{L}) \cap [\tilde{Y}_2]),$$

what had to be proved.

More generally, let $\overline{L}_1, \ldots, \overline{L}_p$ be Hermitian line bundles on X. For any prime cycle $Y \in \mathrm{Z}_p(X)$ one defines

$$\left(\widehat{c}_1(\overline{L}_1) \ldots \widehat{c}_1(\overline{L}_p) \mid Y\right) = \deg(c_1(L_1) \cap \ldots c_1(L_p) \cap Y) \log \operatorname{card}(\mathfrak{o}_K/\mathfrak{p})$$

if Y is vertical; otherwise, Y is flat over $\operatorname{Spec} \mathfrak{o}_K$ and one sets

$$\left(\widehat{c}_1(\overline{L}_1) \ldots \widehat{c}_1(\overline{L}_p) \mid Y\right) = \widehat{\deg} \pi_* \left(\widehat{c}_1(f^*\overline{L}_1) \cap \cdots \cap \widehat{c}_1(f^*\overline{L}_p) \cap [\tilde{Y}]\right),$$

where $\tilde{Y}$ is a generic desingularization of Y, $f\colon \tilde{Y} \to X$ is the composition of the resolution $\tilde{Y} \to Y$ and of the closed embedding of Y in X, and $\pi\colon \tilde{Y} \to \operatorname{Spec} \mathfrak{o}_K$ is the canonical morphism. The above proof shows that this real number does not depend on the choice of $\tilde{Y}$.

When Y is fixed, this expression is multilinear and symmetric in $\overline{L}_1, \dots, \overline{L}_p$. Let $f\colon X' \to X$ be a proper morphism and Y' be a cycle of dimension p on X'. Moreover, according to the projection formula,

$$\left(\widehat{c}_1(f^*\overline{L}_1)\dots\widehat{c}_1(f^*\overline{L}_p) \mid Y'\right) = \left(\widehat{c}_1(\overline{L}_1)\dots\widehat{c}_1(\overline{L}_p) \mid f_*(Y')\right).$$

Now, the definition of the arithmetic intersection with a first arithmetic Chern class as $\widehat{c}_1(\overline{L})$ furnishes us an important inductive formula.

PROPOSITION 3.2.2. *Let X be an arithmetic variety and let Y be an integral closed subscheme of X of dimension $p \geqslant 1$. Let $\overline{L}_1, \dots, \overline{L}_p$ be Hermitian line bundles on X, n a nonzero integer and s a rational section of $L_p^{\otimes n}$ on Y. Then,*

$$\begin{aligned}(\widehat{c}_1(\overline{L}_1)\dots\widehat{c}_1(\overline{L}_p) \mid Y)& \\ = \frac{1}{n}(\widehat{c}_1(\overline{L}_1)\dots\widehat{c}_1(\overline{L}_{p-1}) \mid \operatorname{div}_Y(s)) &+ \sum_{\sigma\in\Sigma}\int_{X_\sigma(\mathbf{C})} \log\|s\|_\sigma^{-1/n} c_1(\overline{L}_1)\wedge\dots c_1(\overline{L}_{p-1})\delta_Y.\end{aligned}$$

Before we embark in proving this proposition, three observations are worth to be made.

a) The first one is that the cycle $\operatorname{div}_Y(s)$ is not flat over $\operatorname{Spec}\mathfrak{o}_K$ in general: there might be vertical components, contained in fibres.

b) Observe then that for a vertical cycle Y, the formula of the Proposition has no integral term, hence coincides with the analogous formula for the degree.

c) Finally, this formula is a generalization of the formula given for the arithmetic degree of an Hermitian line bundle, as well as the formula for the height of a rational point. Namely, let $P \in X(K)$, let $\varepsilon_P\colon \operatorname{Spec}\mathfrak{o}_K \to X$ be the associated section and Y_P be its image. According to the definition,

$$h_{\overline{L}}(Y_P) = \widehat{c}_1(\varepsilon_P^*\overline{L}) \cap [\operatorname{Spec}\mathfrak{o}_K].$$

Now, let s be a non-zero section of ε_P^*L. By definition,

$$\widehat{\deg}(\varepsilon_P^*\overline{L}) = \log\operatorname{card}(\varepsilon_P^*L/\mathfrak{o}_K s) - \sum_{\sigma\in\Sigma}\log\|s\|_\sigma.$$

The Proposition now implies that $h_{\overline{L}}(Y_P) = h_{\overline{L}}(P)$.

Proof. — Let $\tilde{Y}$ be a generic resolution of singularities of X. (...) □

C. Positivity

DEFINITION 3.2.3. *Let $\overline{L}$ be a Hermitian line bundle on X. We say that $\overline{L}$ is arithmetically base-point-free if the following properties are satisfied:*

– *the curvature form $c_1(\overline{L})$ is a nonnegative differential form of bidegree $(1,1)$;*

– *there exists an integer $n \geqslant 1$ and a family of sections $s \in \Gamma(X, L^n)$ generating $L^{\otimes n}$ such that $\|s(x)\|_\sigma \leqslant 1$ for all $x \in X_\sigma(\mathbf{C})$ and all $\sigma \in \Sigma$.*

PROPOSITION 3.2.4. *Let $\overline{L}_1, \dots, \overline{L}_p$ be Hermitian line bundles which are arithmetically base-point-free. Then, for any integral subscheme Y of dimension p in X,*

$$(\widehat{c}_1(\overline{L}_1)\dots\widehat{c}_1(\overline{L}_p)|Y) \geqslant 0.$$

Proof. — We prove the result by induction on p, the statement being obvious if $p = 0$. Let s be a global section of some power $L_p^{\otimes n}$ which does not vanish at the generic point of Y and such that $\|s(x)\|_\sigma \leqslant 1$ for all $x \in X_\sigma(\mathbf{C})$ and all $\sigma \in \Sigma$. According to the inductive formula for heights,

$$\begin{aligned}
&(\widehat{c}_1(\overline{L}_1)\dots\widehat{c}_1(\overline{L}_p)|Y)\\
&\quad = \frac{1}{n}(\widehat{c}_1(\overline{L}_1)\dots\widehat{c}_1(\overline{L}_{p-1})|\operatorname{div}(s)) + \sum_{\sigma\in\Sigma}\int_{X_\sigma(\mathbf{C})} \log\|s\|_\sigma^{-1/n}\, c_1(\overline{L}_1)\dots c_1(\overline{L}_{p-1})\delta_{Y_\sigma(\mathbf{C})}.
\end{aligned}$$

The first term is nonnegative by induction, while the second one is nonnegative since the forms $c_1(\overline{L}_i)$ are nonnegative and $\|s\|_\sigma \leqslant 1$ everywhere. □

COROLLARY 3.2.5. *Let $\overline{L}$ be an Hermitian line bundle on X which is arithmetically base-point-free. Then, for any integral subscheme Y in X, one has $h_{\overline{L}}(Y) \geqslant 0$.*

D. Finiteness

E. Examples

3.2.6. Height of projective spaces. — In this section, we compute the height of $\mathbf{P}^n_{\mathbf{Z}}$ with respect to the line bundle $\overline{\mathscr{O}(1)}$, endowed with the Fubini–Study metric.

We write $\mathbf{P}^n_{\mathbf{Z}} = \operatorname{Proj}\mathbf{Z}[T_0, \dots, T_n]$, and we identify the projective coordinates $T_0, \dots, T_n$ with the global sections $s_0, \dots, s_n$ of $\mathscr{O}(1)$. The space of global sections of $\mathscr{O}(d)$ is identified with the space of homogeneous polynomials of degree d; recall that the Fubini–Study norm of the section s_P corresponding to a polynomial P at a point of homogeneous coordinates $[x_0 : \dots : x_n]$ is given by

$$\|s_P\|([x_0 : \dots : x_n]) = \frac{|P(x_0, \dots, x_n)|}{\left(|x_0|^2 + \dots + |x_n|^2\right)^{d/2}}.$$

This metric is invariant under the action of the unitary group $\mathbf{U}(n+1)$ acting on $\mathbf{P}^n_{\mathbf{C}}$ by homographies.

PROPOSITION 3.2.7. *For any integer n,*

$$h_{\overline{\mathscr{O}(1)}}(\mathbf{P}^n_{\mathbf{Z}}) = \frac{1}{2}\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{1}{m}.$$

Proof. — Let $s_n \in \Gamma(\mathbf{P}^n_{\mathbf{Z}}, \mathscr{O}(1))$ be the global section which corresponds to the polynomial T_n. Observe that the hyperplane section $\operatorname{div}(s_n)$ is isomorphic to $\mathbf{P}^{n-1}_{\mathbf{Z}}$, and that the restriction of $\overline{\mathscr{O}(1)}$ on the hyperplane identifies to the corresponding metrized line bundle on $\mathbf{P}^{n-1}_{\mathbf{Z}}$. The inductive formula for the height (...) then implies

$$h(\mathbf{P}^n_{\mathbf{Z}}) = h(\mathbf{P}^{n-1}_{\mathbf{Z}}) + \int_{\mathbf{P}^n(\mathbf{C})} \log \|s_n\|^{-1} c_1(\overline{\mathscr{O}(1)})^n.$$

It remains to compute the last integral; let us call it I_n.

For $d \geqslant 0$, we write $\mathbf{S}_d$ for the d-dimensional unit sphere in $\mathbf{R}^{d+1}$, so that

$$\mathbf{P}^n(\mathbf{C}) = (\mathbf{C}^{n+1} \setminus \{0\})/\mathbf{C}^* = \mathbf{S}_{2n+1}/\mathbf{U}$$

is the quotient of a $2n+1$-dimensional sphere by the action of the unit group $\mathbf{U} \simeq \mathbf{S}_1$ acting diagonally on $\mathbf{C}^{n+1} = \mathbf{R}^{2n+2}$. Let μ be the invariant probability measure on $\mathbf{S}_{2n+1}$; Since the metric on $\mathscr{O}(1)$ is unitarily invariant, the measure $c_1(\overline{\mathscr{O}(1)})^n$ on $\mathbf{P}^n(\mathbf{C})$ is invariant under $\mathbf{U}(n+1)$; its total mass is equal to 1. Let us write a point $z \in \mathbf{S}_{2n+1}$ as $z = (\sqrt{1-|z_n|^2}z', z_n)$, with $z' \in \mathbf{S}_{2n-1}$ and $z_n \in \mathbf{C}$, $|z_n| \leqslant 1$. In these coordinates, the measure $\mathrm{d}\mu(z)$ takes the following form

$$\mathrm{d}\mu(z) = c\mathrm{d}z_n\mathrm{d}\overline{z_n}\,(1-|z_n|^2)^n\,\mathrm{d}\mu'(z'),$$

where $\mathrm{d}\mu'(z')$ is the invariant measure on $\mathbf{S}_{2n-1}$ and c is a positive real number. We can further write $z_n = \sqrt{r}e^{2\mathrm{i}\pi\theta}$, with $r \in [0,1]$ and $\theta \in \mathbf{R}/\mathbf{Z}$, then

$$\mathrm{d}\mu(z) = c2\pi(1-r)^n\mathrm{d}r\mathrm{d}\theta\mathrm{d}\mu'(z).$$

Since $\mu(\mathbf{S}_{2n+1}) = \mu'(\mathbf{S}_{2n-1}) = 1$,

$$2\pi c = \left(\int_0^1 (1-r)^n\,\mathrm{d}r\right)^{-1} = n+1$$

and

$$I_n = -2\pi c \int_0^1 \frac{1}{2}(1-r)^n \log(r)\,\mathrm{d}r = -\frac{1}{2}(n+1)\int_0^1 (1-r)^n \log(r)\,\mathrm{d}r.$$

Writing $r = 1-t$ and expanding $\log(1-t)$ in power series, we have

$$\int_0^1 \log(r)(1-r)^n\,\mathrm{d}r = -\int_0^1 \log(1-t)\,t^n\,\mathrm{d}t = \sum_{k=1}^{\infty} \int_0^1 \frac{t^k}{k} t^n\,\mathrm{d}t$$
$$= \sum_{k=1}^{\infty} \frac{1}{k(n+k+1)}.$$

In view of the identity

$$\frac{1}{k(n+k+1)} = \frac{1}{n+1}\left(\frac{1}{k} - \frac{1}{n+k+1}\right),$$

telescoping implies

$$\int_0^1 \log(r)(1-r)^n\,\mathrm{d}r = \frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{k}.$$

Finally,

$$I_n = \frac{1}{2}\sum_{k=1}^{n+1}\frac{1}{k}$$

and

$$h(\mathbf{P}^n_{\mathbf{Z}}) = \frac{1}{2}\sum_{m=1}^{n}\sum_{k=1}^{n}\frac{1}{k}.$$

□

3.2.8. Hypersurfaces. —

3.2.9. Abelian varieties. —

§ 3.3
THE ARITHMETIC ANALOGUE OF HILBERT–SAMUEL'S THEOREM

A. The geometric Hilbert–Samuel's theorem

B. Hermitian vector bundles defined by global sections

Let K be a number field. Let X be a projective and flat scheme over $\operatorname{Spec}\mathfrak{o}_K$, let $\overline{L}$ be a hermitian line bundle on X.

Set $E = H^0(X, L)$; since X is projective, this is a flat $\mathfrak{o}_K$-module of finite type, so a projective $\mathfrak{o}_K$-module of finite rank. Let us explain how one can endow it with various structures of normed vector bundle over $\operatorname{Spec}\mathfrak{o}_K$.

Let us fix an archimedean place σ of K. Any section $s \in H^0(X_\sigma, L)$ has a sup-norm:

$$\|s\|_\infty = \sup_{x\in X_\sigma(\mathbf{C})} \|s(x)\|.$$

Moreover, let μ_σ be a measure on $X_\sigma(\mathbf{C})$; then, for any real number $p \geqslant 1$, one can define the L^p-norm of s by the formula

$$\|s\|_{p,\mu_\sigma} = \left(\int_{X_\sigma(\mathbf{C})} \|s(x)\|^p\right)^{1/p}.$$

When $p = 2$, this is a hermitian norm. Assuming that the measures (μ_σ) are invariant by complex conjugation, the family of norms $(\|\cdot\|_{2,\mu_\sigma})$ endows $H^0(X, L)$ with a structure of hermitian vector bundle over $\operatorname{Spec}\mathfrak{o}_K$.

Concerning the choice of these measures, there is a natural choice when the curvature forms $c_1(\overline{L}_\sigma)$ are nonnegative $(1,1)$-forms: it consists in taking $\mu_\sigma = c_1(\overline{L}_\sigma)^{\dim(X_\sigma)}$.

Another possible, canonical, choice of hermitian structures consists in defining the hermitian norm on E_σ so that its unit ball is the John ellipsoid of the unit ball for the sup norm: by definition, this ellipsoid is the ellipsoid of maximal volume contained in the unit ball of E_σ for the sup norm. In other words, the norm $\|\cdot\|_J$ on E_σ is, among all hermitian norms larger than $\|\cdot\|_\infty$, the one whose unit ball has maximal volume. John's theorem then claims that the John ellipsoid is contained in the ball of radius $\sqrt{\dim(E_\sigma)}$.

PROPOSITION 3.3.1. *Let $N = \dim_{\mathbf{Q}} H^0(X, L^n)$. Then, for any $s \in H^0(X, L^n)_\sigma$, and any $q \geqslant p \geqslant 1$,*

$$\|s\|_{p,\mu_\sigma} \leqslant \mu_\sigma(X_\sigma(\mathbf{C}))^{(q-p)/pq} \|s\|_{q,\mu_\sigma}$$
$$\|s\|_{p,\mu_\sigma} \leqslant \|s\|_\infty,$$
$$\|s\|_\infty \leqslant \|s\|_J \leqslant \sqrt{N} \|s\|_\infty.$$

Proof. — The first inequality follows from Hölder's inequality:

$$\int_{X_\sigma(\mathbf{C})} \|s(x)\|^p \,\mathrm{d}\mu_\sigma(x) \leqslant \left(\int_{X_\sigma(\mathbf{C})} \mathrm{d}\mu_\sigma(x)\right)^{1-p/q} \left(\int_{X_\sigma(\mathbf{C})} \|s(x)\|^q \,\mathrm{d}\mu_\sigma(x)\right)^{p/q}$$
$$= \mu_\sigma(X_\sigma(\mathbf{C}))^{(q-p)/q} \|s\|^p_{p,\mu_\sigma}.$$

The second inequality is proved similarly and he third one follows from John's theorem. □

For general hermitian line bundles, these inequalities cannot be improved. However, there are some asymptotic results when L is a large power of a fixed *ample*line bundle which show that, up to an error term, all of these norms are comparable.

PROPOSITION 3.3.2. *Let us assume that $X_\sigma(\mathbf{C})$ is smooth and that μ_σ is a Lebesgue measure. Then, when $n \to \infty$, one has the following inequalities for the norms on $H^0(X, L^n)$:*

$$\|s\|_2 \leqslant \|s\|_\infty \leqslant c(n) \|s\|_2,$$

where $c(n)$ can be estimated as follows:

- *if $\overline{L}$ has a smooth metric with positive curvature, $c(n) = \mathrm{O}(n^{d/2})$;*
- *if $\overline{L}$ has a Lipschitz metric, then $c(n) = \mathrm{O}(n^d)$;*
- *if $\overline{L}$ has a continuous metric, then $c(n) = e^{\mathrm{o}(n)}$;*
- *if $\overline{L}$ has a bounded metric, then $c(n) = e^{\mathrm{O}(n)}$.*

Proof. — We prove the second and third inequalities here. Let (U_i) be a finite open cover of $X_\sigma(\mathbf{C})$ by balls of radius 1 such that the images U_i' of the balls of radius 1/2 still cover $X_\sigma(\mathbf{C})$. Let U_i'' be the closed ball of radius 3/4 in U_i'. The restriction to U_i of the measure μ_σ can be written $h_i\,\mathrm{d}x$, where h_i is continuous and positive.

For any i, let ε_i be a trivialisation of L on U_i, and let φ_i be the real function given by $\|\varepsilon_i\| = e^{-\varphi_i}$.

On U_i, one can write $s = f\varepsilon_i^{\otimes n}$, for some holomorphic function f, so that $\|s(x)\| = \left|f(x)\right| e^{-n\varphi_i(x)}$.

Let x_0 be a point of $X_\sigma(\mathbf{C})$ at which $\|s(x_0)\|$ is maximal. Then, if $x_0 \in U_i'$, one has

$$\int_{X_\sigma(C)} \|s(x)\|^2 \,\mathrm{d}\mu_\sigma(x) \geqslant \int_{U_i} \left|f(x)\right|^2 h_i(x) e^{-2n\varphi_i(x)} \,\mathrm{d}x.$$

Since f is holomorphic, $\left|f\right|^2$ is subharmonic. Let c_i be the infimum of h_i on U_i''. Then, for any positive real number r such that $r < 1/6$,

$$\int_{U_i} \left|f(x)\right|^2 h_i(x) e^{-2n\varphi_i(x)}\,\mathrm{d}x \geqslant \int_{B(x_0,r)} \left|f(x)\right|^2 h_i(x) e^{-2n\varphi_i(x)}\,\mathrm{d}x \geqslant \left|f(x_0)\right|^2 e^{-2n\varphi_i(x_0)} \int_{B(x_0,r)} h_i(x) e^{n(\varphi_i(x_0)-\varphi_i(x))}.$$

Let $\varepsilon > 0$; the functions φ_i being uniformly continuous on U_i'', there exists a positive real number r, independent on x_0, such that one has $\left|\varphi_i(x_0) - \varphi_i(x)\right| \leqslant \varepsilon$ for $|x_0 - x| \leqslant r$. Then,

$$\int_{U_i} \left|f(x)\right|^2 h_i(x) e^{-2n\varphi_i(x)}\,\mathrm{d}x \geqslant \|s(x_0)\|^2 e^{-2n\varepsilon} \int_{B(x_0,r)} h_i(x)\,\mathrm{d}x \gg \|s(x_0)\|^2 e^{-2n\varepsilon}.$$

This concludes the proof of the third inequality.

As for the second one, φ_i being Lipschitz, there exists a real number c suh that $\left|\varphi_i(x) - \varphi_i(x_0)\right| \leqslant c\,\|x - x_0\|$ for any $x, x_0 \in U_i''$. This implies

$$\int_{U_i} \left|f(x)\right|^2 h_i(x) e^{-n\varphi_i(x)}\,\mathrm{d}x \gg \|s(x_0)\|^2 \int_{B(x_0,r)} e^{-2nc\|x-x_0\|}\,\mathrm{d}x.$$

Now, the change of variables $x = x_0 + u/2nc$ gives

$$\int_{B(x_0,r)} e^{-2nc\|x-x_0\|}\,\mathrm{d}x = (2nc)^{-2d} \int_{B(0,2ncr)} e^{-\|u\|}\,\mathrm{d}u,$$

so that, when $n \to \infty$,

$$\|s\|_2 \gg n^{-d}\,\|s(x_0)\| = n^{-d}\,\|s\|_\infty.$$

□

(1)

Let Y be a subvariety of X and assume that L is ample. For n large enough, $H^0(X, L^n)$ surjects onto $H^0(Y, L^n)$. For any norm on $H^0(X, L^n)$, this allows to endow $H^0(Y, L^n)$ with the quotient norm: For $s \in H^0(Y, L^n)$,

$$\|s\|_q = \inf_{\substack{t \in H^0(X,L^n) \\ t|_Y = s}} \|t\|.$$

PROPOSITION 3.3.3. *Assume that the curvature form of $\overline{L}$ is positive. When $n \to \infty$, the following inequality holds for any $s \in H^0(Y, L^n)$:*

$$\|s\|_\infty \leqslant \|s\|_{\infty,q} \leqslant e^{\mathrm{o}(n)}\,\|s\|_\infty.$$

(In fact, one can replace the $e^{\mathrm{o}(n)}$ by $\mathrm{O}(1)$!)

Proof. — Copy that from BOST (2004), Appendix. □

(1) Generalize to admit any measure μ_σ, provided it has full support in X_σ; what about singular varieties?

C. The arithmetic Hilbert–Samuel's theorem

THEOREM 3.3.4. *Let X be a projective flat scheme over* $\operatorname{Spec}\mathbf{Z}$. *Let $\overline{L}$ be an hermitian line bundle on X. We assume that the generic fiber of X is smooth of dimension d and that the curvature form $c_1(\overline{L})$ of $\overline{L}$ at the archimedean place is positive everywhere. Then, when $n\to\infty$,*

$$\widehat{\deg} H^0(X,\overline{L}^n) \sim \frac{1}{d!}(\widehat{c}_1(\overline{L})^d|X)n^d.$$

Proof. — Let us prove the theorem by induction on d; it is true when $X_{\mathbf{Q}}$ is empty so so assume the theorem has been established when $X_{\mathbf{Q}}$ has dimension $< d$. The line bundle L being very ample, one may choose a section s of L such that the generic fiber of $\operatorname{div}(s)$ is smooth. The section s gives rise to a morphism from L^{-1} to $\mathscr{O}_X$; its image J is the ideal sheaf of a subscheme of X whose support coincides with $|\operatorname{div}(s)|$. Considering a primary decomposition of J, there exist ideals I and J of $\mathscr{O}_X$ such that $\mathscr{J}_s = I\cap J$, where $Y=V(I)$ is flat and generically smooth over $\operatorname{Spec}\mathbf{Z}$ and $Z=V(J)$ is vertical.

Since L is ample, the sequence

$$0\to H^0(X,L^n\otimes I)\to H^0(X,L^n)\to H^0(Y,L^n)\to 0$$

is exact, for any large enough integer.

PROPOSITION 3.3.5 (Abbès-Bouche). *When $n\to\infty$,*

$$\frac{1}{\dim H^0(X_{\mathbf{Q}},L^n)}\Big(\widehat{\deg} H^0(X,L^n)-\widehat{\deg} H^0(Y,L^n)-\widehat{\deg} H^0(X,L^n\otimes I)\Big)\to \pm\int_{X(\mathbf{C})}\log\|s\|.$$

Moreover, there are exact sequences

$$0\to IJ\to I\to I\otimes(\mathscr{O}_X/J)\to 0,\qquad 0\to IJ\to I\cap J\to T\to 0,$$

where the support Z' of T is contained in Z, and has empty interior in Z. Since L is ample, these exact sequences induce exact sequences at the level of global sections, for n large enough:

$$0\to H^0(X,L^n\otimes IJ)\to H^0(X,L^n\otimes I)\to H^0(Z,L^n\otimes I)\to 0)$$

and

$$0\to H^0(X,L^n\otimes IJ)\to H^0(X,L^n\otimes(I\cap J))\to H^0(X,L^n\otimes T)\to 0.$$

Let us apply the geometric Hilbert-Samuel formula to $L^n\otimes T$; we get

$$\operatorname{card} H^0(X,L^n\otimes T)=\mathrm{O}(n^{d-1}).$$

Since $I\cap J\simeq L^{-1}$, it follows that

$$\widehat{\deg}(H^0(X,L^n\otimes IJ))-\widehat{\deg}(H^0(X,L^{n-1})=\mathrm{O}(n^{d-1}).$$

Similarly,

$$\widehat{\deg} H^0(X,L^n\otimes I)=\widehat{\deg} H^0(X,L^n\otimes IJ)+\log\operatorname{card} H^0(Z,L^n\otimes I).$$

The cycle Z decomposes as a sum $\sum m_i Z_i$, where Z_i is a vertical component of X lying over an ideal (p_i) of $\operatorname{Spec}\mathbf{Z}$. Then,

$$\log\operatorname{card} H^0(Z, L^n \otimes I) = \sum_i m_i \dim_{\mathbf{F}_{p_i}} H^0(Z_i, L^n \otimes I)\log p_i + \mathrm{O}(n^{d-1}).$$

Moreover, the geometric Hilbert-Samuel formula, as applied to Z_i, claims that

$$\dim_{\mathbf{F}_{p_i}} H^0(Z_i, L^n \otimes I) = \frac{1}{d!}\ell_{Z_i}(I)(c_1(L)^{d-1}|Z_i)n^d + \mathrm{O}(n^{d-1}).$$

Since I is a nonzero subsheaf of $\mathscr{O}_X$, one has $\ell_{Z_i}(I) = 1$, hence

$$\widehat{\deg} H^0(X, L^n \otimes I) = \widehat{\deg} H^0(X, L^{n-1}) + \sum_i m_i (c_1(L)^{d-1}|Z_i)n^d + \mathrm{O}(n^{d-1}).$$

Finally, according to the geometric Hilbert-Samuel formula for $X_{\mathbf{Q}}$, one has

$$\dim_{\mathbf{Q}} H^0(X_{\mathbf{Q}}, L^n) = \frac{1}{d!}(c_1(L)^d|X_{\mathbf{Q}}) = \frac{1}{d!}\int_{X(\mathbf{C})} c_1(\overline{L})^d.$$

Altogether, these estimates furnish the following equality, when $n \to \infty$:

$$\begin{aligned}\widehat{\deg} H^0(X, L^n) &- \widehat{\deg} H^0(X, L^{n-1}) \\ &= \widehat{\deg}(Y, L^n) + \frac{1}{d!}\sum_i m_i (c_1(L)^{d-1}|Z_i)n^d \log p_i - \frac{1}{d!}\int_{X(\mathbf{C})} \log\|s\|\, c_1(\overline{L})^d + \mathrm{o}(n^d).\end{aligned}$$

Then,

$$\widehat{\deg}(Y, L^n) = \frac{1}{d!}(\widehat{c}_1(\overline{L})^d|Y)n^d + \mathrm{o}(n^d),$$

so that

$$\begin{aligned}\widehat{\deg} H^0(X, L^n) - \widehat{\deg} H^0(X, L^{n-1}) &= \frac{1}{d!}\left((\widehat{c}_1(\overline{L})^d|\operatorname{div}(s)) - \int_{X(\mathbf{C})} \log\|s\|\, c_1(\overline{L})^d\right)n^d + \mathrm{o}(n^d) \\ &= \frac{1}{d!}(\widehat{c}_1(\overline{L})^{d+1}|X)n^d + \mathrm{o}(n^d).\end{aligned}$$

It follows that

$$\widehat{\deg} H^0(X, L^n) = \frac{1}{(d+1)!}(\widehat{c}_1(\overline{L})^{d+1}|X)n^{d+1} + \mathrm{o}(n^{d+1}).$$

This concludes the proof. □

THEOREM 3.3.6. *Without assuming that X is generically smooth, and replacing the hermitian norm by the sup norms.*

DEFINITION 3.3.7. *Semipositive line bundles:*

- *restriction to the generic fiber is ample;*
- *metric is continuous, curvature current is nonnegative everywhere;*
- *degree of any vertical curve (subvariety) is nonnegative.*

COROLLARY 3.3.8. *Let X be a flat projective scheme over $\mathbf{Z}$ of relative dimension d, let $\overline{L}$ be a semipositive hermitian line bundle on X.*

Then, for any $\varepsilon > 0$, there exists an integer n and a nonzero section $s \in H^0(X, L^n)$ such that
$$\|s\|_\infty^{1/n} \leqslant \varepsilon - \frac{h_{\overline{L}}(X)}{(d+1)\deg_L(X_{\mathbf{Q}})}.$$

Proof. — (Under the assumption that L is ample with a positive metric.) Let $\overline{E}_n$ be the hermitian vector bundle $H^0(X, \overline{L}^n)$ whose hermitian metric is multiplied by $\exp(nc_n)$, for some real number c_n to be chosen shortly. Then,
$$\begin{aligned}\widehat{\deg}(\overline{E}_n) &= \widehat{\deg}(H^0(X, \overline{L}^n)) - nc_n \operatorname{rank} H^0(X, L^n) \\ &= \left(\frac{1}{(d+1)!} h_{\overline{L}}(X) - c_n \frac{1}{d!} \deg_L(X_{\mathbf{Q}})\right) n^{d+1} + \mathrm{o}(n^{d+1}).\end{aligned}$$
Consequently, if c_n is chosen so that
$$c_n = \frac{1}{d+1} \frac{h_{\overline{L}}(X)}{\deg_L(X_{\mathbf{Q}})} - \varepsilon,$$
we obtain that $\widehat{\deg}(\overline{E}_n)/n^{d+1}$ has a positive limit when $n \to \infty$. Since $\operatorname{rank}(E_n)$ grows like n^d, this implies that
$$\lim_{n\to\infty} \frac{\widehat{\deg}(\overline{E}_n)}{\operatorname{rank}(E_n)\log\operatorname{rank}(E_n)} = +\infty.$$
By Corollary 1.4.3, there exists, for any large enough integer n, a nonzero element $s \in \overline{E}_n$ such that $\|s\| < 1$. That is, $s \in H^0(X, L^n)$ and $\|s\|^{1/n} \leqslant \exp(-c_n)$.

By Proposition **??**,
$$\log\|s\|_\infty^{1/n} \leqslant \log\|s\|^{1/n} + \mathrm{o}(1) \leqslant -c_n + \mathrm{o}(1).$$
For n large enough, we thus have
$$\log\|s\|^{1/n} \leqslant \frac{h_{\overline{L}}(X)}{(d+1)\deg_L(X_{\mathbf{Q}})} + 2\varepsilon,$$
and this concludes the proof of the corollary. □

Let X be an integral flat projective scheme over $\mathbf{Z}$ of relative dimension d. let $\overline{L}$ be a semipositive hermitian line bundle on X. For any integer p such that $0 \leqslant p \leqslant d$, let
$$A_p(X) = \inf_Y \frac{h_{\overline{L}}(Y)}{(p+1)\deg_L(Y_{\mathbf{Q}})},$$
where Y runs among all flat integral subschemes of relative dimension p in X. Observe that $A_d(X) = \frac{h_{\overline{L}}(X)}{(d+1)\deg_L(X_{\mathbf{Q}})}$ and that $A_0(X)$ is the infimum of the heights $h_{\overline{L}}(P)$, where P runs among all algebraic points $P \in X(\overline{Q})$.

COROLLARY 3.3.9. *One has the inequalities*
$$A_d(X) \geqslant A_{d-1}(X) \geqslant \cdots \geqslant A_0(X).$$

Proof (Autissier). — Let $p \in \{1, \dots, d\}$ and let Y be a flat integral subscheme of dimension p in X. Let ε be a positive real number. By the preceding corollary, there exists a positive integer n and a nonzero section $s \in \Gamma(Y, L^n)$ such that

$$\|s\|_\infty^{1/n} \leqslant \varepsilon - \frac{h_{\overline{L}}(Y)}{(p+1)\deg_L(Y_{\mathbf{Q}})}.$$

Let us write the divisor of s as $\sum m_i Z_i + V$, where for each i, Z_i is an integral flat subscheme of Y of dimension $p-1$, m_i is a positive integer, and V is an effective vertical divisor in Y. Then, we have:

$$\begin{aligned} \deg_L(Y_{\mathbf{Q}}) &= \frac{1}{n}\sum_i m_i \deg_L(Zi, \mathbf{Q}) \\ h_{\overline{L}}(Y) &= \frac{1}{n}\sum_i m_i h_{\overline{L}}(Z_i) + \frac{1}{n} h_{\overline{L}}(V) + \int_{Y(\mathbf{C})} \log\|s\|^{-1/n} c_1(\overline{L})^{p-1}. \end{aligned}$$

By definition of $A_{p-1}(\overline{L})$,

$$h_{\overline{L}}(Z_i) \geqslant A_{p-1}(\overline{L})\, p \deg_L(Z_{i,\mathbf{Q}})$$

for all i. Since $\overline{L}$ is semipositive and the divisor V is effective and vertical, $h_{\overline{L}}(V) \geqslant 0$. Moreover,

$$\begin{aligned} \int_{Y(\mathbf{C})} \log\|s\|^{-1/n} c_1(\overline{L})^{p-1} &\geqslant -\|s\|_\infty^{1/n} \int_{Y(\mathbf{C})} c_1(\overline{L})^{p-1} \\ &= -\|s\|_\infty^{1/n} \deg_L(Y_{\mathbf{Q}}) \\ &\geqslant \frac{h_{\overline{L}}(Y)}{p+1} - \varepsilon \deg_L(Y_{\mathbf{Q}}). \end{aligned}$$

Consequently,

$$\begin{aligned} h_{\overline{L}}(Y) &\geqslant A_{p-1}(\overline{L})\frac{n}{p} A_{p-1}(Y) \sum_i m_i \deg_L(Z_{i,\mathbf{Q}}) + \frac{1}{p+1} h_{\overline{L}}(Y) - \varepsilon \deg_L(Y_{\mathbf{Q}}) \\ &\geqslant p A_{p-1}(Y) \deg_L(Y_{\mathbf{Q}}) + \frac{1}{p+1} h_{\overline{L}}(Y) - \varepsilon \deg_L(Y_{\mathbf{Q}}). \end{aligned}$$

This implies that

$$\frac{p}{p+1} h_{\overline{L}}(Y) \geqslant p A_{p-1}(Y) \deg_L(Y_{\mathbf{Q}}) - \varepsilon \deg_L(Y_{\mathbf{Q}}),$$

so that

$$\frac{h_{\overline{L}}(Y)}{(p+1)\deg_L(Y_{\mathbf{Q}})} \geqslant A_{p-1}(Y) - \varepsilon.$$

Make ε tend to 0; we obtain

$$\frac{h_{\overline{L}}(Y)}{(p+1)\deg_L(Y_{\mathbf{Q}})} \geqslant A_{p-1}(Y) \geqslant A_{p-1}(X).$$

Consequently, $A_p(X) \geqslant A_{p-1}(X)$, hence the corollary. □

§ 3.4
THE GENERIC EQUIDISTRIBUTION THEOREM

§ 3.5
THE ARITHMETIC ANALOGUE OF NAKAI–MOISHEZON'S THEOREM

§ 3.6
ADELIC METRICS ON LINE BUNDLES

CHAPTER 4

BOGOMOLOV'S CONJECTURE

APPENDIX A

APPENDIX

§ A.1
NORMED VECTOR SPACES

A. Linear algebra and norms

In this paragraph, we review some constructions of norms on finite dimensional vector spaces given by linear algebra.

A.1. Linear maps. — Let V and W be complex vector spaces endowed with norms. Let $f\colon V \to W$ be a linear map. By definition, its norm $\|f\|$ is the supremum of $\|f(v)\|$ for all $v \in V$ such that $\|v\| \leqslant 1$.

More generally, let $V_1, \dots, V_n$ be complex vector spaces with norms, and let $f\colon V_1 \times \dots \times V_n \to W$ be a multilinear map. Its norm is the supremum of $\|f(v_1, \dots, v_n)\|$ over all families $(v_1, \dots, v_n)$, where $v_i \in V_i$ and $\|v_i\| \leqslant 1$ for each i.

A.2. Subspaces and quotients. — Let V be a complex vector space with a norm. Let W be a subspace of V. It is naturally endowed with the norm induced by restriction of the norm on V.

As for the quotient space V/W, we endow it with the norm defined as follows: let $p\colon V \to V/W$ be the natural projection, then, for any $v \in V$,

$$\|p(v)\|_{V/W} = \inf_{w \in W} \|v + w\|_V = d(v, W) = d(0, v + W).$$

In the usual construction of V/W as cosets of W, observe that $p(v)$ actually *equals* $v + W$, hence $\|p(v)\|_{V/W}$ is the distance of the origin to $p(v)$.

Let $\pi\colon V \to V$ be a projector whose image is W and let $\pi' = \mathrm{id}_V - \pi$ be the projector onto the kernel W' of π, which is a supplementary subspace of W. The projection p induces an isomorphism f from W' to V/W. One has $\|f\| \leqslant 1$. Observe that $\|\pi\| \geqslant 1$ (unless $W = 0$) and $\|\pi'\| \leqslant 1$ (unless $W = V$). If equality holds, *i.e.*, if $\|\pi'\| = 1$, then, then f is an isometry from W', with its induced norm, to V/W, with its quotient norm. This allows to compute the quotient norm on examples. To that aim, it is useful to observe that the map $f^{-1}\colon V/W \to W'$ associates to $p(v)$ the element $\pi'(v)$, for any $v \in V$. In particular, $\|p(v)\|_{V/W} = \|\pi'(v)\|_V$.

A.3. Tensor product. — Let V and W be complex vector spaces endowed with norms. Let t be the canonical bilinear map $V \times W \to V \otimes W$, given by $t(v, w) = v \otimes w$.

We define the π-semi-norm $\|T\|_\pi$ of a tensor $T \in V \otimes W$ as the infimum of all $\sum_i \|v_i\| \|w_i\|$ over all finite families $((v_i, w_i))$ such that $T = \sum v_i \otimes w_i$. In particular, the bilinear map t has norm $\leqslant 1$ when $V \otimes W$ is π-seminormed.

We also define the ε-semi-norm $\|T\|_\varepsilon$ on $V \otimes W$ as the supremum of $\left|\tilde{b}(T)\right|$ over all pairs (f, g) where $f \in V^*$ and $g \in W^*$ are linear forms such that $\|f\|, \|g\| \leqslant 1$, where we have written $\tilde{b}$ for the linear form $V \otimes W \to \mathbf{C}$ such that $\tilde{b}(v \otimes w) = f(v)g(w)$. In other words, $\|T\|_\varepsilon$ is the least positive real number such that

$$\left|\tilde{b}(T)\right| \leqslant \|f\| \|g\| \|T\|_\varepsilon$$

for any two linear forms $f \in V^*$ and $g \in W^*$.

Let us compare the ε and π semi-norms. Let T be any element of $V \otimes W$. Let $f \in V^*$ and $g \in W^*$ be linear forms of norm $\leqslant 1$ and let $f \otimes g$ be the linear form on $V \otimes W$ such that $(f \otimes g)(v \otimes w) = f(v)g(w)$. Let us write $T = \sum v_i \otimes w_i$; then $(f \otimes g)(T) = \sum b(v_i, w_i)$, hence

$$\left|(f \otimes g)(T)\right| \leqslant \sum \left|f(v_i)\right| \left|g(w_i)\right| \leqslant \sum \|v_i\| \|w_i\|.$$

Taking the supremum over all f, g and the infimum over all decompositions $T = \sum v_i \otimes w_i$, we obtain $\|T\|_\varepsilon \leqslant \|T\|_\pi$.

This inequality is very important, especially since it implies that the semi-norm $\|\cdot\|_\pi$ is actually a norm. Indeed, it is clear that $\|\cdot\|_\varepsilon$ is a norm: if T is a non-zero tensor, there exists a linear form φ on $V \otimes W$ such that $\varphi(T) \neq 0$. We can assume that φ is of the form $(v, w) \mapsto f(v)g(w)$, for some linear forms $f \in V^*$ and $g \in W^*$. Then, $\|T\|_\varepsilon \geqslant \left|\varphi(T)\right| / \|f\| \|g\| > 0$.

Duality, *i.e.*, the identification of $V^* \otimes F^*$ with $(V \otimes F)^*$ identifies the norm ε on $V^* \otimes F^*$ with the dual of the norm π on $V \otimes F$, and the norm π on $V^* \otimes F^*$ with the dual of the norm ε on $V \otimes F$.

These constructions can be extended to tensor products of more than two spaces. However, there are no good compatibility results, *e.g.*, no associativity properties.

B. Hermitian norms and linear algebra

Defined for Hermitian spaces, the previous constructions do not generally lead to hermitian norms.

B.1. Subspaces and quotients. — The norm induced on a subspace W of a Hermitian space V is itself a Hermitian norm. The same holds for the quotient. Indeed, the orthogonal projector π on W has norm 1. This identifies the quotient norm on V/W as the hermitian space given by the orthogonal supplement to W in V.

B.2. Linear maps. — Let V and W be Hermitian vector spaces. We define a hermitian scalar product on $\mathrm{Hom}(V, W)$ given by $\langle f, g\rangle_2 = \mathrm{tr}(f^* \circ g) = \mathrm{tr}(g \circ f^*)$, for any linear maps $f, g \colon V \to W$, $f^* \colon W \to V$ being the adjoint of f with respect to the scalar products on V and W.

Let B and C be orthonormal bases of V and W, let B^* be the dual basis of B. Then, the family of rank-one linear maps $\varphi \otimes w$, for $\varphi \in B^*$ and $w \in C$, form a basis of $\operatorname{Hom}(V,W)$, which is formed of unit vectors, two by two orthogonal. In particular, the hermitian scalar product that we have defined is indeed positive definite. The associate norm on $\operatorname{Hom}(V,W)$ is denoted $\|\cdot\|_2$. For any orthonormal basis $(v_1,\dots,v_n)$ of V, one has

$$\|f\|_2^2 = \operatorname{tr}(f^*f) = \sum_{i=1}^n \|f(v_i)\|^2.$$

It is larger than the endomorphism norm. Let indeed v be a unit vector of V such that $\|f(v)\| = \|f\|$; let us complete it into a basis $(v_1,\dots,v_n)$ of V with $v_1 = v$. By the previous formula, $\|f\|_2 \geqslant \|f(v_1)\| = \|f\|$.

B.3. Tensor products. — Let us assume that $V_1,\dots,V_n$ are Hermitian vector spaces and let V be the tensor product $V_1\otimes\cdots\otimes V_n$. One defines an hermitian form on V by the formula

$$\langle v_1\otimes\cdots\otimes v_n, v'_1\otimes\cdots\otimes v'_n\rangle = \prod_{i=1}^n \langle v_i, v'_i\rangle.$$

That it is actually positive definite follows from the observation that if $B_1,\dots,B_n$ are orthonormal bases of $V_1,\dots,V_n$, then the family of vectors $v_1\otimes\cdots\otimes v_n$, where $v_i\in V_i$ for each i, is a basis of V formed of unit vectors which are two-by-two orthogonal. Let us write $\|\cdot\|_2$ for this Hermitian norm.

When $n=2$, let us compare this norm with the norms ε and π.

For $v\in V$, $w\in W$, one has $\|v\otimes w\|_2 = \|v\|\,\|w\|$. Consequently, for any tensor T, decomposed as $T=\sum_i v_i\otimes w_i$, one has $\|T\|_2\leqslant \sum_i \|v_i\|\,\|w_i\|$. This implies that $\|T\|_2\leqslant \|T\|_\pi$.

The other inequality $\|T\|_\varepsilon \leqslant \|T\|_2$ follows by duality. Anyway, let $f\in V^*$ and $g\in W^*$ be linear forms of norm 1. Let (v_i) and (w_j) be orthonormal bases of V, W, let (v_i^*) and (w_j^*) be the dual bases, chosen in such a way that $v_1^* = f$ and $w_1^* = g$. Let $T\in V\otimes W$; we can write $T=\sum T_{i,j}v_i\otimes w_j$. We have

$$\left|(f\otimes g)(T)\right| = \left|T_{1,1}\right| \leqslant \left(\sum \left|T_{i,j}\right|^2\right)^{1/2} = \|T\|_2.$$

Taking the supremum over all couples (f,g) implies that $\|T\|_\varepsilon\leqslant\|T\|_2$.

B.4. Symmetric products. — Let V be a Hermitian vector space. For any integer $p\geqslant 0$, we endow the pth symmetric product $\operatorname{Sym}^p V$ with the quotient norm of the Hermitian norm on $V^{\otimes p}$.

Let us observe that the symmetrization projector s in $V^{\otimes p}$ is given by

$$s = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\sigma,$$

where a permutation σ acts on $V^{\otimes p}$ by permuting the factors. (In formula, $\sigma(v_1\otimes\cdots\otimes v_p) = v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(p)}$.) It is obvious that any such permutation acts as an isometry on the Hermitian space $V^{\otimes p}$. Consequently, its norm is 1 and the norm of the projector s is $\leqslant 1$. As we have seen, this implies that the Hermitian space Sym^p is isometric to the subspace of $E^{\otimes p}$ consisting of symmetric tensors. To give a formula, let $(v_1,\dots,v_n)$ be an orthonormal basis of V. Let $k_1,\dots,k_n$ be integers such that $k_1+\cdots+k_n = p$ and let

$T = v_1^{k_1} \dots v_n^{k_n} \in \mathrm{Sym}^p V$. If $\tilde{T} = v_1^{\otimes k_1} \otimes \cdots \otimes v_n^{\otimes k_n} \in V^{\otimes p}$, we have $\|T\|_2 = \left\| s(\tilde{T}) \right\|$. Moreover, the stabilizer G of $\tilde{T}$ in $\mathfrak{S}_p$ identifies of the product of the symmetric groups $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_n}$, hence

$$s(\tilde{T}) = \frac{k_1! \dots k_n!}{p!} \sum_{\sigma \in \mathfrak{S}_n / G} \sigma(\tilde{T}).$$

We observe that when σ runs over $\mathfrak{S}_n / G$, the vectors $\sigma(\tilde{T})$ are distinct vectors of an orthonormal basis of $V^{\otimes p}$. Consequently,

$$\left\| s(\tilde{T}) \right\|_2 = \left(\frac{p!}{k_1! \dots k_n!} \right)^{-1/2} = \binom{p}{k_1 \dots k_n}^{-1/2}.$$

B.5. Alternate products. — Let V be a Hermitian vector space. For any integer $p \geqslant 0$, we endow the pth alternate product $\mathrm{Alt}^p V$ with the Hermitian form given by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det\left(\langle v_i, w_j \rangle\right).$$

If $(v_1, \dots, v_n)$ is an orthonormal basis of V, then the family of vectors $v_{i_1} \wedge \cdots \wedge v_{i_p}$, where $(i_1, \dots, i_p)$ ranges over all sequences of integers such that $1 \leqslant i_1 < \cdots < i_p \leqslant n$, is an basis of $\mathrm{Alt}^p V$ formed of unit vectors, two by two orthogonal. In particular, this Hermitian form is positive definite. We write $\|\cdot\|_2$ for the associated Hermitian norm.

It is important to note that this Hermitian norm is *not* equal to the quotient norm from $V^{\otimes p}$, it is actually equal to $\sqrt{p!}$ times the quotient norm.

An other way to understand the positivity of the Hermitian norm constructed on $\mathrm{Alt}^p V$ consists in observing that for any vectors $v_1, \dots, v_p \in V$, the determinant $\langle v_1 \wedge \cdots \wedge v_p \rangle^2$ is the Gram determinant of these vectors, *i.e.*, the square of the pth dimensional volume of the parallelepiped formed on these vectors.

The Hadamard inequality writes

$$\left\| v_1 \wedge \dots v_p \right\|_2 \leqslant \|v_1\| \dots \left\| v_p \right\|. \tag{A.1.1}$$

This implies, and is in fact equivalent to this fact, that the Hermitian norm $\|\cdot\|_2$ is less than the quotient norm coming from the norm π on $V^{\otimes p}$. In other words, adding an index q to indicate quotient norms, we have the following inequalities on norms on $\mathrm{Alt}^p V$:

$$\|\cdot\|_{\varepsilon,\mathrm{q}} \leqslant \|\cdot\|_{2,\mathrm{q}} = \frac{1}{\sqrt{p!}} \|\cdot\|_2 \leqslant \|\cdot\|_2 \leqslant \|\cdot\|_{\pi,\mathrm{q}}.$$

§ A.2
VOLUME OF EUCLIDEAN BALLS

LEMMA A.2.1. *The volume of the unit ball in an Euclidean vector space of dimension n is given by*

$$\beta_n = \pi^{n/2} / \Gamma(1 + n/2).$$

The surface of the unit sphere is equal to $n\beta_n$.

Proof. — This is classical. One has

$$\left(\int_{\mathbf{R}} \exp(-u^2)\,\mathrm{d}u\right)^n = \int_{\mathbf{R}^n} \exp(-\|x\|^2)\,\mathrm{d}x = \mathrm{vol}_{n-1}(\mathbf{S}_n)\int_0^\infty \exp(-r^2)\,r^{n-1}\,\mathrm{d}r$$
$$= n\beta_n \int_0^\infty \exp(-t)\,t^{n/2}\,\frac{\mathrm{d}t}{t} = \frac{1}{2n}\beta_n\Gamma(n/2) = \beta_n\Gamma(1+\frac{n}{2}).$$

For $n = 2$, one has $\beta_2 = \pi$ and $\Gamma(2) = 1$, hence $\int_{\mathbf{R}} \exp(-u^2)\,\mathrm{d}u = \sqrt{\pi}$, hence $\beta_n = \pi^{n/2}/\Gamma(1+n/2)$. □

Remark A.2.2. Stirling's formula

$$\Gamma(1+x) \sim x^{x+\frac{1}{2}} e^{-x}\sqrt{2\pi}$$

implies that

$$\log\Gamma(1+x) = x\log x - x + \frac{1}{2}\log(2\pi) + \mathrm{o}(1).$$

Consequently, when $n \to \infty$,

$$\log\beta_n = -\frac{1}{2}n\log n + \frac{1}{2}\log(e\pi)\,n + \mathrm{o}(n).$$

§ A.3
GEOMETRY OF NUMBERS

A. Minkowski First Theorem

THEOREM A.3.1 (Minkowski). *Let L be a lattice in a finite dimensional real vector space V, let $n = \dim V$. Let $\|\cdot\|$ be a norm on V, let B be its unit ball; let also μ be a Haar measure on V. Then,*

$$\mathrm{card}(B\cap L) \geqslant 1 + \left\lfloor \frac{\mu(B)}{2^n\mu(V/L)}\right\rfloor > \frac{\mu(B)}{2^n\mu(V/L)}.$$

In particular, if $\mu(B) \geqslant 2^n\mu(V/L)$, then B contains a non-zero point in L.

It is worth observing that the middle term is equal to the smallest integer which is strictly greater than $2^n\mu(B)/\mu(V/L)$.

Proof. — Let $N = 1 + \lfloor \mu(B)/2^n\mu(V/L)\rfloor$ be the least integer which is *strictly* greater than $\mu(B)/2^n\mu(V/L)$. We first note that it suffices to prove the weaker inequality:

$$\mathrm{card}(B\cap L) \geqslant \mu(B)/2^n\mu(V/L).$$

For $t > 1$, let us indeed consider the norm $\|\cdot\|_t$ on V given by $\|v\|_t = t\|v\|$; its unit ball B_t is equal to tB. Consequently, $\mathrm{card}(B_t\cap L) \geqslant t^n\mu(B)/2^n\mu(V/L)$ which gives at least N distinct points $(v_{t,1},\dots,v_{t,N})$ in $B_t\cap L$ for any $t > 1$. Since B_2 is compact, we may assume that for each u, $v_{t,i}$ has a limit v_i when $t \to 1$; one has $\|v_i\| \leqslant 1$. Since L is a discrete subgroup of V, for each i, the points $v_{t,i}$ coincide with v_i, for t close enough to 1. Therefore, the vectors $(v_1,\dots,v_N)$ are all distinct.

We are now reduced to proving the inequality $\operatorname{card}(B\cap L) \geqslant \mu(B)/2^n\mu(V/L)$, which is the very heart of this Theorem. Due to its importance, we even give four short proofs of it.

First proof (Minkowski). For any subset $A\subset V$, let $\mathbf{1}_A$ be its characteristic function. Let f the the L-periodic function on V defined by $f(x)=\sum_{v\in L}\mathbf{1}_{B/2}(x-v)$. The sum is locally finite because $B/2$ is compact and L is discrete. In particular, f is locally integrable. Let us integrate f on a fundamental domain P for L. We obtain

$$\int_P f = \int_P \sum_{v\in L}\mathbf{1}_{B/2}(x-v)\,\mathrm{d}\mu(x) = \sum_{v\in L}\int_{P+v}\mathbf{1}_{B/2}(x)\,\mathrm{d}\mu(x) = \int_V \mathbf{1}_{B/2}(x)\,\mathrm{d}\mu(x) = \mu(B/2).$$

Consequently, there is at least one point $x\in P$ such that $f(x)\geqslant \mu(B/2)/\mu(P) = \mu(B)/2^n\mu(V/L)$. Let N be the least integer greater or equal than $\mu(B)/2^n\mu(V/L)$. Since f takes it values in $\mathbf{Z}$, it follows that $f(x)\geqslant N$; in other words, there are distinct lattice vectors $v_1,\dots,v_N$ such that $\|x-v_i\|\leqslant\frac{1}{2}$. The vectors $w_i=v_i-v_N$, for $1\leqslant i\leqslant N$ are then lattice vectors such that $\|w_i\|\leqslant 1$.

Second proof (Mordell). When $t\to\infty$, one has $\operatorname{card}(B_t\cap L)\simeq \mu(B_t)/\mu(V/L)\simeq t^n\mu(B)/\mu(V/L)$. (Let P be a fundamental parallepiped for L. When $v\in B_t\cap L$, the domains $P+v$ are disjoint, up to measure 0 and contained in B_{t+r}, where $r=\sup_{x\in P}\|x\|$. Consequently, $\operatorname{card}(B_t\cap L)\mu(P)\leqslant\mu(B_{t+r})$. On the other hand, for any vector $x\in V$ such that $\|x\|\leqslant t-r$, there exists $y\in P$ and $v\in L$ such that $x=v+y$, and $\|y\|\leqslant t$. This shows that $\operatorname{card}(B_t\cap L)\mu(P)\geqslant\mu(B_{t-r})$. These two inequalities imply the result.)

If $\mu(B)/\mu(V/L)>N2^n$, we see that $\operatorname{card}(B_t\cap L)\geqslant N(2t)^n$ when t is large enough. However, we have $(2t)^n=\operatorname{card}(L/2tL)$ for any positive integer t. By Dirichlet's box principle, we can find N distinct vectors $v_1,\dots,v_N\in B_t\cap L$ which have the same image in $L/2tL$. The vectors $w_i=(v_i-v_N)/2t$, for $1\leqslant i\leqslant N$, are distinct, belong to L and satisfy $\|w_i\|\leqslant 1$.

Third proof (Siegel). Let f be the function introduced in Minkowski's proof. We expand it as a Fourier series. Let L^* be the dual lattice, consisting of linear forms $\xi\in V^*$ such that $\langle\xi,v\rangle\in\mathbf{Z}$ for any $v\in L$. (If V is identified to $\mathbf{R}^n$ by a basis of L, so that $L=\mathbf{Z}^n$, then V^* is identified to $\mathbf{R}^n$ and $L^*=\mathbf{Z}^n$.) The Fourier expansion of f is given by

$$f(x)\sim\sum_{\xi\in L^*}c(\xi)\exp(2i\pi\langle\xi,x\rangle),$$

where the Fourier coefficients of f are defined, for $\xi\in L^*$, by the formula

$$c(\xi)=\frac{1}{\mu(P)}\int_P f(x)\exp(-2i\pi\langle\xi,x\rangle)\,\mathrm{d}\mu(x).$$

We observe that $c(0)=\mu(B)/2^n\mu(P)$. Parseval equality holds and states that

$$\frac{1}{\mu(P)}\int_P\left|f(x)\right|^2\,\mathrm{d}\mu(x)=\sum_{\xi\in L^*}|c(\xi)|^2.$$

If $B\cap L=\{0\}$, we see as above that $0\leqslant f(x)\leqslant 1$ everywhere. Consequently,

$$\sum_{\xi\in L^*}|c(\xi)|^2\leqslant 1.$$

In particular, $c(0) \leqslant 1$. More generally, if k is any integer such that $\mu(B) \geqslant k2^n \mu(P)$, we see that there must exist $x \in P$ with $\left|f(x)\right|^2 \geqslant |c(0)|^2$, hence $f(x) \geqslant k$ and we conclude as above that $\operatorname{card}(B \cap L)$ contains at least k distinct points.

Fourth proof (a variant). One has $\operatorname{card}(B \cap L) = \sum_{v \in L} \mathbf{1}_B(v)$. To estimate $\operatorname{card}(B \cap L)$, we thus want to apply the Poisson summation formula to the characteristic function of B. However, $\mathbf{1}_B$ is not continuous. Since we only need a lower bound, we may replace $\mathbf{1}_B$ by any continuous integrable function φ such that $\mathbf{1}_B \geqslant \varphi$ and whose Fourier transform is integrable. The Poisson formula states that

$$\sum_{v \in L} \varphi(v) = \frac{1}{\mu(V/L)} \sum_{\xi \in L^*} \hat{\varphi}(\xi).$$

One has $\hat{\varphi}(0) = \int_V \varphi(x)\, \mathrm{d}\mu(x)$ and the idea is to choose φ of positive type, *i.e.*, such that φ and $\hat{\varphi}$ are nonnegative so that we would obtain

$$\operatorname{card}(B \cap L) \geqslant \sum_{v \in L} \varphi(v) \geqslant \frac{1}{\mu(V/L)} \int_V \varphi(x)\, \mathrm{d}\mu(x).$$

Since $(B/2) - (B/2) \subset B$ and $(B/2) = -(B/2)$, the the convolution product ρ of $\mathbf{1}_{B/2}$ with itself

$$\rho(x) = \int_V \mathbf{1}_{B/2}(y) \mathbf{1}_{B/2}(x-y)\, \mathrm{d}\mu(y) \leqslant \mu(B/2)$$

and $\rho(x) = 0$ if $x \not\in B$. Moreover, by the formula for the Fourier transform of a convolution product, $\hat{\rho} = \widehat{\mathbf{1}_{B/2}} \overline{\widehat{\mathbf{1}_{B/2}}} = \left|\widehat{\mathbf{1}_{B/2}}\right|^2$. We observe that $\hat{\rho}$ is nonnegative and that $\hat{\rho}(0) = \mu(B/2)^2$. We thus take $\varphi = \rho/\mu(B/2)$. Now, the Poisson formula applies to φ and gives the inequality

$$\operatorname{card}(B \cap L) \geqslant \sum_{v \in L} \varphi(v) = \frac{1}{\mu(V/L)} \sum_{\xi \in L^*} \hat{\varphi}(\xi) \geqslant \frac{\mu(B)}{2^n \mu(V/L)}.$$

This concludes the proof. □

B. Successive minima

Let L be a lattice in a finite dimensional real vector space V, let $n = \dim V$; let us fix a Haar measure on L. Let $\|\cdot\|$ be a norm on V and let B be its unit ball.

For any integer $m \in \{1, \ldots, n\}$, let λ_m be the least real number such that $\lambda_m B \cap L$ contains m vectors which are linearly independent. One has

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n.$$

These are called the successive minima of the lattice L (with respect to the norm $\|\cdot\|$).

LEMMA A.3.2. *There exist vectors $v_1, \ldots, v_n \in L$, linearly independent, such that for any $m \in \{1, \ldots, n\}$, $\|v_m\| = \lambda_m$.*

Proof. — For any $\lambda > \lambda_m$, the set $\lambda B \cap L$ is finite (since B is bounded and L discrete) and generates a subspace of dimension $\geqslant m$. Consequently, there exists $\varepsilon > 0$ and a finite set $F \subset L$ such that $\lambda B \cap L = F$ for any $\lambda \in (\lambda_m, \lambda_m + \varepsilon)$. Necessarily, $\|v\| \leqslant \lambda_m$ for any $v \in$

F, so that F is contained in $\lambda_m B$ and generates a sub-vector space of V of dimension at least m. This implies that $\lambda_m B \cap L$ contains at least m linearly independent vectors.

Let us prove by induction on $m \in \{1,\dots,n\}$ that one can find linearly independent vectors $v_1,\dots,v_m$ in L such that $\|v_j\| = \lambda_j$ for all $j \in \{1,\dots,m\}$. The case $m = n$ is the statement to be proved. As we have seen, $\lambda_1 B \cap L$ is nonzero, so that one can find a nonzero vector $v_1 \in L$ such that $\|v_1\| \leqslant \lambda_1$.

Let us now assume that the result holds for m and let us prove it for $m+1$; let $(v_1,\dots,v_m)$ be given by the induction hypothesis. Since $\lambda_{m+1} B \cap L$ generates a subspace of dimension $\geqslant m+1$, there exists a vector $v_{m+1} \in \lambda_{m+1} B \cap L$ which does not belong to the space spanned by $v_1,\dots,v_m$. One has $\|v_{m+1}\| \leqslant \lambda_{m+1}$ and $(v_1,\dots,v_{m+1})$ is linearly independent. Set $\lambda = \|v_{m+1}\|$ and let us show that $\lambda = \lambda_{m+1}$. Assume otherwise and let k be the smallest integer in $\{1,\dots,m+1\}$ such that $\lambda < \lambda_k$. Then, $\lambda B \cap L$ contains k linearly independent vectors $v_1,\dots,v_{k-1},v_{m+1}$, so that $\lambda \geqslant \lambda_k$, contradiction. This concludes the proof, by induction on m. □

Remarks A.3.3. Let $(v_1,\dots,v_n)$ be such a family of minimal vectors.

1) For any $m \in \{0,\dots,n\}$, the intersection $L \cap \operatorname{vect}(v_1,\dots,v_m)$ is contained in $\operatorname{vect}(v_1,\dots,v_{m-1})$. Indeed, we may assume that m is minimal. The family $(v, v_1,\dots,v_{m-1})$ consists of m vectors in L with norms $< \lambda_m$. By definition of λ_m, it cannot be free. Since $(v_1,\dots,v_{m-1})$ are linearly independent, it follows that $v \in \operatorname{vect}(v_1,\dots,v_{m-1})$.

2) By induction on m, one can construct a basis $(e_1,\dots,e_n)$ of L such that, for each integer $m \in \{1,\dots,n\}$,

$$\operatorname{vect}(v_1,\dots,v_m) = \operatorname{vect}(e_1,\dots,e_m).$$

LEMMA A.3.4 (Mahler). *There is a basis $(v_1,\dots,v_n)$ of V consisting of vectors in L such that for any $m \in \{1,\dots,n\}$,*

$$\|v_m\| \leqslant \lambda_1 + \cdots + \lambda_m \leqslant m\lambda_m.$$

Proof. — Let $(v_1,\dots,v_n)$ be a family given by Lemma A.3.2 and let $(e_1,\dots,e_n)$ be a basis of L such that for any integer $m \in \{1,\dots,n\}$, $\operatorname{vect}(v_1,\dots,v_m) = \operatorname{vect}(e_1,\dots,e_m)$.

For each m, let us write $e_m = \sum_{j=1}^m x_{m,j} v_j$, for some rational numbers $x_{m,j}$. without restriction, we may assume that $|x_{m,j}| \leqslant 1$ for $1 \leqslant j \leqslant m-1$. (Otherwise, we can replace e_m by $e_m - \sum_{j=1}^{m-1} \lfloor x_{m,j} \rfloor v_j$.) Let us then prove that $|x_{m,m}| \leqslant 1$ for each integer $m \in \{1,\dots,n\}$. Fix such an integer m; since $(e_1,\dots,e_n)$ is a basis of L, there exists $(y_1,\dots,y_n) \in \mathbf{Z}^n$ such that $v_m = \sum_{j=1}^n y_j e_j$. Since $\operatorname{vect}(e_1,\dots,e_m) = \operatorname{vect}(v_1,\dots,v_m)$, one has $y_j = 0$ for $j > m$; moreover, $y_m = 1/x_{m,m}$. Therefore $|x_{m,m}| \leqslant 1$, as claimed.

Finally, one has

$$\|e_m\| \leqslant \sum_{j=1}^m |x_{m,j}| \, \|v_j\| \leqslant (\lambda_1 + \cdots + \lambda_m) \leqslant m\lambda_m.$$

□

Minkowski's (First) Theorem, Theorem A.3.1 implies that

$$\lambda_1 \leqslant 2\left(\mu(V/L)/\mu(B)\right)^{1/n}.$$

The following Theorem is a strong and important generalization of this inequality.

THEOREM A.3.5 (Minkowski). *With the previous notation, one has*

$$\frac{1}{n!}2^n\mu(V/L)\leqslant\lambda_1\dots\lambda_n\mu(B)\leqslant 2^n\mu(V/L).$$

Proof. — Let $v_1,\dots,v_n$ be n vectors in L, linearly independent, such that $\|v_j\|\leqslant\lambda_j$ for all j.

Let $(w_1,\dots,w_n)$ be a basis of L^*. Their determinant $D=\det(\langle v_i,\rangle w_j)$ is a nonzero integer; therefore $|D|\geqslant 1$. Moreover, if S the simplex with vertices v_i, then $\mu(S)=|D|\mu(V/L)/n!$. The vectors $\pm v_j/\lambda_j$ belong to B; by convexity, B contains 2^n disjoint open simplices with vertices $(0,\varepsilon_1 v_1/\lambda_1,\dots,\varepsilon_n v_n/\lambda_n)$, whose volumes are equal to $|D|\operatorname{vol}(V/L)/n!\lambda_1\dots\lambda_n$. Consequently,

$$\mu(B)\geqslant 2^n\mu(V/L)/n!\lambda_1\dots\lambda_n.$$

Let $(e_1,\dots,e_n)$ be a basis of L such that for all m, $\operatorname{vect}(e_1,\dots,e_m)=\operatorname{vect}(v_1,\dots,v_m)$ and $\lambda_m B^\circ\cap L\subset\operatorname{vect}(e_1,\dots,e_{m-1})$. For any $m\in\{0,\dots,n\}$, let $L_m=\mathbf{Z}e_1+\cdots+\mathbf{Z}e_m$ and let $p_m\colon V\to V/L_m$ the natural projection. We still write μ for the natural measure on V/L_m deduced from μ. By construction of the basis $(e_1,\dots,e_n)$, the quotient map $V/L_{m-1}\to V/L_m$ is injective on $(\lambda_m/2)B^\circ$. Therefore,

$$\mu(p_m((\lambda_m/2)B^\circ))=\mu(p_{m-1}((\lambda_m/2)B^\circ)).$$

By Lemma A.3.6 below applied to $C=(\lambda_m/2)B^\circ$ and $t=\lambda_m/\lambda_{m-1}$,

$$\mu(p_{m-1}((\lambda_m/2)B^\circ)\geqslant(\lambda_m/\lambda_{m_1})^{n-m+1}\mu(p_{m-1}((\lambda_{m-1}/2)B^\circ)).$$

Consequently,

$$\begin{aligned}\mu(V/L)&\geqslant\mu(p_n((\lambda_n/2)B^\circ))\\&\geqslant\frac{\lambda_n}{\lambda_{n-1}}\left(\frac{\lambda_{n-1}}{\lambda_{n-2}}\right)^2\cdots\left(\frac{\lambda_2}{\lambda_1}\right)^{n-1}\left(\frac{\lambda_1}{\lambda_0}\right)^n\mu((1/2)B^\circ)\\&\geqslant\lambda_1\dots\lambda_n 2^{-n}\mu(B).\end{aligned}$$

This concludes the proof of Minkowski's second theorem. □

LEMMA A.3.6. *Let C be a convex subset of V. Let L be a lattice in a subvector space W of V. Let $p\colon V\to V/L$ be the natural projection. For any $t\geqslant 1$, $\mu(p(tC))\geqslant t^{\dim(V)-\dim(W)}\mu(p(C))$.*

Proof. — Let $q\colon V\to V/W$ and $r\colon V/L\to V/W$ be the natural projections. Then,

$$\mu(p(tC))=\int_{V/W}\mu(p(tC)\cap r^{-1}(x))\,\mathrm{d}x=\int_{V/W}\mu(t(p(C)\cap r^{-1}(x/t)))\,\mathrm{d}x.$$

Let $x\in V/W$ and let c be a point in $p(C)\cap r^{-1}(x/t)$. Since this set is convex and $t\geqslant 1$, we see that

$$t(p(C)\cap r^{-1}(x/t))\supset p(C)\cap r^{-1}(x/t)+(t-1)c.$$

In particular,

$$\mu(t(p(C)\cap r^{-1}(x/t))\geqslant\mu(p(C)\cap r^{-1}(x/t)).$$

If $p(C) \cap r^{-1}(x/t)$ is empty, this inequality still holds, so that

$$\begin{aligned}\mu(p(tC)) &\geqslant \int_{V/W} \mu(p(C)\cap r^{-1}(x/t))\,\mathrm{d}x \\ &= t^{\dim(V)-\dim(W)} \int_{V/W} \mu(p(C)\cap r^{-1}(y)\,\mathrm{d}y \\ &= t^{\dim(V)-\dim(W)} \mu(p(C)).\end{aligned}$$

□

C. The Brunn-Minkowski inequality

LEMMA A.3.7. *Let A, B, C be nonempty open subsets of* $\mathbf{R}$ *such that $A+B\subset C$. Then,* $\operatorname{meas}(A)+\operatorname{meas}(B)\leqslant\operatorname{meas}(C)$.

Proof. — The inequality to be proved is invariant under translations of A or B; one thus may assume that $\sup(A)=\inf(B)=0$. Then, $A\subset C$. Indeed, let $a\in A$, since A is open and $\inf(B)=0$, there exists $b\in B$ such that $a-b\in A$. Then, $a=(a-b)+b\in C$. Similarly, $B\subset C$. Moreover, A and B are disjoint. Consequently, $\operatorname{meas}(A)+\operatorname{meas}(B)\leqslant\operatorname{meas}(C)$. □

PROPOSITION A.3.8 (Prékopa-Leindler inequality). *Let $f,g,h\colon \mathbf{R}^n\to[0,\infty)$ be nonnegative lower semi-continuous function with compact support on $\mathbf{R}^n$. Let $\theta\in(0,1)$ be a real number such that*

$$h((1-\theta)x+\theta y)\geqslant f(x)^{1-\theta}g(y)^{\theta} \tag{A.3.9}$$

for all $x,y\in\mathbf{R}^n$. Then,

$$\int_{\mathbf{R}^n} h \geqslant \left(\int_{\mathbf{R}^n} f\right)^{1-\theta}\left(\int_{\mathbf{R}^n} g\right)^{\theta}.$$

Proof. — The proof proceeds by induction on the dimension. The required inequality holds by hypothesis when $n=0$. Let us prove it for $n=1$. For any positive real number α, one can write

$$\int_{\mathbf{R}} h = \int_0^\infty \operatorname{meas}(\{z\in\mathbf{R};\, h(z)>\alpha\})\,\mathrm{d}\alpha,$$

and similarly for f and g. Moreover, for any $\alpha>0$, Equation (A.3.9) implies that

$$(1-\theta)\{x\in\mathbf{R};\, f(x)>\alpha\}+\theta\{y\in\mathbf{R};\, g(y)>\alpha\}\subset\{z\in\mathbf{R};\, h(z)>\alpha\}.$$

Since f, g and h are lower semi-continuous and compactly supported, all three sets are bounded open subsets of $\mathbf{R}$. By Lemma A.3.7,

$$(1-\theta)\operatorname{meas}(\{x\in\mathbf{R};\, f(x)>\alpha\})+\theta\operatorname{meas}(\{y\in\mathbf{R};\, g(y)>\alpha\})\leqslant\operatorname{meas}(\{z\in\mathbf{R};\, h(z)>\alpha\}).$$

Therefore,

$$\begin{aligned}\int_{\mathbf{R}} h &\geqslant (1-\theta)\int_0^\infty \operatorname{meas}(\{x\in\mathbf{R};\, f(x)>\alpha\})\,\mathrm{d}\alpha+\theta\int_0^\infty \operatorname{meas}(\{y\in\mathbf{R};\, g(y)>\alpha\})\,\mathrm{d}\alpha \\ &\geqslant (1-\theta)\int_{\mathbf{R}} f+\theta\int_{\mathbf{R}} g.\end{aligned}$$

By convexity of the exponential function,

$$a^{1-\theta}+b^{\theta}\leqslant(1-\theta)a+\theta b$$

for all positive real numbers a and b. Therefore,

$$\int_{\mathbf{R}} h=\left(\int_{\mathbf{R}} f\right)1-\theta\left(\int_{\mathbf{R}} g\right)^{\theta},$$

which concludes the proof of the one-dimensional case.

Let us assume, by induction, that the result is proved in lower dimensions. Let x_n and $y_n\in\mathbf{R}$; let us define $F,G,H\colon\mathbf{R}^{n-1}\to\mathbf{R}$ by the formulae

$$F(x)=f(x;x_n),\quad G(y)=g(y;y_n),\quad H(z)=h(z;(1-\theta)x_n+\theta y_n).$$

By assumption, for any $x,y\in\mathbf{R}^{n-1}$, one has

$$H((1-\theta)x+\theta y)=h((1-\theta)x+\theta y;(1-\theta)x_n+\theta y_n)\geqslant f(x;x_n)^{1-\theta}g(y;y_n)^{\theta}=F(x)^{1-\theta}G(y)^{\theta}.$$

The induction hypothesis therefore implies

$$\int_{\mathbf{R}^{n-1}} H\geqslant\left(\int_{\mathbf{R}^{n-1}} F\right)^{1-\theta}\left(\int_{\mathbf{R}^{n-1}} F\right)^{\theta}.$$

For any $t\in\mathbf{R}$, let us pose $f_n(t)=\int_{\mathbf{R}^{n-1}} f(x;t)\,\mathrm{d}x$ and let us define g_n and h_n similarly. With this notation, the preceding inequality can be rewritten as

$$h_n((1-\theta)x_n+\theta y_n)\geqslant f_n(x_n)^{1-\theta}g_n(y_n)^{\theta}.$$

These functions are obviously nonnegative and compactly supported; they are also lower semi-continuous. By the one-dimensional case, we thus have

$$\int_{\mathbf{R}} h_n\geqslant\left(\int_{\mathbf{R}} f_n\right)^{1-\theta}\left(\int_{\mathbf{R}} g_n\right)^{\theta},$$

that is,

$$\int_{\mathbf{R}^n} h\geqslant\left(\int_{\mathbf{R}^n} f\right)^{1-\theta}\left(\int_{\mathbf{R}^n} g\right)^{\theta}.$$

□

COROLLARY A.3.10. *Let A, B, C be nonempty bounded open subsets of $\mathbf{R}^n$. Let $\theta\in(0,1)$ such that $C\supset(1-\theta)A+\theta B$. Then,*

$$\operatorname{meas}(C)\geqslant\operatorname{meas}(A)^{1-\theta}\operatorname{meas}(B)^{\theta}.$$

Proof. — Let f, g, h be the the characteristic functions of A, B and C. Since A, B and C are open and bounded, these functions are lower semi-continuous and have compact support. By definition, $\int_{\mathbf{R}^n} f=\operatorname{meas}(A)$, $\int_{\mathbf{R}^n} g=\operatorname{meas}(B)$ and $\int_{\mathbf{R}^n} h=\operatorname{meas}(C)$. Moreover, for any $x\in A$ and any $y\in B$, one has $(1-\theta)x+\theta y\in C$, hence $h((1-\theta)x+\theta y)=1=f(x)^{1-\theta}g(y)^{\theta}$ since $f(x)=g(y)=1$; conversely, if $x\notin A$ or $y\notin B$, then $f(x)^{1-\theta}g(y)^{\theta}=0\leqslant h((1-\theta)x+\theta y)$. Consequently, Proposition **??** applies and gives the desired result. □

COROLLARY A.3.11 (Brunn-Minkowski). *Let A and B be non-empty bounded open subsets of $\mathbf{R}^n$. Then,*

$$\operatorname{meas}(A)^{1/n} + \operatorname{meas}(B)^{1/n} \leqslant \operatorname{meas}(A+B)^{1/n}.$$

Proof. — Let $A' = \operatorname{meas}(A)^{-1/n}A$ and $B' = \operatorname{meas}(B)^{-1/n}B$, so that $\operatorname{meas}(A') = \operatorname{meas}(B') = 1$. Then set

$$\theta = \frac{\operatorname{meas}(B)^{1/n}}{\operatorname{meas}(A)^{1/n} + \operatorname{meas}(B)^{1/n}}.$$

If $x \in A'$ and $y \in B'$, then

$$(1-\theta)x + \theta y \in (\operatorname{meas}(A)^{1/n} + \operatorname{meas}(B)^{1/n})^{-1}(A+B).$$

By the preceding corollary, one obtains

$$(\operatorname{meas}(A)^{1/n} + \operatorname{meas}(B)^{1/n})^{-n} \operatorname{meas}(A+B) \geqslant \operatorname{meas}(A')^{1-\theta} \operatorname{meas}(B')^{\theta} = 1,$$

which proves the claim. □

COROLLARY A.3.12. *Let V be a real vector space, let W be a subspace of V; let $p\colon V \to V/W$ be the natural projection. Let us endow the spaces V, W and V/W with compatible Haar measures. Then, for any symmetric convex open subset B of V,*

$$\operatorname{vol}(B) \leqslant \operatorname{vol}(B \cap W) \operatorname{vol}(p(B)).$$

Proof. — By the compatibility of the chosen Haar measures,

$$\operatorname{vol}(B) = \int_{V/W} \operatorname{vol}(p^{-1}(x) \cap B)\, \mathrm{d}x.$$

For $x \notin p(B)$, $p^{-1}(x) \cap B$ is empty. Fix a point $y \in p^{-1}(x)$ and set $A = p^{-1}(x) \cap B - y$. This is a convex open subset of W; since B is convex symmetric, one has $A - A \subset B + B = 2B$. Moreover, $A - A \subset W$. Let $n = \dim(W)$; by the Brunn-Minkowski inequality (Corollary A.3.11),

$$\operatorname{vol}(A)^{1/n} + \operatorname{vol}(-A)^{1/n} \leqslant \operatorname{vol}(2B \cap W)^{1/n},$$

that is, $\operatorname{vol}(A) \leqslant \operatorname{vol}(B \cap W)$. Therefore,

$$\operatorname{vol}(B) \leqslant \int_{p(B)} \operatorname{vol}(p^{-1}(x) \cap B)\, \mathrm{d}x \leqslant \int_{p(B)} \operatorname{vol}(B \cap W)\, \mathrm{d}x \leqslant \operatorname{vol}(B \cap W) \operatorname{vol}(p(B)).$$

This concludes the proof. □

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