AN INTRODUCTION TO PERVERSE SHEAVES

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CHAPTER 1

CATEGORIES

1.1. Sets, universes and categories

1.1.1. — We wish to work within the classical framework of set theory, as formalized by the ZFC axioms: Zermelo-Fraenkel with choice. However, the inexistence of a "set of all sets" makes this framework not really adequate to consider the usual categories or functors. We thus complement this theory with Grothendieck's concept of universes.

Definition (**1.1.2**). — *An* universe *is a set* U *satisfying the following properties:*

a) *For every set* $x \in \mathrm{U}$ *and every element* $y \in x$*, one has* $y \in \mathrm{U}$*;*
b) *For every* $x, y \in \mathrm{U}$*, one has* $\{x, y\} \in \mathrm{U}$*;*
c) *For every* $x \in \mathrm{U}$*, one has* $\mathfrak{P}(x) \in \mathrm{U}$*;*
d) *For every* $\mathrm{I} \in \mathrm{U}$ *and every family* $(x_i)_{i \in \mathrm{I}}$ *of elements of* U*, one has* $\bigcup_{i\in\mathrm{I}} x_i \in \mathrm{U}$*;*
e) *The set* **N** *belongs to* U.

1.1.3. — In some precise sense, an universe can be seen as a model of set theory: the axioms of a universe precisely guarantee that all classical operations of sets do not leave a given universe. For example, if x, y are elements of a universe U, then the pair (x, y), defined *à la* Kuratowski by $(x, y) = \{\{x\}, \{x, y\}\}$ belongs to U. Then the product set $x \times y$, a subset of $\mathfrak{P}(\mathfrak{P}(x \cup y))$ belongs to U, as well as all of its subsets, so that the graphs of all functions from x to y belong to U. In particular, if f is surjective, then any retraction (whose existence is asserted by the axiom of choice) belongs to U.

Consequently, *existence of universes* does not follow from the axioms of ZFC — this would indeed contradict Gödel's second incompleteness theorem — and ZFC has to be supplemented by an axiom such as the following.

Axiom (**1.1.4**). — *For every set* x*, there exists an universe* U *such that* $x \in \mathrm{U}$.

One checks readily that the intersection of a non-empty family of universes is a universe. Consequently, under axiom 1.1.4, for every set x, there exists a smallest universe U containing x.

Definition **(1.1.5)**. — *A* category $\boldsymbol{C}$ *is the datum of two sets* $\mathrm{ob}(\boldsymbol{C})$ *and* $\mathrm{mor}(\boldsymbol{C})$, *whose elements are respectively called its* objects *and its* morphisms, *of two maps* $o, t\colon \mathrm{mor}(\boldsymbol{C}) \to \mathrm{ob}(\boldsymbol{C})$ *(*origin *and* target*) and of a partial composition map:* $\mathrm{mor}(\boldsymbol{C}) \times \mathrm{mor}(\boldsymbol{C}) \to \boldsymbol{C}$*, denoted* $(f, g) \mapsto g \circ f$ *satisfying the following properties, where* $f, g, h \in \mathrm{mor}(\boldsymbol{C})$*:*

a) *the composition* $g \circ f$ *is defined if and only if* $t(f) = o(g)$*; one has* $o(g \circ f) = o(f)$ *and* $t(g \circ f) = t(g)$*;*
b) *the composition is associative: if* $t(f) = o(g)$ *and* $t(g) = o(h)$*, then* $h \circ (g \circ f) = (h \circ g) \circ f$*;*
c) *for every object* $\mathrm{X} \in \mathrm{ob}(\boldsymbol{C})$*, there exists a morphism* $\mathrm{id}_{\mathrm{X}} \in \mathrm{mor}(\boldsymbol{C})$ *such that* $o(\mathrm{id}_{\mathrm{X}}) = t(\mathrm{id}_{\mathrm{X}}) = \mathrm{X}$, $\mathrm{id}_{\mathrm{X}} \circ f = f$ *for every* $f \in \mathrm{mor}(\boldsymbol{C})$ *such that* $t(f) = \mathrm{X}$*, and* $g \circ \mathrm{id}_{\mathrm{X}} = g$ *for every* $g \in \mathrm{mor}(\boldsymbol{C})$ *such that* $o(g) = \mathrm{X}$.

If $f \in \boldsymbol{C}$, the objects $o(f)$ and $t(f)$ are called the origin and the target of f. For any two objects X, Y in a category $\boldsymbol{C}$, one writes $\boldsymbol{C}(\mathrm{X}, \mathrm{Y})$, or $\mathrm{Hom}_{\boldsymbol{C}}(\mathrm{X}, \mathrm{Y})$ to be the subset of $\mathrm{mor}(\boldsymbol{C})$ consisting of all morphisms f with origin X and target Y.

Example **(1.1.6)**. — Let $\boldsymbol{C}$ be a category. Its *opposite category*, denoted by $\boldsymbol{C}^{\mathrm{o}}$, has the same objects and the same morphisms, but the origin/target maps are exchanged, and the order of composition is switched.

When one writes down a general construction/theorem from category theory both in $\boldsymbol{C}$ and in the opposite category $\boldsymbol{C}^{\mathrm{o}}$, one obtains two related statements, one being obtained from the other by "reversing the arrows".

Example **(1.1.7)**. — Let A be a set and let $\leqslant$ be a *preordering relation* on A, that is, a binary relation on A which is reflexive and transitive. From $(\mathrm{A}, \leqslant)$, one defines a category $\boldsymbol{A}$ as follows: one has $\mathrm{ob}(\boldsymbol{A}) = \mathrm{A}$ and $\mathrm{mor}(\boldsymbol{A})$ is the set of pairs $(a, b) \in \mathrm{A} \times \mathrm{A}$ such that $a \leqslant b$, the maps o and t being the first and second projection respectively, the composition is defined by $(b, c) \circ (a, b) = (a, c)$ for every $a, b, c \in \mathrm{A}$.

Example (**1.1.8**). — All classical mathematical objects, such as sets, abelian groups, topological spaces, k-modules (where k is a ring), k-algebras, etc., give rise to categories.

Since there is no set of all sets, we need to fix a universe U. The category $\boldsymbol{Set}_{\mathrm{U}}$ of U-sets has for objects the elements of U and for morphisms the maps between those sets, the composition being given by the usual composition of maps. Similarly, the abelian groups whose underlying set belongs to U are the objects of a category $\boldsymbol{Ab}_{\mathrm{U}}$, the morphisms of this category being the morphisms of abelian groups between them. One defines analogously categories $\boldsymbol{Top}_{\mathrm{U}}$ or $\boldsymbol{Ring}_{\mathrm{U}}$ whose objects are the topological spaces, or the rings, with underlying set an element of U. Or, if k is an object of $\boldsymbol{Ring}_{\mathrm{U}}$, one defines categories $\boldsymbol{Mod}(k)_{\mathrm{U}}$ or $\boldsymbol{Alg}(k)_{\mathrm{U}}$ of k-modules, or of k-algebras, whose underlying set belongs to U.

In practice, one can often work within an universe U which is fixed once and for all and talk about the category $\boldsymbol{Set}$ of sets, etc.

Definition (**1.1.9**). — *Let* U *be a universe.*

a) *One says that a category* $\boldsymbol{C}$ *is a* U*-category if* $\mathrm{ob}(\boldsymbol{C})$ *and* $\mathrm{mor}(\boldsymbol{C})$ *belong to* U.

b) *One says that a set* X *is* U*-small if there exists a bijection* $f\colon \mathrm{X} \to \mathrm{X}'$ *with an element of* U.

c) *One says that a category* $\boldsymbol{C}$ *is* U*-small if* $\mathrm{ob}(\boldsymbol{C})$ *and* $\mathrm{mor}(\boldsymbol{C})$ *are* U*-small.*

d) *One says that a category* $\boldsymbol{C}$ *is locally* U*-small if for every objects* $\mathrm{X}, \mathrm{Y} \in \mathrm{ob}(\boldsymbol{C})$*, the set* $\boldsymbol{C}(\mathrm{X}, \mathrm{Y})$ *is* U*-small.*

Let U be a universe and $\boldsymbol{C}$ be a category. Since $\mathrm{ob}(\boldsymbol{C})$ can be identified with the subset of $\mathrm{mor}(\boldsymbol{C})$ of all identities, we observe that if $\mathrm{mor}(\boldsymbol{C})$ belongs to U, then $\mathrm{ob}(\boldsymbol{C})$ belongs to U as well.

For example, the category $\boldsymbol{Set}_{\mathrm{U}}$ of U-sets is locally U-small, but not U-small. However, if V is an universe such that $\mathrm{U} \in \mathrm{V}$, then $\boldsymbol{Set}_{\mathrm{U}}$ is a V-category.

1.1.10. — Let $\mathrm{X}, \mathrm{Y} \in \mathrm{ob}(\boldsymbol{C})$ and $f \in \boldsymbol{C}(\mathrm{X}, \mathrm{Y})$.

One says that f is *left-invertible* if there exists $g \in \boldsymbol{C}(\mathrm{Y}, \mathrm{X})$ such that $g \circ f = \mathrm{id}_{\mathrm{X}}$; any such element g is called a left-inverse of f.

One says that g is *right-invertible* if there exists $h \in \boldsymbol{C}(\mathrm{Y}, \mathrm{X})$ such that $f \circ h = \mathrm{id}_{\mathrm{Y}}$, and every such element g is called a right-inverse of g.

One says that f is *invertible*, or an *isomorphism* if it is both left- and right-invertible. In this case, any left-inverse g and any right-inverse h of f coincide, since $g = g \circ \mathrm{id}_Y = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$, and this element is simply called the *inverse* of f.

The definitions of a left-invertible morphism and of a right-invertible are deduced one from the other by passing to the opposite category.

1.1.11. — Let $X, Y \in \mathrm{ob}(\boldsymbol{C})$ and $f \in \boldsymbol{C}(X, Y)$.

One says that f is a *monomorphism* if for every object $Z \in \mathrm{ob}(\boldsymbol{C})$ and every $h, h' \in \boldsymbol{C}(Z, X)$, the equality $f \circ h = f \circ h'$ implies that $h = h'$.

If f is left-invertible, then f is a monomorphism. Let indeed $h, h' \in \boldsymbol{C}(Z, X)$ be such that $f \circ h = f \circ h'$; for every left-inverse g of f, one then has

$$h = (g \circ f) \circ h = g \circ (f \circ h) = g \circ (f \circ h') = (g \circ f) \circ h' = h'.$$

One says that f is an *epimorphism* if for every object $Z \in \mathrm{ob}(\boldsymbol{C})$ and every $g, g' \in \boldsymbol{C}(Y, Z)$, the equality $g \circ f = g' \circ f$ implies that $g = g'$.

If f is right-invertible, then f is an epimorphism. Let indeed $g, g' \in \boldsymbol{C}(Y, Z)$ be such that $g \circ f = g' \circ f$; for every right-inverse h of f, one then has

$$g = g \circ (f \circ h) = (g \circ f) \circ h = (g' \circ f) \circ h = g' \circ (f \circ h) = g'.$$

The definitions of a monomorphism and of an epimorphism are deduced one from the other by passing to the opposite category.

The reader will take care that a morphism can be both a monomorphism and an epimorphism, without being an isomorphism (exercise 1.7.2).

Definition (**1.1.12**). — *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories. A* functor F *from $\boldsymbol{C}$ to $\boldsymbol{D}$ is the datum of two maps* $\mathrm{ob}(F)\colon \mathrm{ob}(\boldsymbol{C}) \to \mathrm{ob}(\boldsymbol{D})$ *and* $\mathrm{mor}(F)\colon \mathrm{mor}(\boldsymbol{C}) \to \mathrm{mor}(\boldsymbol{D})$ *satisfying the following properties:*

a) *For every* $f \in \mathrm{mor}(\boldsymbol{C})$, *one has* $o(\mathrm{mor}(F)(f)) = \mathrm{ob}(F)(o(f))$ *and* $t(\mathrm{mor}(F)(f)) = \mathrm{ob}(F)(t(f))$;

b) *For every pair* (f, g) *in* $\mathrm{mor}(\boldsymbol{C})$ *such that* $t(f) = o(g)$, *one has* $\mathrm{mor}(F)(g \circ f) = \mathrm{mor}(F)(g) \circ \mathrm{mor}(F)(f)$;

c) *For every object* $X \in \mathrm{ob}(\boldsymbol{C})$, *one has* $\mathrm{mor}(F)(\mathrm{id}_X) = \mathrm{id}_{\mathrm{ob}(F)(X)}$.

In practice, the maps $\mathrm{mor}(F)$ and $\mathrm{ob}(F)$ associated with a functor F are simply denoted by F.

Definition (**1.1.13**). — *Let* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *be two categories, and let* F, G *be two functors from* $\boldsymbol{C}$ *to* $\boldsymbol{D}$. *A* morphism of functors $\alpha\colon F \to G$ *is a map* $\alpha\colon \mathrm{ob}(\boldsymbol{C}) \to \mathrm{mor}(\boldsymbol{D})$ *satisfying the following properties:*

a) *For every* $X \in \mathrm{ob}(\boldsymbol{C})$, *the morphism* $\alpha(X)$ *has source* $F(X)$ *and target* $G(X)$;

b) *For every* $X, Y \in \mathrm{ob}(\boldsymbol{C})$ *and every* $f \in \boldsymbol{C}(X, Y)$, *one has* $\alpha(Y) \circ F(f) = G(f) \circ \alpha(X)$.

Morphisms of functors are composed in the obvious way, turning the set $\boldsymbol{Fun}(\boldsymbol{C}, \boldsymbol{D})$ of functors from $\boldsymbol{C}$ to $\boldsymbol{D}$ into a category.

Let U be a universe. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are U-categories, then $\boldsymbol{Fun}(\boldsymbol{C}, \boldsymbol{D})$ is a U-category.

1.2. Limits, colimits, adjunctions

1.2.1. — A *quiver* Q is the datum of a set V (vertices), of a set A (arrows), and of two maps $o, t\colon V \to Q$ (origin and target).

Let $Q = (V, A, o, t)$ be a quiver.

Let $\boldsymbol{C}$ be a category. A Q-*diagram* in $\boldsymbol{C}$ is given by a family $(X_v)_{v\in V}$ of objects of $\boldsymbol{C}$ and of a family $(f_a)_{a\in A}$ of arrows of $\boldsymbol{C}$ such that $o(f_a) = X_{o(a)}$ and $t(f_a) = X_{t(a)}$.

1.2.2. — Let $\boldsymbol{C}$ be a category and let $D = ((X_v), (f_a))$ be a Q-diagram in $\boldsymbol{C}$.

A *cone on the diagram* D is the datum of an object X of $\boldsymbol{C}$ and of a family $(p_v)_{v\in V}$ satisfying the following properties:

a) For every $v \in V$, p_v is a morphism in $\boldsymbol{C}$ with origin X and target X_v;

b) For every $a \in A$, one has $f_a \circ p_{o(a)} = p_{t(a)}$.

One says that a cone $(X, (p_v))$ on a diagram $D = ((X_v), (f_a))$ is a *limit of the diagram* D if for every cone $D' = (X', (p'_v))$ on D, there exists a unique morphism $\varphi \in \boldsymbol{C}(X', X)$ such that $p_v \circ \varphi = p'_v$ for every $v \in V$.

Let $(X, (p_v))$ and $(X', (p'_v))$ be two limits on D. Since X is a limit and X′ is a cone on D, there exists a unique morphism $\varphi \in \boldsymbol{C}(X', X)$ such that $p_v \circ \varphi = p'_v$ for every $v \in V$. Since X′ is a limit and X is a cone on D, there exists a unique morphism $\varphi' \in \boldsymbol{C}(X, X')$ such that $p_v = p'_v \circ \varphi'$ for every $v \in V$. Then $p_v = p'_v \circ \varphi' = p_v \circ \varphi \circ \varphi'$, for every $v \in V$; since X is a limit, one has $\varphi \circ \varphi' = \mathrm{id}_X$. Similarly, $p'_v = p_v \circ \varphi = p'_v \circ \varphi' \circ \varphi$, for every $v \in V$; since X′ is a limit, one has $\varphi' \circ \varphi = \mathrm{id}_{X'}$. Consequently, φ and φ' are isomorphisms, inverse one to the other.

1.2.3. — By passing to the opposite category, one defines the notions of a cocone on a diagram D and of a colimit of D. Explicitly, a *cocone on the diagram* D is the datum of an object X of $\mathcal{C}$ and of a family $(p_v)_{v\in V}$ satisfying the following properties:

a) For every $v \in V$, p_v is a morphism in $\mathcal{C}$ with origin X_v and target X;
b) For every $a \in A$, one has $p_{t(a)} \circ f_a = p_{o(a)}$.

One says that a cocone $(X, (p_v))$ on a diagram $D = ((X_v), (f_a))$ is a *colimit of the diagram* D if for every cocone $D' = (Y, (q_v))$ on D, there exists a unique morphism $\varphi \in \mathcal{C}(Y, X)$ such that $\varphi \circ p_v = q_v$ for every $v \in V$.

If $(X', (p'_v))$ is another colimit of the diagram D, then there exists a unique morphism $\varphi\colon X \to X'$ such that $\varphi \circ p_v = p'_v$ for every $v \in V$, and φ is an isomorphism.

Example (**1.2.4**). — a) Let Q be the empty quiver — no vertex and no arrow. There exists a unique corresponding Q-diagram D in $\mathcal{C}$: it is empty — no object, no morphism. A cone on D is just an object of $\mathcal{C}$; a limit of D is an object X such that for every object X′ in $\mathcal{C}$, there exists a unique morphism $\varphi\colon X' \to X$ in $\mathcal{C}$. Such an object is called a *terminal object* of $\mathcal{C}$. Passing to the opposite category, a colimit of D is called an *initial object*: this is an object X such that for every object $X' \in \mathcal{C}$, there exists a unique morphism $\varphi\colon X \to X'$ in $\mathcal{C}$.

In the case of the category of sets, the empty set is an initial object, and terminal objects are sets with one element; in the case of the category of k-modules, the initial and the terminal objects are the zero module; in the case of the category of groups, the initial and the terminal objects are the groups reduced to the unit element. In the category of rings, the ring $\mathbf{Z}$ is an initial object, and the zero ring is a terminal object. The category of fields has no initial object and no terminal object.

b) Let $Q = (V, A, o, t)$ be a quiver with no arrows. A Q-diagram in $\mathcal{C}$ is just a family $(X_v)_{v\in V}$ of objects, indexed by the set V of vertices of Q. A limit of D is called a *product* of the family (X_v); a colimit of D is called a *coproduct* of the family (X_v).

In the case of the category of sets, one gets the product, resp. the disjoint union; in the case of the category of k-modules, one gets the product, resp. the direct sum; in the case of the category of groups, one gets the product, resp. the free product. In the category of rings, the product is a product, and the tensor product furnishes a coproduct. In the category of fields, products or coproducts rarely exist.

c) Let Q be a quiver with two vertices a, b and two arrows both with origin a and target b. A Q-diagram is given by two objects A, B in $\boldsymbol{C}$ and two morphisms $f, g\colon A \to B$. A limit of this diagram is called an *equalizer* of the pair (f, g); a colimit is called a *coequalizer* of the pair (f, g).

In the case of the category of sets, the equalizer of (f, g) is the subset of A consisting of those elements $a \in A$ such that $f(a) = g(a)$. In fact, the same formula works for the categories of groups, of k-modules, of rings, of fields, etc., the set-theoretical equalizer is a subobject of A and is the equalizer in the given category. In the category of k-modules, $f - g$ is a morphism, and the equalizer of (f, g) is also the kernel of $f - g$.

In the category of sets, the coequalizer of (f, g) is the quotient of B by the finest equivalence relation in B that identifies $f(a)$ and $g(a)$, for every $a \in A$. However, in the category of groups, of k-modules, of rings, of fields, one needs to consider the finest equivalence relation in B which identifies $f(a)$ and $g(a)$, for every $a \in A$, and which moreover is compatible with the given laws. This gives the same set in the category of k-modules, or of rings, but not in the category of groups, where the coequalizer of (f, g) is the quotient of B by the smallest normal subgroup of B that contains $f(a)g(a)^{-1}$, for every $a \in A$.

Definition (**1.2.5**). — *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories and let $G\colon \boldsymbol{C} \to \boldsymbol{D}$ and $F\colon \boldsymbol{D} \to \boldsymbol{C}$ be functors. An* adjunction *for the pair (G, F) is the datum of a pair (η, ε) of functors $\eta\colon \mathrm{id}_{\boldsymbol{C}} \to F \circ G$ and $\varepsilon\colon G \circ F \to \mathrm{id}_{\boldsymbol{D}}$ satisfying the relations*

$$\varepsilon_{G(x)} \circ G(\eta_x) = \mathrm{id}_{G(x)} \quad \textit{and} \quad F(\varepsilon_y) \circ \eta_{F(y)} = \mathrm{id}_{F(y)} .$$

hold for every $x \in \mathrm{ob}(\boldsymbol{C})$ and every $y \in \mathrm{ob}(\boldsymbol{D})$. The morphism η is called the counit *of the adjunction, and the morphism ε is called its* unit.

If the pair (G, F) possesses an adjunction, then one says One says that (G, F) is an adjoint pair, or that G is a left adjoint to F, or that F is a right adjoint to F, and one writes $G \dashv F$.

Proposition (**1.2.6**). — *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories and let $G\colon \boldsymbol{C} \to \boldsymbol{D}$ and $F\colon \boldsymbol{D} \to \boldsymbol{C}$ be functors. The following datas are equivalent:*

a) *An adjunction (η, ε) for the pair (G, F).*

b) *A morphism of functors* $\eta\colon \mathrm{id}_{\boldsymbol{C}} \to \mathrm{F}\circ\mathrm{G}$ *such that for every* $x \in \mathrm{ob}(\boldsymbol{C})$ *and every* $y \in \mathrm{ob}(\boldsymbol{D})$*, the map* $\boldsymbol{D}(\mathrm{G}(x), y) \to \boldsymbol{C}(x, \mathrm{F}(y))$ *given by* $g \mapsto g^\flat = \mathrm{F}(g)\circ\eta_x$ *is bijective.*

c) *A morphism of functors* $\varepsilon\colon \mathrm{G}\circ\mathrm{F} \to \mathrm{id}_{\boldsymbol{D}}$ *such that for every* $x \in \mathrm{ob}(\boldsymbol{C})$ *and every* $y \in \mathrm{ob}(\boldsymbol{D})$*, the map* $\boldsymbol{C}(x, \mathrm{F}(y)) \to \boldsymbol{D}(\mathrm{G}(x), y)$ *given by* $f \mapsto f^\sharp = \varepsilon_y \circ \mathrm{G}(f)$ *is bijective,*

d) *For every object* x *of* $\boldsymbol{C}$ *and every object* y *of* $\boldsymbol{D}$*, a bijection* $f \mapsto f^\sharp$ *from* $\boldsymbol{C}(x, \mathrm{F}(y))$ *to* $\boldsymbol{D}(\mathrm{G}(x), y)$*, with inverse* $g \mapsto g^\flat$*, such that for any objects* x, x' *of* $\boldsymbol{C}$*, any objects* y, y' *of* $\boldsymbol{D}$*, any morphisms* $u \in \boldsymbol{C}(x', x)$ *and* $v \in \boldsymbol{D}(y, y')$*, any morphism* $f \in \boldsymbol{D}(\mathrm{G}(x), y)$ *and any morphism* $g \in \boldsymbol{C}(x, \mathrm{F}(y))$*, one has*

$$(v\circ g\circ \mathrm{G}(u))^\flat = \mathrm{F}(v)\circ g^\flat \circ u \quad \text{and} \quad v\circ f^\sharp \circ \mathrm{G}(u) = (\mathrm{F}(v)\circ f\circ u)^\sharp.$$

In their presence, one has moreover the relations:

$$\begin{aligned} g^\flat &= \mathrm{F}(g)\circ\eta_x \\ f^\sharp &= \varepsilon_y \circ \mathrm{G}(f) \\ \eta_x &= (\mathrm{id}_{\mathrm{G}(x)})^\flat \\ \varepsilon_y &= \mathrm{id}^\sharp_{\mathrm{F}(y)}. \end{aligned}$$

Proof. — To pass from *d*) to *b*), we just set $\eta_x = (\mathrm{id}_{\mathrm{G}(x)})^\flat$ for every object x of $\boldsymbol{C}$. One observes that for every morphism $g \in \boldsymbol{D}(\mathrm{G}(x), y)$, one has $\mathrm{F}(g)\circ\eta_x = \mathrm{F}(g)\circ \mathrm{id}^\flat_{\mathrm{G}(x)} = g^\flat$. Then, for every morphism $u \in \boldsymbol{C}(x, x')$, one has

$$\mathrm{F}\circ\mathrm{G}(u)\circ\eta_x = \mathrm{G}(u)^\flat = \eta_{x'}\circ u.$$

Consequently, the morphisms $\eta_x\colon x \mapsto \mathrm{F}\circ\mathrm{G}(x)$ define a morphism of functors from $\mathrm{id}_{\boldsymbol{C}}$ to $\mathrm{F}\circ\mathrm{G}$.

Conversely, if *b*) holds, we just need to check that the asserted bijection $g \mapsto g^\flat$ satisfies the given formulae of *d*). Indeed, for $u \in \boldsymbol{C}(x', x)$ and $v \in \boldsymbol{C}(y, y')$, one has

$$(v\circ g\circ \mathrm{G}(u))^\flat = \mathrm{F}(v)\circ\mathrm{F}(g)\circ(\mathrm{F}\circ\mathrm{G})(u)\circ\eta_{x'} = \mathrm{F}(v)\circ\mathrm{F}(g)\circ\eta_x\circ u = \mathrm{F}(v)\circ g^\flat\circ u,$$

which proves the second formula. The other follows.

The equivalence between datas *c*) and *d*) is proved similarly, or by considering opposite categories.

Let us now pass from *d*) to *a*). We already dispose of two morphisms of functors $\eta\colon \mathrm{id}_{\boldsymbol{C}} \to \mathrm{F}\circ\mathrm{G}$ and $\varepsilon\colon \mathrm{G}\circ\mathrm{F} \to \mathrm{id}_{\boldsymbol{D}}$. Moreover, one has

$$\varepsilon_{\mathrm{G}(x)} \circ \mathrm{G}(\eta_x) = \eta_x^\sharp = \mathrm{id}_{\mathrm{G}(x)}$$

and

$$\mathrm{F}(\varepsilon_y) \circ \eta_{\mathrm{F}(y)} = \varepsilon_y^\flat = \mathrm{id}_{\mathrm{F}(y)} .$$

Finally, let us pass from *a*) to *d*). For $g \in \boldsymbol{D}(\mathrm{G}(x), y)$, we set $g^\flat = \mathrm{F}(g)\circ\eta_x \in \boldsymbol{C}(x, \mathrm{F}(y))$; for $f \in \boldsymbol{C}(x,\mathrm{F}(y))$, we set $f^\sharp = \varepsilon_y \circ \mathrm{G}(f) \in \boldsymbol{D}(\mathrm{G}(x), y)$. For every $f \in \boldsymbol{C}(x,\mathrm{F}(y))$ and every $g \in \boldsymbol{D}(\mathrm{G}(x),y)$, we then have

$$(f^\sharp)^\flat = (\varepsilon_y \circ \mathrm{G}(f))^\flat = \mathrm{F}(\varepsilon_y)\circ(\mathrm{F}\circ\mathrm{G})(f)\circ\eta_x = \mathrm{F}(\varepsilon_y)\circ\eta_{\mathrm{F}(y)}\circ f = f$$

and

$$(g^\flat)^\sharp = (\mathrm{F}(g)\circ\eta_x)^\sharp = \varepsilon_y\circ(\mathrm{G}\circ\mathrm{F})(y)\circ\mathrm{G}(\eta_x) = g\circ\varepsilon_{\mathrm{G}(x)}\circ\mathrm{G}(\eta_x) = g,$$

so that the defined maps $f \mapsto f^\sharp$ and $g \mapsto f^\flat$ are bijections, inverse one of the other. The remaining formulas follow, as in the passage from *b*) and *c*) to *a*). □

Remark (**1.2.7**). — Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories and let $\mathrm{F}\colon \boldsymbol{D}\to\boldsymbol{C}$ be a functor, and let $\mathrm{G}, \mathrm{G}'\colon \boldsymbol{C}\to\boldsymbol{D}$ be two functors which are both left adjoint to F. Then G and G′ are isomorphic.

More precisely, let (η,ε) and (η',ε') be adjunctions for the pairs (G,F) and (G',F) respectively. For every $x \in \mathrm{ob}(\boldsymbol{C})$, every $y \in \mathrm{ob}(\boldsymbol{D})$, we obtain a bijection

$$\boldsymbol{D}(\mathrm{G}(x),y) \xrightarrow{\sim} \boldsymbol{C}(x,\mathrm{F}(y)) \xrightarrow{\sim} \boldsymbol{D}(\mathrm{G}'(x),y),$$

given explicitly by

$$g \mapsto \tilde g = \varepsilon'_y \circ (\mathrm{G}'\circ\mathrm{F})(g)\circ\mathrm{G}'(\eta_x) = g\circ\varepsilon'_{\mathrm{G}'(x)}\circ\mathrm{G}'(\eta_x) = g\circ\theta_x,$$

where $\theta_x = \varepsilon'_{\mathrm{G}'(x)}\circ\mathrm{G}'(\eta_x) \in \boldsymbol{D}(\mathrm{G}'(x),\mathrm{G}(x))$. This implies that θ_x is an isomorphism, for every $x \in \mathrm{ob}(\boldsymbol{C})$. Moreover, the family $\theta = (\theta_x)$ is an isomorphism of functors from G′ to G.

Similarly, if $\mathrm{F},\mathrm{F}'\colon \boldsymbol{C}\to\boldsymbol{D}$ and $\mathrm{G}\colon \boldsymbol{D}\to\boldsymbol{C}$ are functors such that G is left adjoint both to F and F′, then F and F′ are isomorphic.

Proposition (**1.2.8**). — *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories and let* $\mathrm{G}\colon \boldsymbol{C}\to\boldsymbol{D}$ *and* $\mathrm{F}\colon\boldsymbol{D}\to\boldsymbol{C}$ *be functors. Let* (ε,η) *be an adjunction for the pair* (G,F).

a) *The functor* G *is faithful if and only if η_x is a monomorphism for every object x of $\boldsymbol{C}$;*

b) *The functor* G *if full if and only if* η_x *is right invertible, for every object* x *of* $\boldsymbol{C}$*;*

c) *The functor* G *is fully faithful if and only if* η *is an isomorphism of functors.*

a′) *The functor* F *is faithful if and only if* ε_y *is an epimorphism for every object* y *of* $\boldsymbol{D}$*;*

b′) *The functor* F *if full if and only if* ε_y *is left invertible, for every object* y *of* $\boldsymbol{D}$*;*

c′) *The functor* F *is fully faithful if and only if* ε *is an isomorphism of functors.*

Proof. — a) Let us assume that η_x is a monomorphism, for every object x of $\boldsymbol{C}$, and let us prove that G is faithful. Let x, x' be objects of $\boldsymbol{C}$ and let u and u' be elements of $\boldsymbol{C}(x', x)$ such that $\mathrm{G}(u) = \mathrm{G}(u')$. Then

$$\eta_x \circ u = (\mathrm{F} \circ \mathrm{G})(u) \circ \eta_{x'} = (\mathrm{F} \circ \mathrm{G})(u') \circ \eta_{x'} = \eta_x \circ u',$$

hence $u = u'$. Conversely, let us assume that G is faithful and let $u, u' \in \boldsymbol{C}(x', x)$. One has $(\eta_x \circ u)^\sharp = \varepsilon_{\mathrm{G}(x)} \circ \mathrm{G}(\eta_x) \circ \mathrm{G}(u) = \mathrm{G}(u)$, and $(\eta_x \circ u')^\sharp = \mathrm{G}(u')$. Consequently, if $eta_x \circ u = \eta_x \circ u'$, then $\mathrm{G}(u) = \mathrm{G}(u')$, hence $u = u'$, since G is faithful. This proves *a*).

b) Let us assume that η_x is right invertible. Let x, x' be objects of $\boldsymbol{C}$ and let $v \in \boldsymbol{D}(\mathrm{G}(x'), \mathrm{G}(x))$. Let $\theta_x \in \boldsymbol{C}(\mathrm{F} \circ \mathrm{G}(x), x)$ be such that $\eta_x \circ \theta_x = \mathrm{id}_{\mathrm{F} \circ \mathrm{G}(x)}$. Let us set $u = \theta_x \circ \mathrm{F}(v) \circ \eta_{x'}$; this is an element of $\boldsymbol{C}(x', x)$. Moreover, $\eta_x \circ u = \mathrm{F}(v) \circ \eta_{x'} = v^\flat$, so that $v = (\eta_x \circ u)^\sharp = \mathrm{G}(u)$. This proves that G is full.

Conversely, let us assume that G is full, let x be an object of $\boldsymbol{C}$ and let us choose a morphism $\theta_x \in \boldsymbol{C}(\mathrm{F} \circ \mathrm{G}(x), x)$ such that $\mathrm{G}(\theta_x) = (\mathrm{id}_{\mathrm{F} \circ \mathrm{G}(x)})^\sharp$. Then $(\eta_x \circ \theta_x)^\sharp = \mathrm{G}(\theta_x) = (\mathrm{id}_{\mathrm{F} \circ \mathrm{G}(x)})^\sharp$, so that $\eta_x \circ \theta_x = \mathrm{id}_{\mathrm{F} \circ \mathrm{G}(x)}$. This proves that η_x is right invertible.

c) Let us assume that G is fully faithful. Let $x \in \mathrm{ob}(\boldsymbol{C})$; then η_x has a right inverse θ_x, by *b*). It follows that $\eta_x \circ \theta_x \circ \eta_x = \eta_x$, hence $\theta_x \circ \eta_x = \mathrm{id}_{\mathrm{F} \circ \mathrm{G}(x)}$, because η_x is a monomorphism, by *a*). Consequently, η_x is an isomorphism. This implies that η is an isomorphism of functors.

Conversely, let us assume that η is an isomorphism of functors. In particular, η_x is an isomorphism for every object x of $\boldsymbol{C}$. Then the functor G is faithful, by *a*), and is full, by *b*); it is thus fully faithful.

The primed assertions follow from what has just been proved, by passing to the opposite category. □

Proposition (**1.2.9**). — *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories and let* $G \colon \mathbf{C} \to \mathbf{D}$ *and* $F \colon \mathbf{D} \to \mathbf{C}$ *be functors. Assume that* (G, F) *is an adjoint pair.*

a) *The functor* F *preserves limits and sends monomorphisms to monomorphisms;*
b) *The functor* G *preserves colimits and sends epimorphisms to epimorphisms.*

Proof. — Let (ε, η) be an adjunction for the pair (G, F). Let $Q = (V, A)$ be a quiver and let $D = ((y_v), (g_a))$ be a Q-diagram in $\mathbf{D}$, with limit y; let $g = (g_v)$ be the canonical family of morphisms, where $g_v \in \mathbf{D}(y, y_v)$ for every $v \in V$. Then $F(D) = ((F(y_v)), (F(g_a)))$ is a Q-diagram in $\mathbf{C}$, and $(F(y), (F(g_a)))$ is a cone on $F(D)$. Let us show that it is a limit. Let $(x, (f_v))$ be a cone on $F(D)$: for every $v \in V$, one has $f_v \in \mathbf{C}(x, F(y_v))$; for every edge $a \in A$ with origin v and term v', one has $f_{v'} = F(g_a) \circ f_v$. For every $v \in V$, one has $f_v^\sharp \in \mathbf{D}(G(x), y_v)$; for every edge $a \in A$ with origin v and term v', one has $f_{v'}^\sharp = (F(g_a) \circ f_v)^\sharp = g_a \circ f_v^\sharp$. Consequently, $(G(x), (f_v^\sharp))$ is a cone on D; by definition of a limit, there exists a unique morphism $g \in \mathbf{D}(G(x), y)$ such that $f_v^\sharp = g_v \circ g$ for every $v \in V$. Let $f = g^\flat \in \mathbf{C}(x, F(y))$; for every $v \in V$, one has $(F(g_v) \circ f)^\sharp = g_v \circ f^\sharp = g_v \circ g = f_v^\sharp$, so that $F(g_v) \circ f = f_v$. Conversely, if $f' \in \mathbf{C}(x, F(y))$ satisfies $F(g_v) \circ f' = f_v$ for every $v \in V$, then $f_v^\sharp = g_v \circ (f')^\sharp$, hence $(f')^\sharp = g$ and, finally, $f' = f$. This concludes the proof that the cone $(F(y), (F(g_v)))$ is a limit of the diagram $F(D)$.

Let $v \colon \ \in \mathbf{D}(y, y')$ be a monomorphism and let us prove that $F(v)$ is a monomorphism. Let $u, u' \in \mathbf{C}(x, F(y))$ be such that $F(v) \circ u = F(v) \circ u'$. Then $v \circ \mathrm{id}^\sharp \circ G(u) = (F(v) \circ u)^\sharp = v \circ \mathrm{id}^\sharp \circ G(u')$. Since v is a monomorphism, we get $\mathrm{id}^\sharp \circ G(u) = \mathrm{id}^\sharp \circ G(u')$, hence $u^\sharp = (u')^\sharp$, hence $u = u'$. This proves that $F(v)$ is a monomorphism, as claimed.

The other assertion follows by passing to the opposite categories. □

(1)

1.3. Additive categories

1.3.1. — Let $\mathbf{C}$ be a category. A *zero object* in $\mathbf{C}$ is an object o which is both initial and terminal. Then for every pair (X, Y) of objects in $\mathbf{C}$, there exists a unique morphism in $\mathbf{C}(X, Y)$ which factors through o; it is denoted by o.

(1) To be added: examples, (co)limits as adjoint?

Definition (**1.3.2**). — *Let $\boldsymbol{C}$ be a category. One says that $\boldsymbol{C}$ is* semi-additive *if the following conditions are satisfied:*

a) *$\boldsymbol{C}$ admits finite products and finite coproducts;*

b) *There exists an object of $\boldsymbol{C}$, denoted by 0, which is both initial and terminal;*

c) *Let (X_1, X_2) be a pair of objects of $\boldsymbol{C}$, let $(X_1 \sqcup X_2, (j_1, j_2))$ be a coproduct, let $(X_1 \times X_2, (p_1, p_2))$ and let $\varepsilon \colon X_1 \sqcup X_2 \to X_1 \times X_2$ be a morphism such that $p_a \circ \varepsilon \circ j_b = \mathrm{id}_{X_a}$ if $a = b$, and 0 otherwise. Then ε is an isomorphism.*

Let us detail the third condition a little bit. By definition of a coproduct, the map $f \mapsto (f \circ j_1, f \circ j_2)$ is a bijection from $\boldsymbol{C}(X_1 \sqcup X_2, X_1 \times X_2)$ to $\prod_{b=1}^2 \boldsymbol{C}(X_b, X_1 \times X_2)$. Similarly, for every $b \in \{1, 2\}$, the definition of a product implies that the map $g \mapsto (p_1 \circ g, p_2 \circ g)$ is a bijection from $\boldsymbol{C}(X_b, X_1 \times X_2)$ to $\prod_{a=1}^2 \boldsymbol{C}(X_b, X_a)$. Consquently, the map $f \mapsto (p_a \circ \varepsilon \circ j_b)_{(a,b) \in \{1,2\}^2}$ is a bijection from $\boldsymbol{C}(X_1 \sqcup X_2, X_1 \times X_2)$ to $\prod_{a,b=1}^2 \boldsymbol{C}(X_b, X_a)$. Consequently, there there exists a unique morphism ε as stated, and the assertion is that ε is an isomorphism.

Lemma (**1.3.3**). — *Let $\boldsymbol{C}$ be a semi-additive category. For every pair (X_1, X_2) of objects of $\boldsymbol{C}$ and every pair $f, g \in \boldsymbol{C}(X_1, X_2)$, let $f + g$ be the unique element of $\boldsymbol{C}(X_1, X_2)$ such that*

$$X_1 \xrightarrow{d_{X_1}} X_1 \times X_1 \xrightarrow{(f,g)} X_2 \times X_2 \xrightarrow{\varepsilon^{-1}} X_2 \sqcup X_2 \xrightarrow{\delta_{X_2}} X_2,$$

where d_{X_1} is the unique morphism whose composition with the two canonical morphisms $X_1 \times X_1 \to X_1$ is id_{X_1}, and δ_{X_2} is the unique morphism whose composition with the two canonical morphisms $X_2 \to X_2 \sqcup X_2$ is id_{X_2}. Then the composition law $(f, g) \mapsto f + g$ on $\boldsymbol{C}(X_1, X_2)$ is commutative, associative, the zero morphism is a neutral element.

Moreover, for every triple (X_1, X_2, X_3) of objects of $\boldsymbol{C}$, the composition map $\boldsymbol{C}(X_1, X_2) \times \boldsymbol{C}(X_2, X_3) \to \boldsymbol{C}(X_1, X_3)$ given by $(f, g) \mapsto g \circ f$ is bi-additive: for $f, f' \in \boldsymbol{C}(X_1, X_2)$ and $g, g' \in \boldsymbol{C}(X_2, X_3)$, one has

$$g \circ (f + f') = (g \circ f) + (g \circ f') \quad \textit{and} \quad (g + g') \circ f = (g \circ f) + (g' \circ f).$$

Proof. — To be done. □

Definition (**1.3.4**). — *One says that a semi-additive category $\boldsymbol{C}$ is* additive *if its semi-groups of morphisms $\boldsymbol{C}(X_1, X_2)$ are abelian groups.*

Example (**1.3.5**). — a) Let k be a ring; the category of k-modules is an additive category.

b) Let X be a topological space and let $\mathscr{O}$ be a sheaf of rings on X; the category of $\mathscr{O}$-modules is an additive category.

c) The category of complex Banach spaces, with continuous linear maps for morphisms, is an additive category.

d) The opposite category to a (semi-)additive category is again a (semi-)additive category.

1.3.6. — Let $\boldsymbol{C}$ be an additive category. Let (X, Y) be a pair of objects of $\boldsymbol{C}$ and let $f \in \boldsymbol{C}(\mathrm{X}, \mathrm{Y})$ be a morphism. An equalizer of the pair $(f, 0)$ is called a *kernel of f*, and is denoted by $\mathrm{Ker}(f)$; a coequalizer of the pair $(f, 0)$ is called a *cokernel of f*, and is denoted by $\mathrm{Coker}(f)$. If f is a monomorphism, then 0 is a kernel of f; if f is an epimorphism, then 0 is a cokernel of f.

Let $\mathrm{Ker}(f)$ be a kernel of f, and let $i\colon \mathrm{Ker}(f) \to \mathrm{X}$ be the canonical morphism. By definition of a kernel, the map $\boldsymbol{C}(\mathrm{Z}, \mathrm{Ker}(f)) \to \boldsymbol{C}(\mathrm{Z}, \mathrm{X})$ given by $g \mapsto i \circ g$ is injective. In other words, i is a monomorphism.

By passing to the opposite category, one deduces that the canonical morphism $p\colon \mathrm{Y} \to \mathrm{Coker}(f)$ is an epimorphism.

1.4. Abelian categories

Definition (**1.4.1**). — *One says that an additive category* $\boldsymbol{C}$ *is an* abelian category *if the following properties hold:*

a) *Every morphism has a kernel and a cokernel;*

b) *Every monomorphism is a kernel;*

c) *Every epimorphism is a cokernel.*

Example (**1.4.2**). — a) Let k be a ring. The category of k-modules is an abelian category. Epimorphisms are surjective morphisms, monomorphisms are injective morphisms; kernel and cokernels coincide with the usual notions.

A theorem of Mitchell asserts that for every abelian category $\boldsymbol{C}$, there exists a ring k and an exact fully faithful functor of $\boldsymbol{C}$ into a category of k-modules. In particular, kernels and cokernels are preserved by this embedding. For certain arguments, this allows to pretend objects of $\boldsymbol{C}$ are k-modules and play with their “elements”.

b) Let X be a topological space and let $\mathscr{O}$ be a sheaf of rings on X. The category of $\mathscr{O}$-modules is an abelian category. Monomorphisms, resp. epimorphisms, are the morphisms which induce injective, resp. surjective, morphisms on all stalks. Consequently, monomorphisms are injective morphisms. However, not every epimorphism is surjective (see exercise 3.10.1). Kernels are defined naively; however, the cokernel of a morphism φ of $\mathscr{O}$-modules is the sheaf associated with the presheaf $U \mapsto \operatorname{Coker}(\varphi_U)$.

c) Let $\boldsymbol{A}$ be an abelian category. The additive category $\boldsymbol{C}(\boldsymbol{A})$ of complexes in $\boldsymbol{A}$ is an abelian category. Kernels and cokernels are computed termwise and a morphism of complexes $f\colon X \to Y$ is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if so is $f^n\colon X^n \to Y^n$, for every integer $n \in \mathbf{Z}$.

d) The category of Banach spaces is not an abelian category. Indeed, in this category, monomorphisms are the injective continuous morphisms, while kernels are monomorphisms with closed image.

Proposition **(1.4.3)**. — *Let $f\colon X \to Y$ be a morphism in an abelian category $\boldsymbol{C}$.*

a) *The morphism f is a monomorphism if and only if* $\operatorname{Ker}(f) = 0$*;*

b) *The morphism f is an epimorphism if and only* $\operatorname{Coker}(f) = 0$*;*

c) *The morphism f is an isomorphism if and only it is both a monomorphism and an epimorphism.*

Proof. — a) The conditions "f is a monomorphism" and "$\operatorname{Ker}(f) = 0$" are both equivalent to the statement that for every object Z, the zero morphism is the only morphism $h \in \boldsymbol{C}(Z, X)$ such that $f \circ h = 0$.

b) Similarly, the conditions "f is an epimorphism" and "$\operatorname{Coker}(f) = 0$" are both equivalent to the statement that for every object Z, the zero morphism is the only morphism $g \in \boldsymbol{C}(Y, Z)$ such that $g \circ f = 0$.

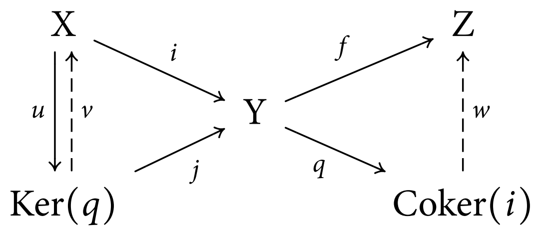
c) If f is an isomorphism, then it is both an epimorphism and a monomorphism. Let us assume, conversely, that f is both an epimorphism and a monomorphism. Since f is a monomorphism, it is the kernel of a morphism $g\colon Y \to Z$. In particular, one has $g \circ f = 0 = 0 \circ f$. Since f is an epimorphism, one has $g = 0$. Since $f\colon X \to Y$ is a kernel of 0, the relation $0 \circ \mathrm{id}_Y = 0$ implies the existence of a unique morphism $h\colon Y \to X$ such that $\mathrm{id}_Y = f \circ h$. In particular, f is right-invertible. By passing to the opposite category, one proves that f is left-invertible. Consequently, f is an isomorphism, as was to be shown. □

Lemma **(1.4.4)**. — *Let $\boldsymbol{C}$ be an abelian category.*

a) *Every monomorphism is a kernel of its cokernel;*
b) *Every epimorphism is a cokernel of its kernel.*

Proof. — *a*) Let $i\colon X \to Y$ be a monomorphism. Let $p\colon Y \to \operatorname{Coker}(i)$ be a cokernel of i and let $j\colon \operatorname{Ker}(p) \to Y$ be a kernel of p. Since $q \circ i = 0$, there exists a unique morphism $u\colon \operatorname{Ker}(f) \to \operatorname{Ker}(q)$ such that $i = j \circ u$. Let us prove that u is an isomorphism.

By definition of an abelian category, there exists a morphism $f\colon Y \to Z$ such that i is a kernel of f. Since $f \circ i = 0$, there exists a (unique) morphism $w\colon \operatorname{Coker}(i) \to Z$ such that $f = w \circ q$.



Since $f \circ j = w \circ q \circ j = 0$, there exists a morphism $v\colon \operatorname{Ker}(q) \to \operatorname{Ker}(f)$ such that $j = i \circ v$. Since i and j are kernels, they are monomorphisms. Then the relations $i = j \circ u = i \circ v \circ u$ and $j = i \circ v = j \circ u \circ v$ imply that $v \circ u = \mathrm{id}_{\operatorname{Ker}(f)}$ and $u \circ v = \mathrm{id}_{\operatorname{Ker}(q)}$. In particular, u is an isomorphism.

The proof of assertion *b*) is similar, and follows from *a*) by passing to the opposite category. □

Proposition (**1.4.5**). — *Let $\mathbf{C}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathbf{C}$. Let $i\colon \operatorname{Ker}(f) \to X$ be a kernel of f and let $p\colon Y \to \operatorname{Coker}(f)$ be a cokernel of f. Let $q\colon X \to \operatorname{Coker}(i)$ be a cokernel of i and let $j\colon \operatorname{Ker}(p) \to Y$ be a kernel of p. There exists a unique morphism $\tilde f\colon \operatorname{Coker}(i) \to \operatorname{Ker}(f)$ such that $f = j \circ \tilde f \circ q$, and $\tilde f$ is an isomorphism.*

A kernel of p is called an *image* of f and is denoted by $\operatorname{Im}(f)$; a cokernel of j is called a *coimage* of f is denoted by $\operatorname{Coim}(f)$. The proposition thus says that any morphism f induces a canonical isomorphism from its image to its coimage.

This is represented by the following diagram:

$$\begin{array}{ccccccc}
\operatorname{Ker}(f) & \xhookrightarrow{\;i\;} & X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;p\;} & \operatorname{Coker}(f) \\
 & & \Big\downarrow q & & \Big\uparrow j & & \\
 & & \operatorname{Coim}(f) & \underset{\tilde f}{\overset{\simeq}{\dashrightarrow}} & \operatorname{Im}(f) & &
\end{array}$$

Observes that when one passes to the opposite category, kernels and cokernels are switched, as are images and coimages.

As an intermediate step, the proof of the proposition uses a result which can be seen as corollary.

Lemma **(1.4.6)**. — a) *There exists a unique morphism* $f_1 : X \to \operatorname{Im}(f)$ *such that* $f = j \circ f_1$.

b) *For every factorization* $f = j' \circ f_1'$, *where* $j' : T \to Y$ *is a* monomorphism *and* $f_1' : X \to T$ *is a morphism, there exists a unique morphism* $u : \operatorname{Im}(f) \to T$ *such that* $f_1' = u \circ f_1$ *and* $j = j' \circ u$.

c) *The morphism* f_1 *is an epimorphism.*

Proof. — a) By definition, $\operatorname{Im}(f)$ is a kernel of $p : Y \to \operatorname{Coker}(f)$, so that $p \circ f = 0$. Consequently, the assertion follows from the definition of a kernel.

b) Let $p' : T \to \operatorname{Coker}(j')$ be a cokernel of j'. Since $p' \circ f = p' \circ j' \circ f_1' = 0$, there exists a unique morphism $u' : \operatorname{Coker}(f) \to \operatorname{Coker}(j')$ such that $p' = u' \circ p$. Then $p' \circ j = u' \circ p \circ j = 0$, so that there exists a unique morphism $u : \operatorname{Ker}(p) \to T$ such that $j = j' \circ u$. Then $j' \circ f_1' = f = f \circ f_1 = j' \circ u \circ f_1$; since j' is a monomorphism, one has $f_1' = u \circ f_1$.

c) Let $s : \operatorname{Ker}(p) \to S$ be a morphism such that $s \circ f_1 = 0$; let us prove that $s = 0$. Let $k : \operatorname{Ker}(s) \to \operatorname{Ker}(p)$ be a kernel of s; there exists a unique morphism $f_1' : X \to \operatorname{Ker}(s)$ such that $f_1 = k \circ f_1'$. Then $f = j \circ f_1 = (j \circ k) \circ f_1'$. Since $j \circ k$ is a monomorphism, part *b*) of lemma 1.4.6 asserts that there exists a unique morphism $u : \operatorname{Ker}(p) \to \operatorname{Ker}(s)$ such that $f_1' = u \circ f_1$ and $j = j \circ k \circ u$. Since j is a monomorphism, this implies that $k \circ u = \mathrm{id}_{\operatorname{Im}(f)}$. Finally, $s = s \circ k \circ u = 0$. ☐

Proof of proposition 1.4.5. — Since $p \circ f = 0$, there exists a unique morphism $f_1 : X \to \operatorname{Ker}(p)$ such that $f = j \circ f_1$; by lemma 1.4.6, f_1 is an epimorphism. Then $j \circ f_1 \circ i = f \circ i = 0$, hence $f_1 \circ i = 0$, because j is a monomorphism. Consequently, there exists a unique morphism $\tilde f : \operatorname{Coker}(i) \to \operatorname{Ker}(p)$ such that $f_1 = \tilde f \circ q$. One then has $f = j \circ \tilde f \circ q$. If $\tilde f'$ is a second morphism such that $f = j \circ \tilde f' \circ q$, one has $\tilde{f}' \circ q = \tilde{f} \circ q$, because j is a monomorphism, hence $\tilde{f}' = \tilde{f}$, because q is an epimorphism.

Since $f_1 = \tilde{f} \circ q$ is an epimorphism, $\tilde{f}$ is an epimorphism as well. By passing to the opposite category, we see that $\tilde{f}$ is a monomorphism. Consequently, $\tilde{f}$ is an isomorphism, as claimed. □

1.4.7. — Grothendieck (1957) introduced additional conditions on an abelian category, pertaining to the existence of arbitrary limits or colimits and their property. Let $\boldsymbol{A}$ be an abelian category. One defines the following axioms:

(AB_3) The category $\boldsymbol{A}$ admits arbitray colimits (one says that it is cocomplete); it is equivalent to the property that it admits arbitrary coproducts.

(AB_4) The category $\boldsymbol{A}$ satisfies the axiom AB_3 and the coproduct of a family of monomorphisms is a monomorphism; this implies that coproducts are exact.

(AB_5) The category $\boldsymbol{A}$ satisfies the axiom AB_3 and colimits are exact: a colimit of a family of monomorphisms is a monomorphism.

The axioms (AB_3^*), (AB_4^*), (AB_5^*) are defined similarly by replacing colimits, cocomplete, coproducts and monomorphisms by limits, complete, products and epimorphisms; they amoint to the initial axioms in the opposite category $\boldsymbol{A}^o$.

Of course, the axiom (AB_3) and its dual should have been stated with more care: for example, assuming that U is a universe such that $\boldsymbol{C}(X, Y)$ belongs to U, for every pair (X, Y) of objects, one should restrict to limits or colimits defined by quivers in U.

One says that a family $(P_i)_{i\in I}$ of objects of $\boldsymbol{C}$ is *generating* (resp. is *cogenerating*) if, for every nonzero object X of $\boldsymbol{C}$, there exists an index $i \in I$ such that $\boldsymbol{C}(P_i, X) = 0$ (resp. $\boldsymbol{C}(X, P_i) = 0$). When the family $(P_i)_{i\in I}$ is reduced to a single object P, one says that P is a generator (resp. a cogenerator).

One says that an abelian category $\boldsymbol{C}$ is a *Grothendieck category* if it satisfies the axiom (AB_5^*) (existence and are exactness of colimits) and if it admits a generator P. Then for every object X of $\boldsymbol{C}$, there exists a set J, a family $(i_j)_{j\in J}$ and an epimorphism $f \colon \bigoplus_{j\in J} P_{i_j} \to X$.

Example **(1.4.8)**. — a) Let k be a ring; the category $\boldsymbol{Mod}(k)$ is an abelian category which satisfies the axioms (AB_3), (AB_4), (AB_5), (AB_3^*) and (AB_4^*), but does not satisfy (AB_5^*).

Moreover, the ring k (viewed as a k-module) is a generator. Consequently, the category $\boldsymbol{Mod}(k)$ is a Grothendieck category.

b) Let X be a topological space and let $\mathscr{O}$ be a sheaf of rings on X. The category $\boldsymbol{Mod}(\mathscr{O})$ of $\mathscr{O}$-modules on X satisfies the axioms (AB_3), (AB_4), (AB_5), (AB_3^*), but not (AB_4^*).

For every open subset, let $\mathscr{O}_U$ be the extension by zero of the ring sheaf $\mathscr{O}|_U$. The family $(\mathscr{O}_U)_U$ is generating.

Definition **(1.4.9)**. — *Let $\boldsymbol{C}$ be an abelian category.*

a) *An object* I *of* $\boldsymbol{C}$ *is said to be* injective *if for every monomorphism* $j\colon X \to Y$ *and every morphism* $f\colon X \to I$, *there exists a morphism* $g\colon Y \to I$ *such that* $f = g \circ j$.

b) *An object* P *of* $\boldsymbol{C}$ *is said to be* projective *if for every epimorphism* $p\colon X \to Y$ *and every morphism* $f\colon P \to Y$, *there exists a morphism* $g\colon P \to X$ *such that* $f = p \circ g$.

In other words, an object I is injective if and only if the left-exact functor $\boldsymbol{C}(\cdot, I)$ is exact; an object P is projective if and only if the left-exact functor $\boldsymbol{C}(P, \cdot)$ is exact.

Theorem **(1.4.10)** (Grothendieck). — *Let $\boldsymbol{C}$ be a Grothendieck abelian category. For every object* X *of* $\boldsymbol{C}$, *there exists an injective object* I *of* $\boldsymbol{C}$ *and a monomorphism* $f\colon X \to I$.

For the proof, see (Grothendieck, 1957, théorème 1.10.1).

When the conclusion of the theorem holds, one says that $\boldsymbol{C}$ admits *enough injectives.*

(2)

1.5. Complexes in additive categories

Let $\boldsymbol{A}$ be an additive category.

(2) To be added: representability of a contravariant additive functor from an abelian category admitting a generator to the category of abelian groups.

1.5.1. — A *complex* in $\boldsymbol{A}$ is a sequence $(d^n : X^n \to X^{n+1})_{n\in\mathbf{Z}}$ of morphisms in $\boldsymbol{A}$ such that $d^{n+1} \circ d^n = 0$ for every $n \in \mathbf{Z}$. These morphisms d^n are called the *differentials* of this complex. One generally denotes such a complex by the letter X, remembering of the objects rather than the differentials, which may then be denoted by d^n_X, writing the name X of the complex as a subscript to avoid possible confusions.

Let X and Y be complexes in $\boldsymbol{A}$. A *morphism of complexes* $f : X \to Y$ is a family $f = (f^n)_{n\in\mathbf{Z}}$ where, for every $n \in \mathbf{Z}$, $f^n \in \boldsymbol{A}(X^n, Y^n)$, such that

$$d^n_Y \circ f^n = f^{n+1} \circ d^{n+1}_X$$

for every $n \in \mathbf{Z}$. Morphisms of complexes are composed in the obvious way.

The complexes in $\boldsymbol{C}$ form an additive category $\boldsymbol{C}(\boldsymbol{A})$: products and coproducts are computed termwise.

One also considers finite or semi-infinite complexes involving sequences indexed by an interval in $\mathbf{Z}$. They amount to extending the sequence of differentials by zero morphisms to/from zero objects.

1.5.2. — Let X be a complex in an additive category $\boldsymbol{A}$. Let $m \in \mathbf{Z}$. The mth *shift* of X is the complex $\Sigma^m X$ defined by: $(\Sigma^m X)^n = X^{m+n}$ and $d^n_{\Sigma^m X} = (-1)^m d^{m+n}_X$ for every $n \in \mathbf{Z}$. For $m = 1$, one simply writes ΣX; this is the complex obtained by shifting X one step to the left.

If $f : X \to Y$ is a morphism of complexes in $\boldsymbol{A}$, the morphism $\Sigma^m f : \Sigma^m X \to \Sigma^m Y$ is defined by $(\Sigma^m f)^n = f^{m+n}$ for every $n \in \mathbf{Z}$.

In this way, the assignment $(X \mapsto \Sigma^m X, f \mapsto \Sigma^m f)$ gives rise to a functor Σ of the category $\boldsymbol{C}(\boldsymbol{A})$ onto itself.

One has $\Sigma^0 = \mathrm{id}$ and $\Sigma^{m+p} = \Sigma^m \circ \Sigma^p$ for every $m, p \in \mathbf{Z}$. In particular, the functors Σ^m are isomorphisms of categories.

1.5.3. — Let $f, g : X \to Y$ be two morphisms of complexes in an additive category $\boldsymbol{A}$. A *homotopy* with origin f and target g is a sequence $\theta = (\theta^n)_{n\in\mathbf{Z}}$ where, for every $n \in \mathbf{Z}$, $\theta^n \in \boldsymbol{A}(X^n, Y^{n-1})$ such that,

$$f^n - g^n = d^{n-1}_Y \circ \theta^n + \theta^{n+1} \circ d^n_X.$$

Let $h : Y \to Z$ be a morphism of complexes; then the family $h \circ \theta = (h^{n-1} \circ \theta^n)$ is a homotopy with origin $h \circ f$ and target $h \circ g$.

Let $k : Z \to X$ be a morphism of complexes; then the family $\theta \circ k = (\theta^n \circ k^{n-1})$ is a homotopy with origin $f \circ k$ and target $g \circ k$.

1.5.4. — Let X and Y be complexes in the additive category $\boldsymbol{A}$; for every $n \in \mathbf{Z}$, let $f^n \in \boldsymbol{A}(X^n, Y^n)$. We define as follows the *cone of* f, denoted by C_f: For every $n \in \mathbf{Z}$, one sets $C_f^n = Y^n \oplus X^{n+1}$, and one defines a morphism $d_{C_f}^n : C_f^n \to C_f^{n+1}$ by the block-matrix $\begin{pmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{pmatrix}$.

Observe that $d_{C_f}^{n+1} \circ d_{C_f}^n$ is given by the product of matrices

$$\begin{pmatrix} d_Y^{n+1} & f^{n+2} \\ 0 & -d_X^{n+2} \end{pmatrix} \begin{pmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & d_Y^{n+1} f^{n+1} - f^{n+2} d_X^{n+1} \\ 0 & 0 \end{pmatrix},$$

so that C_f *is a complex if and only if* f *is a morphism of complexes.*

Let us assume that f is a morphism of complexes. The canonical morphisms $\alpha_f^n : Y^n \to C_f^n = Y^n \oplus X^{n+1}$ define a morphism of complexes $\alpha_f : Y \to C_f$. Similarly, the canonical morphisms $\beta_f^{n+1} : C_f^n = Y^n \oplus X^{n+1} \to X^{n+1}$ define a morphism of complexes $\beta_f : C_f \to \Sigma X$. One has $\beta_f \circ \alpha_f = 0$.

For every $n \in \mathbf{Z}$, let $\theta_f^n : X^n \to Y^{n-1} \oplus X^n = C_f^{n-1}$ be the canonical morphism. One has

$$\begin{aligned} d_{C_f}^{n-1} \circ \theta_f^n + \theta_f^{n+1} \circ d_X^n &= \begin{pmatrix} d_Y^{n-1} & f^n \\ 0 & -d_X^n \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{X^n} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{id}_{X^{n+1}} \end{pmatrix} d_X^n \\ &= \begin{pmatrix} f^n \\ -d_X^n \end{pmatrix} + \begin{pmatrix} 0 \\ d_X^n \end{pmatrix} = \begin{pmatrix} f^n \\ 0 \end{pmatrix} = \alpha_f \circ f. \end{aligned}$$

Consequently, the family $\theta_f = (\theta_f^n)$ is a homotopy with origin $\alpha_f \circ f$ and target 0.

Conversely, let $g : Y \to Z$ be a morphism of complexes and let η be a homotopy with origin $g \circ f$ and target 0. Then the family (γ^n) given, for every $n \in \mathbf{Z}$, by $\gamma^n = \begin{pmatrix} g^n & \varphi^n \end{pmatrix}$ is the unique morphism of complexes $\gamma : C_f \to Z$ such that $\gamma \circ \alpha_f = g$ and $\gamma \circ \theta = \eta$.

In other words, *the triple* $(C_f, \alpha_f, \theta_f)$ *solves the universal problem of making* $\alpha_f \circ f$ *the origin of a homotopy* θ_f *with target* 0.

For every $n \in \mathbf{Z}$, let $\varphi_f^n : C_f^n \to Y^n = (\Sigma Y)^{n-1}$ be the canonical morphism. One has

$$\begin{aligned} d_{\Sigma Y}^{n-1} \circ \varphi_f^n + \varphi_f^{n+1} \circ d_{C_f}^n &= -d_Y^n \begin{pmatrix} \mathrm{id}_{Y^n} & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{id}_{Y^{n+1}} & 0 \end{pmatrix} \begin{pmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} -d_Y^n & 0 \end{pmatrix} + \begin{pmatrix} d_Y^n & f^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & f^{n+1} \end{pmatrix} \\ &= \Sigma f \circ \beta_f, \end{aligned}$$

so that the family $\varphi_f = (\varphi_f^n)$ is a homotopy with origin $\Sigma_f \circ \beta_f$ and target 0.

1.5.5. — Let $f, g : X \to Y$ be two morphisms of complexes and let φ be a homotopy with origin f and target g. For every $n \in \mathbf{Z}$, let $\lambda^n : Y^n \oplus X^{n+1} \to Y^n \oplus X^{n+1}$ be given by the block-matrix $\left(\begin{smallmatrix} 1 & \varphi^{n+1} \\ 0 & 1 \end{smallmatrix}\right)$. The family $\lambda = (\lambda^n)$ is an isomorphism of complexes from C_f to C_g which makes the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha_f} & C_f & \xrightarrow{\beta_f} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle \lambda} & & \| \\ X & \xrightarrow{g} & Y & \xrightarrow{\alpha_f} & C_g & \xrightarrow{\beta_f} & \Sigma X \end{array}$$

commutative.

1.5.6. — Let X, Y be complexes in an additive category $\boldsymbol{A}$. Two morphisms of complexes $f, g : X \to Y$ are said to be *homotopic* if there exists a homotopy with origin f and target g. This is an equivalence relation, compatible with the group structure on $\boldsymbol{C}(\boldsymbol{A})(X, Y)$. Morphisms homotopic to 0 are called *null homotopic*; they form a subgroup $\boldsymbol{C}(\boldsymbol{A})(X, Y)_0$ of $\boldsymbol{C}(\boldsymbol{A})(X, Y)$. Define

$$\boldsymbol{K}(\boldsymbol{A})(X, Y) = \boldsymbol{C}(\boldsymbol{A})(X, Y)/\boldsymbol{C}(\boldsymbol{A})(X, Y)_0.$$

Passing to the quotients, the composition maps in $\boldsymbol{C}(\boldsymbol{A})$ induce composition maps

$$\boldsymbol{K}(\boldsymbol{A})(X, Y) \times \boldsymbol{K}(\boldsymbol{A})(Y, Z) \to \boldsymbol{K}(\boldsymbol{A})(X, Z).$$

These maps define a *category* $\boldsymbol{K}(\boldsymbol{A})$, called the *homotopy category of the additive category* $\boldsymbol{A}$.

It is an additive category. The functors Σ^m, for $m \in \mathbf{Z}$, extend to $\boldsymbol{K}(\boldsymbol{A})$.

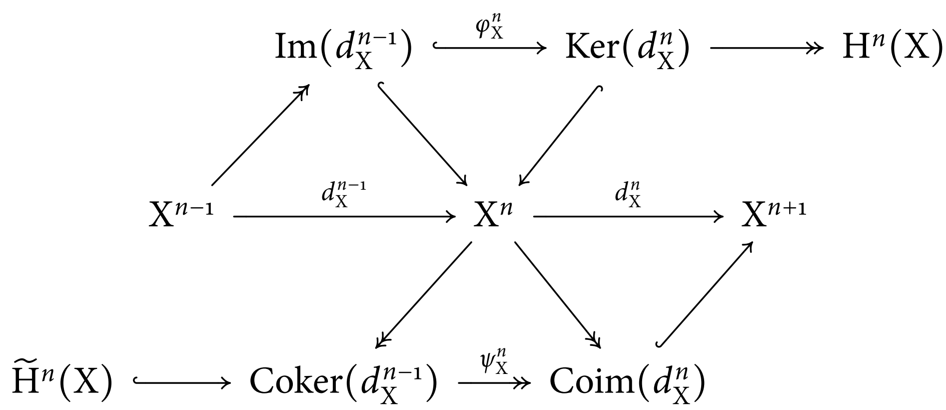
1.6. Complexes in abelian categories

1.6.1. — Let us assume that $\boldsymbol{A}$ an *abelian* category. Let X be a complex in $\boldsymbol{A}$. Let $n \in \mathbf{Z}$. We want to define the nth cohomology object of X. When $\boldsymbol{A}$ is the category of modules over a given ring k, this object is classically defined by $H^n(X) = \operatorname{Ker}(d_X^n)/\mathfrak{I}(d_X^{n-1})$. In the abstract framework of abelian categories, a few other descriptions are available, none of them is obviously preferable to the other.

Since $d_X^n \circ d_X^{n-1} = 0$, the canonical monomorphism $\operatorname{Im}(d_X^{n-1}) \to X^n$ factors uniquely through $\operatorname{Ker}(d^n) \to X^n$. Let $H^n(X)$ be a cokernel of the induced morphism $\varphi_X^n : \operatorname{Im}(d_X^{n-1}) \to \operatorname{Ker}(d_X^n)$.

The consideration of the opposite category furnishes a different description: the canonical morphism $X^n \to \operatorname{Im}(d_X^n)$ factors uniquely through $X^n \to \operatorname{Coker}(d_X^{n-1})$; let $\widetilde{H}^n(X)$ be a kernel of the resulting epimorphism $\psi_X^n\colon \operatorname{Coker}(d_X^{n-1}) \to \operatorname{Im}(d_X^n)$.

(1.6.1.1)



Let $u\colon \operatorname{Ker}(d_X^n) \to \operatorname{Coker}(d_X^{n-1})$ be the composition of the two canonical morphisms indicated on the diagram.

Since $u \circ \varphi_X^n = 0$, the morphism φ_X^n factors uniquely through $\operatorname{Ker}(u)$. Since the following composition of canonical monomorphisms

$$\operatorname{Im}(d_X^{n-1}) \hookrightarrow \operatorname{Ker}(u) \hookrightarrow \operatorname{Ker}(X^n \to \operatorname{Coker}(d_X^{n-1})) \to \operatorname{Im}(d_X^{n-1})$$

is the identity, these monomorphisms are isomorphisms; in particuliar, the morphism φ_X^n induces an isomorphism $\operatorname{Im}(d_X^{n-1}) \xrightarrow{\sim} \operatorname{Ker}(u)$.

Since $\psi_X^n \circ u = 0$, the morphism ψ_X^n factors uniquely through $\operatorname{Coker}(u)$; one checks as above that it induces an isomorphism $\operatorname{Coker}(u) \xrightarrow{\sim} \operatorname{Coim}(d_X^n)$

Using the canonical identification $\operatorname{Coim}(u) \xrightarrow{\sim} \operatorname{Im}(u)$, one then gets canonical isomorphisms

$$\begin{aligned}
\operatorname{Coim}(u) &\simeq \operatorname{Coker}(\operatorname{Ker}(u) \to \operatorname{Ker}(d_X^n)) \\
&\simeq \operatorname{Coker}(\operatorname{Im}(d_X^{n-1}) \to \operatorname{Ker}(d_X^n)) \\
&\simeq \operatorname{Coker}(X^{n-1} \to \operatorname{Ker}(d_X^n)) \\
&\simeq H^n(X) \\
\simeq \operatorname{Im}(u) &\simeq \operatorname{Ker}(\operatorname{Coker}(d_X^{n-1}) \to \operatorname{Coker}(u)) \\
&\simeq \operatorname{Ker}(\operatorname{Coker}(d_X^{n-1}) \to \operatorname{Coim}(d_X^n)) \\
&\simeq \operatorname{Ker}(\operatorname{Coker}(d_X^{n-1}) \to X^{n+1}) \\
&\simeq \widetilde{H}^n(X)
\end{aligned}$$

Once identified via these isomorphisms, any of these objects will be called the *nth cohomology object* of the complex X, and denoted by $H^n(X)$.

One says that the complex X is *acyclic*, or *exact* at X^n if $H^n(X) = 0$. By what precedes, this is equivalent to any of the (equivalent) conditions:

$$\begin{gathered}
u = 0 \\
\operatorname{Coim}(d_X^{n-1}) \xrightarrow{\sim} \operatorname{Ker}(d_X^n) \\
X^{n-1} \twoheadrightarrow \operatorname{Ker}(d_X^n) \\
\operatorname{Coker}(d_X^{n-1}) \xrightarrow{\sim} \operatorname{Im}(d_X^n) \\
\operatorname{Coker}(d_X^{n-1}) \hookrightarrow X^{n+1}.
\end{gathered}$$

If the complex X is exact at X^n, for every $n \in \mathbf{Z}$, then one says that X is acyclic, or exact.

Lemma **(1.6.2)**. — *Let* $\boldsymbol{A}$ *be an abelian category and let* X *be a complex in* $\boldsymbol{A}$*; let* $n \in \mathbf{Z}$*. With the notation of the diagram* (1.6.1.1)*, consider the following morphisms:* (i) *the morphism* $H^n(X) \to \operatorname{Coker}(d_X^{n-1})$ *deduced from the canonical morphism of the diagram and the canonical isomorphism* $\widetilde{H}^n(X) \to \tilde{H}^n(X)$*;* (ii) *the canonical morphism* $\operatorname{Coker}(d_X^{n-1}) \to X^{n+1})$ *factors uniquely through a morphism* $\operatorname{Coker}(d_X^{n-1}) \to \operatorname{Ker}(d_X^{n+1})$*;* (iii) *the canonical morphism* $\operatorname{Ker}(d_X^{n+1}) \to H^{n+1}(X)$*. These morphisms induce an exact sequence:*

$$0 \to H^n(X) \xrightarrow{\text{(i)}} \operatorname{Coker}(d_X^{n-1}) \xrightarrow{\text{(ii)}} \operatorname{Ker}(d_X^{n+1}) \xrightarrow{\text{(iii)}} H^{n+1}(X) \to 0.$$

Proof. — Composed with d_X^{n+1} on the left, and with the canonical epimorphism $X^n \to \operatorname{Coker}(d_X^{n-1})$ on the right, the morphism $\operatorname{Coker}(d_X^{n-1}) \to X^{n+1})$ becomes equal to $d_X^{n+1} \circ d_X^n = 0$. Consequently, Since $X^n \to \operatorname{Coker}(d_X^{n-1})$ is an epimorphism, this implies that the morphism $\operatorname{Coker}(d_X^{n-1}) \to X^{n+1}$ factors through $\operatorname{Ker}(d_X^{n+1})$, hence the existence of the morphism labeled (ii).

Exactness at $H^n(X)$ follows from the fact that the morphism (i), $\widetilde{H}^n(X) \to \operatorname{Coker}(d_X^{n-1})$, is a monomorphism. Similarly, exactness at $H^{n+1}(X)$ follows from the fact that the morphism (iii), $\operatorname{Ker}(d_X^{n+1}) \to H^{n+1}(X)$, is an epimorphism.

Let us show exactness at $\operatorname{Coker}(d_X^{n-1})$: the kernel of the morphism (ii) coincides with that of ψ_X^n, because the morphism $\operatorname{Coim}(d_X^n) \to X^{n+1}$ is a monomorphism, that is, with $\operatorname{Im}(u)$, that is with the image of $H^n(X)$.

Let us finally show exactness at $\operatorname{Ker}(d_X^{n+1})$: by construction, the kernel of the morphism (iii) is the image of d_X^n, which coincides with the image of the morphims (ii). □

1.6.3. — The cohomology objects are functorial: any morphism of complexes $f : X \to Y$ induces morphisms of cohomology objects $H^n(f) : H^n(X) \to H^n(Y)$ in such a way that $H^n(g \circ f) = H^n(g) \circ H^n(f)$ and $H^n(\mathrm{id}_X) = \mathrm{id}_{H^n(X)}$. These functors are also additive.

If $H^n(f)$ is an isomorphism for every $n \in \mathbf{Z}$, then one says that f is a *homologism*, or a *quasi-isomorphism*. We also say that two complexes are homologous, or quasi-isomorphic, if there exists a homologism from one to another. (This is not an equivalence relation in general.)

Lemma **(1.6.4)**. — *Let $\boldsymbol{A}$ be an abelian category.*

a) *Let $f, g : X \to Y$ be morphisms of complexes in $\boldsymbol{A}$. If f and g are homotopic, then $H^n(f) = H^n(g)$ for every $n \in \mathbf{Z}$.*

b) *Let $f : X \to Y$ be a morphism of complexes in $\boldsymbol{A}$. If f induces an isomorphism in the homotopy category $\boldsymbol{K}(\boldsymbol{A})$, then f is a homologism.*

c) *Let X be a complex in $\boldsymbol{A}$. If the identity morphism id_X is null homotopic, then the complex X is acyclic.*

Proof. — a) Let $(\theta^n)_{n\in\mathbf{Z}}$ be a family of morphisms, where for every n, $\theta^n \in \boldsymbol{A}(X^n, Y^{n-1})$, such that $g^n - f^n = d_Y^{n-1} \circ \theta^n + \theta^{n+1} \circ d_X^n$ for every $n \in \mathbf{Z}$. The morphism $H^n(g) - H^n(f) : H^n(X) \to H^n(Y)$ decomposes as the sum of two morphisms respectively induced by $d_Y^{n-1} \circ \theta^n$ and $\theta^{n+1} \circ d_X^n$. The first one is zero, because it factors through the image of d_Y^{n-1} in $H^n(Y)$, which is zero by construction. The second one is zero as well, since d_X^n annihilates $\operatorname{Ker}(d_X^n)$. Consequently, $H^n(g) = H^n(f)$.

b) Let $g\colon Y \to X$ be a morphism of complexes such that $f \circ g$ and $g \circ f$ are homotopic to identity. Then $H^n(f) \circ H^n(g) = \mathrm{id}$ and $H^n(g) \circ H^n(f) = \mathrm{id}$, hence $H^n(f)$ is an isomorphism, for each $n \in \mathbf{Z}$. In other words, f is a homologism.

c) Assume that id_X is null homotopic. Then one has $\mathrm{id}_{H^n(X)} = H^n(\mathrm{id}_X) = H^n(0) = 0$, for every $n \in \mathbf{Z}$. Consequently, $H^n(X) = 0$ for every n, and X is acyclic. □

1.6.5. — Let

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an exact sequence of complexes in an abelian category $\boldsymbol{A}$. For every $n \in \mathbf{Z}$, the morphisms f^n and g^n give rise to an exact sequence

$$0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0$$

in $\boldsymbol{A}$.

Let C_f be the cone of f, let $\alpha_f\colon Y \to C_f$ and $\beta_f\colon C_f \to \Sigma X$ be the canonical morphisms, and let $\theta_f\colon C_f \to \Sigma^{-1}X$ be the canonical homotopy such that $\alpha_f \circ f = d_{C_f} \circ \theta_f + \theta_f \circ d_X$.

Since $g \circ f = 0$, the null homotopy $g \circ f \simeq 0$ induces a unique morphism of complexes $h\colon C_f \to Z$ such that $h \circ \alpha_f = g$ and $h \circ \theta_f = 0$. Explicitly, $\alpha_f = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\theta_f = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$, so that $h = \left(\begin{smallmatrix} g & 0 \end{smallmatrix}\right)$.

Lemma **(1.6.6)**. — a) *The morphisms f, α_f, β_f induce a long exact sequence of objects in $\boldsymbol{A}$:*

$$\xrightarrow{H^{n-1}(\beta_f)} H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(\alpha_f)} H^n(C_f) \xrightarrow{H^n(\beta_f)} H^{n+1}(X) \xrightarrow{H^{n+1}(f)} \dots$$

b) *The morphism $h\colon C_f \to Z$ is a homologism.*

c) *The morphism f is a homologism if and only if C_f is acyclic, if and only if Z is acyclic.*

Proof. — For this proof, we shall pretend that the abelian category $\boldsymbol{A}$ is a category of modules.

a) Since $\alpha_f \circ f$ is null homotopic, one has $H^n(\alpha_f) \circ H^n(f) = 0$. Since $\beta_f \circ \alpha_f = 0$, one has $H^n(\beta_f) \circ H^n(\alpha_f) = 0$. Since $\Sigma f \circ \beta_f$ is null homotopic, one has $H^{n+1}(f) \circ H^n(\beta_f) = 0$.

Let $x \in X^n$ be such that $d_X(x) = 0$ and such that $H^n(f)([x]) = 0$ in $H^n(Y)$. Let $y \in Y^{n-1}$ be such that $f(x) = d_Y(y)$; let $z = \binom{-y}{x}$; one has $d_{C_f}(z) = 0$ and $\beta_f(z) = x$. This proves exactness at $H^n(X)$.

Let $y \in Y^n$ be such that $d_Y(y) = 0$ and such that $H^n(\alpha_f)([y]) = 0$; let $z \in C_f^{n-1}$ be such that $\binom{y}{0} = d_{C_f}(z)$; write $z = \binom{y'}{x'}$. One has $d(y') + f(x') = y$ and $d(x') = 0$, hence $[y] = [f(x')] = H^n(f)([x'])$. This proves exactness at $H^n(Y)$.

Let $z \in H^n(C_f)$ be such that $d_{C_f}(z) = 0$ and $H^n(\beta_f)([z]) = 0$ in $H^{n+1}(X)$; write $z = \binom{y}{x}$. One has $d(y) + f(x) = 0$ and $d(x) = 0$; moreover, there exists $x' \in X^n$ such that $x = d(x')$. Consequently, $d(y + f(x')) = 0$ and

$$z = \binom{y+f(x')}{0} - \binom{f(x')}{-x} = \binom{y+f(x')}{0} - d_{C_f}\left(\binom{0}{x'}\right)$$

so that $[z] = H^n(\alpha_f)([y + f(x')])$. This proves exactness at $H^n(C_f)$.

b) Let $n \in \mathbf{Z}$. Let $z \in C_f^n$ be such that $d(z) = 0$ and $H^n(h)([z]) = 0$. Let $z' \in Z^{n-1}$ be such that $h(z) = d_Z(z')$; write $z = \binom{y}{x}$. Then $d(y) + f(x) = 0$, $d(x) = 0$ and $d(z') = g(y)$. Since $g^{n-1} \colon Y^{n-1} \to Z^{n-1}$ is surjective, there exists $y' \in Y^{n-1}$ such that $z' = g(y')$; then $d(z') = d_Z(g(y')) = g(d_Y(y'))$, so that $g(y - d_Y(y')) = 0$; consequently, there exists $x \in X^n$ such that $y - d_Y(y') = f(x)$. We thus have $z = \binom{d(y')+f(x)}{x} = d\left(\binom{y'}{x}\right)$, so that $[z] = 0$. This proves that $H^n(h)$ is injective.

Let now $z \in Z^n$ be such that $d_Z^n(z) = 0$. Since g^n is surjective, there exists $y \in Y^n$ such that $z = g^n(y)$, hence $0 = d_Z^n(g^n(y)) = g^n(d_Y^n(y))$. Consequently, there exists $x \in X^{n-1}$ such that $d_Y^n(y) = f^{n-1}(x)$. One has $f^n(d_X^{n-1}(x)) = d_Y^n(f^{n-1}(x)) = 0$, hence $d_X(x) = 0$ since f^n is injective. Let $z' = \binom{y}{-x} \in C_f^{n-1}$. One has $d(z') = 0$ and $h(z') = g(y) = z$, so that $H^n(h)([z']) = [z]$. This proves that $H^n(h)$ is surjective.

c) If $H^n(C_f) = 0$, the exact sequence of *a*) shows that $H^n(f)$ is an epimorphism and $H^{n+1}(f)$ is a monomorphism. Consequently, if C_f is acyclic, then $H^n(f)$ is an isomorphism for every n, so that f is a homologism.

Conversely, if $H^n(f)$ is an epimorphism, then $H^n(\alpha_f) = 0$, while if $H^n(f)$ is a monomorphism, then $H^{n-1}(\beta_f) = 0$. In particular, if f is a homologism, then $H^n(\alpha_f) = 0$ and $H^n(\beta_f) = 0$ for every n. Then, $0 = \operatorname{Im}(H^n(\alpha_f)) = \operatorname{Ker}(H^n(\beta_f)) = H^n(C_f)$ for every n, so that C_f is acyclic. □

1.7. Exercises

Exercise **(1.7.1)**. — Let $\mathbf{C}$ be the category of sets.

a) Prove monomorphisms, epimorphisms, and isomorphisms coincide respectively with injective, surjective, and bijective maps.

b) Show that the empty set is the initial object, while singletons are terminal objects.

c) Observe that every morphism to the initial object is an isomorphism, while not every morphism from the terminal object is an isomorphism. Conclude that the category of sets is not equivalent to its opposite category.

d) Compute equalizers, coequalizers, products and coproducts in $\mathbf{C}$. More generally, prove that "all" limits (resp. colimits) exist in $\mathbf{C}$.

Exercise **(1.7.2)**. — In the category of rings, let $f\colon \mathbf{Z} \to \mathbf{Q}$ be the inclusion morphism. Show that f is both a monomorphism and an epimorphism but is not an isomorphism.

Exercise **(1.7.3)**. — *a*) Let $\mathbf{C}$ be a category, let I be a set and, for every $i \in \mathrm{I}$, let $f_i \in \mathbf{C}(x_i, y_i)$ be a morphism in $\mathbf{C}$. Assume that the products $x = \prod_{i\in\mathrm{I}} x_i$ and $y = \prod_{i\in\mathrm{I}} y_i$ exist in $\mathbf{C}$ and let $f \in \mathbf{C}(x, y)$ be the corresponding morphism. If f_i is a monomorphism, for every $i \in \mathrm{I}$, then f is a monomorphism.

b) State the analogous property for coproducts of epimorphisms.

c) Assume that $\mathbf{C}$ is an additive category. Prove that products of finite families of epimorphisms are epimorphisms, and that coproducts of finite families of monomorphisms are monomorphisms.

Exercise **(1.7.4)**. — Let $\mathbf{C}$ be a category.

a) One says that a morphism $u \in \mathbf{C}(x, y)$ is an extremal epimorphism if, for every factorization $u = m \circ v$, where m is a monomorphism, then m is an isomorphism. Prove that an extremal epimorphism is an epimorphism.

b) A morphism which is both an extremal epimorphism, and a monomorphism is an isomorphism.

c) One says that a morphism $u \in \mathbf{C}(x, y)$ is a regular epimorphism if there exists an object t and two morphisms $f, g \in \mathbf{C}(t, x)$ of which u is a coequalizer (that is, a colimit of the diagram $t \overset{f}{\underset{g}{\rightrightarrows}} x$). Prove that a regular epimorphism is an extremal epimorphism.

Exercise **(1.7.5)**. — Let $\boldsymbol{A}$ be an abelian category satisfying both axioms (AB_5) and (AB_5^*).

a) Let X be and let I be a set. Let $(\mathrm{X}_i)_{i\in\mathrm{I}}$ be the family in $\boldsymbol{A}$ where $\mathrm{X}_i = \mathrm{X}$ for every $i \in \mathrm{I}$. Show that the canonical morphism $\mathrm{X}^{(\mathrm{I})} \to \mathrm{X}^{\mathrm{I}}$ from the coproduct of the family $(\mathrm{X}_i)_{i\in\mathrm{I}}$ to its product is an isomorphism.

b) Prove that X is a zero object in $\boldsymbol{A}$.

CHAPTER 2

TRIANGULATED CATEGORIES

2.1. Triangulated categories

Let $\mathcal{C}$ be an additive category endowed with an automorphism Σ of $\mathcal{C}$ (*translation*).

Definition (**2.1.1**). — *A* triangle *in* $\mathcal{C}$ *is a complex* T *such that* $d_T^{n+3} = \Sigma d_T^n$ *for every* $n \in \mathbf{Z}$. *A* morphism of triangles $f : T \to T'$ *is a morphism of complexes such that* $f^{n+3} = \Sigma f^n$ *for every* $n \in \mathbf{Z}$.

Concretely, a triangle only depends on three consecutive objects and morphisms, and is represented as follows:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X,$$

the next differential being $\Sigma u : \Sigma X \to \Sigma Y$, etc. Conversely, a sequence (u, v, w) of three composable morphisms gives rise to a triangle if and only if $v \circ u = 0$, $w \circ v = 0$ and $\Sigma u \circ w = 0$. The datum of a morphism from such a triangle to a similar triangle

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X'$$

is equivalent to the datum of three morphisms $f : X \to X'$, $g : Y \to Y'$ and $h : Z \to Z'$ such that $u' \circ f = g \circ u$, $v' \circ g = h \circ v$ and $w' \circ h = \Sigma f \circ w$.

Morphisms of triangles are composed in the obvious way, and triangles in $\mathcal{C}$ form a category.

Triangles are complexes in $\mathcal{C}$, hence can be shifted; observe that a shift of a triangle is a triangle. However, to avoid confusion with the automorphism Σ of $\mathcal{C}$, we shall not use the letter Σ to indicate shifts of triangles.

2.1.2. — Since triangles in $\boldsymbol{C}$ are complexes, a morphism of triangles $\varphi : \mathrm{T} \to \mathrm{T}'$ gives rise to a cone C_φ. Let us explicit the description of this cone. Let us thus consider a morphism of triangles, as represented by the diagram:

$$\begin{array}{ccccccc} \mathrm{X} & \xrightarrow{u} & \mathrm{Y} & \xrightarrow{v} & \mathrm{Z} & \xrightarrow{w} & \Sigma\mathrm{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ \mathrm{X}' & \xrightarrow{u'} & \mathrm{Y}' & \xrightarrow{v'} & \mathrm{Z}' & \xrightarrow{w'} & \Sigma\mathrm{X}. \end{array}$$

By definition, its cone is the complex

$$\mathrm{X}' \oplus \mathrm{Y} \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} \mathrm{Y}' \oplus \mathrm{Z} \xrightarrow{\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} \mathrm{Z}' \oplus \Sigma\mathrm{X} \xrightarrow{\begin{pmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{pmatrix}} \Sigma\mathrm{X}' \oplus \Sigma\mathrm{Y}.$$

There is also a natural notion of homotopy between morphisms of triangles. If two morphisms of triangles $\mathrm{F} = (f, g, h)$ and $\mathrm{F}' = (f', g', h')$ are homotopic, the choice of a homotopy $\eta = (\theta, \varphi, \psi)$ with origin (f, g, h) and target (f', g', h') gives rise to a morphism of triangles $\lambda : \mathrm{C}_\mathrm{F} \to \mathrm{C}_{\mathrm{F}'}$, explicitly given by the diagram

$$\begin{array}{ccccccccc} \mathrm{C}_\mathrm{F} & = & \mathrm{X}' \oplus \mathrm{Y} & \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} & \mathrm{Y}' \oplus \mathrm{Z} & \xrightarrow{\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} & \mathrm{Z}' \oplus \Sigma\mathrm{X} & \xrightarrow{\begin{pmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{pmatrix}} & \Sigma\mathrm{X}' \oplus \Sigma\mathrm{Y} \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & \Sigma\theta \\ 0 & 1 \end{pmatrix}} \\ \mathrm{C}_{\mathrm{F}'} & = & \mathrm{X}' \oplus \mathrm{Y} & \xrightarrow[\begin{pmatrix} u' & g' \\ 0 & -v \end{pmatrix}]{} & \mathrm{Y}' \oplus \mathrm{Z} & \xrightarrow[\begin{pmatrix} v' & h' \\ 0 & -w \end{pmatrix}]{} & \mathrm{Z}' \oplus \Sigma\mathrm{X} & \xrightarrow[\begin{pmatrix} w' & \Sigma f' \\ 0 & -\Sigma u \end{pmatrix}]{} & \Sigma\mathrm{X}' \oplus \Sigma\mathrm{Y} \end{array}$$

This morphism λ is an isomorphism.

Finally, one says that a triangle T is *contractible* if id_T is null homotopic.

Definition **(2.1.3)**. — *A* triangulated category *is an additive category* $\boldsymbol{C}$ *endowed with an automorphism* Σ *and a set* $\mathscr{T}$ *of triangles such that the following properties hold:*

(2.1.3.1) *A triangle isomorphic to a triangle in* $\mathscr{T}$ *belongs to* $\mathscr{T}$*;*

(2.1.3.2) *A triangle* $\mathrm{X} \xrightarrow{u} \mathrm{Y} \xrightarrow{v} \mathrm{Z} \xrightarrow{w} \Sigma\mathrm{X}$ *in* $\boldsymbol{C}$ *belongs to* $\mathscr{T}$ *if and only if the triangle* $\mathrm{Y} \xrightarrow{-v} \mathrm{Z} \xrightarrow{-w} \Sigma\mathrm{X} \xrightarrow{-\Sigma u} \Sigma\mathrm{Y}$ *belongs to* $\mathscr{T}$*;*

(2.1.3.3) *For every object, the triangle* $\mathrm{X} \xrightarrow{\mathrm{id}_\mathrm{X}} \mathrm{X} \to 0 \to \Sigma\mathrm{X}$ *belongs to* $\mathscr{T}$*;*

(2.1.3.4) *For every morphism* $u : \mathrm{X} \to \mathrm{Y}$ *in* $\boldsymbol{C}$*, there exists a triangle* $\mathrm{X} \xrightarrow{u} \mathrm{Y} \to \mathrm{Z} \to \Sigma\mathrm{X}$ *in* $\mathscr{T}$*;*

(2.1.3.5) *Let* $\mathrm{X} \xrightarrow{u} \mathrm{Y} \xrightarrow{v} \mathrm{Z} \xrightarrow{w} \Sigma\mathrm{X}$ *and* $\mathrm{X}' \xrightarrow{u'} \mathrm{Y}' \xrightarrow{v'} \mathrm{Z} \xrightarrow{w'} \Sigma\mathrm{X}'$ *be triangles of* $\mathscr{T}$*. For every* $f \in \boldsymbol{C}(\mathrm{X}, \mathrm{X}')$ *and every* $g \in \boldsymbol{C}(\mathrm{Y}, \mathrm{Y}')$ *such that* $u' \circ f = g \circ u$*,* *there exists a morphism $h \in \boldsymbol{C}(\mathrm{Z}, \mathrm{Z}')$ such that (f, g, h) induces a morphism of triangles:*

$$\begin{array}{ccccccc} \mathrm{X} & \xrightarrow{u} & \mathrm{Y} & \xrightarrow{v} & \mathrm{Z} & \xrightarrow{w} & \Sigma\mathrm{X} \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ \mathrm{X}' & \xrightarrow{u'} & \mathrm{Y}' & \xrightarrow{v'} & \mathrm{Z}' & \xrightarrow{w'} & \Sigma\mathrm{X} \end{array}$$

whose cone belongs to $\mathscr{T}$.

Let us say that a triangle in $\boldsymbol{C}$ is a *distinguished triangle* if it belongs to $\mathscr{T}$, and that a morphism of triangles is a *distinguished morphism* if its cone is a distinguished triangle.

The axioms a triangulated category thus claim that triangles isomorphic to distinguished triangles are distinguished (2.1.3.1), as well as their shifts (2.1.3.2); they assert the existence of distinguished triangles (2.1.3.3), (2.1.3.4); they finally allow to construct distinguished morphisms between distinguished triangles with two prescribed arrows (2.1.3.5).

Given two morphisms of triangles which are homotopic, if one of them is distinguished, then so is the other.

If one relaxes axiom (2.1.3.5) by only requiring the existence of a morphism h, one gets the weaker notion of a *pretriangulated category.*

2.1.4. — Let $(\boldsymbol{C}, \Sigma, \mathscr{T})$ be a (pre)triangulated category. Let us endow the opposite category $\boldsymbol{C}^{\mathrm{o}}$ with the translation functor Σ^{-1}. Observe that a triangle in $\boldsymbol{C}$, when viewed as a complex in $\boldsymbol{C}^{\mathrm{o}}$, is again a triangle, so that $\mathscr{T}^{\mathrm{o}} = \mathscr{T}$ is a set of triangles in $\boldsymbol{C}$.

Let us prove that $(\boldsymbol{C}^{\mathrm{o}}, \Sigma^{-1}, \mathscr{T}^{\mathrm{o}})$ is a (pre)triangulated category. Axioms (2.1.3.1) and (2.1.3.2) follow formally from their analogue in $\boldsymbol{C}$.

Let X be an object in $\boldsymbol{C}$. Shifting the distinguished triangle $\Sigma^{-1}\mathrm{X} \xrightarrow{\mathrm{id}_{\Sigma^{-1}\mathrm{X}}} \Sigma^{-1}\mathrm{X} \to 0 \to \mathrm{X}$ in $\boldsymbol{C}$, we obtain the distinguished triangle $\Sigma^{-1}\mathrm{X} \to 0 \to \mathrm{X} \xrightarrow{-\mathrm{id}_{\mathrm{X}}} \mathrm{X}$. In the opposite category, this triangle rewrites as $\mathrm{X} \xrightarrow{-\mathrm{id}_{\mathrm{X}}} \mathrm{X} \to 0 \to \Sigma^{-1}\mathrm{X}$. Since it is isomorphic to the triangle $\mathrm{X} \xrightarrow{\mathrm{id}_{\mathrm{X}}} \mathrm{X} \to 0 \to \Sigma^{-1}\mathrm{X}$, axiom (2.1.3.3) holds in $\boldsymbol{C}^{\mathrm{o}}$.

A similar argument shows that axioms (2.1.3.4) and (2.1.3.5) hold as well.

Example **(2.1.5)**. — Let $\boldsymbol{C}$ be an additive category and let $\boldsymbol{K}(\boldsymbol{C})$ be the homotopy category of complexes in $\boldsymbol{C}$. We have seen how the cone of a morphism of complexes $f\colon X \to Y$ sits in a diagram

$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} C_f \xrightarrow{\beta_f} \Sigma X,$$

where $\alpha_f \circ f$ is homotopic to 0, $\beta_f \circ \alpha_f = 0$ and $\Sigma f \circ \beta_f$ is homotopic to 0. Consequently, these diagrams give rise to triangles in the homotopy category. We shall prove below (theorem 2.4.3) that the category $\boldsymbol{K}(\boldsymbol{C})$, endowed with the triangles isomorphic to such a cone triangle, is a triangulated category.

Definition (**2.1.6**). — *Let $\boldsymbol{C}$ be a (pre)triangulated category. An additive functor* $H\colon \boldsymbol{C} \to \boldsymbol{A}$ *to an abelian category is said to be* cohomological *if the complex* $H(T)$ *in $\boldsymbol{A}$ is exact for every distinguished triangle* T.

Let $H\colon \boldsymbol{C} \to \boldsymbol{A}$ be a cohomological functor. For every integer $m \in \mathbf{Z}$, set $H^m = H \circ \Sigma^m$. By definition, every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

gives rise to a long exact sequence

$$\cdots \to H^{-1}(Z) \xrightarrow{H^{-1}(w)} H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \xrightarrow{H^0(w)} H^1(X) \xrightarrow{H^1(u)} \dots$$

On the other hand, a shifting argument shows that to verify that an additive functor is cohomological, it suffices to prove that for every distinguished triangle as above, the complex $H(X) \to H(Y) \to H(Z)$ is exact at $H(Y)$.

Lemma (**2.1.7**). — *Let $\boldsymbol{C}$ be a (pre)triangulated category and let* A *be an object of $\boldsymbol{C}$. The functor $\boldsymbol{C}(A, \cdot)\colon \boldsymbol{C} \to \boldsymbol{Ab}$ is a cohomological functor.*

Proof. — Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle and let us show that the complex

$$\boldsymbol{C}(A, X) \xrightarrow{u_*} \boldsymbol{C}(A, Y) \xrightarrow{v_*} \boldsymbol{C}(A, Z)$$

is exact in the middle, where u_* and v_* are the group morphisms deduced from composition with u and v respectively.

First of all, we recall that $v_* \circ u_* = 0$; indeed $v_* \circ u_*$ maps every $f \in \boldsymbol{C}(A, X)$ to $v \circ u \circ f = 0$.

Let then $f \in \boldsymbol{C}(A, Y)$ be such that $v_*(f) = v \circ f = 0$. By axioms (2.1.3.3) and (2.1.3.2), the triangle $A \to 0 \to \Sigma A \xrightarrow{-\mathrm{id}_{\Sigma A}} \Sigma A$ is distinguished; by axiom (2.1.3.2), the triangle $Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is distinguished. By axiom (2.1.3.5), there exists a morphism $h_1 \in \boldsymbol{C}(\Sigma U, \Sigma X)$ making the following diagram

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & \Sigma A & \xrightarrow{-\mathrm{id}_{\Sigma A}} & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \Sigma f} \\ Y & \xrightarrow{-v} & Z & \xrightarrow{-w} & \Sigma X & \xrightarrow{-\Sigma u} & \Sigma Y \end{array}$$

commutative. In particular, $\Sigma u \circ h_1 = \Sigma f$, so that $f = u \circ \Sigma^{-1} h_1$ belongs to $\mathrm{Im}(u_*)$. □

2.2. Decent cohomological functors, and applications

Definition (**2.2.1**). — *Let $\boldsymbol{A}$ be an abelian category satsfying the axiom* (AB_4^*)*: products exist, and a product of exact sequences is exact. A cohomological functor* $H : \boldsymbol{C} \to \boldsymbol{A}$ *is said to be* decent *if it respects products.*

For example, for every object $A \in \mathrm{ob}(\boldsymbol{C})$, the functor $\boldsymbol{C}(A, \cdot)$, from $\boldsymbol{C}$ to the category $\boldsymbol{Ab}$ of abelian groups, is a decent cohomological functor.

Definition (**2.2.2**). — *A triangle* T *in* $\boldsymbol{C}$ *is said to be* decent *if the complex* H(T) *is exact for every decent cohomological functor* H.

In particular, a distinguished triangle is decent.

In this definition, we should take care about universes. In practice, we will only use decent cohomological functors of the form $\boldsymbol{C}(A, \cdot)$, where A is an object of the (pre)triangulated category $\boldsymbol{C}$. Consequently, if U is a universe containing $\boldsymbol{C}(X, Y)$ for every $X, Y \in \mathrm{ob}(\boldsymbol{C})$, it suffices to consider cohomological functors with values in $\boldsymbol{Ab}_U$. (*Check!*)

Proposition (**2.2.3**). — *Let $\boldsymbol{C}$ be a (pre)triangulated category. Let us consider a morphism*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X. \end{array}$$

of decent triangles.

a) *The cone triangle of this morphism is a decent triangle.*

b) *If f and g are isomorphisms, then h is an isomorphism, so that this morphism of triangles is an isomorphism.*

Proof. — a) By definition, the triangle cone of this morphism is the triangle

$$X' \oplus Y \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{pmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{pmatrix}} \Sigma X' \oplus \Sigma Y.$$

Let us prove that it is decent. Let $H \colon \boldsymbol{C} \to \boldsymbol{A}$ be a decent cohomological functor and let us show that the induced diagram

$$H(X' \oplus Y) \xrightarrow{H\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} H(Y' \oplus Z) \xrightarrow{H\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} H(Z' \oplus \Sigma X) \xrightarrow{H\begin{pmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{pmatrix}} H(\Sigma X' \oplus \Sigma Y).$$

is an exact sequence in the abelian category $\boldsymbol{A}$. Since H respects products, we have $H(X' \oplus Y) = H(X') \oplus H(Y)$, etc., so that this diagram is the cone of the morphism of complexes in $\boldsymbol{A}$ obtained by applying the functor H to the initial morphism of triangles. By assumption, these complexes are acyclic, so that the resulting complex is acyclic as well (lemma 1.6.6; a morphism between acyclic complexes must be a homologism!). This proves that the given morphism of triangles is decent, as was to be shown.

b) Let A be an object of $\boldsymbol{C}$ and let us apply the functor $\boldsymbol{C}(A, \cdot)$; we obtain the diagram of abelian groups

$$\begin{array}{ccccccccc}
\boldsymbol{C}(A,X) & \xrightarrow{u} & \boldsymbol{C}(A,Y) & \xrightarrow{v} & \boldsymbol{C}(A,Z) & \xrightarrow{w} & \boldsymbol{C}(A,\Sigma X) & \xrightarrow{\Sigma u} & \boldsymbol{C}(A,\Sigma Y) \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f & & \downarrow \Sigma g \\
\boldsymbol{C}(A,X') & \xrightarrow[u']{} & \boldsymbol{C}(A,Y') & \xrightarrow[v']{} & \boldsymbol{C}(A,Z') & \xrightarrow[w']{} & \boldsymbol{C}(A,\Sigma X') & \xrightarrow[\Sigma u']{} & \boldsymbol{C}(A,\Sigma Y')
\end{array}$$

whose two rows are exact sequences. The two left vertical morphisms are induced by f and g, hence are isomorphisms, as well as the two right vertical morphisms, which are induced by Σf and Σg. By the *five lemma*, the morphism $h \colon Z \to Z'$ induces an isomorphism $\boldsymbol{C}(A, Z) \to \boldsymbol{C}(A, Z')$. Since this holds for every object A, it then follows from the Yoneda lemma that h is an isomorphism. □

Corollary **(2.2.4)**. — *Let $\boldsymbol{C}$ be a (pre)triangulated category and let $u \colon X \to Y$ be a morphism in $\boldsymbol{C}$. Let $T \colon X \xrightarrow{u} Y \to Z \to \Sigma X$ and $T' \colon X \xrightarrow{u} Y \to Z' \to \Sigma X$ be distinguished triangles. There exists a morphism of triangles of the form $(\mathrm{id}_X, \mathrm{id}_Y, h)$; moreover, for every such morphism, the morphism $h \colon Z \to Z'$ is an isomorphism.*

This corollary furnishes a kind of "uniqueness" to the axiom (2.1.3.4), in the sense that any two triangles extending a given morphism are isomorphic. However, there is no canonical such isomorphism. This is in fact one of the defects of the theory of triangulated categories that it does not functorially complete morphisms into triangles.

Proof. — The existence of such a morphism of triangles follows from axiom (2.1.3.5). Since a distinguished triangle is decent, proposition 2.2.3 implies that h is an isomorphism. □

Corollary (**2.2.5**). — *Let $\boldsymbol{C}$ be a (pre)triangulated category. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism in $\boldsymbol{C}$. The triangle $\mathrm{X} \xrightarrow{f} \mathrm{Y} \to 0 \to \Sigma\mathrm{X}$ is distinguished if and only if f is an isomorphism.*

Proof. — We consider the morphism of triangles:

$$\begin{array}{ccccccc}
\mathrm{X} & \xrightarrow{\mathrm{id}_{\mathrm{X}}} & \mathrm{X} & \longrightarrow & 0 & \longrightarrow & \Sigma\mathrm{X} \\
\| & & \downarrow{\scriptstyle f} & & \| & & \| \\
\mathrm{X} & \xrightarrow{f} & \mathrm{Y} & \longrightarrow & 0 & \longrightarrow & \Sigma\mathrm{X},
\end{array}$$

in which the top triangle is distinguished by construction, and in which two out of three vertical morphisms are isomorphisms. If f is an isomorphism, then this morphism of triangles is an isomorphism, so that the bottom triangle is distinguished as well. Conversely, if both triangles are distinguished, then proposition 2.2.3 (after a shift of the diagram) implies that f is an isomorphism. □

Corollary (**2.2.6**). — *Let us consider the following diagram of distinguished triangles (in which the dashed arrows are not supposed to exist):*

$$\begin{array}{ccccccc}
\mathrm{X} & \xrightarrow{u} & \mathrm{Y} & \xrightarrow{v} & \mathrm{Z} & \xrightarrow{w} & \Sigma\mathrm{X} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
\mathrm{X}' & \xrightarrow[u']{} & \mathrm{Y}' & \xrightarrow[v']{} & \mathrm{Z}' & \xrightarrow[w']{} & \Sigma\mathrm{X}.
\end{array}$$

a) *The following conditions are equivalent: 1) One has $v' \circ g \circ u = 0$; 2) There exists $f : \mathrm{X} \to \mathrm{X}'$ such that $u' \circ f = g \circ u$; 3) There exists $h : \mathrm{Z} \to \mathrm{Z}'$ such that $h \circ v = v' \circ g$; 4) There exists a morphism of triangles (f, g, h).*

b) *If they hold, and if* $\mathcal{C}(X, \Sigma^{-1}Z') = 0$, *then such morphisms* f *and* h *are uniquely determined.*

Proof. — Let us prove that 1)⇒2). Applying the decent functor $\mathcal{C}(X, \cdot)$ to the bottom triangle, one obtains an exact sequence

$$\mathcal{C}(X, \Sigma^{-1}Z') \to \mathcal{C}(X, X') \xrightarrow{u'} \mathcal{C}(X, Y') \xrightarrow{v'} \mathcal{C}(X, Z').$$

Since $v' \circ (g \circ u) = 0$, there exists a morphism $f \in \mathcal{C}(X, X')$ such that $g \circ u = u' \circ f$. If, moreover, $\mathcal{C}(X, \Sigma^{-1}Z') = 0$, there exists exactly one such morphism f.

Conversely, the existence of a morphism $f : X \to X'$ such that $u' \circ f = g \circ u$ implies that $v' \circ g \circ u = v' \circ u' \circ f = 0$. This shows that 1) and 2) are equivalent.

The proof of the equivalence 1)⇔3) follows by passing to the opposite (pre)triangulated category.

The implications 4)⇒2) and 4)⇒3) are obvious.

Finally, if 2) holds, the existence of a morphism of triangles as in 4) follows from axiom (2.1.3.5), so that 2)⇒4). The proof of the implication 3)⇒4) is analogous.

When these conditions hold and $\mathcal{C}(X, \Sigma^{-1}Z') = 0$, the uniqueness of the morphisms f and h has been established during the proof of their equivalence. □

Corollary (**2.2.7**). — *Let* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ *be a distinguished triangle. Assume that* $\mathcal{C}(X, \Sigma^{-1}Z) = 0$. *Then:*

a) *The morphism* w *is the unique morphism* ∂ *such that the triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{\partial} \Sigma X$ *is distinguished;*

b) *For every distinguished triangle of the form* $X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$, *there exists a unique morphism of triangles of the form* $(\mathrm{id}_X, \mathrm{id}_Y, h)$, *and it is an isomorphism.*

Proposition (**2.2.8**). — *Let* $\mathcal{C}$ *be a (pre)triangulated category.*

a) *A product (resp. a coproduct) of a family of distinguished triangles is a distinguished triangle.*

b) *A triangle which is a direct factor of a distinguished triangle is a distinguished triangle.*

Proof. — a) Let $(X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} \Sigma X_i)_{i \in I}$ be a family of distinguished triangles. Assume that the products $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$ and $Z = \prod_{i \in I} Z_i$ exist. Since Σ is an automorphism of the category $\mathcal{C}$, it commutes with products, and ΣX is a product of the family ΣX_i. Let $u\colon X \to Y$ be the unique morphism $p_i^Y \circ u = u_i \circ p_i^X$, for every i; define v and w similarly. Then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle, and the claim is that this triangle is distinguished.

We first prove that it is a decent triangle. To that aim, let A be an object of $\mathcal{C}$ and let us apply the cohomological functor $\mathcal{C}(A, \cdot)$ to this triangle. We get a complex

$$\mathcal{C}(A, \prod_i \Sigma^{-1} Z_i) \xrightarrow{(\Sigma^{-1} w_i)} \mathcal{C}(A, \prod_i X_i) \xrightarrow{(u_i)} \\ \mathcal{C}(A, \prod_i Y_i) \xrightarrow{(v_i)} \mathcal{C}(A, \prod_i Z_i) \xrightarrow{(w_i)} \mathcal{C}(A, \prod_i \Sigma X_i)$$

which, by definition of products in a category, identifies with the product of the complexes

$$\mathcal{C}(A, \Sigma^{-1} Z_i) \xrightarrow{\Sigma^{-1} w_i} \mathcal{C}(A, X_i) \xrightarrow{u_i} \mathcal{C}(A, Y_i) \xrightarrow{v_i} \mathcal{C}(A, Z_i) \xrightarrow{w_i} \mathcal{C}(A, \Sigma X_i),$$

for $i \in I$. Since each of these complexes of abelian groups is exact (lemma 2.1.7), the initial complex is exact as well.

Let us now complete the morphism $u\colon X \to Y$ into a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$. For every $i \in I$, there exists a morphism $h_i\colon Z' \to Z_i$ which gives rise to a morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\ \downarrow{\scriptstyle p_i^X} & & \downarrow{\scriptstyle p_i^Y} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle \Sigma p_i^X} \\ X_i & \xrightarrow{u_i} & Y_i & \xrightarrow{v_i} & Z_i & \xrightarrow{w_i} & \Sigma X_i. \end{array}$$

Let $h\colon Z' \to Z = \prod_{i \in I} Z_i$ be the morphism (h_i). It fits in a morphism of decent triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle h} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w_i} & \Sigma X_i. \end{array}$$

Since two out of three morphisms are isomorphisms (they are identities!), so is h. Consequently, the initial triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is distinguished, as was to be proved.

The case of coproducts follows by considering the opposite triangulated category.

b) Let $T : X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $T' : X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ be two triangles whose direct sum

$$X \oplus X' \xrightarrow{\begin{pmatrix} u & \\ & u' \end{pmatrix}} Y \oplus Y' \xrightarrow{\begin{pmatrix} v & \\ & v' \end{pmatrix}} Z \oplus Z' \xrightarrow{\begin{pmatrix} w & \\ & w' \end{pmatrix}} \Sigma X \oplus \Sigma X'$$

is a distinguished triangle. Let us prove that T is a distinguished triangle.

First of all, it is decent. Indeed, for every object $A \in \boldsymbol{C}$, applying the functor $\boldsymbol{C}(A, \cdot)$ to the triangle $T \oplus T'$ furnishes the exact sequence

$$\begin{aligned} \boldsymbol{C}(A, \Sigma^{-1}Z) \oplus \boldsymbol{C}(A, \Sigma^{-1}Z') \xrightarrow{\begin{pmatrix} \Sigma^{-1}w & \\ & \Sigma^{-1}w' \end{pmatrix}} \boldsymbol{C}(A, X) \oplus \boldsymbol{C}(A, X') \to \\ \xrightarrow{\begin{pmatrix} u & \\ & u' \end{pmatrix}} \boldsymbol{C}(A, Y) \oplus \boldsymbol{C}(A, Y') \xrightarrow{\begin{pmatrix} v & \\ & v' \end{pmatrix}} \boldsymbol{C}(A, Z) \oplus \boldsymbol{C}(A, Z') \to \\ \xrightarrow{\begin{pmatrix} w & \\ & w' \end{pmatrix}} \boldsymbol{C}(A, \Sigma X) \oplus \boldsymbol{C}(A, \Sigma X'). \end{aligned}$$

Consequently, the complex

$$\boldsymbol{C}(A, \Sigma^{-1}Z) \xrightarrow{\Sigma^{-1}w} \boldsymbol{C}(A, X) \xrightarrow{u} \boldsymbol{C}(A, Y) \xrightarrow{v} \boldsymbol{C}(A, Z) \xrightarrow{w} \boldsymbol{C}(A, \Sigma X)$$

is exact, which proves that the triangle T is decent.

Let us now complete the morphism $u : X \to Y$ into a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{\tilde{v}} \tilde{Z} \xrightarrow{\tilde{w}} \Sigma X$. Let $h : \tilde{Z} \to Z$ and $h' : \tilde{Z} \to Z'$ be morphisms that fit in a morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & \Sigma X \\ \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow{\scriptstyle\binom{h}{h'}} & & \downarrow{\scriptstyle\binom{1}{0}} \\ X \oplus X' & \xrightarrow[\begin{pmatrix} u & \\ & u' \end{pmatrix}]{} & Y \oplus Y' & \xrightarrow[\begin{pmatrix} v & \\ & v' \end{pmatrix}]{} & Z \oplus Z' & \xrightarrow[\begin{pmatrix} w & \\ & w' \end{pmatrix}]{} & \Sigma X \oplus \Sigma X'. \end{array}$$

Composing with the projections, we obtain a morphism of decent triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle h} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \end{array}$$

where two vertical arrows out of three are isomorphisms. Consequently, the remaining arrow h is an isomorphism; the triangle T is isomorphic to the top triangle, which is distinguished, hence T is a distinguished triangle. □

Example (**2.2.9**). — Let X and Y be objects of a (pre)triangulated category. The canonical morphisms $X \to X \oplus Y$ and $X \oplus Y \to Y$ fit in a triangle $X \to X \oplus Y \to Y \xrightarrow{0} \Sigma X$. This triangle is distinguished.

Indeed, it is the direct sum of the triangles $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma_X$ and $0 \to Y \xrightarrow{\mathrm{id}_Y} Y \to 0$. The first one is distinguished, by axiom (2.1.3.3). The second one is isomorphic to the triangle $0 \to Y \xrightarrow{-\mathrm{id}_Y} Y \to 0$ which is a shift of the distinguished triangle $Y \xrightarrow{\mathrm{id}_Y} Y \to 0 \to \Sigma Y$, hence is distinguished by axioms (2.1.3.1) and (2.1.3.2).

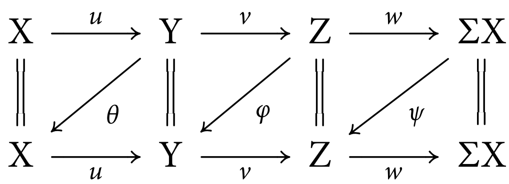
Proposition (**2.2.10**). — *Let* $\mathcal{C}$ *be a (pre)triangulated category.*

a) *A contractible triangle is distinguished.*

b) *A morphism between distinguished triangles which is null homotopic is distinguished.*

Proof. — a) Let T be a contractible triangle. Let us first prove that it is decent. Let A be an object of $\mathcal{C}$ and let us apply the functor $\mathcal{C}(A, \cdot)$. We obtain a complex $\mathcal{C}(A, T)$ of abelian groups. Let $h \colon T \to T[-1]$ be a homotopy with origin id_T and target 0; then $\mathcal{C}(A, h)$ is a homotopy with origin $\mathrm{id}_{\mathcal{C}(A,T)}$ and target 0. Consequently, the complex $\mathcal{C}(A, T)$ is exact.

Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be this triangle and let



be a homotopy with origin id_T and target 0, so that

$$\mathrm{id}_X = \theta \circ u + \Sigma^{-1}(w \circ \psi)$$

$$\mathrm{id}_Y = \varphi \circ v + u \circ \theta$$

$$\mathrm{id}_Z = \psi \circ w + v \circ \varphi.$$

Since $v \circ u = 0$, this implies

$$u = \varphi \circ v \circ u + u \circ \theta \circ u = u \circ \theta \circ u = u + u \circ \Sigma^{-1}(w \circ \psi),$$

so that $u \circ \Sigma^{-1}(w \circ \psi) = 0$, hence $\Sigma u \circ (w \circ \psi) = 0$. Let us complete $u \colon X \to Y$ into a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

and let us apply the cohomological functor $\mathcal{C}(\Sigma X, \cdot)$; one gets an exact sequence

$$\mathcal{C}(\Sigma X, Z') \xrightarrow{w'_*} \mathcal{C}(\Sigma X, \Sigma X) \xrightarrow{\Sigma u_*} \mathcal{C}(\Sigma X, \Sigma Y).$$

consequently, there exists $\psi' \in \mathcal{C}(\Sigma X, Z')$ such that $w' \circ \psi' = w \circ \psi$. Let $\lambda = \psi' \circ w + v' \circ \varphi : Z \to Z'$; one has

$$w' \circ \lambda = w' \circ \psi' \circ w + w' \circ v' \circ \varphi = w \circ \psi \circ w = w - w \circ v \circ \varphi = w$$
$$\lambda \circ v = \psi' \circ w \circ v + v' \circ \varphi \circ v = v' - v' \circ u \circ \theta = v'.$$

Consequently, the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\| & & \| & & \downarrow{\scriptstyle\lambda} & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X
\end{array}$$

depicts a morphism of triangles. Since these triangles are decent and two out of three vertical morphisms are isomorphisms, it is an isomorphism of triangles. In particular, the initial triangle is distinguished.

b) Since homotopical morphisms of triangles give rise to isomorphic cones (§2.1.2), it suffices to show that the null morphism between two distinguished triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

is a distinguished morphism. By definition, the cone of this morphism of triangles is the triangle

$$X' \oplus Y \xrightarrow{\begin{pmatrix} u' & \\ & -v \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} v' & \\ & -w \end{pmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{pmatrix} w' & \\ & -\Sigma u \end{pmatrix}} \Sigma X' \oplus \Sigma Y'.$$

It is isomorphic to the direct sum of the two triangles

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X' \quad \text{and} \quad Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y,$$

the first one is distinguished by hypothesis, and the second one by axiom (2.1.3.2). Thus, proposition 2.2.8 implies that this cone triangle is distinguished, as was to be shown. □

2.3. The octahedral axiom

Let $(\mathbf{C}, \Sigma, \mathscr{T})$ be a triangulated category.

Definition (**2.3.1**). — *One says that a commutative square*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

is homotopically cartesian *if there exists a morphism* $h\colon Z' \to \Sigma Y$ *in* $\mathbf{C}$ *so that the diagram*

$$Y \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} Y' \oplus Z \xrightarrow{(f' \; g')} Z' \xrightarrow{h} \Sigma Y$$

is a distinguished triangle.

Observe that the composition of the first two arrows vanished, since

$$(f' \; g') \circ \begin{pmatrix} g \\ -f \end{pmatrix} = g' \circ f - f' \circ g = 0.$$

We leave to the reader to check that if the above square is homotopically cartesian, witnessed by a morphism $h\colon Z' \to \Sigma Y$, then the morphism $-h$ shows that the square

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xrightarrow{g'} & Z' \end{array}$$

is homotopically cartesian as well.

Lemma (**2.3.2**). — a) *Let*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

be a homotopically cartesian commutative square and let

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \gamma'} \\ Y' & \xrightarrow{\varphi'} & P \end{array}$$

be a commutative square in $\boldsymbol{C}$*. There exists a morphism* $h\colon \mathrm{Z}' \to \mathrm{P}$ *such that* $h\circ g' = \gamma'$ *and* $h \circ f' = \varphi'$*.*

b) *Let* $\mathrm{Y}, \mathrm{Y}', \mathrm{Z}$ *be objects of* $\boldsymbol{C}$ *and let* $f\colon \mathrm{Y}\to\mathrm{Z}$ *and* $g\colon \mathrm{Y}\to\mathrm{Y}'$ *be morphisms. There exist an object* Z' *and morphisms* $g'\colon \mathrm{Z}\to\mathrm{Z}'$ *and* $f'\colon\mathrm{Y}'\to\mathrm{Z}'$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f} & \mathrm{Z} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathrm{Y}' & \xrightarrow{f'} & \mathrm{Z}' \end{array}$$

is a homotopically cartesian square.

c) *Moreover, if* Z'' *is an object of* $\boldsymbol{C}$ *and* $g''\colon \mathrm{Z}\to\mathrm{Z}''$, $f''\colon \mathrm{Y}''\to\mathrm{Z}'$ *are morphisms in* $\boldsymbol{C}$ *such that*

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f} & \mathrm{Z} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g''} \\ \mathrm{Y}' & \xrightarrow{f''} & \mathrm{Z}'' \end{array}$$

is homotopically cartesian, there exists an isomorphism $h\colon\mathrm{Z}'\to\mathrm{Z}''$ *such that* $h\circ f' = f''$ *and* $h\circ g' = g''$*.*

Proof. — a) The functor $\operatorname{Hom}(\cdot,\mathrm{P})$ is a cohomological functor on the opposite triangulated category. Applying it to the distinguished triangle $\mathrm{Y}\to\mathrm{Y}'\oplus\mathrm{Z}\to\mathrm{Z}'\to\Sigma\mathrm{Y}$, we obtain an exact sequence

$$\boldsymbol{C}(\mathrm{Z}',\mathrm{P}) \to \boldsymbol{C}(\mathrm{Y}'\oplus\mathrm{Z},\mathrm{P}) \to \boldsymbol{C}(\mathrm{Y},\mathrm{P}).$$

The image of the morphism $\varphi'-\gamma'$ is $\varphi'\circ g - \gamma'\circ f = 0$. Consequently, there exists a morphism $h\in\boldsymbol{C}(\mathrm{Z}',\mathrm{P})$ such that $\varphi' = h\circ f'$ and $\gamma' = h\circ g'$.

b) It suffices to complete the morphism $\begin{pmatrix} g \\ -f\end{pmatrix}\colon \mathrm{Y}\to\mathrm{Y}'\oplus\mathrm{Z}$ in a distinguished triangle.

c) This follows from the uniqueness property of corollary 2.2.4.

□

Lemma 2.3.2 provides *"homotopy push-outs"* in triangulated categories; by passing to the opposite category, one deduces the following lemma which provides *"homotopy pull-backs"*.

Lemma **(2.3.3)**. — a) *Let*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

be a homotopically cartesian commutative square and let

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & Z \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

be a commutative square in $\mathbf{C}$. *There exists a morphism* $h\colon Q \to Y$ *such that* $f \circ h = \varphi$ *and* $g \circ h = \gamma$.

b) *Let* Y', Z, Z' *be objects of* $\mathbf{C}$ *and let* $f'\colon Y' \to Z'$ *and* $g'\colon Z \to Z'$ *be morphisms. There exist an object* Y *and morphisms* $f\colon Y \to Z$ *and* $g\colon Y \to Y'$ *such that the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

is a homotopically cartesian square.

c) *Moreover, if* $\tilde{Y}$ *is an object of* $\mathbf{C}$ *and* $\tilde{g}\colon \tilde{Y} \to Y$, $\tilde{f}\colon \tilde{Y} \to Y$ *are morphisms in* $\mathbf{C}$ *such that*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & Z \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

is homotopically cartesian, there exists an isomorphism $h\colon \tilde{Y} \to Y$ *such that* $f \circ h = \tilde{f}$ *and* $g \circ h = \tilde{g}$.

Lemma **(2.3.4)**. — *Let*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \| \\ X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \end{array}$$

be a distinguished morphism of distinguished triangles. Then the triangle

$$Y \xrightarrow{\binom{g}{-v}} Y' \oplus Z \xrightarrow{(v'\ h)} Z' \xrightarrow{\Sigma u \circ w'} \Sigma Y$$

is distinguished. In particular, the square

$$\begin{array}{ccc} Y & \xrightarrow{v} & Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow{v'} & Z' \end{array}$$

is homotopically cartesian.

Proof. — By hypothesis, the cone C of this this morphism of triangle,

$$X \oplus Y \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{pmatrix} w' & 1 \\ 0 & -\Sigma u \end{pmatrix}} \Sigma X \oplus \Sigma Y,$$

is a distinguished triangle. Observe that the diagram

$$\begin{array}{ccccccc} X \oplus Y & \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} & Y' \oplus Z & \xrightarrow{\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} & Z' \oplus \Sigma X & \xrightarrow{\begin{pmatrix} w' & 1 \\ 0 & -\Sigma u \end{pmatrix}} & \Sigma X \oplus \Sigma Y' \\ {\scriptstyle \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}}\downarrow & & \| & & \downarrow{\scriptstyle \begin{pmatrix} -w' & -1 \\ 1 & 0 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ \Sigma u & 1 \end{pmatrix}} \\ X \oplus Y & \xrightarrow[\begin{pmatrix} 0 & g \\ 0 & -v \end{pmatrix}]{} & Y' \oplus Z & \xrightarrow[\begin{pmatrix} 0 & 0 \\ v' & h \end{pmatrix}]{} & \Sigma X \oplus Z' & \xrightarrow[\begin{pmatrix} -1 & \\ & \Sigma u \circ w' \end{pmatrix}]{} & \Sigma X \oplus \Sigma Y' \end{array}$$

is commutative. Indeed,

$$\begin{pmatrix} 1 & 0 \\ \Sigma u & 1 \end{pmatrix}\begin{pmatrix} w' & 1 \\ 0 & -\Sigma u \end{pmatrix} = \begin{pmatrix} w' & 1 \\ \Sigma u \circ w' & 0 \end{pmatrix} = \begin{pmatrix} -1 & \\ & \Sigma u \circ w' \end{pmatrix}\begin{pmatrix} -w' & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -w' & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ v' & h \end{pmatrix}$$

$$\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix} = \begin{pmatrix} 0 & g \\ 0 & -v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

Since the vertical morphisms are isomorphisms, it is an isomorphism of triangles from C to the bottom triangle, which is therefore distinguished. On the other hand, the bottom triangle is the direct sum of the triangle T of interest and the triangle $X \to 0 \to \Sigma X \xrightarrow{-1} \Sigma X$. Consequently, the triangle T is a direct factor of a distinguished triangle, hence is distinguished. □

Lemma **(2.3.5)**. — *Let*

$$\begin{array}{ccc} Y & \xrightarrow{v} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow[v']{} & Z' \end{array}$$

be a homotopically cartesian square, and let $\partial : Z' \to \Sigma Y$ *be such that*

$$Y \xrightarrow{\left(\begin{smallmatrix} g \\ -v \end{smallmatrix}\right)} Y' \oplus Z \xrightarrow{(v'\ h)} Z' \xrightarrow{\partial} \Sigma Y$$

is a distinguished triangle. Let

$$Y \xrightarrow{g} Y' \xrightarrow{g'} Y'' \xrightarrow{g''} \Sigma Y$$

be a distinguished triangle. There exists a distinguished triangle

$$Z \xrightarrow{h} Z' \xrightarrow{h'} Y'' \xrightarrow{h''} \Sigma Z$$

such that the diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Y' & \xrightarrow{g'} & Y'' & \xrightarrow{g''} & \Sigma Y \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} & & \| & & \downarrow{\scriptstyle \Sigma v} \\ Z & \xrightarrow[h]{} & Z' & \xrightarrow[h']{} & Y'' & \xrightarrow[h'']{} & \Sigma Z \end{array}$$

is commutative. Moreover, $\partial = g'' \circ h'$.

Proof. — Let $k : Z' \to Y''$ be a morphism such that the morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{\left(\begin{smallmatrix} g \\ -v \end{smallmatrix}\right)} & Y' \oplus Z & \xrightarrow{(v'\ h)} & Z' & \xrightarrow{\partial} & \Sigma Y \\ \| & & \downarrow{\scriptstyle (1\ 0)} & & \downarrow{\scriptstyle k} & & \| \\ Y & \xrightarrow[g]{} & Y' & \xrightarrow[g']{} & Y'' & \xrightarrow[g'']{} & \Sigma Y \end{array}$$

is a distinguished morphism (axiom (2.1.3.5)), so that its cone C is a distinguished triangle. Let us consider the diagram:

$$\begin{array}{ccccccc}
Y \oplus Y' \oplus Z & \xrightarrow{\begin{pmatrix} g & 1 & 0 \\ 0 & -\nu' & -h \end{pmatrix}} & Y' \oplus Z' & \xrightarrow{\begin{pmatrix} g' & k \\ 0 & -\partial \end{pmatrix}} & Y'' \oplus \Sigma Y & \xrightarrow{\begin{pmatrix} g'' & 1 \\ 0 & -\Sigma g \\ 0 & \Sigma\nu \end{pmatrix}} & \Sigma Y \oplus \Sigma Y' \oplus \Sigma Z \\
\downarrow{\begin{pmatrix} 1 & & \\ g & 1 & \\ -\nu & 0 & 1 \end{pmatrix}} & & \downarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \nu' & -1 \end{pmatrix}} & & \downarrow{\begin{pmatrix} -g'' & -1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}} & & \downarrow{\begin{pmatrix} 1 & & \\ \Sigma g & 1 & \\ -\Sigma\nu & 0 & 1 \end{pmatrix}} \\
Y \oplus Y' \oplus Z & \xrightarrow[\begin{pmatrix} 0 & & \\ & 1 & \\ & & h \end{pmatrix}]{} & 0 \oplus Y' \oplus Z' & \xrightarrow[\begin{pmatrix} 0 & & \\ & 0 & \\ & & k \end{pmatrix}]{} & \Sigma Y \oplus 0 \oplus Y'' & \xrightarrow[\begin{pmatrix} 1 & & \\ & 0 & \\ & & \Sigma\nu\circ g \end{pmatrix}]{} & \Sigma Y \oplus \Sigma Y' \oplus \Sigma Z.
\end{array}$$

Its first line is the triangle C; its bottom line is the direct sum of the three triangles

$$Y \to 0 \to \Sigma Y \xrightarrow{1} \Sigma Y$$

$$Y' \xrightarrow{1} Y' \to 0 \to \Sigma Y'$$

$$Z \xrightarrow{h} Z' \xrightarrow{k} Y'' \xrightarrow{\Sigma\nu\circ g''} \Sigma Z;$$

the vertical morphisms are isomorphisms. Let us check that is is commutative:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \nu' & -1 \end{pmatrix} \begin{pmatrix} g & 1 & 0 \\ 0 & -\nu' & -h \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ g & 1 & 0 \\ -\nu'\circ g & 0 & h \end{pmatrix} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & h \end{pmatrix} \begin{pmatrix} 1 & & \\ g & 1 & \\ -\nu & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -g'' & -1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \nu' & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -g' & -k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \nu' & -1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & k \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ \Sigma g & 1 & \\ -\Sigma\nu & 0 & 1 \end{pmatrix} \begin{pmatrix} g'' & 1 \\ 0 & -\Sigma g \\ 0 & \Sigma\nu \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \Sigma\nu\circ g \end{pmatrix} \begin{pmatrix} -g'' & -1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Consequently, the bottom triangle is distinguished; in particular, the triangle $Z \xrightarrow{h} Z' \xrightarrow{k} Y'' \xrightarrow{\Sigma\nu\circ g''} \Sigma Z$ is distinguished. Set $Z'' = Y''$, $h' = k$ and $h'' = \Sigma\nu \circ g''$; one has $h \circ \nu = \nu' \circ g$, by hypothesis; one has $\partial = g'' \circ k = g'' \circ h'$ and $h' \circ \nu' = k \circ \nu' = g'$, by definition of k; and one has $\Sigma\nu \circ g'' = h''$ by definition of h''. This concludes the proof of the lemma. □

Theorem **(2.3.6)** (Verdier's "octahedral axiom"). — *Let*

$$X_1 \xrightarrow{u_3} X_2 \xrightarrow{v_3} Z_3 \xrightarrow{w_3} \Sigma X_1$$
$$X_2 \xrightarrow{u_1} X_3 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} \Sigma X_2$$
$$X_1 \xrightarrow{u_2} X_3 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} \Sigma X_1,$$

be three distinguished triangles, where $u_2 = u_1 \circ u_3$. *There exist two morphisms* $m_1 : Z_3 \to Z_2$ *and* $m_3 : Z_2 \to Z_1$ *such that*

$$Z_3 \xrightarrow{m_1} Z_2 \xrightarrow{m_3} Z_1 \xrightarrow{\Sigma v_3 \circ w_1} \Sigma Z_3$$

is a distinguished triangle, fitting in a commutative diagram

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{u_3} & X_2 & \xrightarrow{v_3} & Z_3 & \xrightarrow{w_3} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle m_1} & & \| \\
X_1 & \xrightarrow{u_2} & X_3 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \\
 & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle m_3} & & \\
 & & Z_1 & = & Z_1 & & \\
 & & \downarrow{\scriptstyle w_1} & & \downarrow{\scriptstyle \Sigma v_3 \circ w_1} & & \\
\Sigma X_1 & \xrightarrow{\Sigma u_3} & \Sigma X_2 & \xrightarrow{\Sigma v_3} & \Sigma Z_3 & \xrightarrow{-\Sigma w_3} & \Sigma^2 X_1
\end{array}$$

Proof. — By axiom (2.1.3.5) of triangulated categories (definition 2.1.3), there exists a morphism $m_1 : Z_3 \to Z_2$ fitting in a distinguished morphism of distinguished triangles

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{u_3} & X_2 & \xrightarrow{v_3} & Z_3 & \xrightarrow{w_3} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle m_1} & & \| \\
X_1 & \xrightarrow{u_2} & X_3 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2.
\end{array}$$

By lemma 2.3.4, the commutative square

$$\begin{array}{ccc}
X_2 & \xrightarrow{v_3} & Z_3 \\
\downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle m_1} \\
X_3 & \xrightarrow{v_2} & Z_2
\end{array}$$

is homotopically cartesian, and the triangle

$$X_2 \xrightarrow{\binom{u_1}{-v_3}} X_3 \oplus Z_3 \xrightarrow{(v_2\ m_1)} Z_2 \xrightarrow{\Sigma u_3 \circ w_2} \Sigma X_2$$

is distinguished. Lemma 2.3.5 then furnishes a morphism $m_3 \colon Z_2 \to Z_1$ giving rise to a morphism of distinguished triangles

$$\begin{array}{ccc} X_2 & \xrightarrow{v_3} & Z_3 \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle m_1} \\ X_3 & \xrightarrow{v_2} & Z_2 \\ \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle m_3} \\ Z_1 & = & Z_1 \\ \downarrow{\scriptstyle w_1} & & \downarrow{\scriptstyle \Sigma v_3 \circ w_1} \\ X_2 & \xrightarrow{\Sigma v_3} & Z_3. \end{array}$$

This concludes the proof of the theorem. □

Remark **(2.3.7)**. — The name of this axiom comes from a particular way of representing its final diagram as an octahedron. Indeed, if one identifies the vertex X_i and its shift ΣX_i, for each i, as well as the two vertices of an identity morphisms, one gets a figure with eight triangles: four of them are the distinguished triangles, and the four other are the commutative triangles.

The reader shall find in Hartshorne (1966), Beĭlinson *et al* (1982), Neeman (1990), or in May (2001) alternative representations of the diagram, which s·he may find more appealing.

Remark **(2.3.8)**. — Verdier's definition of a triangulated category amounts to a pretriangulated category satisfying the octahedral axiom (theorem 2.3.6). Conversely, theorem 1.8 of (Neeman, 1991) proves that a triangulated category in Verdier's sense is a triangulated category according to definition 2.1.3.

To conclude this section, let us quote a strengthening of the octahedral axiom, refering to (Beĭlinson *et al*, 1982, prop. 1.1.11) for its proof.

Proposition **(2.3.9)** (“3×3 lemma”). — *Every commutative square*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[u']{} & Y' \end{array}$$

can be completed to a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f'} \\ X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' & \xrightarrow{w''} & \Sigma X'' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle h''} & (-1) & \downarrow{\scriptstyle -\Sigma f''} \\ \Sigma X & \xrightarrow{\Sigma u} & \Sigma Y & \xrightarrow{\Sigma v} & \Sigma Z & \xrightarrow{-\Sigma w} & \Sigma^2 Z \end{array}$$

where the rows and columns are distinguished triangles, with commutative squares except for the right bottom one which is (-1)*-commutative.*

2.4. The homotopy category of an additive category

2.4.1. — Let $\boldsymbol{A}$ be an additive category and let $\boldsymbol{K}(\boldsymbol{A})$ be its homotopy category; recall that the objects of $\boldsymbol{K}(\boldsymbol{A})$ are complexes in $\boldsymbol{A}$, and morphisms are homotopy classes of morphisms of complexes. It is an additive category. Let Σ be the translation functor of $\boldsymbol{K}(\boldsymbol{A})$.

For every morphism $f \colon X \to Y$ of complexes in $\boldsymbol{A}$, we have constructed in §1.5.4 its cone C_f which is a complex, together with morphisms of complexes $\alpha_f \colon Y \to C_f$, $\beta_f \colon C_f \to \Sigma X$, such that $\alpha_f \circ f$, $\beta_f \circ \alpha_f$ and $\Sigma f \circ \alpha_f$ are null homotopic. In the homotopy category $\boldsymbol{K}(\boldsymbol{A})$, this construction gives rise to a *triangle*:

$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} C_f \xrightarrow{\beta_f} \Sigma X.$$

We have also proved in §1.5.5 that two homotopic morphisms $f, g \colon X \to Y$ give rise to isomorphic triangles.

Definition **(2.4.2)**. — *Let $\mathscr{T}$ be the set of all triangles in $\boldsymbol{K}(\boldsymbol{A})$ which are isomorphic to the cone triangle of a morphism of complexes.*

Theorem **(2.4.3)**. — *Let* $\boldsymbol{A}$ *be an additive category. Together with its translation automorphism, and its set of triangles* $\mathscr{T}$*, the homotopy category* $\boldsymbol{K}(\boldsymbol{A})$ *is a triangulated category.*

Let us say that a triangle is distinguished if it belongs to $\mathscr{T}$.

Proof. — 1) By definition, a triangle which is isomorphic to a distinguished triangle is distinguished.

2) Let us consider a distinguished triangle T and let us prove that the shift of T is distinguished as well. We may assume that T is the triangle $X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X$ (where $g = \alpha_f$ and $h = \beta_f$), so that its shift is the triangle $Y \xrightarrow{-g} C_f \xrightarrow{-h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$. By definition, $C_{-g} = C_f \oplus \Sigma Y = Y \oplus \Sigma X \oplus \Sigma Y$, endowed with the differential

$$d_{C_{-g}} = \begin{pmatrix} d_{C_f} & -g \\ 0 & -d_Y \end{pmatrix} = \begin{pmatrix} d_Y & \Sigma f & 1 \\ 0 & -\Sigma d_X & 0 \\ 0 & 0 & -\Sigma d_Y \end{pmatrix}.$$

Let $u = \begin{pmatrix} 0 \\ 1 \\ -\Sigma f \end{pmatrix} : \Sigma X \to C_{-g}$; let $v = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} : C_{-g} \to \Sigma X$. Observe that u is a morphism of complexes because

$$d_{C_{-g}} \circ u = \begin{pmatrix} d_Y & \Sigma f & 1 \\ 0 & -\Sigma d_X & 0 \\ 0 & 0 & -\Sigma d_Y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -\Sigma f \end{pmatrix} = \begin{pmatrix} 0 \\ -\Sigma d_X \\ \Sigma d_Y \circ \Sigma f \end{pmatrix},$$

$$u \circ d_{\Sigma X} = -\begin{pmatrix} 0 \\ 1 \\ -\Sigma f \end{pmatrix} \Sigma d_X = \begin{pmatrix} 0 \\ -\Sigma d_X \\ \Sigma f \circ \Sigma d_X \end{pmatrix}$$

and $\Sigma f \circ \Sigma d_X = \Sigma d_Y \circ \Sigma f$, since $f : X \to Y$ is a morphism of complexes. Similarly,

$$d_{\Sigma X} \circ v = -\Sigma d_X \circ \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\Sigma d_X & 0 \end{pmatrix},$$

$$v \circ \Sigma d_{C_{-g}} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} d_Y & \Sigma f & 1 \\ 0 & -\Sigma d_X & 0 \\ 0 & 0 & -\Sigma d_Y \end{pmatrix} = \begin{pmatrix} 0 & -\Sigma d_X & 0 \end{pmatrix},$$

hence v is a morphism of complexes. One has

$$v \circ u = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -\Sigma f \end{pmatrix} = 1.$$

On the other hand,

$$u \circ v = \begin{pmatrix} 0 \\ 1 \\ -\Sigma f \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\Sigma f & 0 = \end{pmatrix} = \mathrm{id} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma f & 1 \end{pmatrix}.$$

Let $\theta : \mathrm{C}_{-g} \to \Sigma^{-1}\mathrm{C}_{-g}$ be defined by the matrix $\left(\begin{smallmatrix} 0 & & \\ & 0 & \\ 1 & 0 & 0 \end{smallmatrix}\right)$. One has

$$\begin{aligned} d_{\mathrm{C}_{-g}}\theta + \theta d_{\mathrm{C}_{-g}} &= \begin{pmatrix} d_{\mathrm{Y}} & \Sigma f & 1 \\ 0 & -\Sigma d_{\mathrm{X}} & 0 \\ 0 & 0 & -\Sigma d_{\mathrm{Y}} \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_{\mathrm{Y}} & \Sigma f & 1 \\ 0 & -\Sigma d_{\mathrm{X}} & 0 \\ 0 & 0 & -\Sigma d_{\mathrm{Y}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\Sigma d_{\mathrm{Y}} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{\mathrm{Y}} & \Sigma f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma f & 1 \end{pmatrix} \\ &= \mathrm{id} - u \circ v. \end{aligned}$$

Consequently, $u \circ v - \mathrm{id}$ is null homotopic. This proves that u induces an isomorphism from C_f to C_{-g} in the homotopy category, with inverse isomorphism v. Let us now prove that the diagram

$$\begin{array}{ccccccc} \mathrm{Y} & \xrightarrow{-g} & \mathrm{C}_f & \xrightarrow{-h} & \Sigma\mathrm{X} & \xrightarrow{-\Sigma f} & \Sigma\mathrm{Y} \\ \| & & \| & & \uparrow{\scriptstyle v} & & \| \\ \mathrm{Y} & \xrightarrow[-g]{} & \mathrm{C}_f & \xrightarrow[\alpha_{-g}]{} & \mathrm{C}_{-g} & \xrightarrow[\beta_{-g}]{} & \Sigma\mathrm{Y} \end{array}$$

is commutative. Indeed, one has

$$v \circ \alpha_{-g} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} = \beta_f.$$

We have proved that the triangle $\Sigma\mathrm{T}$ is isomorphic do the cone triangle of C_{-g}; it is thus distinguished.

One proves similarly, or by considering the opposite category of $\mathcal{C}$, that $\Sigma^{-1}\mathrm{T}$ is distinguished as well.

3) Let X be a complex and let $f\colon 0 \to X$ be the unique morphism. One has $C_f = X$, $\alpha_f = \mathrm{id}$, and $\beta_f = 0$. This leads to a distinguished triangle $0 \to X \xrightarrow{1} X \to 0$. This triangle is isomorphic to the triangle $0 \to X \xrightarrow{-1} X \to 0$, which is therefore distinguished. Since a shift of a distinguished triangle is distinguished, this proves that the triangle $X \xrightarrow{1} X \to 0 \to \Sigma X$ is distinguished.

4) For every morphism $f\colon X \to Y$, the cone triangle $X \xrightarrow{f} Y \xrightarrow{\alpha_f} C_f \xrightarrow{\beta_f} \Sigma X$ is distinguished, by definition.

5) Let us finally prove axiom (2.1.3.5): given two distinguished triangles and a partial morphism between them, we need to show that it can be completed into a distinguished morphism of distinguished triangles in $\boldsymbol{K}(\boldsymbol{A})$. We may assume that the two given triangles are cone triangles, and choose representatives in $\boldsymbol{C}(\boldsymbol{A})$. Let thus consider a diagram of complexes in $\boldsymbol{A}$:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{\alpha_u} & C_u & \xrightarrow{\beta_u} & \Sigma X \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{\alpha_{u'}} & C_{u'} & \xrightarrow{\beta_{u'}} & \Sigma X'
\end{array}$$

where u, u', f, g are morphisms of complexes such that $g \circ u$ is homotopic to $u' \circ f$; let $\theta\colon X \to \Sigma^{-1}Y'$ be a homotopy such that $g \circ u - u' \circ f = d\theta + \theta d$. Let us show the existence of a morphism $h\colon Y \oplus \Sigma X \to Y' \oplus \Sigma X'$ such that the triangle

$$X' \oplus Y \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -\alpha \end{pmatrix}} Y' \oplus (Y \oplus \Sigma X) \xrightarrow{\begin{pmatrix} \alpha' & h \\ 0 & -\beta \end{pmatrix}} (Y' \oplus \Sigma X') \oplus \Sigma X \xrightarrow{\begin{pmatrix} \beta' & \Sigma f \\ 0 & -\Sigma u \end{pmatrix}} \Sigma X' \oplus \Sigma Y$$

is distinguished. We set $h = \begin{pmatrix} g & \Sigma^{-1}\theta \\ 0 & \Sigma f \end{pmatrix}$. Then

$$h \circ \alpha - \alpha' \circ g = \begin{pmatrix} g & \Sigma^{-1}\theta \\ 0 & \Sigma f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} g = 0,$$

$$\Sigma f \circ \beta - \beta' \circ h = \Sigma f \begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} g & \Sigma^{-1}\theta \\ 0 & \Sigma f \end{pmatrix} = \begin{pmatrix} 0 & \Sigma f \end{pmatrix} - \begin{pmatrix} 0 & \Sigma f \end{pmatrix} = 0,$$

so that the preceding diagram is indeed a morphism of distinguished triangles. To prove that its cone is a distinguished morphism, we shall show that there exists morphisms of complexes φ and ψ, as represented by the diagram below, that give rise to isomorphisms, inverse one of the other:

$$\begin{array}{ccccccc}
X'\oplus Y & \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -1 \\ 0 & 0 \end{pmatrix}} & Y'\oplus Y\oplus \Sigma X & \xrightarrow{\begin{pmatrix} 1 & g & \theta \\ 0 & 0 & f \\ 0 & 0 & -1 \end{pmatrix}} & Y'\oplus \Sigma X'\oplus \Sigma X & \xrightarrow{\begin{pmatrix} 0 & 1 & f \\ 0 & 0 & -u \end{pmatrix}} & \Sigma X'\oplus \Sigma Y \\
\Vert & & \Vert & & \psi \uparrow \downarrow \varphi & & \Vert \\
X'\oplus Y & \xrightarrow[\begin{pmatrix} u' & g \\ 0 & -1 \\ 0 & 0 \end{pmatrix}]{} & Y'\oplus Y\oplus \Sigma X & \xrightarrow[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}]{} & Y'\oplus Y\oplus \Sigma X\oplus \Sigma X'\oplus \Sigma Y & \xrightarrow[\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}]{} & \Sigma X'\oplus \Sigma Y
\end{array}$$

Set

$$\varphi = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & f \\ 0 & 0 & -u \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 1 & 0 \\ \theta & f & -1 & 0 & 0 \end{pmatrix}.$$

Write A, B, C for the matrices of the first row, and A', B', C' for those of the second row. One has

$$\psi A' = A, \quad B'\varphi = B, \quad \psi \circ \varphi = \mathrm{id}\,.$$

Moreover,

$$\varphi \circ \varphi = \begin{pmatrix} 1 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & u & 0 & 0 \end{pmatrix} = \mathrm{id} + \begin{pmatrix} 0 & g & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & -1 \end{pmatrix}.$$

Let

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} : Y'\oplus Y\oplus \Sigma X\oplus \Sigma X'\oplus \Sigma Y \to \Sigma^{-1}Y'\oplus \Sigma^{-1}Y\oplus X\oplus X'\oplus Y.$$

The differential of the cone $Y' \oplus Y \oplus \Sigma X \oplus \Sigma X' \oplus \Sigma X$ is given by

$$\delta = \begin{pmatrix} d & 0 & 0 & u' & g \\ 0 & d & u & 0 & -1 \\ 0 & 0 & -d & 0 & 0 \\ 0 & 0 & 0 & -d & 0 \\ 0 & 1 & 0 & 0 & -d \end{pmatrix},$$

and one checks that $\delta H + H\delta = \varphi\circ\psi - \mathrm{id}$. Consquently, φ and ψ are isomorphisms in the homotopy category, inverse one of the other. Consequently, the first row is isomorphic to the bottom row, which is a cone, hence it is is a cone.

This concludes the proof of the theorem. □

2.5. Localization

Definition **(2.5.1)**. — *Let $\boldsymbol{C}, \boldsymbol{D}$ be triangulated categories. A functor* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ *is called a* triangulated functor *if it is additive, commutes with translations, and if it maps a distinguished triangle in $\boldsymbol{C}$ to a distinguished triangle in $\boldsymbol{D}$.*

2.5.2. — Let $\boldsymbol{C}$ be a triangulated category. A subcategory $\boldsymbol{D}$ of $\boldsymbol{C}$ is called a *triangulated subcategory* if the following properties hold:

(i) Every object of $\boldsymbol{C}$ which is isomorphic to an object of $\boldsymbol{D}$ belongs to $\boldsymbol{D}$;

(ii) For every objects X, Y of $\boldsymbol{D}$, one has $\boldsymbol{D}(X, Y) = \boldsymbol{C}(X, Y)$;

(iii) The subcategory $\boldsymbol{D}$ is stable under the translation functor of $\boldsymbol{C}$ and under finite coproducts;

(iv) For every morphism $f\colon X \to Y$ in $\boldsymbol{D}$, there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\boldsymbol{C}$, and Z is an object of $\boldsymbol{D}$.

These axioms imply that $\boldsymbol{D}$ is a triangulated category when endowed with the restriction of the translation functor Σ and the set of triangles of $\boldsymbol{C}$ whose vertices belong to $\boldsymbol{D}$, and that the inclusion functor is a fully faithful triangulated functor.

Moreover, in a distinguished triangle $X \to Y \to Z \to \Sigma X$, if two objects out of three belong to $\boldsymbol{D}$, then so does the third one. This holds by hypothesis if X and Y belong to $\boldsymbol{D}$, and the two other cases follow by considering translated triangles.

The triangulated subcategory $\boldsymbol{D}$ is said to be *thick* if for every objects Y, Y' of $\boldsymbol{C}$ such that $Y \oplus Y'$ is an object of $\boldsymbol{D}$, then $Y, Y' \in \mathrm{ob}(\boldsymbol{D})$.

Lemma **(2.5.3)**. — *Let* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ *be a triangulated functor. Let* $\mathrm{Ker}(F)$ *be the full subcategory of* $\boldsymbol{C}$ *whose objects are the objects* $X \in \mathrm{ob}(\boldsymbol{C})$ *such that* $F(X) \simeq 0$. *Then* $\mathrm{Ker}(F)$ *is a thick triangulated subcategory.*

Proof. — a) Let $X \in \mathrm{ob}(\mathrm{Ker}(F))$; then $F(X) \simeq 0$, hence $F(\Sigma X) = \Sigma(F(X)) \simeq 0$, so that $\Sigma X \in \mathrm{ob}(\mathrm{Ker}(F))$. One proves similarly that if $X \in \mathrm{ob}(\mathrm{Ker}(F))$, then $\Sigma^{-1}X \in \mathrm{ob}(\mathrm{Ker}(F))$ too.

b) Let $X, Y \in \mathrm{ob}(\boldsymbol{C})$ be such that $X \simeq Y$; if $F(X) \simeq 0$, then $F(Y) \simeq F(X) \simeq 0$;

c) Let $Y, Y' \in \mathrm{ob}(\boldsymbol{C})$; if $Y \oplus Y' \in \mathrm{Ker}(F)$, then $0 \simeq F(Y \oplus Y') \simeq F(Y) \oplus F(Y')$, hence $F(Y) \simeq 0$ and $Y \in \mathrm{Ker}(F)$;

d) Let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle in $\boldsymbol{C}$; assume that $X, Y \in \mathrm{ob}(\mathrm{Ker}(F))$, so that $F(X) \simeq F(Y) \simeq 0$; then $0 \to 0 \to F(Z) \to 0$ is a distinguished triangle in $\boldsymbol{D}$, so that $F(Z) \simeq 0$, hence $Z \in \mathrm{ob}(\mathrm{Ker}(F))$. □

The following theorem asserts that every thick triangulated subcategory appears in this way.

Theorem **(2.5.4)** (Verdier). — *Let* $\boldsymbol{C}$ *be a triangulated category and let* $\boldsymbol{N}$ *be a triangulated subcategory of* $\boldsymbol{C}$. *There exist a triangulated category* $\boldsymbol{D}$ *and a triangulated functor* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ *such that:*

a) $\boldsymbol{N} \subset \mathrm{Ker}(F)$*;*

b) *If* $F'\colon \boldsymbol{C} \to \boldsymbol{D}'$ *is a triangulated functor such that* $\boldsymbol{N} \subset \mathrm{Ker}(F')$*, there exists a unique triangulated functor* $G\colon \boldsymbol{D} \to \boldsymbol{D}'$ *such that* $F' = G \circ F$.

Moreover, $\mathrm{Ker}(F)$ *is the smallest thick triangulated subcategory of* $\boldsymbol{C}$ *containing* $\boldsymbol{N}$.

The proof of the theorem will occupy the rest of the section. We consider throughout a triangulated category $\boldsymbol{C}$ and a triangulated subcategory $\boldsymbol{N}$.

Definition **(2.5.5)**. — *One says that a morphism* $f\colon X \to Y$ *in* $\boldsymbol{C}$ *is an isomorphism* $(\mathrm{mod}\ \boldsymbol{N})$ *if there exists a distinguished triangle* $X \xrightarrow{f} Y \to Z \to \Sigma X$ *in* $\boldsymbol{C}$ *such that* $Z \in \mathrm{ob}(\boldsymbol{N})$.

By corollary 2.2.4, this implies that $Z \in \mathrm{ob}(\boldsymbol{N})$ for every such distinguished triangle.

Lemma **(2.5.6)**. — a) *If a morphism in $\boldsymbol{C}$ is an isomorphism, then it is an isomorphism* (mod $\boldsymbol{N}$)*;*

b) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\boldsymbol{C}$; if two morphisms among f, g and $g\circ f$ are isomorphisms* (mod $\boldsymbol{N}$)*, then so is the third one;*

c) *Let $f\colon X \to Y$ be a morphism in $\boldsymbol{C}$. Then f is an isomorphism* (mod $\boldsymbol{N}$) *if and only if Σf is an isomorphism* (mod $\boldsymbol{N}$).

As a consequence, there exists a unique subcategory $\boldsymbol{S}$ of $\boldsymbol{C}$ whose set of morphisms is the set of isomorphisms (mod $\boldsymbol{N}$). Its set of objects is the set of objects of $\boldsymbol{C}$.

Proof. — a) Indeed, if f is an isomorphism, then the triangle $X \xrightarrow{f} Y \to 0 \to \Sigma X$ is distinguished, and $0 \in \boldsymbol{N}$.

b) Let us consider an octahedral diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & U & \longrightarrow & \Sigma X \\
\| & & \downarrow{\scriptstyle g} & & \downarrow & & \| \\
X & \xrightarrow{g\circ f} & Z & \longrightarrow & V & \longrightarrow & \Sigma X \\
& & \downarrow & & \downarrow & & \\
& & W & = & W & & \\
& & \downarrow & & \downarrow & & \\
\Sigma X & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma U & \longrightarrow & \Sigma^2 X.
\end{array}$$

By definition, f, resp. g, resp. $g\circ f$, is an isomorphism (mod $\boldsymbol{N}$) if and only if U, resp. W, resp. V, is an object of $\boldsymbol{N}$. Since $\boldsymbol{N}$ is a triangulated subgcategory of $\boldsymbol{N}$, if, in the distinguished triangle $U \to V \to W \to \Sigma U$, two objects out of three belong to $\boldsymbol{N}$, then so does the third one. This implies the claim.

c) Let us assume that f is an isomorphism (mod $\boldsymbol{N}$) and let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle, where $Z \in \operatorname{ob}(\boldsymbol{N})$. Translating this triangle three times, one obtains a distinguished triangle $\Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2 X$, which is isomorphic to the triangle $\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2 X$. Since ΣZ is an object of $\boldsymbol{N}$, this shows that Σg is an isomorphism (mod $\boldsymbol{N}$). The other direction is analogous. □

2.5.7. — Let X, Y be objects of $\boldsymbol{C}$. Let $D(X,Y)$ be the set of triples (Z, f, g) where $Z \in ob(\boldsymbol{C})$, $f \in \boldsymbol{S}(Z,X)$ is an isomorphism (mod $\boldsymbol{N}$) and $g \in \boldsymbol{C}(Z,Y)$; we represent such a triple by the diagram

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array}$$

Let $\sim$ be the relation in $D(X,Y)$ defined as follows: $(Z', f', g') \sim (Z'', f'', g'')$ if there exists a triple $(Z, f, g) \in D(X,Y)$ and isomorphisms (mod $\boldsymbol{N}$) $u' \in \boldsymbol{S}(Z,Z')$ and $u'' \in \boldsymbol{S}(Z,Z'')$ such that $f = f' \circ u' = f'' \circ u''$ and $g = g' \circ u' = g'' \circ u''$. We represent this by the diagram:

$$\begin{array}{ccccc} & & Z' & & \\ & {}^{f'}\swarrow & \uparrow{\scriptstyle u'} & \searrow^{g'} & \\ X & \xleftarrow[f]{} & Z & \xrightarrow{g} & Y \\ & {}_{f''}\nwarrow & \downarrow{\scriptstyle u''} & \nearrow_{g''} & \\ & & Z'' & & \end{array}$$

Lemma **(2.5.8)**. — a) *Let*

$$\begin{array}{ccc} Y & \xrightarrow{v} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow{v'} & Z' \end{array}$$

be a homotopically cartesian square. Then $v \in mor(\boldsymbol{S})$ *if and only if* $v' \in mor(\boldsymbol{S})$*; and* $g \in mor(\boldsymbol{S})$ *if and only if* $h \in mor(\boldsymbol{S})$.

b) *The relation* $\sim$ *in* $D(X,Y)$ *is an equivalence relation.*

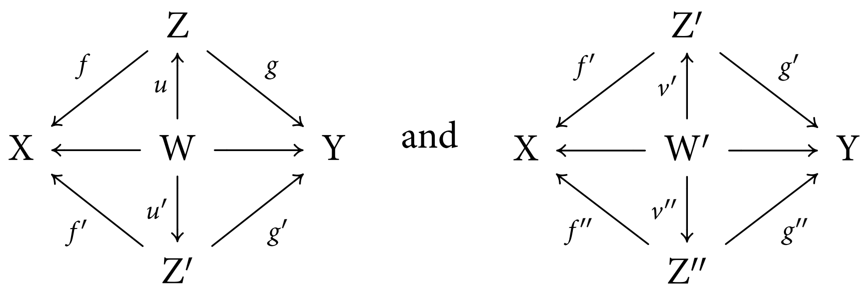
c) *Let* $(W_1, f_1, g_1) \in D(X,Y)$ *and* $(W_2, f_2, g_2) \in D(Y,Z)$*; there exists a triple* $(W, f, g) \in D(W_1, W_2)$ *such that the square*

$$\begin{array}{ccc} W & \xrightarrow{g} & W_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ W_1 & \xrightarrow{g_1} & Y \end{array}$$

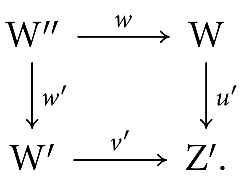
is commutative; for such (W, f, g)*, the triple* $(W, f_1 \circ f, g_2 \circ g)$ *belongs to* $D(X,Z)$ *and its equivalence class of only depends on the equivalence classes of the triples* (W_1, f_1, g_1) *and* (W_2, f_2, g_2).

Proof. — a) With the notation of lemma 2.3.5, the morphism g is an isomorphism (mod $\boldsymbol{N}$) if and only if $Y'' \in \mathrm{ob}(\boldsymbol{N})$, if and only if h is an isomorphism (mod $\boldsymbol{N}$). The other assertion follows by symmetry.

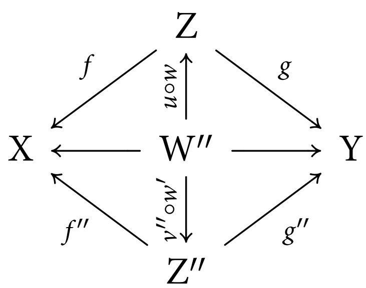
b) It is obvious that the given relation is reflexive and symmetric; let us establish that it is transitive. Let (Z, f, g), (Z', f', g'), (Z'', f'', g'') be elements of $D(X, Y)$ such that $(Z, f, g) \sim (Z', f', g')$ and $(Z', f', g') \sim (Z'', f'', g'')$; by definition, there exist two diagrams



as above, where u, u', v', v'' are isomorphisms (mod $\boldsymbol{N}$). By lemma 2.3.3, there exists an object W'' and morphisms $w \in \boldsymbol{C}(W'', W)$ and $w' \in \boldsymbol{C}(W'', W')$ that give rise to a homotopically cartesian square



By *a*), the morphisms w and w' belong to $\mathrm{mor}(\boldsymbol{S})$. Consequently, the diagram



proves that $(Z, f, g) \sim (Z'', f'', g'')$, as was to be shown.

It then follows from the definition of $\sim$ that it is the equivalence relation generated by the relation given by $(Z, f, g) \sim (Z', f', g')$ if there exists a morphism $u \in \boldsymbol{S}(Z', Z)$ such that $f' = f \circ u$ and $g' = g \circ u$.

c) By lemma 2.3.3, there exists a homotopically cartesian commutative square:

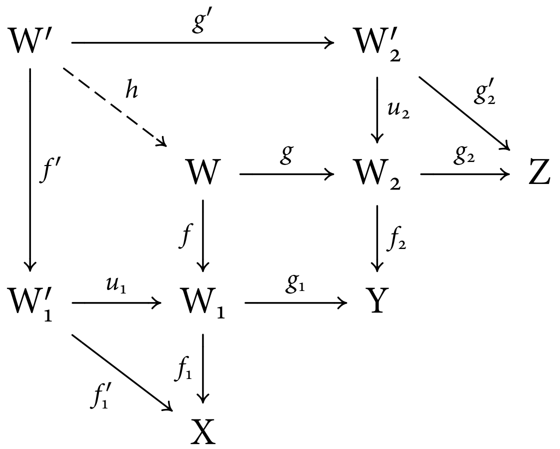
$$\begin{array}{ccc} W & \xrightarrow{g} & W_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ W_1 & \xrightarrow{g_1} & Y. \end{array}$$

By *a*), the morphism f is an isomorphism (mod $\boldsymbol{N}$), so that $(W, f, g) \in D(W_1, W_2)$. This proves that $D(W_1, W_2)$ is not empty.

Let (W, f, g) be any element of $D(W_1, W_2)$; observe that $f_1 \circ f \in \boldsymbol{S}(W, X)$, so that $(W, f_1 \circ f, g_2 \circ g)$ belongs to $D(X, Z)$. Let us now show that for every such triple (W, f, g), the equivalence class of $(W, f_1 \circ f, g_2 \circ g)$ modulo $\sim$ only depends on the equivalence classes of (W_1, f_1, g_1) and (W_2, f_2, g_2).

More generally, let $(W'_1, f'_1, g'_1) \in D(X, Y)$ and $(W'_2, f'_2, g'_2) \in D(Y, Z)$ be equivalent to (W_1, f_1, g_1) and (W_2, f_2, g_2) respectively. Let us choose $(W', f', g') \in D(W'_1, W'_2)$ such that $g'_1 \circ f' = f'_2 \circ g'$ and let us prove that the elements $(W', f'_1 \circ f', g'_2 \circ g')$ and $(W, f_1 \circ f, g_2 \circ g)$ of $D(X, Z)$ are equivalent. By the definition of the equivalence relation $\sim$, we may assume that there exists $u_1 \in \boldsymbol{S}(W'_1, W_1)$ and $u_2 \in \boldsymbol{S}(W'_2, W_2)$ such that $f_1 \circ u_1 = f'_1$ and $g_1 \circ u_1 = g'_1$ on the one side, and that $f_2 \circ u_2 = f'_2$ and $g_2 \circ u_2 = g'_2$ on the other side.

By lemma 2.3.3, there exists a morphism $h : W' \to W$ such that $f \circ h = u_1 \circ f'$ and $g \circ h = u_2 \circ g'$, so that the following diagram is commutative:



Recall that f, f', u_1 are isomorphisms (mod $\boldsymbol{N}$). By lemma 2.5.6, *b*), $u_1 \circ f'$ is an isomorphism (mod $\boldsymbol{N}$), so that $f \circ h$ is an isomorphism (mod $\boldsymbol{N}$). Applying lemma 2.5.6, *b*), again, we conclude that h is an isomorphism (mod $\boldsymbol{N}$).

Consequently, $(W, f_1 \circ f, g_2 \circ g)$ is equivalent to $(W', f_1 \circ f \circ h, g_2 \circ g \circ h)$ which is equal to $(W', f'_1 \circ f', g'_2 \circ g')$. □

Proposition **(2.5.9)**. — a) *There exists a unique category* $\boldsymbol{D}$ *such that* $\mathrm{ob}(\boldsymbol{D}) = \mathrm{ob}(\boldsymbol{C})$, $\boldsymbol{D}(\mathrm{X},\mathrm{Y}) = \mathrm{D}(\mathrm{X},\mathrm{Y})/{\sim}$ *for every objects* X, Y, *and such that the composition of the classes of triples* $(\mathrm{W}_1, f_1, g_1) \in \mathrm{D}(\mathrm{X},\mathrm{Y})$ *and* $(\mathrm{W}_2, f_2, g_2) \in \mathrm{D}(\mathrm{Y},\mathrm{Z})$ *is the class of a triple* $(\mathrm{W}, f_1 \circ f, g_2 \circ g)$ *where* (W, f, g) *is any element of* $\mathrm{D}(\mathrm{W}_1,\mathrm{W}_2)$ *such that* $g_1 \circ f = f_2 \circ g$.

b) *There exists a unique functor* $\mathrm{F}\colon \boldsymbol{C} \to \boldsymbol{D}$ *such that for every morphism* $f\colon \mathrm{X} \to \mathrm{Y}$ *in* $\boldsymbol{C}$, *the morphism* $\mathrm{F}(f)$ *is the equivalence class of the triple* $(\mathrm{X}, \mathrm{id}_{\mathrm{X}}, f)$.

c) *For every morphism* $f \in \boldsymbol{S}(\mathrm{X},\mathrm{Y})$, $\mathrm{F}(f)$ *is an isomorphism in* $\boldsymbol{D}$, *and its inverse is the class of the triple* $(\mathrm{X}, f, \mathrm{id}_{\mathrm{X}})$. *Moreover, for every triple* $\varphi = (\mathrm{W}, f, g) \in \mathrm{D}(\mathrm{X},\mathrm{Y})$, *one has* $\mathrm{F}([\varphi]) = \mathrm{F}(g) \circ \mathrm{F}(f)^{-1}$.

d) *Let* $\boldsymbol{D}'$ *be a category, let* $\mathrm{F}'\colon \boldsymbol{C} \to \boldsymbol{D}'$ *be a functor such that* $\mathrm{F}'(f)$ *is invertible, for every morphism* $f \in \mathrm{mor}(\boldsymbol{S})$. *Then there exists a unique functor* $\mathrm{G}\colon \boldsymbol{D} \to \boldsymbol{D}'$ *such that* $\mathrm{F}' = \mathrm{G} \circ \mathrm{F}$.

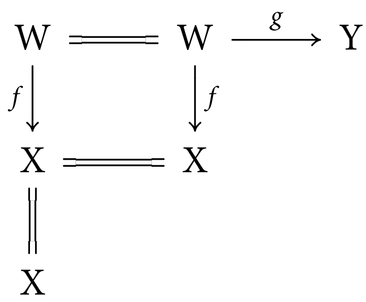
Proof. — a) The set of objects, the set of morphisms and the composition law are prescribed; it thus remains to prove that the composition law is associative and the existence of neutral elements at each object.

Let X, Y, Z, T be objects of $\boldsymbol{C}$, let $u \in \mathrm{D}(\mathrm{X},\mathrm{Y})$, $v \in \mathrm{D}(\mathrm{Y},\mathrm{Z})$ and $w \in \mathrm{D}(\mathrm{Z},\mathrm{T})$; write $[u]$ for the class of u in $\boldsymbol{D}(\mathrm{X},\mathrm{Y})$, etc. We build objects P, Q, R as depicted by the diagram, all of whose vertical arrows are isomorphisms (mod $\boldsymbol{N}$):

$$
\begin{array}{ccccccc}
\mathrm{R} & \longrightarrow & \mathrm{Q} & \longrightarrow & \cdot & \longrightarrow & \mathrm{T} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{P} & \longrightarrow & \cdot & \longrightarrow & \mathrm{Z} & & \\
\downarrow & & \downarrow & & & & \\
\cdot & \longrightarrow & \mathrm{Y} & & & & \\
\downarrow & & & & & & \\
\mathrm{X} & & & & & &
\end{array}
$$

By construction, $[v] \circ [u]$ is the class of the triple $(\mathrm{P}, \mathrm{P} \to \mathrm{X}, \mathrm{P} \to \mathrm{Z})$, hence $[w] \circ ([v] \circ [u])$ is the class of the triple $(\mathrm{R}, \mathrm{R} \to \mathrm{X}, \mathrm{R} \to \mathrm{T})$. Similarly, $[w] \circ [v]$ is the class of the triple $(\mathrm{Q}, \mathrm{Q} \to \mathrm{Y}, \mathrm{Q} \to \mathrm{T})$, hence $([w] \circ [v]) \circ [u]$ is the class of the triple $(\mathrm{R}, \mathrm{R} \to \mathrm{X}, \mathrm{R} \to \mathrm{T})$. The composition law is thus associative.

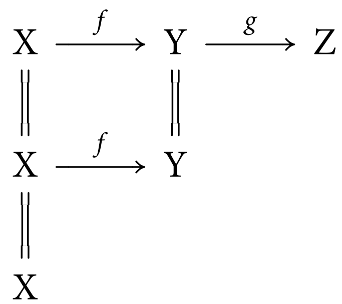
Let us show that the class ε_X of the triple (X, id_X, id_X) is an identity at X. Let $\varphi = (W, f, g) \in D(X, Y)$. By construction, the diagram



shows that the composition $[\varphi] \circ [\varepsilon_X]$ is represented by φ, so that $[\varphi] \circ [\varepsilon_X] = [\varphi]$. One proves similarly that $[\varepsilon_Y] \circ [\varphi] = [\varphi]$.

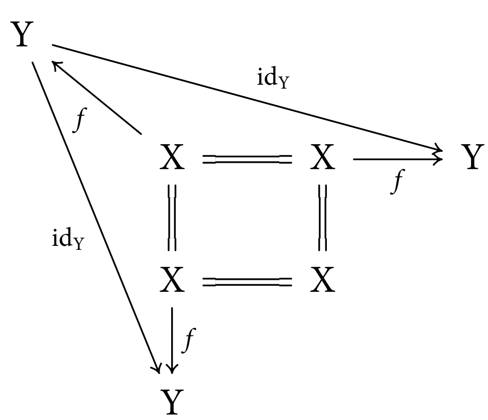
b) For $f \in \boldsymbol{C}(X, Y)$, the origin of $F(f)$ is X, and the target of $F(f)$ is Y. Consequently, there exists at most one such functor, and the map on objects has to be the identity.

By construction, $F(id_X) = \varepsilon_X$. Let $f \in \boldsymbol{C}(X, Y)$ and $g \in \boldsymbol{C}(Y, Z)$; the diagram



and the definition of the composition law show that $F(g \circ f) = F(g) \circ F(f)$. Consequently, F is a functor.

c) Let $f \in \boldsymbol{S}(X, Y)$; let φ be the class of (X, f, id_X). The diagram



proves that $F(f) \circ [\varphi]$ is represented by (X, f, f) and that (X, f, f) is equivalent to $(Y, \mathrm{id}_Y, \mathrm{id}_Y)$. Consequently, $F(f) \circ [\varphi] = \varepsilon_Y$. One proves similarly that $[\varphi] \circ F(f) = \varepsilon_X$.

Finally, if $\varphi = (W, f, g) \in D(X, Y)$, the diagram

$$\begin{array}{ccccc} W & = & W & \xrightarrow{g} & Y \\ \| & & \| & & \\ W & = & W & & \\ {\scriptstyle f}\downarrow & & & & \\ X & & & & \end{array}$$

shows that $F(g) \circ F(f)^{-1}$ is represented by (W, f, g), hence $[\varphi] = F(g) \circ F(f)^{-1}$.

d) Necessarily, $G(X) = F'(X)$ for every object X of $\boldsymbol{C}$. Moreover, for every triple $\varphi = (W, f, g)$, one has $[\varphi] = F(g) \circ F(f)^{-1}$ so that necessarily, $G([\varphi]) = F'(g)^{-1} \circ F'(f)$. It remains to show that these formulae define a functor G such that $G \circ F = F'$.

Let $\varphi = (W, f, g)$ and (W', f', g') be equivalent triples; let us show that $F'(g) \circ F'(f)^{-1} = F'(g') \circ F'(f')^{-1}$. By the definition of the equivalence relation on $D(X, Y)$, we may assume that there exists $h \in \boldsymbol{S}(W, W')$ such that $f = f' \circ h$ and $g = g' \circ h$. Then

$$F'(g) = F'(g') \circ F'(h) = F'(g') \circ F'(f')^{-1} \circ F'(f') \circ F'(h) = F'(g') \circ F'(f')^{-1} \circ F'(f),$$

hence $F'(g) \circ F'(f)^{-1} = F'(g') \circ F'(f')^{-1}$. Consequently, G is well defined. One has $F' = G \circ F$ by construction.

To prove that G is a functor, we need to check that it maps unit elements to unit elements, and that it is compatible with composition. Since $\varepsilon_X = F(\mathrm{id}_X)$, one has $G(\varepsilon_X) = F'(\mathrm{id}_X) = \mathrm{id}_{F'(X)}$. Let then $\varphi = (W_1, f_1, g_1) \in D(X, Y)$ and $\psi = (W_2, f_2, g_2) \in D(Y, Z)$; let $(W, f, g) \in D(W_1, W_2)$ be such that $g_1 \circ f = f_2 \circ g$. By definition, $[\psi] \circ [\varphi]$ is the class of the triple $(W, f_1 \circ f, g_2 \circ g)$. Consequently,

$$\begin{aligned} G([\psi] \circ [\varphi]) &= G([(W, f_1 \circ f, g_2 \circ g)]) \\ &= F'(g_2 \circ g) \circ F'(f_1 \circ f)^{-1} \\ &= F'(g_2) \circ F'(f_2)^{-1} \circ F'(f_2) \circ F'(g) \circ F'(f)^{-1} \circ F'(f_1)^{-1} \\ &= G([\psi]) \circ F'(f_2 \circ g) \circ F'(f)^{-1} \circ F'(f_1)^{-1} \\ &= G([\psi]) \circ F'(g_1) \circ F'(f_1)^{-1} = G([\psi]) \circ G([\varphi]), \end{aligned}$$

hence G is a functor.

This concludes the proof of the proposition. □

Remark (**2.5.10**). — The universal property stated in part *d*) of proposition 2.5.9 is preserved by passing to the opposite categories, so that the category $\boldsymbol{D}^{\mathrm{o}}$ is canonically isomorphic to the category obtained by this construction by starting from $\boldsymbol{C}^{\mathrm{o}}$ and its triangulated subcategory $\boldsymbol{N}^{\mathrm{o}}$.

Proposition (**2.5.11**). — a) *Let $f, g\colon \mathrm{X} \to \mathrm{Y}$ be morphisms in $\boldsymbol{C}$. The following conditions are equivalent:*

(i) $\mathrm{F}(f) = \mathrm{F}(g)$;

(ii) *There exist an isomorphism h $(\mathrm{mod}\ \boldsymbol{N})$ such that $f \circ h = g \circ h$;*

(iii) *There exists an isomorphism h $(\mathrm{mod}\ \boldsymbol{N})$ such that $h \circ f = h \circ g$;*

(iv) *The morphism $f - g$ factors through an object of $\boldsymbol{N}$.*

b) *Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a morphism in $\boldsymbol{C}$. The following conditions are equivalent:*

(i) $\mathrm{F}(f)$ *is an isomorphism;*

(ii) *There exists morphisms $g\colon \mathrm{W} \to \mathrm{X}$ and $h\colon \mathrm{Y} \to \mathrm{Z}$ such that $f \circ g$ and $h \circ f$ are isomorphisms $(\mathrm{mod}\ \boldsymbol{N})$;*

(iii) *For every distinguished triangle $\mathrm{X} \xrightarrow{f} \mathrm{Y} \to \mathrm{Z} \to \Sigma\mathrm{X}$ in $\boldsymbol{C}$, there exists an object $\mathrm{Z}' \in \boldsymbol{C}$ such that $\mathrm{Z} \oplus \mathrm{Z}' \in \boldsymbol{N}$;*

(iv) *There exists a distinguished triangle $\mathrm{X} \xrightarrow{f} \mathrm{Y} \to \mathrm{Z} \to \Sigma\mathrm{X}$ in $\boldsymbol{C}$ and an object $\mathrm{Z}' \in \boldsymbol{C}$ such that $\mathrm{Z} \oplus \mathrm{Z}' \in \boldsymbol{N}$.*

Proof. — a) (i)⇒(ii). By hypothesis, the triples $(\mathrm{X}, \mathrm{id}_{\mathrm{X}}, f)$ and $(\mathrm{X}, \mathrm{id}_{\mathrm{X}}, g)$ in $\mathrm{D}(\mathrm{X}, \mathrm{Y})$ are equivalent; there exist a triple (W, f', g') in $\mathrm{D}(\mathrm{X}, \mathrm{Y})$ and morphisms $u \in \boldsymbol{S}(\mathrm{W}, \mathrm{X})$ and $v \in \boldsymbol{S}(\mathrm{W}, \mathrm{Y})$ such that $u = f' = v$ and $f \circ u = g' = g \circ v$. In particular, the morphism f' is an isomorphism $(\mathrm{mod}\ \boldsymbol{N})$.

The implication (i)⇒(iii) follows from the same argument by passing to the opposite category.

The implications (ii)⇒(i) and (iii)⇒(i) hold, because $\mathrm{F}(h)$ is an isomorphism.

Let us prove that (ii)⇒(iv). Let $h\colon \mathrm{W} \to \mathrm{X}$ be an isomorphism $(\mathrm{mod}\ \boldsymbol{N})$ such that $f \circ h = g \circ h$; by definition of $\boldsymbol{S}$, we may complete h into a distinguished triangle $\mathrm{W} \xrightarrow{h} \mathrm{X} \xrightarrow{k} \mathrm{N} \to \Sigma\mathrm{W}$, where $\mathrm{N} \in \boldsymbol{N}$. Let us apply the cohomological functor $\boldsymbol{C}(\cdot, \mathrm{Y})$ on $\boldsymbol{C}^{\mathrm{o}}$: this gives an exact sequence

$$\boldsymbol{C}(\mathrm{N}, \mathrm{Y}) \xrightarrow{k^*} \boldsymbol{C}(\mathrm{X}, \mathrm{Y}) \xrightarrow{h^*} \boldsymbol{C}(\mathrm{W}, \mathrm{Y}).$$

By assumption, $h^*(f-g) = (f-g)\circ h = 0$; consequently, there exists a morphism $j\colon \mathrm{N} \to \mathrm{Y}$ such that $f - g = j \circ h$: the morphism $f - g$ factors through an object of $\boldsymbol{N}$.

Conversely, assume that (iv) holds and let us consider an object N of $\boldsymbol{N}$ and a factorization $f - g = v \circ u$, where $u \in \boldsymbol{C}(\mathrm{X}, \boldsymbol{N})$ and $v \in \boldsymbol{C}(\mathrm{N}, \mathrm{Y})$. There exists a distinguished triangle $\mathrm{X} \xrightarrow{u} \mathrm{N} \to \mathrm{W} \xrightarrow{w} \Sigma\mathrm{X}$, hence, by translation, a distinguished triangle $\Sigma^{-1}\mathrm{W} \xrightarrow{-\Sigma^{-1}w} \mathrm{X} \xrightarrow{-u} \mathrm{N} \to \mathrm{W}$. Consequently, $\Sigma^{-1}w \in \boldsymbol{S}(\Sigma^{-1}\mathrm{W}, \mathrm{X})$. Moreover, $(f - g) \circ \Sigma w = v \circ u \circ \Sigma w = 0$. This proves (ii).

b) (i)⇒(ii). Let $(\mathrm{W}, s, g) \in \mathrm{D}(\mathrm{Y}, \mathrm{X})$ be a triple whose equivalence class is an inverse of $\mathrm{F}(f)$. By definition of the composition, $(\mathrm{W}, s, f \circ g)$ is equivalent to $(\mathrm{Y}, \mathrm{id}_\mathrm{Y}, \mathrm{id}_\mathrm{Y})$. Consequently, there exists a triple (Z, h, k) and isomorphisms $(\mathrm{mod}\ \boldsymbol{N})$, $u\colon \mathrm{Z} \to \mathrm{W}$ and $v\colon \mathrm{Z} \to \mathrm{Y}$, such that $s \circ u = h$, $f \circ g \circ u = k$, $h = \mathrm{id}_\mathrm{Y} \circ v = v$ and $k = \mathrm{id}_\mathrm{Y} \circ v = v$. Then, $h = k = v$ and u are isomorphisms $(\mathrm{mod}\ \boldsymbol{N})$ and $(f \circ g) \circ u = k$, so that $f \circ g$ is an isomorphism $(\mathrm{mod}\ \boldsymbol{N})$. The other part of (ii) follows by passing to the opposite category.

(ii)⇒(i). These assumptions imply that $\mathrm{F}(f)$ is left-invertible and right-invertible; consequently, $\mathrm{F}(f)$ is invertible.

(ii)⇒(iii). Let $h\colon \mathrm{Y} \to \mathrm{T}$ be such that $h \circ f$ is an isomorphism $(\mathrm{mod}\ \boldsymbol{N})$. By the implication (ii)⇒(i), $\mathrm{F}(f)$ is an isomorphism, as well as $\mathrm{F}(h \circ f)$, so that $\mathrm{F}(h)$ is an isomorphism as well. The commutative diagram

$$\begin{array}{ccccccc} \mathrm{X} & \xrightarrow{f} & \mathrm{Y} & \xrightarrow{u} & \mathrm{Z} & \longrightarrow & \Sigma\mathrm{X} \\ \downarrow{\scriptstyle h\circ f} & & \downarrow{\scriptstyle \binom{h}{u}} & & \Vert & & \downarrow{\scriptstyle \Sigma h\circ\Sigma f} \\ \mathrm{T} & \xrightarrow[\binom{1}{0}]{p} & \mathrm{T}\oplus\mathrm{Z} & \xrightarrow[(0\ 1)]{q} & \mathrm{Z} & \xrightarrow[0]{} & \Sigma\mathrm{T}, \end{array}$$

corresponds to a morphism of distinguished triangles; since its bottom row is a contractible triangle, this morphism is distinguished (proposition 2.2.10, *b*)), hence the commutative square

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{f} & \mathrm{Y} \\ \downarrow{\scriptstyle h\circ f} & & \downarrow{\scriptstyle \binom{h}{u}} \\ \mathrm{T} & \xrightarrow[\binom{1}{0}]{} & \mathrm{T}\oplus\mathrm{Z} \end{array}$$

is homotopically cartesian (lemma 2.3.4).

Since $h \circ f$ is an isomorphism (mod $\boldsymbol{N}$), so is $\binom{h}{u}$. Let then

$$p = \binom{1}{0} : \mathrm{T} \to \mathrm{T} \oplus \mathrm{Z} \quad \text{and} \quad q = \begin{pmatrix} 1 & 0 \end{pmatrix} : \mathrm{T} \oplus \mathrm{Z} \to \mathrm{T}.$$

One has

$$\binom{h}{u} \circ f = \begin{pmatrix} h\circ f \\ u \circ f \end{pmatrix} = \begin{pmatrix} h\circ f \\ 0 \end{pmatrix} = p \circ (h \circ f).$$

Since $\mathrm{F}(f)$, $\mathrm{F}(\binom{h}{u})$ and $\mathrm{F}(h \circ f)$ are isomorphisms in $\boldsymbol{D}$, this shows that $\mathrm{F}(p)$ is an isomorphism in $\boldsymbol{D}$. One has $q \circ p = \mathrm{id}_{\mathrm{T}}$, so that $\mathrm{F}(q)$ is the left-inverse of $\mathrm{F}(p)$. Consequently, it is also its right-inverse and the image of $p \circ q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by F coincides with that of id.

By *a*), there exists an isomorphism (mod $\boldsymbol{N}$), $\tilde{s} = \binom{s}{t} : \mathrm{W} \to \mathrm{T} \oplus \mathrm{Z}$, such that $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ \tilde{s} = 0$. Consequently, $t = 0$. Let $\mathrm{W} \xrightarrow{s} \mathrm{T} \to \mathrm{N} \to \Sigma\mathrm{W}$ be a distinguished triangle; its coproduct with the distinguished triangle $0 \to \mathrm{Z} \xrightarrow{-1} \mathrm{Z} \to 0$ is the distinguished triangle

$$\mathrm{W} \xrightarrow{\binom{s}{0}} \mathrm{T} \oplus \mathrm{Z} \to \mathrm{N} \oplus \mathrm{Z} \to \Sigma\mathrm{W},$$

so that $\mathrm{N} \oplus \mathrm{Z}$ is an object of $\boldsymbol{N}$, as was to be shown.

The implication (iii)⇒(iv) is obvious since there exists a distinguished triangle of the form $\mathrm{X} \xrightarrow{f} \mathrm{Y} \to \mathrm{Z} \to \Sigma\mathrm{X}$.

Let us finally prove that (iv)⇒(ii). Consider a distinguished triangle $\mathrm{X} \xrightarrow{f} \mathrm{Y} \to \mathrm{Z} \to \Sigma\mathrm{X}$. Since $0 \to \mathrm{Z}' \xrightarrow{-1} \mathrm{Z}' \to 0$ is a distinguished triangle, the triangle

$$\mathrm{X} \xrightarrow{\binom{f}{0}} \mathrm{Y} \oplus \mathrm{Z}' \to \mathrm{Z} \oplus \mathrm{Z}' \to \Sigma\mathrm{X}$$

is distinguished. Since $\mathrm{Z} \oplus \mathrm{Z}'$ is an object of $\boldsymbol{N}$, by assumption, the morphism $\binom{f}{0} : \mathrm{X} \to \mathrm{Y} \oplus \mathrm{Z}'$ is an isomorphism (mod $\boldsymbol{N}$). Let $g = \binom{1}{0} : \mathrm{X} \to \mathrm{Y} \oplus \mathrm{Z}'$. One has $\binom{f}{0} = g \circ f$, which proves the first part of (ii). The second one is proved analogously.

□

Corollary **(2.5.12)**. — *Let* X *be an object of* $\boldsymbol{C}$. *The object* F(X) *is isomorphic to* 0 *if and only if there exists an object* Y *of* $\boldsymbol{C}$ *such that* $\mathrm{X} \oplus \mathrm{Y} \in \mathrm{ob}(\boldsymbol{N})$.

Proof. — Apply part *b*) of the proposition to the zero morphism $f \in \boldsymbol{C}(0, \mathrm{X})$. Then $\mathrm{F}(\mathrm{X}) \simeq 0$ if and only if $\mathrm{F}(f)$ is an isomorphism. Given the distinguished triangle $0 \to \mathrm{X} \to \mathrm{X} \to 0$, the corollary follows from the equivalence (i)⇔(iii).

□

Proposition (**2.5.13**). — *The category $\boldsymbol{D}$ is an additive category and the functor* F *is an additive functor.*

Proof. — a) Let us show that 0 is both an initial and a terminal object in $\boldsymbol{D}$. We thus need to show that for every object X of $\boldsymbol{C}$, the sets $D(X,0)$ and $D(0,X)$ have exactly one equivalence class.

Let $(V,f,g),(V',f',g')\in D(X,0)$; since 0 is a terminal object in $\boldsymbol{C}$, one has $g=g'=0$. Let

$$\begin{array}{ccc} W & \xrightarrow{u} & V \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle f} \\ V' & \xrightarrow[f']{} & X \end{array}$$

be an homotopically cartesian square (lemma 2.3.3). Since f and f' are isomorphisms (mod $\boldsymbol{N}$), so are u and u' (lemma 2.5.8). The triple $(W,f\circ u,0)$ and the morphisms u,u' imply that $(V,f,g)\sim(V',f',g')$, as was to be shown. Consequently, 0 is a terminal object in $\boldsymbol{D}$. By considering the opposite category, 0 is an initial object in $\boldsymbol{D}$.

b) Let X, Y be objects of $\boldsymbol{C}$, and let us show that $X\oplus Y$ is a product of X and Y in the category $\boldsymbol{D}$. Let $i\colon X\to X\oplus Y$, $j\colon Y\to X\oplus Y$, $p\colon X\oplus Y\to X$ and $q\colon X\oplus Y\to Y$ be the canonical morphisms. Given an object P and two morphisms $\varphi\colon P\to X$ and $\psi\colon P\to Y$ in $\boldsymbol{D}$, we need to show that there exists a unique morphism $\theta\colon P\to X\oplus Y$ such that $F(p)\circ\theta=\varphi$ and $F(q)\circ\theta=\psi$. The morphisms φ,ψ are represented by triples $(W,f,g)\in D(P,X)$ and $(W',f',h)\in D(P,Y)$. Considering a homotopically cartesian square

$$\begin{array}{ccc} W'' & \longrightarrow & W' \\ \downarrow & & \downarrow{\scriptstyle f'} \\ W & \xrightarrow{f} & P \end{array}$$

we reduce to the case where $W=W'$ and $f=f'$. Let $k=\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right)$ and let $\theta\in\boldsymbol{D}(P,X\oplus Y)$ be the class of the triple (W,f,k). One has $F(p)\circ\theta=\varphi$ and $F(q)\circ\theta=\psi$. Let $\theta'\in\boldsymbol{D}(P,X\oplus Y)$ be a morphism such that $F(p)\circ\theta'=\varphi$ and $F(q)\circ\theta'=\psi$. Up to replacing W, we may assume that θ' is represented by a triple of the form (W,f,k'), where $k'=\left(\begin{smallmatrix} g' \\ h' \end{smallmatrix}\right)$. Then (W,f,g) and (W,f,g') are equivalent, so that, in particular, $F(g)=F(g')$; by proposition 2.5.11, *a*), there exists an isomorphism (mod $\boldsymbol{N}$), $u\colon U\to W$, such that $g\circ u=g'\circ u$. Similarly, there exists an isomorphism (mod $\boldsymbol{N}$), $v\colon V\to W$, such that $g\circ v=g'\circ v$.

Considering a homotopy pull-back of u and v, we may assume that $U = V$ and $u = v$. Then $k \circ u = k' \circ u$, hence $F(k) = F(k')$.

Similarly, one proves that $X \oplus Y$ is a product.

c) By the two preceding paragraphs, the category $\boldsymbol{D}$ is semi-additive, and the functor F is additive. It remains to prove that every morphism $\varphi \in \boldsymbol{D}(X, Y)$ has an opposite. Let (W, f, g) be a triple representing φ and let $\varphi' = [(W, f, -g)]$. One checks that $\varphi + \varphi' = [(W, f, 0)] = 0$. This concludes the proof that the category $\boldsymbol{D}$ is additive. ☐

2.5.14. — If f is an isomorphism (mod $\boldsymbol{N}$), then so are Σf and $\Sigma^{-1} f$; consequently, there exists unique endofunctors of $\boldsymbol{D}$, stil denoted by Σ and Σ^{-1}, such that $\Sigma \circ F = F \circ \Sigma$ and $\Sigma^{-1} \circ F = F \circ \Sigma^{-1}$. One has $\Sigma \circ \Sigma^{-1} \circ = F = \mathrm{id}_{\boldsymbol{D}} \circ F$, so that $\Sigma \circ \Sigma^{-1} = \mathrm{id}_{\boldsymbol{D}}$; similarly, $\Sigma^{-1} \circ \Sigma = \mathrm{id}_{\boldsymbol{D}}$. In particular, Σ is an automorphism of the category $\boldsymbol{D}$.

Lemma **(2.5.15)**. — *Any diagram of distinguished triangles in* $\boldsymbol{C}$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where f, g are isomorphisms (mod $\boldsymbol{N}$) *can be extended to a morphism of triangles, where h is an isomorphism* (mod $\boldsymbol{N}$).

Proof. — Let us complete the morphism $u' \circ f : X \to Y'$ to a distinguished triangle $X \xrightarrow{u' \circ f} Y' \xrightarrow{v''} Z'' \xrightarrow{w''} \Sigma X$. We then decompose the given diagram as the composition of two diagrams of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h''} & & \| \\ X & \xrightarrow{u' \circ f} & Y' & \xrightarrow{v''} & Z'' & \xrightarrow{w''} & \Sigma X \\ {\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

Starting with the first two triangles, let us now apply the octahedral axiom:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\Vert & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h''} & & \Vert \\
X & \xrightarrow{u'\circ f} & Y & \xrightarrow{v} & Z'' & \longrightarrow & \Sigma X \\
 & & \downarrow & & \downarrow & & \\
 & & N & = & N & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma X & \xrightarrow{\Sigma u} & Y & \xrightarrow{\Sigma v} & Z & \xrightarrow{-\Sigma w} & \Sigma^2 X
\end{array}$$

Since g is an isomorphism (mod $\boldsymbol{N}$), the object N belongs to $\boldsymbol{N}$; consequently, h'' is an isomorphism (mod $\boldsymbol{N}$) as well.

We then apply the octahedral axiom in the opposite category to the last two triangles, after having shifted them to the left:

$$\begin{array}{ccccccc}
\Sigma^{-1}Y' & \xrightarrow{-\Sigma^{-1}v'} & \Sigma^{-1}Z' & \xrightarrow{-\Sigma^{-1}w'} & X' & \xrightarrow{-u'} & Y' \\
 & & \downarrow & & \downarrow & & \\
 & & N' & = & N' & & \\
 & & \downarrow & & \downarrow & & \\
Y' & \xrightarrow{-v''} & Z'' & \xrightarrow{-w''} & \Sigma X & \xrightarrow{-\Sigma u'\circ\Sigma f} & \Sigma Y \\
\Vert & & \downarrow{\scriptstyle h'} & & {\scriptstyle \Sigma f}\downarrow & & \Vert \\
Y' & \xrightarrow{-v'} & Z' & \xrightarrow{-w'} & \Sigma X' & \xrightarrow{-\Sigma u'} & \Sigma Y'
\end{array}$$

Again, since the morphism Σf is an isomorphism (mod $\boldsymbol{N}$), the object N′ belongs to $\boldsymbol{N}$, so that h' is an isomorphism (mod $\boldsymbol{N}$). Finally, we may let $h = h' \circ h''$; it is an isomorphism (mod $\boldsymbol{N}$). □

Theorem **(2.5.16)**. — *Let $\mathscr{T}$ be the set of triangles in $\boldsymbol{D}$ which are isomorphic to the image under* F *of a distinguished triangle of $\boldsymbol{C}$. The category $\boldsymbol{D}$ is a triangulated category, when endowed with its endofunctor* Σ *and the set $\mathscr{T}$ of triangles, and the functor* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ *is a triangulated functor.*

Let us remark that the given set of distinguished triangles in $\boldsymbol{D}$ is the smallest possible one for which F is a triangulated functor. Indeed, this condition implies that the image of a distinguished triangle is again distinguished, and axiom (2.1.3.1) of a triangulated category imposes that the triangles in $\mathscr{T}$ be distinguished.

Proof. — 1) By construction, a triangle which is isomorphic to a distinguished triangle is distinguished.

2) It follows from the definition and the analogous property for the triangulated category $\boldsymbol{C}$ that if T is a triangle in $\boldsymbol{D}$, then the shift of T is distinguished if and only if T is distinguished.

3) For every object X, the triangle $X \xrightarrow{1} X \to 0 \to \Sigma X$ in $\boldsymbol{D}$ is the image under F of the "same" triangle in $\boldsymbol{C}$, hence is distinguished.

4) Let $\varphi\colon X \to Y$ be a morphism in $\boldsymbol{D}$; let it be represented by a triple (W, s, f), where s is an isomorphism (mod $\boldsymbol{N}$). Let $W \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma W$ be a distinguished triangle in $\boldsymbol{C}$. The diagram in $\boldsymbol{D}$

$$\begin{array}{ccccccc}
W & \xrightarrow{F(f)} & Y & \xrightarrow{F(g)} & Z & \xrightarrow{F(h)} & \Sigma W \\
{\scriptstyle F(s)}\downarrow & & \| & & \| & & \downarrow{\scriptstyle \Sigma s} \\
X & \xrightarrow[\varphi]{} & Y & \xrightarrow[F(g)]{} & Z & \xrightarrow[\Sigma F(s)\circ F(h)]{} & \Sigma X
\end{array}$$

shows that the bottom triangle is distinguished.

5) Let us consider a partial diagram of distinguished triangles in $\boldsymbol{D}$:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

and let us show that there exists a morphism $h\colon Z \to Z'$ in $\boldsymbol{D}$ that gives rise to a distinguished morphism of distinguished triangles.

We may assume that both horizontal triangles are images by F of distinguished triangles in $\boldsymbol{C}$.

Let us show that the leftmost commutative square is isomorphic to the image of a commutative square in $\boldsymbol{C}$. Let $(U, s_1, u_1) \in D(X, Y)$ and $(V, t_1, g_1) \in D(Y, Y')$ be representatives of u and g. Let $(W, t_2, u_2) \in D(U, V)$ be a triple built from a homotopy pull-back of u_1 and t_1. Similarly, let $(U', t'_1, f_1) \in D(X, X')$ and $(V', s'_1, u'_1) \in D(X', Y')$ be representatives of f and g'. Let (W', s'_2, f_2) be a triple built from a homotopy pull-back of f_1 and s'_1. Let finally $(P, s_3, t_3) \in D(W, W')$ be a triple built from a homotopy pull-back of $t'_1 \circ s'_2$ and $s_1 \circ t_2$. These constructions are summarized by the diagrams:

$$
\begin{array}{ccccc}
W & \xrightarrow{u_2} & V & \xrightarrow{g_1} & Y' \\
{\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_1} & & \\
U & \xrightarrow{u_1} & Y & & \\
{\scriptstyle s_1}\downarrow & & & & \\
X & & & &
\end{array}
\qquad
\begin{array}{ccccc}
W' & \xrightarrow{f_2} & V' & \xrightarrow{u'_1} & Y' \\
{\scriptstyle s'_2}\downarrow & & \downarrow{\scriptstyle s'_1} & & \\
U' & \xrightarrow{f_1} & X' & & \\
{\scriptstyle t'_1}\downarrow & & & & \\
X & & & &
\end{array}
\qquad
\begin{array}{ccc}
P & \xrightarrow{t_3} & W \\
{\scriptstyle s_3}\downarrow & & \downarrow{\scriptstyle s_1\circ t_2} \\
W' & \xrightarrow{t'_1\circ s'_2} & X
\end{array}
$$

Moreover, t_2, s'_1, s_3 and t_3 are isomorphisms (mod $\boldsymbol{N}$).

Let us complete $u_2 \circ t_3 : P \to V$ into a distinguished triangle $P \to V \to Q \to \Sigma P$ in $\boldsymbol{C}$; let us complete $u'_1 : V' \to Y'$ into a distinguished triangle $V' \to Y' \to Q' \to \Sigma V'$ in $\boldsymbol{C}$. This furnishes the diagram of distinguished triangles in $\boldsymbol{C}$:

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\uparrow{\scriptstyle s} & & {\scriptstyle t}\uparrow & & {\scriptstyle r}\uparrow & & {\scriptstyle \Sigma s}\uparrow \\
P & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & \Sigma P \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f'} \\
V' & \longrightarrow & Y' & \longrightarrow & Q' & \longrightarrow & \Sigma V' \\
{\scriptstyle s'}\downarrow & & \| & & \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle \Sigma s'} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}
$$

where s, t, s' are morphisms in $\boldsymbol{S}$ such that $f = F(s') \circ F(f') \circ F(s)^{-1}$ and $g = F(g') \circ F(t)^{-1}$. Let us choose a morphism $h' : Q \to Q'$ that gives rise to a distinguished morphism of distinguished triangles in $\boldsymbol{C}$. By lemma 2.5.15, there exist morphisms $r : Q \to Z$ and $r' : Q' \to Z'$ in $\boldsymbol{S}$ that give rise to isomorphisms of triangles in $\boldsymbol{D}$. The morphism $h = F(r') \circ F(h') \circ F(r)^{-1} : Z \to Z'$ in $\boldsymbol{D}$ induces a distinguished isomorphism of distinguished triangles.

This shows that the category $\boldsymbol{D}$ satisfies the five axioms of a triangulated category. By construction, the functor F is triangulated. □

Proof of theorem 2.5.4. — We have constructed a triangulated category $\boldsymbol{D}$ and a triangulated functor $F : \boldsymbol{C} \to \boldsymbol{D}$ such that $\boldsymbol{N} \subset \operatorname{Ker}(F)$. Let now $F' : \boldsymbol{C} \to \boldsymbol{D}'$ be any triangulated functor to a triangulated category $\boldsymbol{D}'$ be such that $\boldsymbol{N} \subset \operatorname{Ker}(F')$. Let $f : X \to Y$ be an isomorphism (mod $\boldsymbol{N}$) and let $X \xrightarrow{f} Y \to Z \to \Sigma X$ be a distinguished triangle in $\boldsymbol{C}$; by definition, Z is an object of $\boldsymbol{N}$. Then

$F'(X) \xrightarrow{F'(f)} F'(Y) \to 0 \to \Sigma F'(X)$ is a distinguished triangle, because F' is a triangulated functor and $\boldsymbol{N} \subset \operatorname{Ker}(F')$. By corollary 2.2.5, the morphism $F'(f)$ is an isomorphism.

By proposition 2.5.9, there exists a unique functor $G\colon \boldsymbol{D} \to \boldsymbol{D}'$ such that $F' = G \circ F$.

The functor G is additive: let indeed $\varphi, \psi\colon X \to Y$ be morphisms in $\boldsymbol{D}$; there exists an object W, an isomorphism (mod $\boldsymbol{N}$), $s\colon W \to X$, and morphisms $f, g\colon W \to Y$ in $\boldsymbol{C}$ such that $\varphi = F(f) \circ F(s)^{-1}$ and $\psi = F(g) \circ F(s)^{-1}$. Then, $\varphi + \psi = F(f+g) \circ F(s)^{-1}$, hence

$$\begin{aligned} G(\varphi+\psi) &= G(F(f+g)\circ F(s)^{-1}) = G(F(f+g))\circ G(F(s))^{-1} \\ &= F'(f+g)\circ F'(s)^{-1} = F'(f)\circ F'(s)^{-1} + F'(g)\circ F'(s)^{-1} \\ &= G(\varphi) + G(\psi), \end{aligned}$$

as was to be shown.

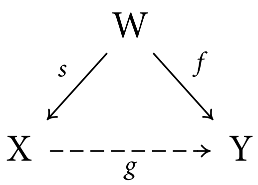
The functor G is triangulated. Indeed, if T is a distinguished triangle in $\boldsymbol{D}$, it is isomorphic to the image by F of a distinguished triangle T_1 in $\boldsymbol{C}$. Then $G(T) = G(F(T_1)) = F'(T_1)$ is a distinguished triangle, because F' is a triangulated functor.

Finally, it follows from corollary 2.5.12 that $\operatorname{Ker}(F)$ is the smallest thick triangulated subcategory of $\boldsymbol{C}$ containing $\boldsymbol{N}$. This concludes the proof of theorem 2.5.4. ☐

Proposition (**2.5.17**). — *We keep the notation of theorem 2.5.4. For an object* Y *in* $\boldsymbol{C}$, *the following assertions are equivalent:*

(i) *One has* $\boldsymbol{C}(X, Y) = 0$ *for every object* $X \in \boldsymbol{N}$;

(ii) *For every diagram*



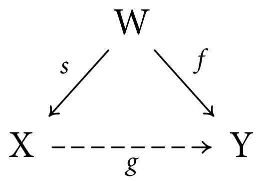
in $\boldsymbol{C}$, *where* $s \in \boldsymbol{S}$ *is an isomorphism* (mod $\boldsymbol{N}$), *there exists a unique morphism* $g\colon X \to Y$ *in* $\boldsymbol{C}$ *such that* $g \circ s = f$;

(iii) *For every object* X *in* $\boldsymbol{C}$, *the functor* F *induces an isomorphism* $\boldsymbol{C}(X, Y) \to \boldsymbol{D}(F(X), F(Y))$;

(iv) *For every object* Z *in* $\boldsymbol{C}$ *and every morphism* $f\colon \mathrm{Y}\to\mathrm{Z}$ *which is an isomorphism* (mod $\boldsymbol{N}$), *there exists a morphism* $g\colon \mathrm{Z}\to\mathrm{Y}$ *such that* $g\circ f=\mathrm{id}_{\mathrm{Y}}$.

We leave to the reader to state the analogous statement in the opposite category.

Proof. — (i)⇒(ii). Let



be a diagram in $\boldsymbol{C}$, where s is an isomorphism (mod $\boldsymbol{N}$). Let $\mathrm{W}\xrightarrow{s}\mathrm{X}\to\mathrm{Z}\to\Sigma\mathrm{W}$ be a distinguished triangle in $\boldsymbol{C}$; since s is an isomorphism (mod $\boldsymbol{N}$), the object Z belongs to $\boldsymbol{N}$. Applying the (contravariant) cohomological functor $\boldsymbol{C}(\cdot,\mathrm{Y})$, we obtain an exact sequence

$$\boldsymbol{C}(\mathrm{Z},\mathrm{Y})\to\boldsymbol{C}(\mathrm{X},\mathrm{Y})\xrightarrow{s}\boldsymbol{C}(\mathrm{W},\mathrm{Y})\to\boldsymbol{C}(\Sigma^{-1}\mathrm{Z},\mathrm{Y}).$$

Since Z and $\Sigma^{-1}\mathrm{Z}$ belong to $\boldsymbol{N}$, one has $\boldsymbol{C}(\mathrm{Z},\mathrm{Y})=\boldsymbol{C}(\Sigma^{-1}\mathrm{Z},\mathrm{Y})=0$. Consequently, the morphism s induces an isomorphism $\boldsymbol{C}(\mathrm{X},\mathrm{Y})\xrightarrow{\sim}\boldsymbol{C}(\mathrm{W},\mathrm{Y})$. In particular, there exists a unique morphism $g\colon\mathrm{X}\to\mathrm{Y}$ in $\boldsymbol{C}$ such that $g\circ s=f$.

(ii)⇒(iii). By definition of morphisms in $\boldsymbol{D}$, assertion (ii) implies that the canonical morphism $\boldsymbol{C}(\mathrm{X},\mathrm{Y})\to\boldsymbol{D}(\mathrm{X},\mathrm{Y})$ is surjective. On the other hand, let $g\in\boldsymbol{C}(\mathrm{X},\mathrm{Y})$ be such that $\mathrm{h}(g)=0$. By proposition 2.5.11, there exists a morphism $s\colon\mathrm{W}\to\mathrm{X}$ in $\boldsymbol{S}$ such that $g\circ s=0$ in $\boldsymbol{C}$. Assertion (i) applied with $f=0$ then implies that $g=0$.

(iii)⇒(iv). Let $f\colon\mathrm{Y}\to\mathrm{Z}$ be an isomorphism (mod $\boldsymbol{N}$). Since it induces an isomorphism in $\boldsymbol{D}$, assertion (iii) implies that there exists a morphism $g\in\boldsymbol{C}(\mathrm{Z},\mathrm{Y})$ such that $\mathrm{h}(g)=\mathrm{h}(f)^{-1}$. One has $\mathrm{h}(g\circ f)=\mathrm{id}_{\mathrm{Y}}$. By (iii) again, $g\circ f=\mathrm{id}_{\mathrm{Y}}$ in $\boldsymbol{C}(\mathrm{Y},\mathrm{Y})$, as was to be shown.

(iv)⇒(i). Let X be an object in $\boldsymbol{N}$ and let $f\colon\mathrm{X}\to\mathrm{Y}$ be a morphism in $\boldsymbol{C}$. Let $\mathrm{X}\xrightarrow{f}\mathrm{Y}\xrightarrow{g}\mathrm{Z}\to\Sigma\mathrm{X}$ be a distinguished triangle. The triangle $\mathrm{Y}\xrightarrow{g}\mathrm{Z}\to\Sigma\mathrm{X}\xrightarrow{\Sigma f}\Sigma\mathrm{Y}$ is distinguished as well, so that g is an isomorphism (mod $\boldsymbol{N}$). By (iv), there exists a morphism $h\colon\mathrm{Z}\to\mathrm{Y}$ such that $h\circ g=\mathrm{id}_{\mathrm{Y}}$ in $\boldsymbol{C}$. Consequently, $f=\mathrm{id}_{\mathrm{Y}}\circ f=h\circ g\circ f=0$ in $\boldsymbol{C}$. This proves that $\boldsymbol{C}(\mathrm{X},\mathrm{Y})=0$, as claimed. □

Corollary **(2.5.18)**. — *Let* $\boldsymbol{N}^{\perp}$ *be the subcategory of* $\boldsymbol{C}$ *consisting of objects* Y *such that* $\boldsymbol{C}(\mathrm{X},\mathrm{Y})=0$ *for every* $\mathrm{X}\in\boldsymbol{N}$. *Then* $\boldsymbol{N}^{\perp}$ *is a thick triangulated subcategory of* $\boldsymbol{C}$ *the restriction to which the functor* F *induces a fully faithful functor to* $\boldsymbol{D}$.

Proof. — The subcategory $\boldsymbol{N}^{\perp}$ is stable under translation. Let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle, where X and Z belong to $\boldsymbol{N}^{\perp}$. Let W be an object in $\boldsymbol{N}$. Applying the cohomological functor $\boldsymbol{C}(W, \cdot)$, we obtain an exact sequence

$$\boldsymbol{C}(W, X) \to \boldsymbol{C}(W, Y) \to \boldsymbol{C}(W, Z)$$

of abelian groups. Consequently, $\boldsymbol{C}(W, Y) = 0$. This proves that $Y \in \boldsymbol{N}^{\perp}$.

Consequently, the full subcategory $\boldsymbol{N}^{\perp}$ is stable under forming distinguished triangles. In particular, it is stable under forming finite coproducts, hence it is an additive subcategory.

Finally, let X and Y be objects in $\boldsymbol{C}$ such that $X \oplus Y \in \boldsymbol{N}^{\perp}$. For every object $W \in \boldsymbol{N}$, one has $0 = \boldsymbol{C}(W, X \oplus Y) = \boldsymbol{C}(W, X) \oplus \boldsymbol{C}(W, Y)$, so that $\boldsymbol{C}(W, X) = \boldsymbol{C}(W, Y) = 0$. Consequently, X and Y belong to $\boldsymbol{N}^{\perp}$.

This shows that $\boldsymbol{N}^{\perp}$ is a thick triangulated subcategory.

The last assertion follows from proposition 2.5.17. □

2.6. Derived categories

2.6.1. — Let $\boldsymbol{A}$ be an abelian category and let $\boldsymbol{K}(\boldsymbol{A})$ be its homotopy category. By lemma 1.6.6, the cohomology functors $H^n : K(\boldsymbol{A}) \to \boldsymbol{A}$ are cohomological functors, for all $n \in \mathbf{Z}$. One has $H^n = H^0 \circ \Sigma^n$.

Lemma (**2.6.2**). — *Let $\boldsymbol{N}$ be the full subcategory of $\boldsymbol{K}(\boldsymbol{A})$ whose objects are the acyclic complexes.*

a) *The subcategory $\boldsymbol{N}$ is a thick triangulated subcategory of $\boldsymbol{K}(\boldsymbol{A})$.*

b) *Let $f : X \to Y$ be a morphism of complexes in $\boldsymbol{A}$. Then f is an isomorphism $(\mathrm{mod}\ \boldsymbol{N})$ if and only if its cone C_f is acyclic, if and only if f is a homologism.*

Proof. — a) First of all, $\boldsymbol{N}$ is an additive subcategory of $\boldsymbol{K}(\boldsymbol{A})$, invariant under the translation automorphism. Let then X, Y be acyclic complexes and $u : X \to Y$ be a morphism in $\boldsymbol{K}(\boldsymbol{A})$; let us extend it to a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in $\boldsymbol{K}(\boldsymbol{A})$. We need to show that Z is acyclic as well. By definition of the distinguished triangles of $\boldsymbol{K}(\boldsymbol{A})$, we may assume that Z is the cone of a morphism of complexes representing u. Since X and Y are acyclic, u is a homologism. By lemma 1.6.6, Z is acyclic.

b) This follows from lemma 1.6.6.

□

Definition **(2.6.3)**. — *Let* $\boldsymbol{A}$ *be an abelian category. The* derived category $\boldsymbol{D}(\boldsymbol{A})$ *is the triangulated category quotient of the homotopy category* $\boldsymbol{K}(\boldsymbol{A})$ *by its thick triangulated subcategory of acyclic complexes.*

There are canonical functors

$$\boldsymbol{A} \to \boldsymbol{C}(\boldsymbol{A}) \xrightarrow{\mathrm{k}} \boldsymbol{K}(\boldsymbol{A}) \xrightarrow{\mathrm{h}} \boldsymbol{D}(\boldsymbol{A}).$$

The functor $\boldsymbol{A} \to \boldsymbol{C}(\boldsymbol{A})$ considers an object X of $\boldsymbol{A}$ as the unique complex such that $X^n = 0$ for $n \neq 0$ and $X^0 = X$, all differentials being 0. It is fully faithful.

Let f be a morphism in $\boldsymbol{K}(\boldsymbol{A})$. By lemma 2.5.11, *b*), the morphism $\mathrm{h}(f)$ is an isomorphism if and only if f is a homologism. More generally, if $H : \boldsymbol{K}(\boldsymbol{A}) \to \boldsymbol{B}$ is a functor, there exists a functor $G : \boldsymbol{D}(\boldsymbol{A}) \to \boldsymbol{B}$ such that $H = G \circ \mathrm{h}$ if and only if H maps homologisms to isomorphisms. The necessity of the condition is clear, by what precedes, and the converse assertion follows from proposition 2.5.9, *d*). If, moreover, the functor H is additive (resp. a cohomological functor), then so is G.

2.6.4. — Let $\boldsymbol{A}$ be an abelian category and let $\boldsymbol{C}$ be a triangulated category. A ∂-functor $F : \boldsymbol{A} \to \boldsymbol{C}$ is an additive functor endowed, for every exact sequence $S = (0 \to X \to Y \to Z \to 0)$ in $\boldsymbol{A}$, with a morphism $\partial(S) : F(Z) \to \Sigma F(X)$, satisfying the following properties:

a) For every exact sequence $S = (0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0)$, the diagram

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\partial(S)} \Sigma F(X)$$

is a distinguished triangle;

b) For every morphism of exact sequences

$$\begin{array}{ccccccccccc} S & & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ S' & & 0 & \longrightarrow & X' & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0, \end{array}$$

the diagram

$$\begin{array}{ccc} F(Z) & \xrightarrow{\partial(S)} & \Sigma F(X) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle \Sigma F(f)} \\ F(Z') & \xrightarrow{\partial(S)} & \Sigma F(X') \end{array}$$

is commutative.

Lemma (**2.6.5**). — *The functor* $\boldsymbol{A} \to \boldsymbol{D}(\boldsymbol{A})$ *is a* ∂*-functor, when, with every exact sequence* $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$, *one associates the composition in* $\boldsymbol{D}(\boldsymbol{A})$ *of the canonical morphism* $\beta_u : C_u \to \Sigma X$ *and of the inverse of the canonical homologism* $C_u \to Z$ *(see lemma 1.6.6).*

Proof. — □

2.6.6. — Let X be a complex in an abelian category $\boldsymbol{A}$. The naïve (or stupid) truncations of X are obtained by replacing X^m by zero outside of a given range, for example:

$$\sigma_{\leqslant n}(X) = \cdots \to X^{n-1} \to X^n \to 0 \to \dots$$

There is a canonical morphism from X to $\sigma_{\leqslant n}(X)$, induced by the identity maps $X^m \to X^m$ for $m \leqslant n$, and by the zero maps otherwise. This morphism induces isomorphisms $H^m(X) \to H^m(\sigma_{\leqslant n}(X))$ for $m < n$; for $m = n$, one gets the morphism

$$H^n(X) = \operatorname{Ker}(d^n_X)/\operatorname{Im}(d^{n-1}_X) \to X^n/\operatorname{Im}(d^{n-1}_X) = H^n(\sigma_{\leqslant n}(X))$$

which is not an epimorphism unless $d^n_X = 0$. Moreover, homologous complexes may have non-homologous stupid truncations.

The correct *truncations* of X are the following complexes:[1]

$$\begin{aligned}
\tau_{\leqslant n}(X) &= \cdots \to X^{n-1} \to \operatorname{Ker}(d^n_X) \to 0 \to 0 \to \dots \\
\tau'_{\leqslant n}(X) &= \cdots \to X^{n-1} \to X^n \to \operatorname{Im}(d^n_X) \to 0 \to \dots \\
\tau'_{\geqslant n}(X) &= \cdots \to 0 \to \operatorname{Im}(d^{n-1}_X) \to X^n \to X^{n+1} \to \dots \\
\tau_{\geqslant n}(X) &= \cdots \to 0 \to 0 \to \operatorname{Coker}(d^{n-1}_X) \to X^{n+1} \to \dots
\end{aligned}$$

There are canonical morphisms of complexes

$$\tau_{\leqslant n}(X) \to \tau'_{\leqslant n}(X) \to X \to \tau'_{\geqslant n}(X) \to \tau_{\geqslant n}(X),$$

of which non-obvious morphisms are induced by the differential.

The morphisms $\tau'_{\geqslant n}(X) \to \tau_{\geqslant n}(X)$ and $\tau_{\leqslant n}(X) \to \tau'_{\leqslant n}(X)$ are homologisms, hence they induce isomorphisms in the derived category $\boldsymbol{D}(\boldsymbol{A})$.

The morphism $\tau'_{\leqslant n}(X) \to X$ induces an isomorphism $H^i(\tau_{\leqslant n}(X)) \xrightarrow{\sim} H^i(X)$ for every integer i such that $i \leqslant n$, while one has $H^i(\tau'_{\leqslant n}(X)) = 0$ for $i > n$. The similar property holds for the morphism $\tau_{\leqslant n}(X) \to X$.

[1] There still are mistakes there...

The morphism $X \to \tau'_{\geqslant n}(X)$ induces an isomorphism $H^i(X) \xrightarrow{\sim} H^i(\tau_{\geqslant n}(X))$ for every integer i such that $i \geqslant n$, and one has $H^i(\tau_{\geqslant n}(X)) = 0$ for $i < n$. The similar property holds for the morphism $X \to \tau_{\geqslant n}(X)$.

Moreover, the diagrams

$$0 \to \tau'_{\leqslant n-1}(X) \to \tau_{\leqslant n}(X) \to \Sigma^{-n} H^n(X) \to 0 \tag{2.6.6.1}$$

$$0 \to \Sigma^{-n} H^n(X) \to \tau_{\geqslant n}(X) \to \tau'_{\geqslant n+1}(X) \to 0 \tag{2.6.6.2}$$

$$0 \to \tau_{\leqslant n}(X) \to X \to \tau'_{\geqslant n+1}(X) \to 0 \tag{2.6.6.3}$$

$$0 \to \tau'_{\leqslant n}(X) \to X \to \tau_{\geqslant n+1}(X) \to 0 \tag{2.6.6.4}$$

are exact sequence of complexes.

These exact sequences of complexes induce distinguished triangles in the homotopy category $\boldsymbol{K}(\boldsymbol{A})$ and in the derived category $\boldsymbol{D}(\boldsymbol{A})$:

$$\tau'_{\leqslant n-1}(X) \to \tau_{\leqslant n}(X) \to \Sigma^{-n} H^n(X) \to \Sigma\tau'_{\leqslant n-1}(X) \tag{2.6.6.5}$$

$$\Sigma^{-n} H^n(X) \to \tau_{\geqslant n}(X) \to \tau'_{\geqslant n+1}(X) \to \Sigma^{1-n} H^n(X) \tag{2.6.6.6}$$

$$\tau_{\leqslant n}(X) \to X \to \tau'_{\geqslant n+1}(X) \to \Sigma\tau_{\leqslant n}(X) \tag{2.6.6.7}$$

$$\tau'_{\leqslant n}(X) \to X \to \tau_{\geqslant n+1}(X) \to \Sigma\tau'_{\leqslant n}(X). \tag{2.6.6.8}$$

Every morphism of complexes $f: X \to Y$ induces in an obvious way a morphism of complexes $\tau_{\leqslant n}(f): \tau_{\leqslant n}(X) \to \tau_{\leqslant n}(Y)$, and $\tau_{\leqslant n}$ defines a functor from the category $\boldsymbol{C}(\boldsymbol{A})$ to itself. If f is null homotopic (resp. a homologism), then so is $\tau_{\leqslant n}(f)$. Consequently, the functor $\tau_{\leqslant n}$ extends to an endofunctor of the homotopy category $\boldsymbol{K}(\boldsymbol{A})$ (resp. of the derived category $\boldsymbol{D}(\boldsymbol{A})$).

Similar properties hold for the other truncations.

The diagrams (2.6.6.1–2.6.6.4), the distinguished triangles (2.6.6.5–2.6.6.8), are functorial.

2.6.7. — By imposing the vanishing of appropriate cohomology objects, we can define various full subcategories of $\boldsymbol{K}(\boldsymbol{A})$:

– $\boldsymbol{K}^{\geqslant a}(\boldsymbol{A})$, whose objects X satisfy $H^n(X) = 0$ for $n < a$;

– $\boldsymbol{K}^{+}(\boldsymbol{A}) = \bigcup_{a\in \mathbf{Z}} \boldsymbol{K}^{\geqslant a}(\boldsymbol{A})$, whose objects X satisfy $H^n(X) = 0$ for n smaller than some integer (depending on X);

– $\boldsymbol{K}^{\leqslant a}(\boldsymbol{A})$, whose objects X satisfy $H^n(X) = 0$ for $n > a$;

– $\boldsymbol{K}^{-}(\boldsymbol{A}) = \bigcup_{a\in \mathbf{Z}} \boldsymbol{K}^{\leqslant a}(\boldsymbol{A})$, whose objects X satisfy $H^n(X) = 0$ for n larger than some integer (depending on X);

– $\boldsymbol{K}^{\mathrm{b}}(\boldsymbol{A}) = \boldsymbol{K}^{+}(\boldsymbol{A}) \cap \boldsymbol{K}^{-}(\boldsymbol{A})$, whose objects X satisfy $\mathrm{H}^n(\mathrm{X}) = 0$ outside of some bounded interval (depending on X).

Their images in $\boldsymbol{D}(\boldsymbol{A})$ are denoted by $\boldsymbol{D}^{\geqslant a}(\boldsymbol{A})$, etc.

The categories $\boldsymbol{K}^{-}(\boldsymbol{A})$, $\boldsymbol{K}^{+}(\boldsymbol{A})$ and $\boldsymbol{K}^{\mathrm{b}}(\boldsymbol{A})$ are triangulated subcategories of $\boldsymbol{K}(\boldsymbol{A})$ containing the subcategory of acyclic objects in $\boldsymbol{K}(\boldsymbol{A})$. The categories $\boldsymbol{D}^{-}(\boldsymbol{A})$, $\boldsymbol{D}^{+}(\boldsymbol{A})$ and $\boldsymbol{D}^{\mathrm{b}}(\boldsymbol{A})$ are triangulated subcategories of $\boldsymbol{D}(\boldsymbol{A})$; they can also be defined as the localization of the corresponding subcategories of $\boldsymbol{K}(\boldsymbol{A})$ under their subcategory of acyclic objects.

The categories $\boldsymbol{K}^{\leqslant a}(\boldsymbol{A})$ and $\boldsymbol{K}^{\geqslant a}(\boldsymbol{A})$ are *not* triangulated subcategories of $\boldsymbol{K}(\boldsymbol{A})$; they are not even stable under translation.

Remark **(2.6.8)**. — Let $\mathrm{X} \in \boldsymbol{K}^{\leqslant 0}(\boldsymbol{A})$ and $\mathrm{Y} \in \boldsymbol{K}^{\geqslant 0}(\boldsymbol{A})$. The cohomological functor $\mathrm{H}^0 : \boldsymbol{D}(\boldsymbol{A}) \to \boldsymbol{A}$ induces an isomorphism

$$\boldsymbol{D}(\boldsymbol{A})(\mathrm{X},\mathrm{Y}) \to \boldsymbol{A}(\mathrm{H}^0(\mathrm{X}),\mathrm{H}^0(\mathrm{Y})).$$

To begin with, an object of $\mathrm{D}(\mathrm{X},\mathrm{Y})$ is a triple (Z, f, g), where f and g are homotopy classes of an homologism $f : \mathrm{Z} \to \mathrm{X}$ and of a morphism of complexes $g : \mathrm{Z} \to \mathrm{Y}$, which, by abuse, we still denote by the same letter. Then $\mathrm{Z} \in \boldsymbol{K}^{\leqslant 0}(\boldsymbol{A})$, so that the canonical morphism $\tau_{\leqslant 0}(\mathrm{Z}) \to \mathrm{Z}$ maps to isomorphisms under all functors $\mathrm{H}^n(\cdot)$; consequently, it is an isomorphism. Similarly, the canonical morphism $\mathrm{Y} \to \tau_{\geqslant 0}(\mathrm{Y})$ is an isomorphism. We may thus assume that $\mathrm{Z}^n = 0$ for $n > 0$ and that $\mathrm{Y}^n = 0$ for $n < 0$. In particular, one has $\mathrm{H}^0(\mathrm{Z}) = \mathrm{Coker}(d_{\mathrm{Z}}^{-1})$ and $\mathrm{H}^0(\mathrm{Y}) = \mathrm{Ker}(d_{\mathrm{Y}}^0)$. Then all components g^n vanish, for $n \neq 0$, so that g is given by a single morphism $g^0 : \mathrm{Z}^0 \to \mathrm{Y}^0$, subject to the conditions $g^0 \circ d_{\mathrm{X}}^{-1} = 0$ and $d_{\mathrm{Y}}^0 \circ g^0 = 0$. Consequently, the datum of g is equivalent to that of the morphism $\mathrm{H}^0(g) : \mathrm{H}^0(\mathrm{Z}) \to \mathrm{H}^0(\mathrm{Y})$. Composed with the isomorphism $f : \mathrm{Z} \to \mathrm{X}$ in $\boldsymbol{D}(\boldsymbol{A})(\mathrm{Z},\mathrm{X})$, this implies the claim.

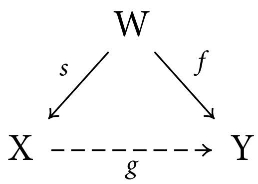
In particular, the canonical functor $\boldsymbol{A} \to \boldsymbol{D}(\boldsymbol{A})$ is fully faithful.

Definition **(2.6.9)**. — *Let* X *be a complex in an abelian category* $\boldsymbol{A}$. *One says that* X *is* homotopically injective *if* $\boldsymbol{K}(\boldsymbol{A})(\mathrm{N},\mathrm{X}) = 0$ *for every acyclic complex* N.

By definition, this notion only depends on the isomorphism class of X in the homotopy category $\boldsymbol{K}(\boldsymbol{A})$.

Proposition **(2.6.10)**. — *Let* Y *be a complex in an abelian category* $\boldsymbol{A}$. *The following assertions are equivalent:*

(i) *The complex* Y *is homotopically injective;*
(ii) *For every diagram*



of morphisms of complexes, where s is a homologism, there exists a unique morphism $g\colon X \to Y$ *in* $\boldsymbol{K}(\boldsymbol{A})$ *such that* $g \circ s$ *is homotopic to* f*;*
(iii) *For every complex* X*, the functor* h *induces an isomorphism* $\boldsymbol{K}(\boldsymbol{A})(X, Y) \to \boldsymbol{D}(\boldsymbol{A})(X, Y)$*;*
(iv) *For every complex* Z *and every homologism* $f\colon Y \to Z$*, there exists a morphism* $g\colon Z \to Y$ *such that* $g \circ f$ *is homotopic to* id_Y*.*

Proof. — This is a particular case of proposition 2.5.17. □

Proposition **(2.6.11)**. — *Homotopically injective complexes form a thick triangulated subcategory* $\boldsymbol{I}$ *of* $\boldsymbol{K}(\boldsymbol{A})$ *and the restriction to* $\boldsymbol{I}$ *of the localization functor* $\boldsymbol{K}(\boldsymbol{A}) \to \mathbf{D}(\boldsymbol{A})$ *is fully faithful.*

Proof. — This follows from corollary 2.5.18. □

Theorem **(2.6.12)**. — *If* $\boldsymbol{A}$ *is a Grothendieck abelian category (see §1.4.7), then every complex is homologous to a homotopically injective complex.*

This theorem is due to Spaltenstein (1988) in the particular case where $\boldsymbol{A}$ is the category of abelian sheaves on a topological space. His methods have then be extended to reach the result in this form by Alonso Tarrío *et al* (2000) and Serpé (2003). More precisely, these authors show that the functor $h\colon \boldsymbol{K}(\boldsymbol{A}) \to \boldsymbol{D}(\boldsymbol{A})$ admits a right adjoint.

Corollary **(2.6.13)**. — *Let* $\boldsymbol{A}$ *be a Grothendieck abelian category and let* $\boldsymbol{I}$ *be the thick triangulated category of* $\boldsymbol{K}(\boldsymbol{A})$ *consisting of all homotopically injective complexes. The triangulated functor* $\boldsymbol{I} \to \boldsymbol{D}(\boldsymbol{A})$ *is an equivalence of triangulated categories.*

Corollary **(2.6.14)**. — *Let* $\boldsymbol{A}$ *be a Grothendieck abelian category. If* U *is an universe such that* $\boldsymbol{A}$ *is locally* U*-small, then* $\boldsymbol{D}(\boldsymbol{A})$ *is locally* U*-small.*

Example **(2.6.15)**. — Let U be a universe, let k be a ring whose underlying set belongs to U and let $\boldsymbol{Mod}(k)_{\mathrm{U}}$ be the category of k-modules whose underlying set belongs to U. For every objects X, Y of $\boldsymbol{Mod}(k)_{\mathrm{U}}$, one has $\mathrm{Hom}(\mathrm{X},\mathrm{Y}) \in \mathrm{U}$, so that the category $\boldsymbol{Mod}(k)_{\mathrm{U}}$ is locally U-small. Consequently, the derived category $\boldsymbol{D}(\boldsymbol{Mod}(k)_{\mathrm{U}})$ is locally U-small.

Proposition **(2.6.16)**. — *Let* Y *be a complex in an abelian category.*

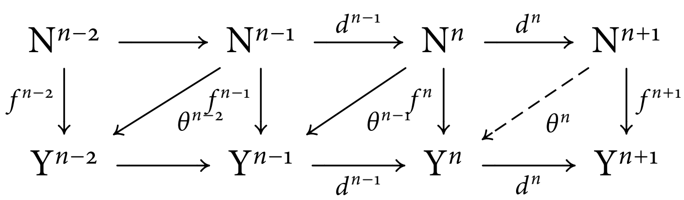
a) *Assume that* $\mathrm{Y}^n = 0$ *for* $n \neq 0$. *Then* Y *is homotopically injective if and only if* Y^0 *is an injective object of* $\boldsymbol{A}$.

b) *Assume that* $\mathrm{Y}^n = 0$ *for* $n < 0$ *and that* Y^n *is an injective object of* $\boldsymbol{A}$ *for* $n \geqslant 0$. *Then* Y *is homotopically injective.*

Proof. — a) Let us assume that Y is homotopically injective and let us show that Y^0 is injective. Let $j\colon \mathrm{X}' \to \mathrm{X}$ be a monomorphism in $\boldsymbol{A}$, let $f'\colon \mathrm{X}' \to \mathrm{Y}^0$ be a morphism; we need to show that there exists $f\colon \mathrm{X} \to \mathrm{Y}^0$ such that $f \circ j = f'$. Let $k\colon \mathrm{X} \to \mathrm{X}''$ be a cokernel of j, so that $\mathrm{N} = (0 \to \mathrm{X}' \xrightarrow{j} \mathrm{X} \xrightarrow{k} \mathrm{X}'' \to 0)$ is an acyclic complex in $\boldsymbol{A}$ (we put the term X′ in degree 0). Morphisms of complexes $u\colon \mathrm{N} \to \mathrm{Y}$ correspond to morphisms $u^0\colon \mathrm{X}' \to \mathrm{Y}^0$; a morphism is null homotopic if and only if extends to X. Since $\boldsymbol{K}(\boldsymbol{A})(\mathrm{N},\mathrm{Y}) = 0$, the morphism f' extends to X, as claimed.

The converse assertion follows from b).

b) Let us now assume that $\mathrm{Y}^n = 0$ for $n < 0$ and that Y^n is injective for $n \geqslant 0$, and let us prove that Y is homotopically injective. Let N be an acyclic complex and let $f\colon \mathrm{N} \to \mathrm{X}$ be a morphism of complexes. In order to show that f is null homotopic, we will construct morphisms $\theta^n\colon \mathrm{N}^{n+1} \to \mathrm{Y}^n$ such that $f^n = \theta^n \circ d_{\mathrm{X}}^n + d_{\mathrm{Y}}^{n-1} \circ \theta^{n-1}$, by induction on n. If $n < 0$, it suffices to define $\theta^n = 0$. We then assume that θ^m is constructed for $m < n$ and proceed to defining θ^n. Consider the diagram



One has

$$f^n \circ d_{\mathrm{N}}^{n-1} = d_{\mathrm{Y}}^{n-1} \circ f^{n-1} = d_{\mathrm{Y}}^{n-1}(\theta^{n-1} \circ d_{\mathrm{N}}^{n-1} + d_{\mathrm{Y}}^{n-2} \circ \theta^{n-2}) = d_{\mathrm{Y}}^{n-1} \circ \theta^{n-1} \circ d_{\mathrm{N}}^{n-1},$$

hence $f^n - d_Y^{n-1}\theta^{n-1} \colon N^n \to Y^n$ factors through $N^n/\operatorname{Im}(d_N^{n-1})$. Since N is acyclic, $\operatorname{Im}(d_N^{n-1}) = \operatorname{Ker}(d_N^n)$. The morphism d_N^n induces a monomorphism $N^n/\operatorname{Ker}(d_N^n) \to N^{n+1}$. Since Y^n is injective, there exists a morphism $\theta^n \colon N^{n+1} \to Y^n$ such that $f^n - d_Y^{n-1}\theta^{n-1} = \theta^n \circ d_N^n$. This provides the required morphism.

This shows that the morphism f is null homotopic and concludes the proof of the proposition. □

Proposition **(2.6.17)**. — *Let $\boldsymbol{A}$ be a Grothendieck abelian category and let $\boldsymbol{I}$ be its additive full subcategory of injective objects. Let* $X \in \boldsymbol{C}(\boldsymbol{A})$ *be a complex, let* $p \in \mathbf{Z}$ *be such that* $H^n(X) = 0$ *for every integer* $n \leqslant p$.

a) *There exists a complex* $Y \in \boldsymbol{C}(\boldsymbol{I})$ *such that* $Y^n = 0$ *for every* $n \leqslant p$ *and a homologism* $f \colon X \to Y$.

b) *If, moreover,* $X^n = 0$ *for* $n \leqslant p$, *then one can moreover choose* Y *and* f *in such a way that* f^n *is a monomorphism for every* $n \in \mathbf{Z}$.

Proof. — We first observe that asssertion *a*) follows from *b*). Indeed, by hypothesis, the canonical morphism $X \to \tau_{\geqslant p}(X)$ is a homologism, and the complex $\tau_{\geqslant p}(X)$ satisfies the hypothesis of *b*), so that X is homologous to a complex in $\boldsymbol{C}^{>p}(\boldsymbol{I})$.

Let us now prove *b*). We shall construct the complex Y and the morphism $i \colon X \to Y$ by induction, degree by degree, in such a way that for every integer $n \in \mathbf{N}$, the induced morphism $X \to \sigma_n(Y)$ induces isomorphisms $H^m(X) \to H^m(Y)$ for $m < n$, and a monomorphism $H^n(X) \to \operatorname{Coker}(d_Y^{n-1})$.

For $n < p$, we take $Y^n = 0$, the morphisms i^n and d_Y^n are taken equal to zero. Assuming that Y^m and i^m are defined for $m \leqslant n$, and that d_Y^m is defined for $m \leqslant n-1$, let us define Y^{n+1}, $i^{n+1} \colon X^{n+1} \to Y^{n+1}$ and $d_Y^n \colon Y^n \to Y^{n+1}$. We first define

$$Z^{n+1} = \operatorname{Coker}(d_Y^{n-1}) \oplus_{\operatorname{Coker}(d_X^{n-1})} \operatorname{Ker}(d_X^{n+1})$$

so to have a cartesian diagram

$$\begin{array}{ccc} \operatorname{Coker}(d_X^{n-1}) & \longrightarrow & \operatorname{Ker}(d_X^{n+1}) \\ \downarrow & & \downarrow \\ \operatorname{Coker}(d_Y^{n-1}) & \longrightarrow & Z^{n+1}. \end{array}$$

This furnishes a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^n(X) & \longrightarrow & \mathrm{Coker}(d_X^{n-1}) & \longrightarrow & \mathrm{Ker}(d_X^{n+1}) & \longrightarrow & H^{n+1}(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^n(X) & \longrightarrow & \mathrm{Coker}(d_Y^{n-1}) & \longrightarrow & Z^{n+1} & \longrightarrow & H^{n+1}(X) & \longrightarrow & 0 \end{array}$$

in which the first line is the exact sequence from lemma 1.6.2.

Let us prove that the second line is exact as well. The morphism $H^n(X) \to \mathrm{Coker}(d_Y^{n-1})$ is a monomorphism, by the induction hypothesis, and the morphism from Z^{n+1} to $H^{n+1}(X)$ is an epimorphism by construction of Z^{n+1}. To prove the remaining two other exactness properties, we pretend that $\boldsymbol{A}$ is a category of modules.

Let us consider the class $\bar{y}$ in $\mathrm{Coker}(d_Y^{n-1})$ of an element y in Y^n which is mapped to 0 in Z^{n+1}; by construction, the element $(\bar{y}, 0)$ of $\mathrm{Coker}(d_Y^{n-1}) \oplus \mathrm{Ker}(d_X^{n+1})$ belongs to the submodule $\mathrm{Coker}(d_X^{n-1})$: there exists an element $x \in X^n$ such that $(\bar{y}, 0) = (\bar{x}, -\bar{x})$, so that $x \in \mathrm{Ker}(d_X^{n+1})$ and y comes from the class of x in $H^n(X)$.

Let us then prove exactness at Z^{n+1}. Let $y \in \mathrm{Coker}(d_Y^{n-1})$ and $x \in \mathrm{Ker}(d_X^{n+1})$ such that the class z of $(\bar{y}, x)$ in Z^{n+1} is mapped to 0 in $H^{n+1}(X)$. This means that x belongs to $\mathrm{Im}(d_X^n)$, so that there exists $x' \in X^n$ such that $x = d_X^n(x')$. In the right hand side of the equality $(y, x) = (y + i^n(x), 0) - (i^n(x), -d_X^n(x'))$, the class in Z^{n+1} of the first term comes from $\mathrm{Coker}(d_Y^{n-1})$, while that of the second term is zero. Consequently, z comes from $\mathrm{Coker}(d_Y^{n-1})$, as was to be shown.

Let now Y^{n+1} be an object of $\boldsymbol{I}$ and $j \colon Z^{n+1} \hookrightarrow Y^{n+1}$ be a monomorphism; these exist, since $\boldsymbol{A}$ is a Grothendieck abelian category. We then define $d_Y^n \colon Y^n \to Y^{n+1}$ to be the compositions with j of the canonical morphisms $Y^n \to \mathrm{Coker}(d_Y^{n-1}) \to Z^{n+1}$. Similarly, we consider the composition $\mathrm{Ker}(d_X^{n+1}) \to Z^{n+1} \xrightarrow{j} Y^{n+1}$; since Y^{n+1} is injective, this morphism can be extended to a morphism $i^{n+1} \colon X^{n+1} \to Y^{n+1}$.

TO BE FINISHED

□

As a corollary, we have the following partial result in the direction of theorem 2.6.12.

Corollary **(2.6.18)**. — *Let $\boldsymbol{A}$ be an abelian category possessing enough injectives, for example, a Grothendieck abelian category. Then the canonical functor $\boldsymbol{K}^+(\boldsymbol{I}) \to \boldsymbol{D}^+(\boldsymbol{A})$ is an equivalence of triangulated categories.*

Proof. — By proposition 2.6.17, every complex in $\boldsymbol{C}^+(\boldsymbol{A})$ is homologous to a complex in $\boldsymbol{K}^+(\boldsymbol{I})$. Such a complex is homotopically injective, by proposition 2.6.16. The corollary follows from that. □

2.7. Derived functors

2.7.1. — Let $\boldsymbol{A}, \boldsymbol{B}$ be abelian categories and let $\mathrm{F}\colon \boldsymbol{A} \to \boldsymbol{B}$ be an additive functor. The functor F induces an additive functor on complexes, $\mathrm{F}\colon \boldsymbol{C}(\boldsymbol{A}) \to \boldsymbol{C}(\boldsymbol{B})$; since it preserves homotopies, it also induces a triangulated functor on $\mathrm{F}\colon \boldsymbol{K}(\boldsymbol{A}) \to \boldsymbol{K}(\boldsymbol{B})$ between the associated homotopy categories. However, if X is an acyclic complex in $\boldsymbol{A}$, the complex F(X) may not be acyclic (unless F is *exact*) so that this functor $\mathrm{F}\colon \boldsymbol{K}(\boldsymbol{A}) \to \boldsymbol{K}(\boldsymbol{B})$ does not induce a natural triangulated functor between the corresponding derived categories.

The theory of right (resp. left) derived functors aims at associating with F a functor $\boldsymbol{R}\mathrm{F}$ (resp. $\boldsymbol{L}\mathrm{F}$) between the corresponding derived categories which reflects the properties of the initial functor F.

We assume that $\boldsymbol{A}$ is a Grothendieck category and let $\boldsymbol{I}$ be the thick triangulated subcategory of $\boldsymbol{K}(\boldsymbol{A})$ of homotopically injective complexes. Then the functor $\boldsymbol{I} \to \boldsymbol{D}(\boldsymbol{A})$ is an equivalence of triangulated categories (corollary 2.6.13). The derived functor $\boldsymbol{R}\mathrm{F}$ is defined as the composition of a (chosen) quasi-inverse of this equivalence, the functor F, and the canonical functor to $\boldsymbol{D}(\boldsymbol{B})$. This is depicted by the diagram

$$\begin{array}{ccccc} \boldsymbol{I} & \longrightarrow & \boldsymbol{K}(\boldsymbol{A}) & \xrightarrow{\mathrm{F}} & \boldsymbol{K}(\boldsymbol{B}) \\ & \searrow^{\sim} & \downarrow \mathrm{h} & & \downarrow \mathrm{h} \\ & & \boldsymbol{D}(\boldsymbol{A}) & \xrightarrow{\boldsymbol{R}\mathrm{F}} & \boldsymbol{D}(\boldsymbol{B}) \end{array}$$

in which, we insist, the square is usually *not* commutative! Explicitly, given a complex $\mathrm{X} \in \boldsymbol{K}(\boldsymbol{A})$, the "recipe" to compute $\boldsymbol{R}\mathrm{F}(\mathrm{X})$ consists in taking the chosen homotopically injective *resolution* of X, namely a homologism $\varepsilon\colon \mathrm{X} \to \mathrm{I}$ to a homotopically injective complex I, and defining $\boldsymbol{R}\mathrm{F}(\mathrm{X}) = \mathrm{F}(\mathrm{I})$. Moreover, $\boldsymbol{R}\mathrm{F}(\mathrm{X})$ is endowed with a canonical morphism $\mathrm{F}(\varepsilon)\colon \mathrm{F}(\mathrm{X}) \to \boldsymbol{R}\mathrm{F}(\mathrm{X})$ in $\boldsymbol{K}(\boldsymbol{B})$.

Let $f : X \to X'$ be a morphism in $\boldsymbol{K}(\boldsymbol{A})$, let $\varepsilon' : X' \to I'$ be the chosen homotopically injective resolution of X'. Since I' is homotopically injective, there exists a unique morphism $f' : I \to I'$ in $\boldsymbol{K}(\boldsymbol{A})$ such that $f' \circ \varepsilon = f \circ \varepsilon'$ (proposition 2.6.10); let us define $\boldsymbol{R}F(f) = F(f') : F(I) \to F(I')$.

These maps define a *functor* $\boldsymbol{K}(\boldsymbol{A}) \to \boldsymbol{D}(\boldsymbol{B})$.

If $f : X \to X'$ is a homologism, then the morphism $f' : I \to I'$ constructed above is an isomorphism in $\boldsymbol{K}(\boldsymbol{A})$ (proposition 2.6.10), so that $\boldsymbol{R}F(f)$ is an isomorphism. Consequently, the functor we have just defined extends to a functor

$$\boldsymbol{R}F : \boldsymbol{D}(\boldsymbol{A}) \to \boldsymbol{D}(\boldsymbol{B}),$$

called the *right derived functor* of F.

Remark **(2.7.2)**. — a) For every object X of $\boldsymbol{K}(\boldsymbol{A})$ and every homologism $\varepsilon : X \to I$, where I is homotopically injective, the morphism $F(\varepsilon) : F(X) \to F(I) = \boldsymbol{R}F(X)$ in $\boldsymbol{K}(\boldsymbol{B})$ satisfies an universal property: for every object Y of $\boldsymbol{K}(\boldsymbol{B})$, the canonical morphism $F(\varepsilon) : h(F(X)) \to \boldsymbol{R}F(X)$ induces an isomorphism

$$\operatorname*{colim}_{X \xrightarrow{s} X'} \boldsymbol{K}(\boldsymbol{A})(Y, F(X')) \xrightarrow{\sim} \boldsymbol{D}(\boldsymbol{B})(h(Y), \boldsymbol{R}F(X)).$$

(In fact, the system $(F(X'))_{X \to_s X'}$ is "eventually constant".)

This universal property of the right derived functor $\boldsymbol{R}F$ is also formulated by saying that $\boldsymbol{R}F$ is a left Kan extension of the functor $h \circ \boldsymbol{K}(F)$ with respect to the localization functor $h : \boldsymbol{K}(\boldsymbol{A}) \to \boldsymbol{D}(\boldsymbol{A})$.

b) If we remove the hypothesis that the abelian category $\boldsymbol{A}$ is a Grothendieck category, then the right derived functor $\boldsymbol{R}F$ may not exist. However, the left hand side of this formula furnishes a definition, for every object X, of a "functor $\boldsymbol{R}F(X)$" on the category $\boldsymbol{D}(\boldsymbol{B})$. One then may say that F is right derivable at X if this "functor $\boldsymbol{R}F(X)$" is representable and denote by $\boldsymbol{R}F(X)$ an object that represents it. The previous construction shows that the functor F is right derivable at every object that admits a homotopically injective resolution.

Lemma **(2.7.3)**. — *Let* $F : \boldsymbol{A} \to \boldsymbol{B}$ *be an additive functor between abelian categories; assume that* $\boldsymbol{A}$ *is a Grothendieck abelian category. Let* $X \in \boldsymbol{D}^{+}(\boldsymbol{A})$ *and let* $a \in \mathbf{Z}$.

a) *If* $X \in \boldsymbol{D}^{\geqslant a}(\boldsymbol{A})$, *then* $\boldsymbol{R}F(X) \in \boldsymbol{D}^{\geqslant a}(\boldsymbol{B})$. *In particular,* $\boldsymbol{R}F(X) \in \boldsymbol{D}^{+}(\boldsymbol{B})$;

b) *For every integer n such that* $n \leqslant a$, *the canonical morphism* $\tau_{\leqslant a}(X) \to X$ *induces an isomorphism* $H^{i}(\boldsymbol{R}F(\tau_{\leqslant a}(X))) \to H^{i}(\boldsymbol{R}F(X))$.

Proof. — a) The construction of a homotopically injective resolution of X is a complex I such that $I^n = 0$ for $n < a$. Consequently, $\boldsymbol{R}F(X) = F(I)$ is a complex whose terms of degree $< a$ vanish. This implies that $H^n(\boldsymbol{R}F(X)) = H^n(F(I))) = 0$ for $n < a$. Consquently, $\boldsymbol{R}F(X) \in \boldsymbol{D}^{\geqslant a}(\boldsymbol{B})$. The second assertion follows readily since there exists an integer a such that $X \in \boldsymbol{D}^{\geqslant a}(\boldsymbol{A})$.

b) Let us consider the distinguished triangle $\tau_{\leqslant a}(X) \to X \to \tau_{>a}(X) \to \Sigma\tau_{\leqslant a}(X)$ in $\boldsymbol{D}(\boldsymbol{A})$. Applying the triangulated functor $\boldsymbol{R}F$ and a shift, we obtain a distinguished triangle

$$\Sigma^{-1}\boldsymbol{R}F(\tau_{>a}(X)) \to \boldsymbol{R}F(\tau_{\leqslant a}(X)) \to \boldsymbol{R}F(X) \to \boldsymbol{R}F(\tau_{>a}(X)).$$

Let n be an integer and let us apply the cohomological functor H^n: we obtain an exact sequence

$$H^{n-1}(\boldsymbol{R}F(\tau_{>a}(X))) \to H^n(\boldsymbol{R}F(\tau_{\leqslant a}(X))) \to H^n(\boldsymbol{R}F(X)) \to H^n(\boldsymbol{R}F(\tau_{>a}(X))).$$

The complex $\tau_{>a}(X)$ belongs to $\boldsymbol{D}^{>a}(\boldsymbol{A})$, hence its image by $\boldsymbol{R}F$ belongs to $\boldsymbol{D}^{>a}(\boldsymbol{B})$, by *a*). If $n < a$, then the extreme terms of this exact sequence vanish, so that the considered morphism

$$H^n(\boldsymbol{R}F(\tau_{\leqslant a}(X))) \to H^n(\boldsymbol{R}F(X))$$

is an isomorphism. □

2.7.4. — Assume that the functor F is left exact. For every integer n, let $\boldsymbol{R}^nF\colon \boldsymbol{A} \to \boldsymbol{B}$ be the composition of the inclusion functor $\boldsymbol{A} \to \boldsymbol{D}(\boldsymbol{A})$, of $\boldsymbol{R}F$, and of H^n. Every exact sequence $0 \to X \to Y \to Z \to 0$ in $\boldsymbol{A}$ gives rise to a distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\boldsymbol{D}^+(\boldsymbol{A})$. Applying the triangulated functor $\boldsymbol{R}F$ and the cohomology functor $H^n = H \circ \Sigma^n$, we obtain to a long exact sequence

$$\cdots \to \boldsymbol{R}^{n-1}F(Z) \to \boldsymbol{R}^n(F(X)) \to \boldsymbol{R}^nF(Y) \to \boldsymbol{R}^nF(Z) \to \boldsymbol{R}^{n+1}F(X) \to \dots$$

Let X be an object in $\boldsymbol{A}$. Viewing X as a complex concentrated in degree 0, we already know that $\boldsymbol{R}F(X) \in \boldsymbol{D}^{\geqslant 0}(X)$. Consequently, $\boldsymbol{R}^n\boldsymbol{F}(X) = 0$ for $n < 0$.

Let us then choose an injective resolution $\varepsilon\colon X \to I$ of X, where I is a complex with injective terms such that $I^n = 0$ for $n < 0$. Since F is left exact, the exact sequence $0 \to X \to I^0 \to I^1$ gives rise to an exact sequence $0 \to F(X) \to F(I^0) \to F(I^1)$, which shows that the morphism $F(\varepsilon)\colon F(X) \to \boldsymbol{F}(I) = \boldsymbol{R}F(X)$ induces an isomorphism $F(X) \simeq \boldsymbol{R}^0F(X)$.

Definition (**2.7.5**). — *Let* $F : \boldsymbol{A} \to \boldsymbol{B}$ *be a left exact additive functor between abelian categories. One says that a full additive subcategory* $\boldsymbol{A}_0$ *of* $\boldsymbol{A}$ *is* injective with respect to F, *or is* F-injective, *if the following conditions hold:*

(i) *Every object of* $\boldsymbol{A}$ *admits a monomorphism into an object of* $\boldsymbol{A}_0$*;*

(ii) *For every exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $\boldsymbol{A}$*, where* X *and* Y *are objects of* $\boldsymbol{A}_0$*, the object* Z *belongs to* $\boldsymbol{A}_0$*;*

(iii) *For every exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $\boldsymbol{A}$*, with objects in* $\boldsymbol{A}_0$*, the complex* $0 \to F(X) \to F(Y) \to F(Z) \to 0$ *is exact.*

Example (**2.7.6**). — If the category $\boldsymbol{A}$ has enough injectives, then the full subcategory of injectives objects of $\boldsymbol{A}$ is F-injective for every left-exact functor F.

Assertion (i) is the definition of having enough injectives. Let $0 \to X \xrightarrow{j} Y \xrightarrow{p} Z \to 0$ be an exact sequence of objects in $\boldsymbol{A}$, where X and Y are injective objects. Since X is an injective object, the identity id_X extends along the monomorphism $j : X \to Y$, hence there exists a morphism $r : Y \to X$ such that $r \circ j = \mathrm{id}_X$; consequently, the morphism $(r, p) : Y \to X \oplus Z$ is then an isomorphism and the given exact sequence is split. It first follows that Z, a direct summand of an injective object, is injective as well, and then that the image of the given exact sequence under any additive morphism is again exact.

Proposition (**2.7.7**). — *Let* $F : \boldsymbol{A} \to \boldsymbol{B}$ *be a left exact additive functor between abelian categories. Let* $\boldsymbol{A}_0$ *be an* F*-injective subcategory of* $\boldsymbol{A}$.

a) *For every complex* $Y \in \boldsymbol{K}^+(\boldsymbol{A})$ *with terms in* $\boldsymbol{A}_0$*, the canonical morphism* $F(Y) \to \boldsymbol{R}F(Y)$ *in* $\boldsymbol{D}^+(\boldsymbol{B})$ *is an isomorphism;*

b) *For every complex* $X \in \boldsymbol{K}^+(\boldsymbol{A})$*, there exists a complex* $Y \in \boldsymbol{K}^+(\boldsymbol{A}_0)$ *and a homologism* $\varepsilon : X \to Y$ *in* $K^+(\boldsymbol{A})$.

In particular, one can define $\boldsymbol{R}F(X)$ by considering an arbitrary F-injective resolution of X.

Proof. — □

Remark (**2.7.8**). — Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a left-exact functor between abelian categories. Let $\boldsymbol{A}_0$ be an F-injective subcategory of $\boldsymbol{A}$. Then, every object X of $\boldsymbol{A}_0$ satisfies $\boldsymbol{R}^nF(X) = 0$ for $n > 0$ — one says that X is F-*acyclic.*

2.7.9. — Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be Grothendieck abelian categories, let $\mathrm{F}\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\mathrm{G}\colon \boldsymbol{B} \to \boldsymbol{C}$ be additive functors, so that the three derived functors $\boldsymbol{R}\mathrm{F}$, $\boldsymbol{R}\mathrm{G}$ and $\boldsymbol{R}(\mathrm{G} \circ \mathrm{F})$ are defined. Let us construct a canonical morphism of functors

$$\boldsymbol{R}(\mathrm{G} \circ \mathrm{F}) \to \boldsymbol{R}\mathrm{G} \circ \boldsymbol{R}\mathrm{F}$$

Let X be any object in $\boldsymbol{K}(\boldsymbol{A})$ and let $\varepsilon\colon \mathrm{X} \to \mathrm{I}$ be a homologism from X to a homotopically injective object I. By construction, $\boldsymbol{R}\mathrm{F}(\mathrm{X}) = \mathrm{F}(\mathrm{I})$ and $\boldsymbol{R}(\mathrm{G} \circ \mathrm{F})(\mathrm{X}) = (\mathrm{G} \circ \mathrm{F})(\mathrm{I})$. Let J be a homotopically injective object in $\boldsymbol{K}(\boldsymbol{B})$ and let $\eta\colon \mathrm{F}(\mathrm{I}) \to \mathrm{J}$ be a homologism, so that $\boldsymbol{R}\mathrm{G}(\mathrm{F}(\mathrm{I})) = \mathrm{G}(\mathrm{J})$. Evaluated at X, the canonical morphism

$$\boldsymbol{R}(\mathrm{G} \circ \mathrm{F})(\mathrm{X}) \to \boldsymbol{R}\mathrm{G} \circ \boldsymbol{R}\mathrm{F}(\mathrm{X})$$

is then the morphism

$$\mathrm{G}(\eta)\colon \boldsymbol{R}(\mathrm{G} \circ \mathrm{F})(\mathrm{X}) = (\mathrm{G} \circ \mathrm{F})(\mathrm{I}) \to \mathrm{G}(\mathrm{J}) = \boldsymbol{R}\mathrm{G}(\mathrm{F}(\mathrm{I})) = \boldsymbol{R}\mathrm{G} \circ \boldsymbol{R}\mathrm{F}(\mathrm{X}).$$

We leave to the reader to check that this construction furnishes a morphism of functors.

Corollary (**2.7.10**). — *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be Grothendieck categories and let $\mathrm{F}\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\mathrm{G}\colon \boldsymbol{B} \to \boldsymbol{C}$ be left exact functors. Let $\boldsymbol{A}_0$ be an* F*-injective subcategory of $\boldsymbol{A}$ and let $\boldsymbol{B}_0$ be an* G*-injective subcategory of $\boldsymbol{B}$ such that $\mathrm{F}(\mathrm{X}) \in \mathrm{ob}(\boldsymbol{B}_0)$ for every object* X *of $\boldsymbol{A}_0$. Then the canonical morphism of functors*

$$\boldsymbol{R}(\mathrm{G} \circ \mathrm{F}) \to \boldsymbol{R}\mathrm{G} \circ \boldsymbol{R}\mathrm{F}, \qquad \boldsymbol{D}^+(\boldsymbol{A}) \to \boldsymbol{D}^+(\boldsymbol{C}),$$

is an isomorphism.

Proof. — Let X be any object in $\boldsymbol{K}^+(\boldsymbol{A})$ and let $\varepsilon\colon \mathrm{X} \to \mathrm{I}$ be a homologism from X to a homotopically injective object I. We may assume that X is a bounded below complex, with terms in $\boldsymbol{A}_0$. Since $\boldsymbol{A}_0$ is F-injective, the morphism $\mathrm{F}(\varepsilon)\colon \mathrm{F}(\mathrm{X}) \to \mathrm{F}(\mathrm{I})$ in $\boldsymbol{K}^+(\boldsymbol{B})$ is a homologism and induces an isomorphism in $\boldsymbol{D}(\boldsymbol{B})$ from F(X) to $\mathrm{F}(\mathrm{I}) = \boldsymbol{R}\mathrm{F}(\mathrm{X})$. Moreover, one has $\boldsymbol{R}(\mathrm{G} \circ \mathrm{F})(\mathrm{X}) = (\mathrm{G} \circ \mathrm{F})(\mathrm{I})$. Let J be a homotopically injective object in $\boldsymbol{K}^+(\boldsymbol{B})$ and let $\eta\colon \mathrm{F}(\mathrm{I}) \to \mathrm{J}$ be a homologism, so that $\boldsymbol{R}\mathrm{G}(\mathrm{F}(\mathrm{I})) = \mathrm{G}(\mathrm{J})$. On the other hand, since F(X) belongs to $\boldsymbol{K}^+(\boldsymbol{B}_0)$, the morphism $\mathrm{G}(\eta)\colon \mathrm{G}(\mathrm{F}(\mathrm{X})) \to \mathrm{G}(\mathrm{J})$ is an isomorphism, because $\boldsymbol{B}_0$ is an F-injective subcategory of $\boldsymbol{B}$, as was to be shown. □

2.8. Exercises

Exercise **(2.8.1)**. — Let $\mathbf{C}$ be an abelian category and let X be a complex in $\mathbf{C}$. One says that X is contractible if id_X is null homotopic.

a) Prove that X is contractible if and only if it is acyclic and if the exact sequence $0 \to \mathrm{Ker}(d^n_X) \to X^n \to \mathrm{Im}(d^n_X) \to 0$ is split, for every integer $n \in \mathbf{Z}$.

b) Let k be a ring and let $\mathbf{C}$ be the category of k-modules. Assume that X is an acyclic complex such that X^n is a free k-module, for every $n \in \mathbf{Z}$.

Prove that X is contractible if, moreover, k is a field or a principal ideal domain.

c) Prove that X is contractible if, moreover one has $X^n = 0$ for every $n < 0$.

d) Let $k = \mathbf{Z}/4\mathbf{Z}$ and $X^n = \mathbf{Z}/4\mathbf{Z}$ for every $n \in \mathbf{Z}$, let d^n_X be given by the multiplication by 2. Prove that X is acyclic but not contractible.

Exercise **(2.8.2)**. — Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle in a (pre)triangulated category. Establish the equivalence of the following conditions: (i) u is an isomorphism; (ii) $v = 0$; (iii) $w = 0$. If they hold, prove that $Z = 0$.

Exercise **(2.8.3)**. — (From (Gelfand & Manin, 2003, Exercise III.4.1).) Let $\mathbf{A}$ be an abelian category and let $f : X \to Y$ be a morphism of complexes in $\mathbf{A}$. Consider the four following statements: (i) $f = 0$ in $\mathbf{C}(\mathbf{A})$; (ii) $f = 0$ in $\mathbf{K}(\mathbf{A})$; (iii) $f = 0$ in $\mathbf{D}(\mathbf{A})$; (iv) $H^n(f) = 0$ for every $n \in \mathbf{Z}$.

a) Establish the following implications: (i)⇒(ii)⇒(iii)⇒(iv).

b) Give examples where (ii) holds but not (i), and (iii) holds but not (ii).

c) Let $\mathbf{A} = \mathbf{Ab}$ be the category of abelian groups and let f be given by the following morphism of complexes

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}/3\mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

where the horizontal maps are induced by multiplication by 2, while the vertical maps are the canonical ones. Prove that $H^n(f) = 0$ for all $n \in \mathbf{Z}$ but that $f \neq 0$ in $\mathbf{D}(\mathbf{A})$.

CHAPTER 3

COHOMOLOGY OF SHEAVES

3.1. General topology

3.1.1. — Let X be a topological space. We say that X is *separated* (equivalently, Hausdorff) if any two distinct points admit disjoint neighborhoods.

We say that X is *compact* if it is separated and if it satisfies the Borel–Lebesgue covering property: if a family of open subsets covers X, then a finite subfamily already covers X.

We say that X is *locally compact* if it is separated and if every point of X has a compact neighborhood.

Let X be a topological space and let A be a subset of X. One says that A is *locally closed* if for every point $a \in A$, there exists a neighborhood U of a in X such that $A \cap U$ is a closed subset of U.

Lemma **(3.1.2)**. — *Let* X *be topological space and let* A *be a subset of* X.

a) *Assume that* A *is locally closed in* X. *The union* U *of all open subsets* V *of* X *such that* $A \cap V$ *is a closed subset of* V *is an open subset of* X *of which* A *is a closed subset, and it is the largest such open subset.*

b) *The following properties are equivalent: (i) The set* A *is locally closed in* X*; (ii) There exist an open subset* U *and a closed subset* Z *of* X *such that* $A = U \cap Z$*; (iii) The set* A *is open in its closure* $\overline{A}$.

c) *Assume that* X *is locally compact. Then* A *is locally closed in* X *if and only if it is locally compact for the induced topology.*

Proof. — a) The set U is open in X, because it is the union of a family open subsets of X. By definition of a locally closed subset of X, one has $A \subset U$. Let us prove that A is closed in U. Let $x \in U - A$; by construction of U, there exists a open subset V of X, containing x, such that $A \cap V$ is closed in V; then $V - (A \cap V)$

is an open neighborhood of x in U, which proves the claim. Any open subset V of X of which A is a closed subset is a member of the family of which U is the union, so that U is indeed the largest such open subset of X.

b) (i)⇒(ii). By *a*), there exists an open subset U of X of which A is a closed subset. By definition of the induced topology of U, there exists a closed subset Z of X such that $A = Z \cap U$, as was to be shown.

(ii)⇒(iii). Under the assumptions of (ii), the relation $U - A = U - (U \cap Z)$ shows that A is closed in U, so that $\overline{A} \cap U = A$. By definition of the induced topology of $\overline{A}$, this proves that A is open in $\overline{A}$.

(iii)⇒(i). Let V be an open subset of X such that $A = V \cap \overline{A}$. One observes that for every $a \in A$, the set V is an open neighborhood of a such that $A \cap V = A$ is closed in V. Consequently, A is locally closed in X.

c) First assume that A is locally closed in X and let U be an open subset of X of which A is a closed subset. Then U is locally compact, hence A is locally compact as well, because open subsets and closed subsets of a locally compact space are locally compact. Let us now assume that A is locally compact. Let $a \in A$ and let W be a compact neighborhood of a in A which is compact, hence closed because X is separated (being locally compact). Let V be a closed neighborhood of a in X such that $W = A \cap V$. Since W is compact, it is closed in V; then $\mathring{V}$ is an open neighborhood of a in X and

$$A \cap \mathring{V} = (A \cap V) \cap \mathring{V} = W \cap \mathring{V}$$

is closed in $\mathring{V}$. This proves that A is locally closed in X.

□

Definition (**3.1.3**). — *A continuous map $f : X \to Y$ is* separated *if for every pair (x, x') of points of X such that $f(x) = f(x')$ and $x \neq x'$, there exist disjoint open subsets U and U′ of X such that $x \in U$ and $x' \in U'$.*

If X is separated, then every continuous map with origin X is separated.

Definition (**3.1.4**). — *A continuous map $f : X \to Y$ is* closed *if $f(A)$ is a closed subset of Y, for every closed subset A of X. It is* proper *if it is universally closed, that is, if the map $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed, for every topological space Z.*

Proposition (**3.1.5**). — *Let $f : X \to Y$ be a continuous map; let us assume that X is separated.*

a) *The map* f *is proper if and only if* $f^{-1}(y)$ *is compact, for every* $y \in \mathrm{Y}$.

b) *If* X *and* Y *are locally compact (in particular, separated), the map* f *is proper if and only if* $f^{-1}(\mathrm{A})$ *is a compact subset of* X*, for every compact subset* A *of* Y.

Proposition **(3.1.6)**. — *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a continuous map.*

a) *Assume that* f *is proper (resp. separated). Then for every subspace* A *of* Y*, the map* $f_{\mathrm{A}} : f^{-1}(\mathrm{A}) \to \mathrm{A}$ *induced by* f *by restriction is proper (resp. separated).*

b) *Assume that* f *is proper (resp. separated). Then for every closed subspace* Z *of* X*, the map* $f|_{\mathrm{Z}} : \mathrm{Z} \to \mathrm{Y}$ *is proper (resp. separated).*

c) *Assume that there exists an open covering* $\mathscr{V}$ *of* V *such that for every* $\mathrm{V} \in \mathscr{V}$*, the map* $f_{\mathrm{V}} : f^{-1}(\mathrm{V}) \to \mathrm{V}$ *is proper (resp. separated). Then* f *is proper (resp. separated).*

Definition **(3.1.7)**. — *A topological space* X *is* paracompact *if for every open covering* $\mathscr{U}$ *of* X*, there exists an open covering* $\mathscr{V}$ *of* X *satisfying the following properties:*

a) *For every* $\mathrm{V} \in \mathscr{V}$*, there exists* $\mathrm{U} \in \mathscr{U}$ *such that* $\mathrm{V} \subset \mathrm{U}$ *(the covering* $\mathscr{V}$ refines *the covering* $\mathscr{U}$*);*

b) *Every point of* X *has an open neighborhood* A *such that the set of* $\mathrm{V} \in \mathscr{V}$ *such that* $\mathrm{A} \cap \mathrm{V} \neq \varnothing$ *is finite.*

A compact topological space is paracompact; a metrizable topological space is paracompact; every subspace of a cellular space is paracompact.

Lemma **(3.1.8)**. — *Let* X *be a locally compact topological space.*

a) *For every open subset* U *of* X *and every point* $a \in \mathrm{U}$*, there exists an open neighborhood* V *of* a *such that* $\overline{\mathrm{V}} \subset \mathrm{U}$.

b) *Let* $(\mathrm{U}_1, \dots, \mathrm{U}_n)$ *be a finite family of open subsets of* X*, let* $\mathrm{U} = \mathrm{U}_1 \cup \dots \cup \mathrm{U}_n$ *and let* A *be a compact subset of* X *which is contained in* U*. There exists a family* $(\mathrm{V}_1, \dots, \mathrm{V}_n)$ *of open subsets of* X *such that* $\mathrm{A} \subset \mathrm{V}_1 \cup \dots \cup \mathrm{V}_n$ *and* $\overline{\mathrm{V}_i} \subset \mathrm{U}_i$ *for every* $i \in \{1, \dots, n\}$.

c) *In particular, for every compact subset* A *of* X *and every open neighborhood* U *of* A*, there exists an open neighborhood* V *of* A *such that* $\overline{\mathrm{V}} \subset \mathrm{U}$.

Proof. — a) By definition of a locally compact topological space, the point a admits a compact neighborhood C. Replacing U by $\mathrm{U} \cap \mathring{\mathrm{C}}$, we may assume that $\overline{\mathrm{U}}$ is compact. Let then $\mathrm{A} = \partial(\mathrm{U}) = \overline{\mathrm{U}} - \mathrm{U}$; it is a compact subset of X which does not contain the point a. Since X is separated, there exists, for every point $x \in A$, an open neighborhood V_x of a and an open neighborhood W_x of x such that $V_x \cap W_x = \varnothing$. Since A is compact, there exists a finite family S of A such that the W_x, for $x \in S$, cover A; let then $W = \bigcup_{x\in S} W_x$ and $V = U \cap \bigcap_{x\in S} V_x$. By construction, W is an open neighborhood of A, V is an open neighborhood of a contained in U, and $V \cap W = \varnothing$. Moreover,

$$\overline{V} \subset \overline{U} \cap \bigcap_{x\in S} \overline{V_x} \subset \overline{U} - \bigcup_{x\in S} W_x \subset U.$$

b) For every $a \in A$, let V_a be a compact neighborhood of a which is contained in U. When $a \in A$, the interiors $\mathring{V}_a$ form an open covering of A; since A is compact, there exists a finite subset S of A such that $A \subset \bigcup_{a\in S} \mathring{V}_a$. The latter set is an open neighborhood V of A; since S is finite, $\overline{V} \subset \bigcup_{a\in S} V_a \subset U$.

c) For every $a \in A$, let us choose an index $i(a) \in \{1, \dots, n\}$ such that $a \in U_{i(a)}$ and an open neighborhood W_a of a such that $\overline{W_a} \subset U_{i(a)}$. The family $(W_a)_{a\in A}$ of open subsets of X covers A; consequently, there exists a finite subset S of A such that the family $(W_a)_{a\in S}$ covers A. For every $i \in \{1, \dots, n\}$, let V_i be the union of the open sets W_a, for $a \in S$ such that $i = i(a)$; it is an open subset of X such that $\overline{V_i} = \bigcup_{\substack{a\in S\\ i(a)=i}} \overline{W_a} \subset \overline{U_i}$. Moreover, $\bigcup_{i\in I} V_i = \bigcup_{a\in S} W_a$ is a neighborhood of A in X. □

3.2. Abelian categories of abelian sheaves

3.2.1. — Let X be a topological space. The set Op(X) of open subsets of X is ordered by inclusion; we consider the associated category $\boldsymbol{Op}(X)$.

Let $\boldsymbol{C}$ be a category (for instance, the category of sets, or the category of abelian groups). A $\boldsymbol{C}$*-presheaf* on X is a contravariant functor $\mathscr{F}$ from $\boldsymbol{Op}(X)$ to the category $\boldsymbol{C}$. A morphism of presheaves is a morphism of functors.

It the category $\boldsymbol{C}$ admits limits (resp. colimits), then the category of $\boldsymbol{C}$-presheaves on X admits limits (resp. colimits), which are computed pointwise.

3.2.2. — Let U be an open subset of X and let $\mathscr{U}$ be an open covering of U. To these data, we attach a quiver whose vertices are the pairs $\{V, V'\}$ of elements of $\mathscr{U}$, this vertex being the target of two arrows of respective origins $\{V\}$ and $\{V'\}$. Every $\boldsymbol{C}$-presheaf $\mathscr{F}$ defines a diagram, with value $\mathscr{F}(V \cap V')$ at the vertex $\{V, V'\}$, and with morphisms induced by the inclusions $V \cap V' \subset V$ (resp. $V \cap V' \subset V'$), for $V, V' \in \mathscr{U}$.

One says that a $\boldsymbol{C}$-presheaf is a *sheaf* if for every open subset U of X, and every open covering $\mathscr{U}$ of U, the cone $(\mathscr{F}(U) \to \mathscr{F}(V \cap V'))_{V,V' \in \mathscr{U}}$ is a limit.

3.2.3. — Assume that the category $\boldsymbol{C}$ admits limits. Then the natural functor from the category of sheaves on X to the category of presheaves admit a left adjoint $\mathscr{F} \to \mathscr{F}^{\dagger}$, inducing, for every presheaf $\mathscr{F}$ and every sheaf $\mathscr{G}$, a bijection

$$\operatorname{Hom}(\mathscr{F}^{\dagger}, \mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}, \mathscr{G})).$$

Let us consider a diagram of sheaves. Viewed as a diagram of presheaves, its limit is a sheaf, and is its limit in the category of sheaves.

However, the presheaf-colimit of this diagram is usually not a sheaf; the associated sheaf furnishes a colimit in the category of sheaves.

3.2.4. — We denote by $\boldsymbol{Ab}(X)$ the category of sheaves of abelian groups on X. If $\mathscr{O}_X$ is sheaf of rings on X, we denote by $\boldsymbol{Mod}(\mathscr{O}_X)$ the category of $\mathscr{O}_X$-modules on X. Let k be an abelian group (resp. a ring); we write k_X for the constant sheaf on X associated with k. One has $\boldsymbol{Mod}(\mathbf{Z}_X) = \boldsymbol{Ab}(X)$.

These categories admit limits and colimits. Limits (for example products or equalizers), are computed as presheaves. Colimits, for example coproducts or coequalizers, require to consider the sheaf associated with the colimit-presheaf.

Theorem **(3.2.5)**. — *Let* X *be a topological space and let* $\mathscr{O}$ *be a sheaf of rings on* X*. The category* $\boldsymbol{Mod}(\mathscr{O})$ *of sheaves of* $\mathscr{O}$*-modules on* X *is a Grothendieck abelian category. In particular, the category* $\boldsymbol{Ab}(X)$ *of abelian sheaves on* X *is a Grothendieck abelian category.*

As a consequence, these categories have enough injective objects (theorem 1.4.10).

3.2.6. — Assume that $\mathscr{O}_X$ is a sheaf of commutative rings. Then the category of $\mathscr{O}_X$-modules admits an internal Hom bifunctor $\mathit{Hom}(\cdot,\cdot)$, and a tensor product. These are additive functors.

3.2.7. — Let $x \in X$. The fiber of a an abelian sheaf $\mathscr{F}$ at a point x is the colimit

$$\mathscr{F}_x = \operatorname*{colim}_{U \ni x} \mathscr{F}(U).$$

The functor $\boldsymbol{Ab}(X) \to \boldsymbol{Ab}$ given by $\mathscr{F} \mapsto \mathscr{F}_x$ is exact.

3.2.8. — Let $f\colon X \to Y$ be a continuous map. For every open subset V of Y, $f^{-1}(V)$ is an open subset of X. Consequently, f induces a functor $\boldsymbol{Op}(Y) \to \boldsymbol{Op}(X)$.

By composition, every presheaf $\mathscr{F}$ on X induces a presheaf $f_*\mathscr{F}$ on Y. If $\mathscr{F}$ is a sheaf, then $f_*\mathscr{F}$ is a sheaf as well.

The mapping $\mathscr{F} \mapsto f_*\mathscr{F}$ gives rise to a functor $f_*\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}(Y)$.

This functor has a left adjoint, denoted f^*, which can be defined as follows. For every sheaf $\mathscr{G}$ on Y and every open subset U of X, let $f^*_{\mathrm{pre}}\mathscr{G}(U) = \operatorname{colim}_{V \supset f(U)} \mathscr{G}(V)$. The universal maps endow $f^*_{\mathrm{pre}}\mathscr{G}$ with the structure of a presheaf on Y. Let $f^*\mathscr{G}$ be the associated sheaf.

For every open subset V of Y, one has $f(f^{-1}(V)) \subset V$. The canonical morphism $\mathscr{G}(V) \to \operatorname{colim}_{W \supset f(f^{-1}(W))} \mathscr{G}(W) = f^*_{\mathrm{pre}}\mathscr{G}(f^{-1}(V))$. furnishes a morphism of sheaves $\varepsilon_{\mathscr{G}}\colon \mathscr{G} \to f_*f^*\mathscr{G}$. This morphism is functorial in $\mathscr{G}$.

On the other hand, for every open subset U of X and every open subset V of Y containing $f(U)$, one has $U \subset f^{-1}(V)$; then, the canonical map $f^*_{\mathrm{pre}}(f_*\mathscr{F})(U) = \operatorname{colim}_{V \supset f(U)} f_*\mathscr{F}(V) = \operatorname{colim}_{V \supset f(U)} \mathscr{F}(f^{-1}(V)) \to \mathscr{F}(U)$ defines a morphism of presheaves $f^*_{\mathrm{pre}}(f_*\mathscr{F}) \to \mathscr{F}$, hence a morphism of sheaves $\varepsilon_{\mathscr{F}}\colon f^*(f_*\mathscr{F}) \to \mathscr{F}$. This morphism is functorial in $\mathscr{F}$.

The morphisms $\varepsilon_\cdot$ and $\eta_\cdot$ are the unit and the counit of an adjunction $f^* \dashv f_*$, and furnish functorial bijections:

$$\operatorname{Hom}_X(f^*\mathscr{G}, \mathscr{F}) \simeq \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

for every sheaf $\mathscr{F}$ on X and every sheaf $\mathscr{G}$ on Y.

At the level of fibers, one has functorial isomorphisms $(f^*\mathscr{G})_x \simeq \mathscr{G}_{f(x)}$, for every $x \in X$. This implies in particular that the functor $f^*\colon \boldsymbol{Ab}(Y) \to \boldsymbol{Ab}(X)$ is exact.

3.2.9. — When $\boldsymbol{C}$ is the category of sets, sheaves have an alternative, topological, definition in terms of étale spaces over X, that is, a topological space E endowed with a continuous map $p\colon E\to X$ which is a local homeomorphism (every point of E has an open neighborhood U such that $p|_U$ is a homeomorphism from U to an open subset of X).

To every étale space $p\colon E\to X$ is associated its sheaf $\mathscr{F}$ of continuous sections. The fiber $\mathscr{F}_x$ is identified with $p^{-1}(x)$. Conversely, given a sheaf $\mathscr{F}$, one endowes the set $E_{\mathscr{F}} = \coprod_{x\in X}\mathscr{F}_x$ with a topology so that the projection $E_{\mathscr{F}}\to X$ is étale and its sheaf of continuous sections identifies with $\mathscr{F}$.

In the setting of étale spaces, the functor f^* corresponds to the fiber product of topological spaces.

3.2.10. — Let $\mathscr{F}$ be an abelian sheaf on X. Let U be an open subset of X and let $s\in\mathscr{F}(U)$. By the sheaf property of $\mathscr{F}$, if $\mathscr{V}$ is a family of open subsets of U with union V, such that $s|_W = 0$ for every $W\in\mathscr{V}$, then $s|_V = 0$. The *support* of s, $\operatorname{supp}(s)$, is the intersection of all closed subsets Z of U such that $s|_{U-Z} = 0$. It is a closed subset of U, and the restriction of s to its complement is 0; consequently, it is the smallest such closed subset.

Morphisms of sheaves respect supports. Precisely, let $u\colon\mathscr{F}\to\mathscr{G}$ be a morphism of abelian sheaves on X. For every open subset U of X and every section $s\in\mathscr{F}(U)$, one has $\operatorname{supp}(u(s))\subset\operatorname{supp}(s)$. Indeed, the restriction of s to $U-\operatorname{supp}(s)$ is the zero section, so that the restriction of $u(s)$ to $U-\operatorname{supp}(s)$ is zero as well.

Theorem (**3.2.11**). — *Let $\mathscr{F}$ be sheaf on X, let A be a subspace of X and let $j\colon A\to X$ be the canonical inclusion. Let us make one of the following hypotheses:*

(i) *The subspace A admits a basis of paracompact neighborhoods;*
(ii) *The space X is paracompact and A is closed;*
(iii) *The space X is metrizable;*
(iv) *The space X is separated and A is compact.*

Then the canonical morphism of presheaves $j^*_{\mathrm{pre}}\mathscr{F}\to j^*\mathscr{F}$ induces a bijection

$$\operatorname*{colim}_{U\supset A}\mathscr{F}(U)\to j^*\mathscr{F}(A).$$

For the proof, I refer to (Godement, 1973, théorème 3.3.1, p. 150) or (Bourbaki, 2016, I, p. 37, théorème 2).

Corollary (**3.2.12**). — *Let X and Y be topological spaces and let $f\colon X\to Y$ be a continuous map. Let $\mathscr{F}$ be a sheaf on X. Let $y\in Y$ and write $X_y = f^{-1}(y)$.*

The restriction maps $\Gamma(f^{-1}(\mathrm{V}),\mathscr{F})\to\Gamma(\mathrm{X}_y,\mathscr{F}|_{\mathrm{X}_y})$*, when* V *ranges over all open neighborhoods of* y *define a map* $(f_*\mathscr{F})_y\to\Gamma(\mathrm{X}_y,\mathscr{F}|_{\mathrm{X}_y})$*. If* f *is proper and* X *is separated, then this map is a bijection.*

Proof. — Since f is proper, the set X_y is compact. By case (iv) of theorem 3.2.11, the canonical map

$$\operatorname*{colim}_{\mathrm{U}\supset\mathrm{X}_y}\Gamma(\mathrm{U},\mathscr{F})\to\Gamma(\mathrm{X}_y,\mathscr{F}|_{\mathrm{X}_y})$$

is bijective, where U ranges over all open neighborhoods of X_y in X.

Let U be an open neighborhood of X_y in X. Since f is proper, it is closed, hence $f(\mathrm{X}-\mathrm{U})$ is a closed subset of Y; since it does not contain y, there exists an open neighborhood V of y in Y such that $f(\mathrm{X}-\mathrm{U})\subset\mathrm{Y}-\mathrm{V}$. Consequently, $f^{-1}(\mathrm{V})\subset\mathrm{U}$. Neighborhoods of X_y of the form $f^{-1}(\mathrm{V})$ are thus cofinal in the ordered set of all neighborhoods of X_y, and the canonical map

$$(f_*\mathscr{F})_y=\operatorname*{colim}_{\mathrm{V}\ni y}\Gamma(f^{-1}(\mathrm{V}),\mathscr{F})\to\operatorname*{colim}_{\mathrm{U}\supset\mathrm{X}_y}\Gamma(\mathrm{U},\mathscr{F})$$

is bijective. This implies the corollary. □

3.3. Extensions by zero

3.3.1. — Let $j\colon\mathrm{W}\to\mathrm{X}$ be the inclusion of a *locally closed* subset of X. Let $\mathscr{F}$ be a sheaf on W. For every open subset U of X, let $j_!\mathscr{F}(\mathrm{U})$ be the subset of $j_*\mathscr{F}(\mathrm{U})=\mathscr{F}(\mathrm{W}\cap\mathrm{U})$ consisting of all sections s whose support (which is closed in $\mathrm{W}\cap\mathrm{U}$) is closed in U. This is a sub-presheaf of $j_*\mathscr{F}$, actually a subsheaf. Moreover, the construction $j_!$ gives rise to a functor $\boldsymbol{Ab}(\mathrm{W})\to\boldsymbol{Ab}(\mathrm{X})$.

Lemma (**3.3.2**). — a) *For every* $x\in\mathrm{X}-\mathrm{W}$*, one has* $j_!(\mathscr{F})_x=0$.

b) *For every* $x\in\mathrm{W}$*, the canonical morphism* $(j_*\mathscr{F})_x\to\mathscr{F}_x$ *induces an isomorphism* $(j_!\mathscr{F})_x\to\mathscr{F}_x$.

Proof. — a) Let $x\in\mathrm{X}-\mathrm{W}$. Let U be an open subset of X such that $x\in\mathrm{U}$ and let $s\in\mathscr{F}(\mathrm{W}\cap\mathrm{U})$ be an element of $j_!\mathscr{F}(\mathrm{U})$. By hypothesis, one has $\operatorname{supp}(s)\subset\mathrm{W}$, so that $x\notin\operatorname{supp}(s)$, hence there exists an open neighborhood V of x such that $\mathrm{V}\subset\mathrm{U}$ and $s|_{\mathrm{V}}=0$. This implies that the germ s_x of s at x is 0. We thus have $(j_!\mathscr{F})_x=0$.

b) Let $x \in \mathrm{W}$. Since W is locally closed, there exists an open neighborhood V of x in X such that W∩V is closed in V. For every an open subset U of V and every $s \in j_*\mathscr{F}(\mathrm{U}) = \mathscr{F}(\mathrm{W}\cap\mathrm{U})$, the support $\mathrm{supp}(s)$ of x is closed in W∩U, hence in U, because W∩U is closed in U. Consequently, $j_!\mathscr{F}(\mathrm{U}) = j_*\mathscr{F}(\mathrm{U})$ for every such U. In particular, the inclusion $j_!\mathscr{F} \to j_*\mathscr{F}$ induces an isomorphism $(j_!\mathscr{F})|_{\mathrm{U}} \to (j_*\mathscr{F})|_{\mathrm{U}}$. In particular, the canonical morphisms $(j_!\mathscr{F})_x \to (j_*\mathscr{F})_x \to \mathscr{F}_x$ are isomorphisms.

□

Proposition **(3.3.3)**. — *Let* X *be a topological space and let* $j\colon \mathrm{W} \to \mathrm{X}$ *be the inclusion of a locally closed subset of* X. *The functor* $j_!$ *is exact and fully faithful. It induces an equivalence of categories from* $\mathbf{Ab}(\mathrm{W})$ *to the full subcategory of* $\mathbf{Ab}(\mathrm{X})$ *consisting of abelian sheaves* $\mathscr{G}$ *such that* $\mathscr{G}_x = 0$ *for every* $x \in \mathrm{X} - \mathrm{W}$. *On that subcategory, the functor* j^* *induces a quasi-inverse.*

Proof. — At the level of fibers, the functor $j_!$ induces the identity functor, or the zero functor; it is in particular exact.

Let $\mathscr{G}$ be an abelian sheaf on X such that $\mathscr{G}_x = 0$ for every $x \in \mathrm{X}-\mathrm{W}$. Let us show that the canonical morphism $\eta_{\mathscr{G}}\colon \mathscr{G} \to j_*j^*\mathscr{G}$ factors through $j_!j^*\mathscr{G}$. Let indeed U be an open subset of X. Then $j_*j^*\mathscr{G}(\mathrm{U}) = j^*\mathscr{G}(\mathrm{W}\cap\mathrm{U})$ and the morphism $\eta_{\mathscr{G}}(\mathrm{U})\colon \mathscr{G}(\mathrm{U}) \to j^*\mathscr{G}(\mathrm{W}\cap\mathrm{U})$ factors through the morphism $s \mapsto (s|_{\mathrm{V}})_{\mathrm{V}}$ from $\mathscr{G}(\mathrm{U})$ to $j^*_{\mathrm{pre}}\mathscr{G}(\mathrm{W}\cap\mathrm{U}) = \mathrm{colim}_{\mathrm{V}\supset\mathrm{W}\cap\mathrm{U}}\mathscr{G}(\mathrm{V}) \simeq \mathrm{colim}_{\mathrm{U}\supset\mathrm{V}\supset\mathrm{W}\cap\mathrm{U}}\mathscr{G}(\mathrm{V})$. Moreover, for every $s \in \mathscr{G}(\mathrm{U})$, the support of s is closed in U, hence the support of $s|_{\mathrm{V}}$ is closed in V, for every open subset V of X such that $\mathrm{U} \supset \mathrm{V} \supset \mathrm{W}\cap\mathrm{U}$. This implies that the image of s belongs to $j_!j^*\mathscr{G}(\mathrm{U})$. The resulting morphism of sheaves, $\mathscr{G} \to j_!j^*\mathscr{G}$, induces an isomorphism on fibers: this is tautological for $x \in \mathrm{W}$, and follows from the fact that $\mathscr{G}_x = 0$ otherwise. This morphism is thus an isomorphism.

Let then $\mathscr{F}$ be an abelian sheaf on W. The canonical morphism $\varepsilon_{\mathscr{F}}\colon j^*j_*\mathscr{F} \to \mathscr{F}$ induces a morphism $j^*j_!\mathscr{F} \to \mathscr{F}$. Let O be an open subset of X containing W such that W is closed in O. Let V be an open subset of W. The canonical morphisms

$$j^*_{\mathrm{pre}}j_!\mathscr{F}(\mathrm{V}) = \operatorname*{colim}_{\mathrm{U}\supset\mathrm{V}} j_!\mathscr{F}(\mathrm{U}) \simeq \operatorname*{colim}_{\mathrm{O}\supset\mathrm{U}\supset\mathrm{V}} j_!\mathscr{F}(\mathrm{U}) \to \operatorname*{colim}_{\mathrm{O}\supset\mathrm{U}\supset\mathrm{V}} j_*\mathscr{F}(\mathrm{U}) = \operatorname*{colim}_{\mathrm{O}\supset\mathrm{U}\supset\mathrm{V}} \mathscr{F}(\mathrm{W}\cap\mathrm{U}) \to \mathscr{F}(\mathrm{V})$$

are isomorphisms. They induce an isomorphism from the presheaf $j^*_{\mathrm{pre}} j_! \mathscr{F}$ to the sheaf $\mathscr{F}$, so that the corresponding morphism from $j^* j_! \mathscr{F}$ to $\mathscr{F}$ is an isomorphism as well.

Let $\mathscr{F}$ and $\mathscr{G}$ be abelian sheaves on W and let $v\colon j_!\mathscr{F} \to j_!\mathscr{G}$ be a morphism of abelian sheaves. There exists a a unique morphism of sheaves $u\colon \mathscr{F} \to \mathscr{G}$ the diagram

$$\begin{array}{ccc} j^* j_! \mathscr{F} & \xrightarrow{j^* v} & j^* j_! \mathscr{G} \\ {\scriptstyle \varepsilon_{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\mathscr{G}}} \\ \mathscr{F} & \xrightarrow{u} & \mathscr{G}. \end{array}$$

Since $j_! u$ induces the morphism v_x on the fibers at x, one then has $j_! u = v$.

This concludes the proof of the proposition. □

3.3.4. — We still consider the inclusion $j\colon \mathrm{W} \to \mathrm{X}$ of a locally closed subset of X. Let $\mathscr{G}$ be a sheaf on X. For every open subset U of X, let $\Gamma_{\mathrm{W}}(\mathscr{G})(\mathrm{U})$ be the set of all sections $s \in \mathscr{G}(\mathrm{U})$ such that $\mathrm{supp}(s) \subset \mathrm{W}$. This is a subsheaf of $\mathscr{G}$; moreover, the construction $\mathscr{G} \mapsto \Gamma_{\mathrm{W}}(\mathscr{G})$ is functorial.

Let $x \in \mathrm{X} - \mathrm{W}$, let U be an open neighborhood of x and let $s \in \Gamma_{\mathrm{W}}(\mathscr{G})$. By assumption, $x \notin \mathrm{supp}(s)$; by definition of the support, there exists an open neighborhood V of x which is contained in U such that $s|_{\mathrm{V}} = 0$. This shows that $\Gamma_{\mathrm{W}}(\mathscr{G})_x = 0$. Moreover, if $\mathscr{G}_x = 0$ for every $x \in \mathrm{X} - \mathrm{W}$, then $\Gamma_{\mathrm{W}}(\mathscr{G}) = \mathscr{G}$.

Let us define $j^! \mathscr{G} = j^* \Gamma_{\mathrm{W}}(\mathscr{G})$. This construction defines a functor $\boldsymbol{Ab}(\mathrm{X}) \to \boldsymbol{Ab}(\mathrm{W})$. The morphisms

$$\mathscr{F} \to j^* j_! \mathscr{F} = j^! j_! \mathscr{F}$$

and

$$j_! j^! \mathscr{G} \to \Gamma_{\mathrm{W}}(\mathscr{G}) \to \mathscr{G}$$

are the unit and the counit of an adjunction $(j_!, j^!)$: they provide functorial isomorphisms

$$\mathrm{Hom}_{\mathrm{X}}(j_! \mathscr{F}, \mathscr{G}) \simeq \mathrm{Hom}_{\mathrm{W}}(\mathscr{F}, j^! \mathscr{G})$$

for every abelian sheaf $\mathscr{F}$ on W and every abelian sheaf $\mathscr{G}$ on X.

3.3.5. — If W is closed in X, then $j_! = j_*$.

3.3.6. — Let us assume that W is open in X; let Z = X − W and let $i : Z \to X$ be the inclusion.

Let $\mathscr{G}$ be an abelian sheaf on X. For every open subset U of W and every section $s \in \mathscr{G}(U)$, the support of s is closed in U, hence $\Gamma_W(\mathscr{G})|_W = \mathscr{G}$. Consequently, one has the equality $j^! = j^*$ of functors from $\boldsymbol{Ab}(X)$ to $\boldsymbol{Ab}(W)$.

We now show that *the diagram*

$$0 \to j_! j^* \mathscr{G} \to \mathscr{G} \to i_* i^* \mathscr{G} \to 0 \tag{3.3.6.1}$$

is an exact sequence, where the morphisms are induced by the counit of the adjunction $j_! \dashv j^*$ and the unit of the adjunction $i^* \dashv i_*$. It is called the *glueing exact sequence.*

We first show that the map $j_! j^* \mathscr{G} \to \mathscr{G}$ is injective. Let indeed U be an open subset of X and let $s \in j_! j^* \mathscr{G}(U)$. By definition, s is a section of $j^*\mathscr{G}(U) = \mathscr{G}(W \cap U)$ whose support is closed in U, and the counit is induced by the identity map, hence it is injective.

The image of s in $i_* i^* \mathscr{G}(U) = i^*\mathscr{G}(Z \cap U)$ is the germ of s along $Z \cap U$; since $\operatorname{supp}(s)$ is a closed subset of U contained in $W \cap U$, it does not meet $Z \cap U$, and this germ is zero. Conversely, let $s \in \mathscr{G}(U)$ be a section whose germ along $Z \cap U$ vanishes. This implies that there exists an open neighborhood V of $Z \cap U$ in U such that $s|_V = 0$. Consequently, the support of s is contained in U − V, hence is closed in U, so that s is induced by a section of $j_! j^* \mathscr{G}(U)$.

Finally, let U be an open subset of X and let $s \in i_* i^* \mathscr{G}(U) = i^* \mathscr{G}(Z \cap U)$. By construction, $Z \cap U$ is covered by open subsets V of U on which s is induced by a section t of $\mathscr{G}(V)$. This proves the surjectivity of the unit morphism $\mathscr{G} \to i_* i^* \mathscr{G}$.

On the level of fibers, the glueing exact sequence (3.3.6.1) induces the exact sequences

$$0 \to \mathscr{G}_x \to \mathscr{G}_x \to 0 \to 0, \qquad \text{for } x \in W,$$
$$0 \to 0 \to \mathscr{G}_x \to \mathscr{G}_x \to 0, \qquad \text{for } x \in Z.$$

3.3.7. — We keep the same context. Let X be a topological space, let $j : W \to X$ be the inclusion of an open subset W of X, and let $i : Z \to X$ be the inclusion of the complementary closed subset Z = X − W.

Between the categories of abelian sheaves on X, Z, and W, we have six functors: i^*, i_*, $i_!$, $j_!$, j^*, j_* forming four adjoint pairs $i^* \dashv i_*$, $i_* \dashv i^!$, $j_! \dashv j^*$ and $j^* \dashv j_*$:

$$\boldsymbol{Ab}(\mathrm{Z}) \;\underset{\underset{i^!}{\longleftarrow}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{\;i_*\;}}}\; \boldsymbol{Ab}(\mathrm{X}) \;\underset{\underset{j_*}{\longleftarrow}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{\;j^*\;}}}\; \boldsymbol{Ab}(\mathrm{W}).$$

Moreover, one has $j^* \circ i_* = 0$, $i^* \circ j_! = 0$ and $i^! \circ j_* = 0$.

The functor i_* is fully faithful. In fact, the counit of the pair (i^*, i_*) and the unit of the pair $(i_*, i^!)$ are isomorphisms.

Moreover, the functors $j_!$ and j_* are fully faithful: the counit of the pair (j^*, j_*) and the unit of the pair $(j_!, j^*)$ are isomorphisms.

3.3.8. — Every sheaf $\mathscr{F}$ on X furnishes a sheaf $j^*\mathscr{F}$ on W, a sheaf $i^*\mathscr{F}$ on Z, and a morphism of sheaves $i^*\eta_{\mathscr{F}} \colon i^*\mathscr{F} \to i^*j_*(j^*\mathscr{F})$ on Z. The assignment $\mathscr{F} \mapsto (j^*\mathscr{F}, i^*\mathscr{F}, i^*\eta_{\mathscr{F}})$ induces a functor from the category $\boldsymbol{Ab}(\mathrm{X})$ to the category of triples $(\mathscr{F}_{\mathrm{W}}, \mathscr{F}_{\mathrm{Z}}, \varphi)$ of triples consisting of an abelian sheaf $\mathscr{F}_{\mathrm{W}}$ on W, an abelian sheaf $\mathscr{F}_{\mathrm{Z}}$ on Z, and of a morphism of abelian sheaves $\varphi \colon \mathscr{F}_{\mathrm{Z}} \to i^*j_*\mathscr{F}_{\mathrm{W}}$. Let us show that *this functor is an equivalence of categories.*

We first show that it is fully faithful. Let indeed $\mathscr{F}, \mathscr{G}$ be sheaves on X and let us consider $v \colon i^*\mathscr{F} \to i^*\mathscr{G}$ and $w \colon j^*\mathscr{F} \to j^*\mathscr{G}$ be morphisms of sheaves making the diagram

$$\begin{array}{ccc} i^*\mathscr{F} & \xrightarrow{\;i^*\eta_{\mathscr{F}}\;} & i^*j_*j^*\mathscr{F} \\ \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle i^*j_*w} \\ i^*\mathscr{G} & \xrightarrow{\;i^*\eta_{\mathscr{G}}\;} & i^*j_*j^*\mathscr{G} \end{array}$$

commutative. Let U be an open subset of X and let $s \in \mathscr{F}(\mathrm{U})$; let us show that there exists a unique section $u(s) \in \mathscr{G}(\mathrm{U})$ such that $u(s)|_{\mathrm{Z}\cap\mathrm{U}} = v(s|_{\mathrm{Z}\cap\mathrm{U}})$ and $u(s)|_{\mathrm{W}\cap\mathrm{U}} = w(s|_{\mathrm{W}\cap\mathrm{U}})$. These conditions impose $u(s)_x = v_x(s_x)$ for $x \in \mathrm{Z}$, and $u(s)_x = w_x(s_x)$ for $x \in \mathrm{W}$, so that there is at most one such section $u(s)$.

Conversely, let $t = v(s|_{\mathrm{Z}\cap\mathrm{U}})$. For every point $x \in \mathrm{Z} \cap \mathrm{U}$, there exists an open neighborhood V^x of x in U and a section $t^x \in \mathscr{G}(\mathrm{V}_x)$ such that $t^x|_{\mathrm{Z}\cap\mathrm{V}^x} = t|_{\mathrm{Z}\cap\mathrm{V}^x}$. Observe that we have the following equalities in $i^*\mathscr{G}(\mathrm{Z}\cap\mathrm{U})$:

$$i^*j_*w \circ i^*\eta_{\mathscr{F}}(s|_{\mathrm{Z}\cap\mathrm{U}}) = (j_*w \circ i^*\eta_{\mathscr{F}}(s|_{\mathrm{U}}))|_{\mathrm{Z}\cap\mathrm{U}} = j_*w(s|_{\mathrm{W}\cap\mathrm{U}})|_{\mathrm{Z}\cap\mathrm{U}},$$

where $j_* w(s|_{W\cap U})$ is just the section $w(s|_{U\cap W}) \in \mathscr{G}(W\cap U)$, but considered as a section of $j_*\mathscr{G}(U)$. Then, the commutation of the above diagram implies that

$$\begin{aligned} t^x|_{Z\cap V^x} &= i^*\eta_{\mathscr{G}}(t|_{Z\cap V^x}) \\ &= i^*\eta_{\mathscr{G}} \circ v(s|_{Z\cap V^x}) \\ &= i^* j_* w \circ i^*\eta_{\mathscr{F}}(s|_{Z\cap V^x}) \\ &= j_* w(s|_{W\cap V^x})|_{Z\cap V^x} \end{aligned}$$

in $i^* j_* j^* \mathscr{G}(Z\cap V^x)$. Consequently, there exists an open neighborhood U^x of $Z\cap V^x$ in V^x such that $t^x|_{W\cap U^x} = w(s|_{W\cap U^x})$. Up to replacing V^x by U^x, we thus assume that $t^x|_{W\cap V^x} = w(s|_{W\cap V^x})$ for every $x \in Z$. This shows that t^x satisfies the properties of the required section $u(s|_{V^x})$. By the uniqueness property, t^x and t^y coincide on $V^x \cap V^y$, for all $x, y \in Z$; and t^x coincides with $v(s|_{W\cap U})$ on $W\cap V^x$, by construction. Consequently, there exists a section $u(s) \in \mathscr{G}(U)$ such that $u(s)|_{V^x} = t^x$ for every $x \in Z$, and $u(s)|_{W\cap U} = v(s|_{W\cap U})$. It satisfies the desired requirements, and this concludes the proof that the considered functor is fully faithful.

We now prove that it is essentially surjective. Let $(\mathscr{F}_W, \mathscr{F}_Z, \varphi)$ be a triple consisting of a sheaf $\mathscr{F}_W$ on W, a sheaf $\mathscr{F}_Z$ on Z, and of a morphism of sheaves $\varphi\colon F_Z \to i^* j_* \mathscr{F}_W$. For every open subset U of X, one defines $\mathscr{F}(U)$ as set set of pairs (s, t), where $s \in \mathscr{F}_W(W\cap U)$, $t \in \mathscr{F}_Z(Z\cap U)$ satisfy $\varphi(t) = i^*s$. In fact, $\mathscr{F}$ is the kernel of the morphism of sheaves

$$j_*\mathscr{F}_W \times i_*\mathscr{F}_Z \to i^* j_* \mathscr{F}_W, \qquad (s,t) \mapsto i^*(s) - \varphi(t).$$

It is thus a sheaf on $\mathscr{X}$, and one checks readily that it maps to the given triple by the considered functor. This concludes the proof.

3.4. Direct images

3.4.1. — Let X and Y be topological spaces, let $f\colon X \to Y$ be a continuous map. The functor $f_*\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}(Y)$ is left exact, as is the functor $\Gamma\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}$. As usual, we denote by $\boldsymbol{R}f_*\colon \boldsymbol{D}(\boldsymbol{Ab}(X)) \to \boldsymbol{D}(\boldsymbol{Ab}(Y))$ and $\boldsymbol{R}\Gamma\colon \boldsymbol{D}(\boldsymbol{Ab}(X)) \to \boldsymbol{D}(\boldsymbol{Ab})$ their derived functors.

For $n \in \mathbf{Z}$, we also denote by $\boldsymbol{R}^n f_*\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}$ and $H^n\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}$ the functors $H^n \circ \boldsymbol{R}f_*$ and $H^n \circ \boldsymbol{R}\Gamma$.

Remark (3.4.2). — Let $\mathscr{F}$ be a sheaf on X and let n be an integer.

Let $\varepsilon\colon \mathscr{F} to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. By definition, one has $\boldsymbol{R}^n\mathscr{F} = \operatorname{Ker}(d^n_{\mathscr{I}})/\operatorname{Im}(d^{n-1}_{\mathscr{I}})$. It follows from the definition of kernels, images and quotients in the category of sheaves that $\boldsymbol{R}^n\mathscr{F}$ is the sheaf associated with the presheaf $V \mapsto \operatorname{Ker}(d^n_{\mathscr{I}}|_V)/\operatorname{Im}(d^{n-1}_{\mathscr{I}}|_V) = H^n(f^{-1}(V), \mathscr{F})$.

Consequently, the sheaf $\boldsymbol{R}^n\mathscr{F}$ on X. is the sheaf associated with the presheaf $V \mapsto H^n(f^{-1}(V), \mathscr{F})$.[1]

Proposition (3.4.3). — *Let $f\colon X \to Y$ be a continuous map of topological spaces and let $\mathscr{F}$ be a sheaf of abelian groups on* X. *If the sheaf $\mathscr{F}$ is injective, then so is $f_*\mathscr{F}$.*

Proof. — This follows from the fact that the functor f_* has a left adjoint which is exact. Explicitly, let $u\colon \mathscr{M} \to \mathscr{N}$ be a monomorphism of abelian sheaves on Y and let $\varphi\colon \mathscr{M} \to f_*\mathscr{F}$ be a morphism of abelian sheaves on Y. Let $u^\flat\colon f^*\mathscr{M} \to f^*\mathscr{N}$ and $\varphi^\flat\colon f^*\mathscr{M} \to \mathscr{F}$ be the corresponding morphism of abelian sheaves on X. Since f^* is exact, the morphism $u^\flat$ is a monomorphism. Since $\mathscr{F}$ is injective, there exists a morphism $v\colon f^*\mathscr{N} \to \mathscr{F}$ such that $v\circ u^\flat = \varphi^\flat$. Let $v^\sharp\colon \mathscr{N} \to f_*\mathscr{F}$ be the morphism of sheaves on Y corresponding to v. One has $v^\sharp \circ u = \varphi$. Consequently, $\mathscr{F}$ is injective. ☐

Corollary (3.4.4). — *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. The canonical morphism of functors $\boldsymbol{R}(g\circ f)_* \to \boldsymbol{R}g_* \circ \boldsymbol{R}f_*$ from $\boldsymbol{D}^+(\boldsymbol{Ab}(X))$ to $\boldsymbol{D}^+(\boldsymbol{Ab}(Z))$ is an isomorphism.*

Proof. — Consider an element of $\boldsymbol{D}^+(\boldsymbol{Ab}(X))$, represented by a complex $\mathscr{F}^\bullet$ in $\boldsymbol{K}^+(\boldsymbol{Ab}(X))$ with injective terms. Then the canonical morphisms $(g\circ f)_*\mathscr{F}^\bullet \to \boldsymbol{R}(g\circ f)_*\mathscr{F}^\bullet$ and $f_*\mathscr{F}^\bullet \to \boldsymbol{R}f_*\mathscr{F}^\bullet$ are homologisms. Since $\mathscr{F}^j$ is injective for every j, so is $f_*\mathscr{F}^j$; consequently, the canonical morphism $\boldsymbol{g}_*(f_*\mathscr{F}^\bullet) \to \boldsymbol{R}g_*(f_*\mathscr{F}^\bullet)$ is an isomorphism. The corollary follows. ☐

Theorem (3.4.5). — [2] *Let $f\colon X \to Y$ be a map of topological spaces; assume that* X *is separated and that f is proper. Let $y \in Y$ and let $X_y = f^{-1}(y)$.*

a) *For every sheaf $\mathscr{F}$ on* X, *the canonical map $(f_*\mathscr{F})_y \to \Gamma(X_y, \mathscr{F}|_{X_y})$ is an isomorphism;*

[1] Trouver une formulation correcte.

[2] Pas correctement énoncé; ajouter la preuve.

b) *This isomorphism extends uniquely to an isomorphism of cohomological functors* $(\boldsymbol{R}^j f_* \cdot)_y \simeq \boldsymbol{R}^j\Gamma(X_y, \cdot|_{X_y})$ *on* $\boldsymbol{D}^+(\boldsymbol{Ab}(X))$.

Definition **(3.4.6)**. — *One says that a sheaf* $\mathscr{F}$ *on* X *is* flasque *if for every open subset* U *of* X, *the restriction map* $\mathscr{F}(X) \to \mathscr{F}(U)$ *is surjective.*

Lemma **(3.4.7)**. — *Let* $\mathscr{F}$ *be a flasque sheaf on* X.

a) *The sheaf* $\mathscr{F}|_U$ *is flasque, for every open subset* U *of* X;

b) *Let* $f : X \to Y$ *be a continuous map. The sheaf* $f_*\mathscr{F}$ *is flasque.*

Proof. — a) Let V be an open subset of U and let $s \in \mathscr{F}(V)$. Since $\mathscr{F}$ is flasque, there exists $s' \in \mathscr{F}(X)$ such that $s'|_V$. Then $t = s'|_U$ is an element of $\mathscr{F}(U)$ such that $t|_V = s$. This proves that $\mathscr{F}|_U$ is flasque.

b) Let U be an open subset of Y and let $s \in f_*\mathscr{F}(U)$. By definition, s is a section t of $\mathscr{F}(f^{-1}(U))$. Since $\mathscr{F}$ is flasque, there exists a section $t' \in \mathscr{F}(X)$ such that $t'|_{f^{-1}(U)} = t$. Then t' can be viewed as a section s' of $\mathscr{F}(Y)$ and $t'|_U = t$. This proves that $f_*\mathscr{F}$ is flasque.

□

Example **(3.4.8)**. — If X is discrete, every section of an étale space is continuous, so that a sheaf $\mathscr{F}$ on X is flasque if and only if its fibers $\mathscr{F}_x$ are non-empty, for all $x \in X$. Indeed, In particulier, every abelian sheaf on X is flasque in this case.

Let X^δ be the set X endowed with the discrete topology. The identity $p : X^\delta \to X$ is continuous. For every abelian sheaf $\mathscr{F}$ on X, one lets $G(\mathscr{F}) = p_*p^*\mathscr{F}$. This is a flasque sheaf on X, and the unit $\eta_{\mathscr{F}} : \mathscr{F} \to G(\mathscr{F})$ is a monomorphism. Explicitly, one has $G(\mathscr{F})(U) = \prod_{x\in U} \mathscr{F}_x$, for every open subset U of X, the restriction morphisms are the morphisms $\prod_{x\in U} \mathscr{F}_x \to \prod_{x\in V} \mathscr{F}_x$, for $V \subset U$ which are not only surjective, but have a section.

Example **(3.4.9)**. — *An injective sheaf* $\mathscr{F}$ *on* X *is flasque.*

Let indeed U be an open subset of X and let $s \in \mathscr{F}(U)$. Let $j : U \to X$ be the canonical inclusion; let $f : j_!\mathbf{Z}_U \to \mathscr{F}$ be the unique morphism corresponding to the morphism $\mathbf{Z}_U \to j^*\mathscr{F}$ which maps 1 to s. The canonical morphism $u : j_!\mathbf{Z}_U \to \mathbf{Z}_X$ is injective; since $\mathscr{F}$ is injective, there exists a unique morphism $g : \mathbf{Z}_X \to \mathscr{F}$ such that $g \circ u = f$; it corresponds to a section $t \in \mathscr{F}(X)$ such that $t|_U = s$. Consequently, the restriction morphism $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective, as as to be shown.

Proposition **(3.4.10)**. — *Let $\mathscr{F}$ be a sheaf on X. Assume that for every open subset U of X and every $s \in \mathscr{F}(U)$, there exists an open covering $\mathscr{V}$ of X such that for every $V \in \mathscr{V}$, there exists $t_V \in \mathscr{F}(V)$ such that $t_V|_{U\cap V} = s|_{U\cap V}$. Then $\mathscr{F}$ is flasque.*

Proof. — Let U be an open subset of X and let $s \in \mathscr{F}(U)$. Let us show that there exists $t \in \mathscr{F}(X)$ such that $t|_U = s$. Let $\mathscr{R}$ be the set of pairs (V, t), where V is an open subset of X and $t \in \mathscr{F}(V)$. The relation $\leq$ defined by $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$ is an ordering relation on $\mathscr{R}$; moreover, the ordered set $\mathscr{R}$ is inductive. By Zorn's lemma, we may consider a maximal element (W, t) of $\mathscr{R}$ such that $(U, s) \leq (W, t)$; let us show that $W = X$. By hypothesis, there exists an open covering $\mathscr{V}$ of X and, for every $V \in \mathscr{V}$, an element $t_V \in \mathscr{F}(V)$ such that $t_V|_{U\cap W} = t|_{W\cap V}$. If $W \neq X$, there exists an open subset $V \in \mathscr{V}$ such that $V \not\subset W$; then, there exists a unique section $t' \in (W \cup V)$ which restricts to t on W and to t_V on V. In particular, $(W \cup V, t')$ is an element of $\mathscr{R}$ such that $(W, t) \leq (W \cup V, t')$, contradicting the hypothesis that (W, t) were maximal. □

Corollary **(3.4.11)**. — *Let $\mathscr{F}$ be a sheaf on X. Assume that there exists an open covering $\mathscr{V}$ of X such that $\mathscr{F}|_V$ is flasque, for every $V \in \mathscr{F}$. Then $\mathscr{F}$ is flasque.*

Proof. — Indeed, the condition of the proposition is satisfied: since $\mathscr{F}|_V$ is flasque, there exists $t \in \mathscr{F}(V)$ which restricts to $s|_{U\cap V}$. □

Proposition **(3.4.12)**. — *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of abelian sheaves. Assume that $\mathscr{F}'$ is flasque.*

a) *For every open subset U of X, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact.*

b) *If $\mathscr{F}$ is flasque, then $\mathscr{F}''$ is flasque as well.*

Proof. — [3] It is a general fact that the sequence $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is exact, except possibly at $\mathscr{F}''(X)$.

Let $s \in \mathscr{F}''(X)$. Let us show that there exists $t \in \mathscr{F}(X)$ with image s. Let $\mathscr{R}$ be the set of pairs (V, t), where V is an open subset of X and $t \in \mathscr{F}(V)$ maps to $s|_V$ in $\mathscr{F}''(V)$. The relation $\leq$ defined by $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$ is an ordering relation on $\mathscr{R}$; moreover, the ordered set $\mathscr{R}$ is inductive.

(3) This seems to be another instance of the same reasoning. Find a unifying statement?

By Zorn's lemma, we may consider a maximal element (W, t) of $\mathscr{R}$; let us show that $W = X$. By hypothesis, there exists an open covering $\mathscr{V}$ of X and, for every $V \in \mathscr{V}$, an element $t_V \in \mathscr{F}(V)$ which maps to $s|_V$ in $\mathscr{F}''(V)$. If $W \neq X$, there exists an open subset $V \in \mathscr{V}$ such that $V \notin W$. Then the elements $t|_{V\cap W}$ and $t_V|_{V\cap W}$ of $\mathscr{F}(V \cap W)$ both map to $s|_{V\cap W}$ in $\mathscr{F}''(V \cap W)$; consequently, their difference belongs to $\mathscr{F}'(V \cap W)$. Since $\mathscr{F}'$ is flasque, there exists $u \in \mathscr{F}'(X)$ such that $t|_{V\cap W} - t_V|_{V\cap W} = u|_{V\cap W}$. In other words, the elements $t \in \mathscr{F}(W)$ and $t_V + u|_V \in \mathscr{F}(V)$ agree on $V \cap W$; consequently, there exists a unique section $t' \in (W \cup V)$ which restricts to t on W and to $t_V + u|_V$ on V; it maps to $s|_{W\cup V}$ in $\mathscr{F}'(W \cup V)$. In particular, $(W \cup V, t')$ is an element of $\mathscr{R}$ such that $(W, t) \leq (W \cup V, t')$, contradicting the hypothesis that (W, t) were maximal.

Let U be an open subset of X. By restriction to U, the initial exact sequence of sheaves on X furnishes an exact sequence of sheaves on U, and $\mathscr{F}'|_U$ is flasque. By the case already treated, the diagram $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact.

Assume now that $\mathscr{F}$ is flasque as well; let us show that $\mathscr{F}''$ is flasque. Let U be an open subset of X and let $s \in \mathscr{F}''(U)$. By what precedes, there exists $t \in \mathscr{F}(U)$ which maps to s. Since $\mathscr{F}$ is flasque, there exists $t' \in \mathscr{F}(X)$ such that $t'|_U = t$. Then the image s' of t' in $\mathscr{F}''(X)$ satisfies $s'|_U = s$. Consequently, $\mathscr{F}''$ is flasque. □

Corollary (**3.4.13**). — *Let $f : X \to Y$ be a continuous map of topological spaces. The full subcategory of $\boldsymbol{Ab}(X)$ consisting of flasque sheaves is injective with respect to the functor f_*.*

In particular, the category of flasque sheaves on X is injective with respect to the global sections functor $\Gamma(X, \cdot)$.

As a consequence, if $\mathscr{F}^\bullet$ is a complex in $\boldsymbol{K}^+(\boldsymbol{Ab}(X))$ such that $\mathscr{F}^j$ is flasque, for every j, then the canonical morphism $f_*\mathscr{F}^\bullet \to \boldsymbol{R}f_*\mathscr{F}^\bullet$ is a homologism of complexes of abelian sheaves, hence it induces an isomorphism in $\boldsymbol{D}^+(\boldsymbol{Ab}(Y))$.

Proof. — We need to check the three properties of definition 2.7.5:

(i) Every abelian sheaf on X embeds in a flasque sheaf.

(ii) For every exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of abelian sheaves on X, where $\mathscr{F}'$ and $\mathscr{F}''$ are flasque, the sheaf $\mathscr{F}''$ is flasque.

(iii) For every exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of flasque abelian sheaves on X, the diagram $0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}'' \to 0$ of abelian sheaves on Y is exact.

The first property follows from example 3.4.8, and the second one follows from proposition 3.4.12, *b*). Let us finally prove the last one. Let V be an open subset of Y. Taking sections on V, the given diagram induces a diagram $0 \to f_*\mathscr{F}'(V) \to f_*\mathscr{F}(V) \to f_*\mathscr{F}''(V) \to 0$ of abelian groups, which identifies with the diagram $0 \to \mathscr{F}'(f^{-1}(V)) \to \mathscr{F}(f^{-1}(V)) \to \mathscr{F}''(f^{-1}(V)) \to 0$. By proposition 3.4.12, the latter diagram is exact. In particular, the initial diagram is exact, as was to be shown. □

3.5. Cohomology with compact support

3.5.1. — Let $f \colon X \to Y$ be a continuous map. Let $\mathscr{F}$ be an abelian sheaf on X. For every open subset V of Y, let $f_!\mathscr{F}(V)$ be the set of all sections $s \in \mathscr{F}(f^{-1}(V))$ whose support is proper and separated over V, that is, such that the map f induces by restriction a proper and separated map $f|_{\operatorname{supp}(s)} \colon \operatorname{supp}(s) \to f^{-1}(V)$. This is a subgroup of $f_*\mathscr{F}(V)$.

Proposition (**3.5.2**). — *Let $f \colon X \to Y$ be a continuous map. For every abelian sheaf $\mathscr{F}$ on X, $f_!\mathscr{F}$ is an abelian subsheaf of $f_*\mathscr{F}$.*

Proof. — Let V be an open subset of Y. The support of the zero section of $f_*\mathscr{F}$ is empty, hence is proper over V. Let then $s, s' \in \mathscr{F}(f^{-1}(V))$; consider them as elements the support of the element $s + s'$ is contained in the union $\operatorname{supp}(s) \cup \operatorname{supp}(s')$; it is proper over V; consequently, $s + s' \in f_!\mathscr{F}(V)$. Similarly, the support of $-s$ is equal to the support of s, so that $-s \in f_!\mathscr{F}(V)$ if $s \in f_!\mathscr{F}(V)$. This shows that $f_!\mathscr{F}(V)$ is a subgroup of $f_*\mathscr{F}(V)$.

Let U, V be open subsets of Y such that $V \subset U$. Let $s \in f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. One has $\operatorname{supp}(s|_{f^{-1}(V)}) = \operatorname{supp}(s) \cap f^{-1}(V)$. Consequently, $\operatorname{supp}(s)$ is proper over U, then $\operatorname{supp}(s|_{f^{-1}(V)})$ is proper over V. Consequently, the restriction morphism $f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$ maps $f_!\mathscr{F}(U)$ into $f_!\mathscr{F}(V)$. In other words, $f_!\mathscr{F}$ is a sub-presheaf of $f_*\mathscr{F}$.

Let finally U be an open subset of Y, let $\mathscr{V}$ be an open covering of U and let $(s_V)_{V\in\mathscr{V}}$ be a family, where $s_V \in f_!\mathscr{F}(V)$, such that $s_V|_{V\cap W} = s_W|_{V\cap W}$ for every $V, W \in \mathscr{V}$. Viewing the section s_V as an element of $f_*\mathscr{F}(V)$, for every $V \in \mathscr{V}$, we see that there exists a unique element $s \in f_*\mathscr{F}(U)$ such that $s|_V = s_V$ for every $V \in \mathscr{V}$; indeed, $f_*\mathscr{F}$ is a sheaf. Then, for every $V \in \mathscr{V}$, the intersection $\operatorname{supp}(s) \cap f^{-1}(V)$ is proper over V; consequently, $\operatorname{supp}(s)$ is proper over U, hence $s \in f_!\mathscr{F}(U)$. This proves that $f_!\mathscr{F}$ is a subsheaf of $f_*\mathscr{F}$, and concludes the proof of the proposition. □

3.5.3. — Let $f: X \to Y$ be a continuous map of topological spaces.

Let $u: \mathscr{F} \to \mathscr{G}$ be a morphism of abelian sheaves on X. For every open subset V of Y, the morphism $f_*(u): f_*\mathscr{F}(V) \to f_*\mathscr{G}(V)$ maps $f_!\mathscr{F}(V)$ to $f_!\mathscr{G}(V)$. Indeed, the image $f_*(u)(s)$ of a section $s \in f_!\mathscr{F}(V)$ is the section $u(s)$ of $f_*\mathscr{G}(f^{-1}(V))$; its support is a closed subset of the support of s, hence is proper over V.

Consequently, the maps $\mathscr{F} \mapsto f_!\mathscr{F}$ and $u \mapsto f_!(u)$ define a functor from the category $\boldsymbol{Ab}(X)$ to the category $\boldsymbol{Ab}(Y)$.

Lemma **(3.5.4)**. — *The functor $f_!$ is a left-exact additive functor.* [4]

Proof. — It follows from its definition that the functor $f_!$ is additive.

Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ be an exact sequence of abelian sheaves on X and let us show that the diagram $0 \to f_!\mathscr{F}' \to f_!\mathscr{F} \to f_!\mathscr{F}''$ of abelian sheaves on Y is an exact sequence.

Let V be an open subset of Y and let $s \in f_!\mathscr{F}'(V)$ map to 0 in $f_!\mathscr{F}(V)$. Then s, viewed as an element of $\mathscr{F}'(f^{-1}(V))$, maps to 0 in $\mathscr{F}(f^{-1}(V))$. Since the morphism from $\mathscr{F}'$ to $\mathscr{F}$ is a monomorphism, one has $s = 0$. This shows that the morphism $f_!\mathscr{F}' \to f_!\mathscr{F}$ is a monomorphism.

Let then $s \in f_!\mathscr{F}(V)$ map to 0 in $f_!\mathscr{F}''(V)$. Again, the section s, when viewed as an element of $\mathscr{F}(f^{-1}(V))$ maps to 0 in $\mathscr{F}''(f^{-1}(V))$. By definition of the exactness of the sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$, there exists a unique section $s' \in \mathscr{F}'(f^{-1}(V))$ which maps to s. Since the morphism $\mathscr{F}' \to \mathscr{F}$ is a monomorphism, the support of s' is equal to the support of s, hence is proper over V, since $s \in f_!\mathscr{F}(V)$. This proves that the section s of $f_!\mathscr{F}(V)$ is the image of a section of $f_!\mathscr{F}'(V)$, as was to be shown. □

3.5.5. — By the general theory of derived functors, the functor $f_!$, for a continuous map $f: X \to Y$, gives rise to a functor $\boldsymbol{R}f_!: \boldsymbol{D}(\boldsymbol{Ab}(X)) \to \boldsymbol{D}(\boldsymbol{Ab}(Y))$.

(4) Prove the general commutation with filtered colimits, as well as commutation with arbitrary coproducts.

When f is the canonical map from X to a point, the functor $f_!$ identifies with the functor Γ_c of sections with compact support.

3.5.6. — To be able to compute conveniently cohomology with compact support, it is important to find a suitable category of abelian sheaves which is adapted to these functors and is as large as possible. On locally compact topological spaces, a convenient category is that of *soft* sheaves.

Definition (**3.5.7**). — *Let* X *be a locally compact topological space. An abelian sheaf* $\mathscr{F}$ *is* soft[(5)] *if for every compact subspace* A *of* X, *the canonical morphism* $\mathscr{F}(X) \to \mathscr{F}|_A(A)$ *is surjective.*

Example (**3.5.8**). — *Every flasque sheaf on a locally compact topological space is soft.*

Let indeed X be a locally compact topological space and let $\mathscr{F}$ be a flasque sheaf on X. Let A be a compact subset of X and let $s \in \Gamma(A, \mathscr{F}|_A)$. By the extension theorem 3.2.11, there exists an open neighborhood U of A and a section $s' \in \Gamma(U, \mathscr{F})$ such that $s'|_A = s$. By the definition of a flasque sheaf, there exists $t \in \Gamma(X, \mathscr{F})$ such that $t|_U = s'$. Then $t|_A = (t|_U)|_A = s'|_A = s$. This proves that $\mathscr{F}$ is soft.

Example (**3.5.9**). — Let X be a locally compact topological space. *The sheaf* $\mathscr{C}_X$ *of continuous functions on* X *is soft.*

Let indeed A be a compact subset of X and let $f \in \mathscr{C}_X|_A(A)$. The sheaf $\mathscr{C}_X|_A$ is the sheaf of germs of continuous functions on a neighborhood of A, but the extension theorem 3.2.11 implies that f is induced by a continuous function f' defined on an open neighborhood W of A. We may assume that $\overline{W}$ is compact. By Urysohn's theorem (which is valid on compact spaces), there exists a continuous function $h : \overline{W} \to [0;1]$ such that $h \equiv 1$ in a neighborhood of A and $h \equiv 0$ on ∂W. Let $g : X \to \mathbf{R}$ be defined by $g(x) = h(x)f'(x)$ for $x \in W$ and $g(x) = 0$ for $x \in X - W$. Its restriction to $\overline{W}$ is continuous; since $X - W$ and $\overline{W}$ is a covering of X by closed subsets, the function g is continuous. By construction, it coincides with f' on neighborhood of A, so that the section $g|_A$ of $\mathscr{C}_X|_A(A)$ is equal to f.

A similar argument shows that the sheaf of $\mathscr{C}^\infty$-functions on a closed subset X of a manifold is soft.

(5) In the book of (Godement, 1973), which considers general support conditions, this notion is called c-soft.

Lemma (**3.5.10**). — *Let* $\mathscr{F}$ *be a soft abelian sheaf on a locally compat topological space* X. *For every locally compact*[6] *subspace* W *of* X, *the sheaf* $\mathscr{F}|_W$ *is soft*

Proof. — Let A be a compact subset of W and let $s \in (\mathscr{F}|_W|_A)(A) = \mathscr{F}|_A(A)$. Then A is a compact subset of X, hence there exists $t \in \mathscr{F}(X)$ such that $t|_A = s$. Then $t|_W$ is a section of $\mathscr{F}|_W(W)$ which restricts to s on A. □

Proposition (**3.5.11**). — *Let* X *be a locally compact topological space and let* $\mathscr{F}$ *be a soft abelian sheaf on* X. *Let* A *be a compact subset of* X *and let* U *be an open neighborhood of* A. *The canonical morphism* $\Gamma_c(U, \mathscr{F}) \to \Gamma(A, \mathscr{F}|_A)$ *is surjective.*

Proof. — Let V be an open neighborhood of A such that $\overline{V}$ is compact and contained in U. Let $s \in \Gamma(A, \mathscr{F}|_A)$. Let $B = A \cup \partial V$; since $A \cap \partial V = \varnothing$, there exists a unique section s' of $\mathscr{F}|_B(B)$ whose restriction to A is equal to s and whose restriction to ∂V is zero. Since $\mathscr{F}$ is soft, there exists a section $t' \in \Gamma(X, \mathscr{F})$ such that $t'|_A = s$ and $t'|_{\partial V} = 0$. There exists an open neighborhood W′ of ∂V such that $t'|_{W'} = 0$; let $W = W' \cup (X - V)$. Then there exists a unique section $t \in \mathscr{F}(X)$ such that $t|_V = t'$ and $t|_W = 0$; indeed, $W \cap V = W' \cap V$ and $t'|_{W' \cap V} = 0$. By construction, the support of t is contained in $X - W \subset V$. Since $\overline{V}$ is compact and contained in U, so is the support of t. □

Corollary (**3.5.12**). — *Let* $\mathscr{O}$ *be a soft sheaf of rings on a locally compact topological space* X, *then any sheaf of* $\mathscr{O}$*-modules on* X *is soft.*

Proof. — Let indeed $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on X and let A be a compact subset of X. Let $s \in \Gamma(A, \mathscr{F}|_A)$. By the extension theorem 3.2.11, there exists a neighborhood U of A and a section $s' \in \Gamma(U, \mathscr{F})$ such that $s'|_A = s$. Let $f \in \Gamma_c(U, \mathscr{O})$ be any section with compact support such that $f|_A = 1$. The section $fs' \in \Gamma(U, \mathscr{F})$ has compact support, hence can be extended by zero to a section $t \in \Gamma(X, \mathscr{F})$. By construction, one has $t|_A = f|_A s'|_A = s$. □

Proposition (**3.5.13**). — *Let* X *be a locally compact topological space and let* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *be an exact sequence of abelian sheaves on* X *Assume that* $\mathscr{F}'$ *is soft.*

[6] Recall from lemma 3.1.2 that a subset W of a locally compact space is locally compact if and only if it is locally closed.

a) *For every open subset* U *of* X*, the sequence* $0 \to \Gamma_c(U, \mathscr{F}') \to \Gamma_c(U, \mathscr{F}) \to \Gamma_c(U, \mathscr{F}'') \to 0$ *is exact.*

b) *If, moreover,* $\mathscr{F}$ *is soft, then* $\mathscr{F}''$ *is soft as well.*

Proof. — By left-exactness of the functor $\Gamma(U, \cdot)$, the given sequence is exact except possibly at $\Gamma(U, \mathscr{F}'')$, so that we just need to prove the surjectivity of the morphism $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$.

Let us first prove this surjectivity in the case where $U = X$ is compact.

Let $s'' \in \mathscr{F}''(X)$. There exists a finite covering $(U_i)_{1\leqslant i\leqslant n}$ of X and a family (s_i), where $s_i \in \mathscr{F}(U_i)$ for every i, such that s_i lifts $s''|_{U_i}$. Let $(V_i)_{1\leqslant i\leqslant n}$ be an open covering of X such that $\overline{V_i} \subset U_i$ for every i (lemma 3.1.8). For $m \in \{1, \dots, n\}$, let $W_m = \overline{V_1} \cup \dots \cup \overline{V_m}$. Let us show by induction on m that $s''|_{W_m}$ lifts to a section of $\mathscr{F}$. For $m = 1$, the section $s|_{W_1}$ lifts $s''|_{W_1}$. Let $s \in \mathscr{F}(W_m)$ be a section that lifts $s''|_{W_m}$; the restrictions of s and s_{m+1} to $W_m \cap V_{m+1}$ both lift $s''|_{W_m \cap V_{m+1}}$, so that their differences belong to $\mathscr{F}'(W_m \cap V_{m+1})$. Since $\mathscr{F}'$ is soft, there exists a section $s' \in \mathscr{F}'(X)$ such that $s|_{W_m\cap V_{m+1}} - s_{m+1}|_{W_m\cap V_{m+1}} = s'|_{W_m\cap V_{m+1}}$. Then s and $s_{m+1} + s'|_{V_{m+1}}$ coincide on $W_m \cap V_{m+1}$, hence can be glued to a section of $\mathscr{F}(W_{m+1})$ that lifts $s''|_{W_{m+1}}$. This proves the desired surjectivity when $U = X$ is compact.

Let now U be an open subset of X and let us prove that the morphism $\Gamma_c(U, \mathscr{F}) \to \Gamma_c(U, \mathscr{F}'')$ is surjective. Let $s'' \in \Gamma_c(U, \mathscr{F}'')$ and let V be an open neighborhood of s'' such that $\overline{V}$ is compact. Since $\mathscr{F}'|_{\overline{V}}$ is soft and $\overline{V}$ is compact, the section s'' lifts to a section $s \in \mathscr{F}(\overline{V})$. The restriction $s|_{\partial V}$ maps to 0 in $\mathscr{F}''(\partial V)$, hence belongs to $\mathscr{F}'(\partial V)$. Since $\mathscr{F}'$ is soft, there exists a section $t \in \mathscr{F}'(\overline{V})$ such that $t|_{\partial V} = s|_{\partial V}$. The section $s - t \in \mathscr{F}(\overline{V})$ lifts $s''|_{\overline{V}}$ and restricts to 0 on ∂V. Consequently, its extension by 0 is an element of $\mathscr{F}(U)$ which lifts s''; moreover, its support is contained in $\overline{V}$, hence is compact. This concludes the proof of *a*).

Let us finally assume moreover that $\mathscr{F}$ is soft and let us prove that $\mathscr{F}''$ is soft as well. Let A be a compact subset of X and let $s'' \in \Gamma(A, \mathscr{F}''|_A)$. Let V be an open neighborhood of A such that $\overline{V}$ is compact and s'' is induced by a section of $\mathscr{F}''$ on $\overline{V}$; we still denote it by s''. Since $\mathscr{F}'|_{\overline{V}}$ is soft, there exists $s \in \mathscr{F}(\overline{V})$ that lifts s''. Since $\mathscr{F}$ is soft, there exists $t \in \Gamma_c(X, \mathscr{F})$ that extends s. The image t'' of t in $\Gamma_c(X, \mathscr{F}'')$ extends s''. That concludes the proof of the proposition. □

Corollary **(3.5.14)**. — *Let $f\colon X \to Y$ be a continuous map of topological spaces. Assume that X is locally compact. Then the functor $f_!\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}(Y)$ is left exact and the full subcategory of $\boldsymbol{Ab}(X)$ consisting of soft abelian sheaves is injective with respect to $f_!$.*

In particular, this subcategory is injective with respect to the functor $\Gamma_c(X,\cdot)$ of global sections with compact support.

Proof. — Every abelian sheaf can be embedded into a flasque sheaf, hence into a soft sheaf. Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of abelian sheaves on X, where $\mathscr{F}'$ and $\mathscr{F}$ are soft; by the proposition, the sheaf $\mathscr{F}''$ is soft as well, so that we just need to prove that the sequence $0 \to f_!\mathscr{F}' \to f_!\mathscr{F} \to f_!\mathscr{F}'' \to 0$ of abelian sheaves on Y is exact. By left exactness of the functor $f_!$, it suffices to prove the surjectivity of the morphism $f_!\mathscr{F} \to f_!\mathscr{F}''$. By proposition **??**, its fiber at a point $y \in Y$ identifies with the morphism $\Gamma_c(X_y, \mathscr{F}|_{X_y}) \to \Gamma_c(X_y, \mathscr{F}''_{X_y})$. The sequence $0 \to \mathscr{F}'|_{X_y} \to \mathscr{F}|_{X_y} \to \mathscr{F}''|_{X_y} \to 0$ is exact, and $\mathscr{F}'_{X_y}$ is flasque. Consequently, the sequence obtained by applying the functor $\Gamma_c(X_y,\cdot)$ is exact, as was to be shown. □

Corollary **(3.5.15)**. — *Let $j\colon U \to X$ be the inclusion of an open subset. There exists a unique isomorphism of ∂-functors $\boldsymbol{R}\Gamma_c(U,\cdot) \to \boldsymbol{R}\Gamma_c(X,\cdot) \circ j_!$ which extends the isomorphism of functors $\Gamma_c(U,\cdot) \to \Gamma_c(X, j_!(\cdot))$.*

Proof. — The functor $j_!$ is exact, because it induces either the identity, or 0, on the fibers. Consequently, the composition $\boldsymbol{R}\Gamma_c(X,\cdot) \circ j_!$ is indeed a ∂-functor. Moreover, applying the functor $j_!$ to an injective resolution $\mathscr{G}_0 \to \dots$ of $\mathscr{F}$, one gets a soft resolution of $j_!\mathscr{F}$. Applying the functor $\Gamma_c(X,\cdot)$, we obtain the the desired isomorphism.(7) □

Corollary **(3.5.16)**. — *Let X be a locally compact topological space. Let $\mathscr{F}$ be an abelian sheaf on X. The following properties are equivalent:*

(i) *The abelian sheaf $\mathscr{F}$ is soft;*
(ii) *One has $H^j_c(U, \mathscr{F}) = 0$ for every open subset U of X and every integer $j \geqslant 1$;*
(iii) *One has $H^1_c(U, \mathscr{F}) = 0$ for every open subset U of X.*

(7) Uniqueness?

Proof. — The implication (i)⇒(ii) follows from the preceding corollary, and the implication (ii)⇒(iii) is obvious. Let us thus assume that $H^1_c(U, \mathscr{F}) = 0$ for every open subset U of X and let us prove that $\mathscr{F}$ is soft.

Let now A be a compact subset of X, let $i\colon A \hookrightarrow X$ and $j\colon X - A \to X$ be the inclusions. Let us consider the long exact sequence of cohomology with compact supports associated with the canonical exact sequence

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0\,;$$

we obtain

$$0 \to \Gamma_c(X - A, \mathscr{F}|_U) \to \Gamma_c(X, \mathscr{F}) \to \Gamma_c(A, \mathscr{F}|_A) \to H^1_c(X - A, \mathscr{F}|_U).$$

The assumption (iii) implies that the canonical morphism $\Gamma_c(X, \mathscr{A}) \to \Gamma(A, \mathscr{F}|_A)$ is surjective. This proves that $\mathscr{F}$ is soft. □

3.5.17. — To be added: $f_!$ of soft is soft; composition $(g \circ f)_! = g_! \circ f_!$, and similarly after derivation.

Theorem (**3.5.18**). — [8] *Let $f\colon X \to Y$ be a continuous map of topological spaces and let $\mathscr{F}$ be a sheaf on X. Let $y \in Y$ and let $X_y = f^{-1}(y)$. By restriction, the canonical map $(f_*\mathscr{F})_y \to \Gamma(X_y, \mathscr{F}|_{X_y})$ induces a bijection $(f_!\mathscr{F})_y \xrightarrow{\sim} \Gamma_c(X_y, \mathscr{F}|_{X_y})$.*

Moreover, one has isomorphisms of ∂-functors $(\boldsymbol{R}^n f_!(\cdot))_y \to H^n_c(X_y, (\cdot)|_{X_y})$.

3.6. Tensor products

3.7. Verdier duality

3.8. The six operations

3.9. Constructible sheaves

3.10. Exercises

Exercise (**3.10.1**). — Let X be a complex manifold, for example $X = \mathbf{C}$, and let $\mathscr{O}_X$ be the sheaf of holomorphic functions on X. By associating with a holomorphic function f on an open subset U of X the (non-vanishing) function $\exp(f)$, one defines a morphism of sheaves $\varepsilon\colon \mathscr{O}_X \to \mathscr{O}_X^\times$.

[8] Donner un énoncé plus précis, et la preuve.

a) Prove that if U is simply connected (say, contractible), then the induced morphism $\varepsilon_{\mathrm{U}} : \mathscr{O}_{\mathrm{X}}(\mathrm{U}) \to \mathscr{O}_{\mathrm{X}}^{\times}(\mathrm{U})$ is surjective.

b) Prove that the morphism ε is surjective.

c) Let $\mathrm{X} = \mathbf{C}$ and $\mathrm{U} = \mathbf{C}^{\times}$. Prove that the function $z \in \mathscr{O}_{\mathrm{X}}^{\times}(\mathbf{C}^{\times})$ does not belong to the image of ε_{U}.

Exercise **(3.10.2)**. — Let X be a topological space.

a) Let U and V be open subsets of X; compute $\mathrm{Hom}(\mathbf{Z}_{\mathrm{U}}, \mathbf{Z}_{\mathrm{V}})$.

b) Let $\mathscr{U}$ be a family of open subsets of X and let $\mathscr{F}_{\mathscr{U}}$ be the sheaf $\bigoplus_{\mathrm{U}\in\mathscr{U}} \mathbf{Z}_{\mathrm{U}}$ on X. If $\mathscr{U}$ is a covering of X, construct an epimorphism of sheaves $f_{\mathscr{U}} : \mathscr{F}_{\mathscr{U}} \to \mathbf{Z}_{\mathrm{X}}$.

c) Let $(\mathscr{U}_n)$ be a sequence of open coverings of X such that, for every $n \in \mathbf{N}$, the covering $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$, and such that the empty set is the only open subset U of X which is finer that every $\mathscr{U}_n$. Let $f_n : \mathscr{F}_n \to \mathbf{Z}_{\mathrm{X}}$ be the epimorphism constructed in the preceding question. Prove that the corresponding product morphism, $f : \prod \mathscr{F}_n \to \prod \mathbf{Z}_{\mathrm{X}}$, is not an epimorphism.

CHAPTER 4

TRUNCATION STRUCTURES

4.1. Definition of truncation structures

Definition (**4.1.1**). — *Let $\boldsymbol{D}$ be a triangulated category. A* truncation structure, *in short* t-structure, *on $\boldsymbol{D}$ is the datum of two full subcategories $\boldsymbol{D}^{\leqslant 0}$ and $\boldsymbol{D}^{\geqslant 1}$ of $\boldsymbol{D}$ satisfying the following conditions:*

(i) *Every object isomorphic to an object of $\boldsymbol{D}^{\leqslant 0}$ (resp. of $\boldsymbol{D}^{\leqslant 0}$) belongs to $\boldsymbol{D}^{\leqslant 0}$ (resp. of $\boldsymbol{D}^{\leqslant 0}$);*

(ii) *One has $\boldsymbol{D}(\mathrm{X}, \mathrm{Y}) = 0$ for every $\mathrm{X} \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$ and every $\mathrm{Y} \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$;*

(iii) *One has $\Sigma\boldsymbol{D}^{\leqslant 0} \subset \boldsymbol{D}^{\leqslant 0}$ and $\boldsymbol{D}^{\geqslant 1} \subset \Sigma\boldsymbol{D}^{\geqslant 1}$;*

(iv) *For every object $\mathrm{X} \in \mathrm{ob}(\boldsymbol{D})$, there exists a distinguished triangle $\mathrm{A} \to \mathrm{X} \to \mathrm{B} \to \Sigma\mathrm{A}$ in $\boldsymbol{D}$, where $\mathrm{A} \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$ and $\mathrm{B} \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$.*

Remark (**4.1.2**). — Let $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ be a truncation structure on the triangulated category $\boldsymbol{D}$. We introduce the notation $\boldsymbol{D}^{\leqslant n} = \Sigma^{-n}\boldsymbol{D}^{\leqslant 0}$ and $\boldsymbol{D}^{\geqslant n+1} = \Sigma^{-n}\boldsymbol{D}^{\geqslant 1}$, for every integer n.

Condition (iii) of the definition can thus be written $\boldsymbol{D}^{\leqslant 0} \subset \boldsymbol{D}^{\leqslant 1}$ and $\boldsymbol{D}^{\geqslant 1} \subset \boldsymbol{D}^{\geqslant 0}$. In fact, for every pair (m, n) of integers such that $m \leqslant n$, one has $\boldsymbol{D}^{\leqslant m} \subset \boldsymbol{D}^{\leqslant n}$ and $\boldsymbol{D}^{\geqslant n} \subset \boldsymbol{D}^{\geqslant m}$.

Example (**4.1.3**). — Let $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ be a truncation structure on the triangulated category $\boldsymbol{D}$. For every integer n, the pair $(\boldsymbol{D}^{\leqslant n}, \boldsymbol{D}^{\geqslant n+1})$ is a translation structure on $\boldsymbol{D}$, called the translation structure deduced by translation.

Example (**4.1.4**). — Let $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ be a truncation structure on the triangulated category $\boldsymbol{D}$. Then $(\boldsymbol{D}^{\geqslant 1,0}, \boldsymbol{D}^{\leqslant 0,0})$ is a truncation structure on the opposite triangulated category $\boldsymbol{D}^{\mathrm{o}}$.

Example **(4.1.5)**. — Let $\boldsymbol{D}$ be a triangulated category. Then the pairs $(\boldsymbol{D}, 0)$ and $(0, \boldsymbol{D})$ are "degenerate" truncation structures on $\boldsymbol{D}$.

Example **(4.1.6)**. — [1] Let $\boldsymbol{A}$ be an abelian category and let $\mathbf{D}(\boldsymbol{A})$ be its derived category. Let $n \in \mathbf{Z}$. Recall that $\mathbf{D}^{\leqslant n}(\boldsymbol{A})$ is the full subcategory of $\mathbf{D}(\boldsymbol{A})$ consisting of complexes X such that $\mathrm{H}^j(\mathrm{X}) = 0$ for $j > 0$, while $\mathbf{D}^{\geqslant n+1}(\boldsymbol{A})$ is the full subcategory of $\mathbf{D}(\boldsymbol{A})$ consisting of complexes Y such that $\mathrm{H}^j(\mathrm{Y}) = 0$ for $j \leqslant 0$. Given a complex $\mathrm{X} \in \mathbf{D}(\boldsymbol{A})$, recall that $\tau_{\leqslant 0}\mathrm{X} \in \mathbf{D}^{\leqslant 0}(\boldsymbol{A})$, $\tau_{\geqslant 1}\mathrm{X} \in \mathbf{D}^{\geqslant 1}(\boldsymbol{A})$, and that there is a distinguished triangle $\tau_{\leqslant 0}\mathrm{X} \to \mathrm{X} \to \tau_{\geqslant 1}\mathrm{X} \to \Sigma\tau_{\leqslant 0}\mathrm{X}$. This shows that the pair $(\mathbf{D}^{\leqslant 0}(\boldsymbol{A}), \mathbf{D}^{\geqslant 1}(\boldsymbol{A}))$ is a truncation structure on $\mathbf{D}(\boldsymbol{A})$. Moreover, for every integer n, one has $\mathbf{D}^{\leqslant n}(\boldsymbol{A}) = \Sigma^{-n}\mathbf{D}^{\leqslant 0}(\boldsymbol{A})$ and $\mathbf{D}^{\geqslant n+1}(\boldsymbol{A}) = \Sigma^{-n}\mathbf{D}^{\geqslant 1}(\boldsymbol{A})$.

Proposition **(4.1.7)**. — *Let $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ be a truncation structure on the triangulated category $\boldsymbol{D}$.*

a) *The inclusion $\boldsymbol{D}^{\leqslant 0} \to \boldsymbol{D}$ admits a right adjoint $\tau_{\leqslant 0}$ with counit η;*

b) *The inclusion $\boldsymbol{D}^{\geqslant 1} \to \boldsymbol{D}$ admits a left adjoint $\tau_{\geqslant 1}$ with unit ε;*

c) *For every object X of $\boldsymbol{D}$, there exists a unique morphism $\partial_{\mathrm{X}} : \tau_{\geqslant 1}\mathrm{X} \to \Sigma\tau_{\leqslant 0}\mathrm{X}$ such that the triangle $\tau_{\leqslant 0}\mathrm{X} \xrightarrow{\eta_{\mathrm{X}}} \mathrm{X} \xrightarrow{\varepsilon_{\mathrm{X}}} \tau_{\geqslant 1}\mathrm{X} \xrightarrow{\partial_{\mathrm{X}}} \Sigma\tau_{\leqslant 0}\mathrm{X}$ is distinguished.*

d) *Let $\mathrm{A} \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$, $\mathrm{B} \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$ and let $\mathrm{A} \to \mathrm{X} \to \mathrm{B} \to \Sigma\mathrm{A}$ be a distinguished triangle. There exists a unique morphism of distinguished triangles*

$$\begin{array}{ccccccc}
\mathrm{A} & \longrightarrow & \mathrm{X} & \longrightarrow & \mathrm{B} & \longrightarrow & \Sigma\mathrm{A} \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\tau_{\leqslant 0}\mathrm{X} & \xrightarrow{\eta_{\mathrm{X}}} & \mathrm{X} & \xrightarrow{\varepsilon_{\mathrm{X}}} & \tau_{\geqslant 1}\mathrm{X} & \xrightarrow{\partial_{\mathrm{X}}} & \Sigma\tau_{\leqslant 0}\mathrm{X}.
\end{array}$$

Proof. — a) We need to find, for every object $\mathrm{Y} \in \mathrm{ob}(\boldsymbol{D})$, an object $\tau_{\leqslant 0}$ of $\boldsymbol{D}^{\leqslant 0}$, a morphism $\eta_{\mathrm{Y}} : \tau_{\leqslant 0}\mathrm{Y} \to \mathrm{Y}$ in $\boldsymbol{D}$ inducing bi-functorial isomorphisms $\boldsymbol{D}(\mathrm{X}, \mathrm{Y}) \simeq \boldsymbol{D}(\mathrm{X}, \tau_{\leqslant 0}\mathrm{Y})$, for $\mathrm{X} \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$. Let $\mathrm{A} \to \mathrm{Y} \to \mathrm{B} \to \Sigma\mathrm{A}$ be a distinguished triangle, where $\mathrm{A} \in \boldsymbol{D}^{\leqslant 0}$ and $\mathrm{B} \in \boldsymbol{D}^{\geqslant 1}$. Applying the cohomological functor $\boldsymbol{D}(\mathrm{X}, \cdot)$ to the translated distinguished triangle $\Sigma^{-1}\mathrm{B} \to \mathrm{A} \to \mathrm{Y} \to \mathrm{B}$, we obtain an exact sequence

$$\boldsymbol{D}(\mathrm{X}, \Sigma^{-1}\mathrm{B}) \to \boldsymbol{D}(\mathrm{X}, \mathrm{A}) \to \boldsymbol{D}(\mathrm{X}, \mathrm{Y}) \to \boldsymbol{D}(\mathrm{X}, \mathrm{B}).$$

[1] Preuve incomplète

Since $\Sigma^{-1}B \in \boldsymbol{D}^{\geqslant 1}$ and $B \in \boldsymbol{D}^{\geqslant 1}$, the two extreme groups vanish, so that the morphism $A \to Y$ induces an isomorphism $\boldsymbol{D}(X, A) \to \boldsymbol{D}(X, Y)$ for every $X \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$.

For every object Y in $\boldsymbol{D}$, let us choose a distinguished triangle $A \xrightarrow{u} Y \to B \to \Sigma A$ and let us set $\tau_{\leqslant 0}Y = A$ and $\eta_Y = u$. Let $f \colon Y \to Z$ be a morphism in $\boldsymbol{D}$. By what precedes, the morphism η_Z induces an isomorphism $\boldsymbol{D}(\tau_{\leqslant 0}Y, \tau_{\leqslant 0}Z) \to \boldsymbol{D}(\tau_{\leqslant 0}Y, Z)$. Let then $\tau_{\leqslant 0}(f) \colon \tau_{\leqslant 0}Y \to \tau_{\leqslant 0}Z$ be the unique morphism such that $\eta_Z \circ \tau_{\leqslant 0}(f) = f \circ \eta_Y$.

One checks readily that $\tau_{\leqslant 0}$ is a functor and that the morphisms η_Y, for $Y \in \mathrm{ob}(\boldsymbol{D})$, are the counits of an adjunction, making $\tau_{\leqslant 0}$ a right adjoint of the inclusion of $\boldsymbol{D}^{\leqslant 0}$ in $\boldsymbol{D}$.

b) This is proved analogously to *a*), or can be deduced from *a*) by passing to the opposite category and shifting. In fact, one may choose for every object Y a distinguished triangle $A \to Y \to B \to \Sigma A$ as above and set $\tau_{\geqslant 1}Y = B$.

c) Let Y be an object of $\boldsymbol{D}$. By construction, the counit η_Y of the adjunction $(\cdot, \tau_{\leqslant 0})$ and the unit ε_Y of the adjunction $(\tau_{\geqslant 1}, \cdot)$ stand in a distinguished triangle

$$\tau_{\leqslant 0}Y \xrightarrow{\eta_Y} Y \xrightarrow{\varepsilon_Y} \tau_{\geqslant 1} \xrightarrow{\partial} \Sigma\tau_{\leqslant 0}Y.$$

Since $\boldsymbol{D}(\tau_{\leqslant 0}Y, \tau_{\geqslant 1}Y) = 0$, the uniqueness of the differential ∂ follows from corollary 2.2.6.

d) Let Y be an object of $\boldsymbol{D}$, let $A \xrightarrow{u} Y \xrightarrow{v} B \xrightarrow{w} \Sigma A$ be a distinguished triangle, where $A \in \boldsymbol{D}^{\leqslant 0}$ and $B \in \boldsymbol{D}^{\geqslant 1}$. Let us show that there exist a unique morphism of distinguished triangles of the form (f, id_Y, h):

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & Y & \xrightarrow{v} & B & \xrightarrow{w} & \Sigma A \\
\downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
\tau_{\leqslant 0}Y & \xrightarrow{\eta_Y} & Y & \xrightarrow{\varepsilon_Y} & \tau_{\geqslant 1}Y & \xrightarrow{\partial_Y} & \Sigma\tau_{\leqslant 0}Y.
\end{array}$$

Since $A \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$ and $\tau_{\geqslant 1}Y \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$, one has $\varepsilon_Y \circ \mathrm{id}_Y \circ u \in \boldsymbol{D}(A, \tau_{\geqslant 1}Y) = 0$, by definition of a truncation structure. Consequently, the assertion follows from corollary 2.2.6. □

Corollary **(4.1.8)**. — a) *Let* $X \in \mathrm{ob}(\boldsymbol{D})$. *The following properties are equivalent: 1) One has* $\tau_{\leqslant 0}X = 0$*; 2) One has* $\boldsymbol{D}(A, X) = 0$ *for every object* $A \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$*; 3) The morphism* $\eta_X \colon X \to \tau_{\geqslant 1}X$ *is an isomorphism; 4) One has* $X \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$.

b) *The category* $\boldsymbol{D}^{\geqslant 1}$ *is a thick additive subcategory of* $\boldsymbol{D}$*; it is stable under products and under extensions (if* $X \to Y \to Z \to \Sigma X$ *is a distinguished triangle and* $X, Z \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$*, then* $Y \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$*).*

Proof. — a) Assume that $\tau_{\leqslant 0}X = 0$; by adjunction of $(\cdot, \tau_{\leqslant 0})$, one then has $\boldsymbol{D}(A, X) \simeq \boldsymbol{D}(A, \tau_{\leqslant 0}X) = 0$. Assume conversely that $\boldsymbol{D}(A, \tau_{\leqslant 0}X) = 0$. By adjunction of $(\cdot, \tau_{\leqslant 0})$, one has $0 = \boldsymbol{D}(\tau_{\leqslant 0}X, X) = \boldsymbol{D}(\tau_{\leqslant 0}X, \tau_{\leqslant 0}X)$, so that $\mathrm{id}_{\tau_{\leqslant 0}X} = 0$ and $\tau_{\leqslant 0}X = 0$. This proves that 1)⇔2).

1)⇒3). Assume that $\tau_{\leqslant 0}X = 0$. The canonical distinguished triangle $0 \to X \xrightarrow{\varepsilon_X} \tau_{\geqslant 1}X \to 0$ then implies that ε_X is an isomorphism.

3)⇒4). If ε_X is an isomorphism, it follows from the definition of a truncation structure that $X \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$.

4)⇒1). Finally, let us assume that $X \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$. One thus has $0 = \boldsymbol{D}(\tau_{\leqslant 0}X, X) = \boldsymbol{D}(\tau_{\leqslant 0}X, \tau_{\leqslant 0}X)$, hence $\tau_{\leqslant 0}X = 0$.

b) The characterization 2)⇔4) implies that it is stable under products. More precisely, for every family $(X_i)_{i\in I}$ with product X and for every object $A \in \boldsymbol{D}^{\leqslant 0}$, the isomorphism $\boldsymbol{D}(A, X) \simeq \prod_i \boldsymbol{D}(A, X_i)$ implies that $X \in \boldsymbol{D}^{\geqslant 1}$ if and only if $X_i \in \boldsymbol{D}^{\geqslant 1}$ for every i. In particular, $\boldsymbol{D}^{\geqslant 1}$ is a thick additive subcategory of $\boldsymbol{D}$.

Let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle, where X and Z belong to $\boldsymbol{D}^{\geqslant 1}$. Let $A \in \boldsymbol{D}^{\leqslant 0}$; let us apply the cohomological functor $\boldsymbol{D}(A, \cdot)$ to this triangle. This furnishes an exact sequence $\boldsymbol{D}(A, X) \to \boldsymbol{D}(A, Y) \to \boldsymbol{D}(A, Z)$. Since $\boldsymbol{D}(A, X) = \boldsymbol{D}(A, Z) = 0$, we thus have $\boldsymbol{D}(A, Y) = 0$, hence $Y \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$ by *a*).

Recall that if $Y \simeq X \oplus Z$, then there exists a distinguished triangle $X \to Y \to Z \to \Sigma X$; in particular Y belongs to $\boldsymbol{D}^{\geqslant 1}$ if both X and Z do. ☐

Either by a similar reasoning, or by passing to the opposite category, one has the following corollary.

Corollary **(4.1.9)**. — a) *Let* $X \in \mathrm{ob}(\boldsymbol{D})$*. The following properties are equivalent: 1) One has* $\tau_{\geqslant 1}X = 0$*; 2) One has* $\boldsymbol{D}(X, B) = 0$ *for every object* $A \in \mathrm{ob}(\boldsymbol{D}^{\geqslant 1})$*; 3) The morphism* $\eta_X : \tau_{\leqslant 0}X \to X$ *is an isomorphism; 4) One has* $X \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$*.*

b) *The category* $\boldsymbol{D}^{\leqslant 0}$ *is a thick additive subcategory of* $\boldsymbol{D}$*; it is stable under coproducts and under extensions (if* $X \to Y \to Z \to \Sigma X$ *is a distinguished triangle and* $X, Z \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$*, then* $Y \in \mathrm{ob}(\boldsymbol{D}^{\leqslant 0})$*).*

4.1.10. — In category theory, an adjoint is only unique up to a canonical isomorphism, and the construction of the functors $\tau_{\leqslant 0}$ and $\tau_{\geqslant 1}$ involved the choice of a distinguished triangle $A \to X \to B \to \Sigma A$, for every object $X \in \boldsymbol{D}$, where $A \in \boldsymbol{D}^{\leqslant}$ and $B \in \boldsymbol{D}^{\geqslant 1}$.

If $X \in \boldsymbol{D}^{\leqslant 0}$, we may assume that the chosen distinguished triangle is $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$, where 0 is a zero object, chosen to be X if $X \simeq 0$. In this case, one has $\tau_{\leqslant 0}X = X$ for every object $X \in \boldsymbol{D}^{\leqslant 0}$, and $\eta_X = \mathrm{id}_X$.

Similarly, if $X \in \boldsymbol{D}^{\geqslant 1}$, we assume that the chosen distinguished triangle is $0 \to X \xrightarrow{\mathrm{id}_X} X \to \Sigma 0$, where 0 is a zero object chosen to be X if $X \simeq 0$, so that $\tau_{\geqslant 1}X = X$ and $\varepsilon_X = \mathrm{id}_X$.

When $X \in \boldsymbol{D}^{\leqslant 0} \cap \boldsymbol{D}^{\geqslant 1}$, it is a zero object and the two chosen distinguished triangles coincide.

4.1.11. — Let n be an integer. The functor $\tau_{\leqslant n} = \Sigma^{-n}\tau_{\leqslant 0}\Sigma^{n}$ is a right adjoint of the inclusion functor $\boldsymbol{D}^{\leqslant n} \to \boldsymbol{D}$. The functor $\tau_{\geqslant n+1} = \Sigma^{-n}\tau_{\geqslant 1}\Sigma^{n}$ is a left adjoint of the inclusion functor $\boldsymbol{D}^{\geqslant n+1} \to \boldsymbol{D}$.

The functors are called the *truncation functors* associated with the given truncation structure on $\boldsymbol{D}$.

To simplify the notation, we also let $\tau_{<n} = \tau_{\leqslant n-1}$ and $\tau_{>n} = \tau_{\geqslant n+1}$, for every integer n. In particular, an object X of $\boldsymbol{D}$ belongs to $\boldsymbol{D}^{\leqslant n}$ if and only if $\tau_{>n}X = 0$; it belongs to $\boldsymbol{D}^{\geqslant n}$ if and only if $\tau_{<n}X = 0$.

4.1.12. — Let a, b be integers such that $a \leqslant b$. One has $\boldsymbol{D}^{\leqslant a} \subset \boldsymbol{D}^{\leqslant b}$. Given our construction of the functors $\tau_{\leqslant n}$, we thus have $\tau_{\leqslant b} \circ \tau_{\leqslant a} = \tau_{\leqslant a}$. On the other hand, the composition $\tau_{\leqslant a} \circ \tau_{\leqslant b}$ is a right adjoint of the inclusion of $\boldsymbol{D}^{\leqslant a}$ into $\boldsymbol{D}$, so that there is a canonical isomorphism of functors $\tau_{\leqslant a} \simeq \tau_{\leqslant b} \circ \tau_{\leqslant a}$.

Similarly, we have $\tau_{\geqslant a+1} \circ \tau_{\geqslant b+1} = \tau_{\geqslant b+1} \simeq \tau_{\geqslant b+1} \circ \tau_{\geqslant a+1}$.

4.1.13. — Let a, b be integers and let $X \in \mathrm{ob}(\boldsymbol{D}^{\leqslant b})$. Then one has $\tau_{>b}\tau_{\geqslant a}X \simeq \tau_{\geqslant a}\tau_{>b}X = 0$, hence $\tau_{\geqslant a}X \in \boldsymbol{D}^{\leqslant b}$.

Similarly, if $X \in \boldsymbol{D}^{\geqslant a}$, then $\tau_{\leqslant b}X \in \boldsymbol{D}^{\geqslant a}$ as well.

Proposition (**4.1.14**). — *Let a and b be integers such that $a \leqslant b$. For every object $X \in \boldsymbol{D}$, there exists a unique morphism $f_X \colon \tau_{\geqslant a} \circ \tau_{\leqslant b}X \to \tau_{\leqslant b} \circ \tau_{\geqslant a}X$ such that the* *following diagram is commutative*

$$\begin{array}{ccccc}
\tau_{\leqslant b}X & \xrightarrow{\eta^b_X} & X & \xrightarrow{\varepsilon^a_X} & \tau_{\geqslant a}X \\
{\scriptstyle \varepsilon^a_{\tau_{\leqslant b}X}}\downarrow & & & & \uparrow{\scriptstyle \eta^b_{\tau_{\geqslant a}X}} \\
\tau_{\geqslant a}\tau_{\leqslant b}X & & \overset{}{\underset{f_X}{\dashrightarrow}} & & \tau_{\leqslant b}\tau_{\geqslant a}X
\end{array}$$

Moreover, the morphisms f_X *give rise to an isomorphism of functors* $f : \tau_{\geqslant a}\circ\tau_{\leqslant b} \to \tau_{\leqslant b}\circ\tau_{\geqslant a}$.

Proof. — Since $\tau_{\leqslant b}X \in \boldsymbol{D}^{\leqslant b}$, the morphism $\varepsilon^a_X \circ \eta^b_X$ factors uniquely through $\tau_{\leqslant b}\tau_{\geqslant a}X$: there exists a unique morphism $f'_X : \tau_{\leqslant b}X \to \tau_{\leqslant b}\tau_{\geqslant a}X$ such that $\eta^b_{\tau_{\geqslant a}X} \circ f'_X = \eta^a_X \circ \varepsilon^b_X$. As we have seen, one has $\tau_{\leqslant b}\tau_{\geqslant a}X \in \boldsymbol{D}^{\geqslant a}$; consequently, the morphism f'_X factors uniquely through $\tau_{\geqslant a}\tau_{\leqslant b}X$: there exists a unique morphism $f_X : \tau_{\geqslant a}\tau_{\leqslant b}X \to \tau_{\leqslant b}\tau_{\geqslant a}X$ such that $f_X \circ \varepsilon^a_{\tau_{\leqslant b}X} = f'_X$. The morphism f_X satisfies

$$\eta^b_{\tau_{\geqslant a}X} \circ f_X \circ \varepsilon^a_{\tau_{\leqslant b}X} = \eta^b_{\tau_{\geqslant a}X} \circ f'_X = \varepsilon^a_X \circ \eta^b_X.$$

The uniqueness of such a morphism is established by the same argument, reversed: the relation $\eta^b_{\tau_{\geqslant a}X} \circ (f_X \circ \varepsilon^a_{\tau_{\leqslant b}X}) = \varepsilon^a_X \circ \eta^b_X$ implies that $f_X \circ \varepsilon^a_{\tau_{\leqslant b}X} = f'_X$, and this in turns characterizes f_X.

Let us show that f_X is an isomorphism. We build an octahedron

$$\begin{array}{ccccccc}
\tau_{<a}X & \longrightarrow & \tau_{\leqslant b}X & \xrightarrow{\varepsilon^a_{\tau_{\leqslant b}X}} & \tau_{\geqslant a}\tau_{\leqslant b}X & \longrightarrow & \Sigma\tau_{<a}X \\
\| & & \downarrow{\scriptstyle \eta^b_X} & & \downarrow & & \| \\
\tau_{<a}X & \xrightarrow{\eta^{a-1}_X} & X & \xrightarrow{\varepsilon^a_X} & \tau_{\geqslant a}X & \longrightarrow & \Sigma\tau_{<a}X \\
 & & \downarrow{\scriptstyle \varepsilon^{b+1}_X} & & \downarrow & & \\
 & & \tau_{>b}X & = & \tau_{>b}X & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma\tau_{<a}X & \longrightarrow & \Sigma\tau_{\leqslant b}X & \longrightarrow & \Sigma\tau_{\geqslant a}\tau_{\leqslant b}X & \longrightarrow & \Sigma^2\tau_{<a}X
\end{array}$$

where both horizontal triangles are the canonical truncation triangles, as well as the left vertical triangle. This furnishes a distinguished triangle

$$\tau_{\geqslant a}\tau_{\leqslant b}X \to \tau_{\geqslant a}X \to \tau_{>b}X \to \Sigma\tau_{\geqslant a}\tau_{\leqslant b}X.$$

One has $\tau_{\geqslant a}\tau_{\leqslant b}X \in \boldsymbol{D}^{\leqslant b}$ and $\tau_{>b}X \in \boldsymbol{D}^{>b}$, so that there exists a unique isomorphism of distinguished triangles

$$\begin{array}{ccccccc}
\tau_{\geqslant a}\tau_{\leqslant b}X & \longrightarrow & \tau_{\geqslant a}X & \longrightarrow & \tau_{>b}X & \longrightarrow & \Sigma\tau_{\geqslant a}\tau_{\leqslant b}X \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\tau_{\leqslant b}\tau_{\geqslant a}X & \xrightarrow{\eta^b_{\tau_{\geqslant a}X}} & \tau_{\geqslant a}X & \longrightarrow & \tau_{>b}\tau_{\geqslant a}X & \longrightarrow & \Sigma\tau_{\leqslant b}\tau_{\geqslant a}X.
\end{array}$$

The left vertical morphism is equal to f_X. Indeed, the morphism

$$\eta^b_{\tau_{\geqslant a}X} \circ f_X : \tau_{\geqslant a}\tau_{\leqslant b}X \to \tau_{\geqslant a}X$$

is the only one that makes the middle upper square of the octahedron commute, and $u = f_X$ is the only morphism such that $\eta^b_{\tau_{\geqslant a}X} \circ u = \eta^b_{\tau_{\geqslant a}X} \circ f_X$. Consequently, f_X is an isomorphism.

Finally, one deduces from the characterization of the morphism f_X that it induces an isomorphism of functors. □

4.2. The heart of a truncation structure

Definition **(4.2.1)**. — *Let* $\boldsymbol{D}$ *be a triangulated category. The* heart *of a truncation structure* $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ *on* $\boldsymbol{D}$ *is the full subcategory* $\boldsymbol{D}^{\leqslant 0} \cap \boldsymbol{D}^{\geqslant 0}$.

Theorem **(4.2.2)**. — *Let* $\boldsymbol{D}$ *be a triangulated category and let* $\boldsymbol{C}$ *be the heart of a truncation structure* $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ *on* $\boldsymbol{D}$.

a) *The category* $\boldsymbol{C}$ *is an abelian category; as a subcategory of* $\boldsymbol{D}$, *it is thick and stable under finite products and extensions.*

b) *A complex* $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ *in* $\boldsymbol{C}$ *is an exact sequence if and only if there exists a morphism* $w : Z \to \Sigma X$ *such that* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ *is a distinguished triangle in* $\boldsymbol{D}$.

c) *The functor* $H^0 = \tau_{\geqslant 0}\tau_{\leqslant 0} : \boldsymbol{D} \to \boldsymbol{C}$ *is a cohomological functor.*

Proof. — a) First of all, the category $\boldsymbol{C}$ is a thick additive subcategory of $\boldsymbol{D}$, because both $\boldsymbol{D}^{\leqslant 0}$ and $\boldsymbol{D}^{\geqslant 0}$ are themselves thick additive subcategories of $\boldsymbol{D}$.

Let us show that any morphism in $\boldsymbol{C}$ admits a kernel and a cokernel. Let thus $u : X \to Y$ be a morphism in $\boldsymbol{C}$ and let us choose a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in $\boldsymbol{D}$. The vertices Y and ΣX of the translated triangle $Y \to Z \to \Sigma X \to \Sigma Y$ belong to $\boldsymbol{D}^{\leqslant 0}$ and to $\boldsymbol{D}^{\geqslant -1}$, hence $Z \in \boldsymbol{D}^{\leqslant 0} \cap \boldsymbol{D}^{\geqslant -1}$.

Let us then prove that the composition $v' = \varepsilon_Z \circ v : Y \to \tau_{\geqslant 0}Z$ is a cokernel of u. One has $v' \circ u = \varepsilon_Z \circ v \circ u = 0$. Let moreover $f : Y \to W$ be a morphism in $\boldsymbol{C}$ such that $f \circ u = 0$; applying the (contravariant) cohomological functor $\boldsymbol{D}(\cdot, W)$ to the previous distinguished triangle, we obtain an exact sequence

$$\boldsymbol{D}(\Sigma X, W) \xrightarrow{w^*} \boldsymbol{D}(Z, W) \xrightarrow{v^*} \boldsymbol{D}(Y, W) \xrightarrow{u^*} \boldsymbol{D}(X, W).$$

Since $\Sigma X \in \boldsymbol{D}^{\leqslant -1}$ and $W \in \boldsymbol{D}^{\geqslant 0}$, one has $\boldsymbol{D}(\Sigma X, W) = 0$. Since $u^*(f) = f \circ u = 0$, there exists a unique morphism $g' \in \boldsymbol{D}(Z, W)$ such that $f = v^*(g') = g' \circ v$. Since $W \in \boldsymbol{D}^{\geqslant 0}$, there exists a unique morphism $g \in \boldsymbol{D}(\tau_{\geqslant 0}Z, W)$ such that $g' = g \circ \varepsilon_Z$. The morphism g satisfies $f = g' \circ v = g \circ \varepsilon_Z \circ v = g \circ v'$, and it is the unique such morphism.

In a similar manner, we show that the composition $w' = \Sigma^{-1}w \circ \eta_{\Sigma^{-1}Z} : \tau_{\leqslant 0}\Sigma^{-1}Z \to X$ is a kernel of f.

Retaining the notation, let us moreover assume that u is a monomorphism. Then its kernel vanishes, one has $\tau_{\leqslant 0}\Sigma^{-1}Z = 0$, hence $\Sigma^{-1}Z \in \boldsymbol{D}^{\geqslant 1}$, hence $Z \in \boldsymbol{D}^{\geqslant 0}$. Since we had $Z \in \boldsymbol{D}^{\leqslant 0}$, this shows that $Z \in \boldsymbol{C}$. The preceding construction then shows that $u : X \to Y$ is a kernel of v.

Similarly, if u is an epimorphism, its cokernel vanishes, hence $\tau_{\geqslant 0}Z = 0$. This implies that $Z \in \boldsymbol{D}^{\leqslant -1} \cap \boldsymbol{D}^{\geqslant -1}$, hence $\Sigma^{-1}Z \in \boldsymbol{C}$. The preceding construction shows that u is the cokernel of the morphism $\Sigma^{-1}w : \Sigma^{-1}Z \to X$.

We have shown that $\boldsymbol{C}$ is an additive category in which every morphism admits a kernel and a cokernel, such that every monomorphism is a kernel, and every epimorphism is a cokernel. Consequently, $\boldsymbol{C}$ is an abelian category.

b) By definition, a complex $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ in $\boldsymbol{C}$ is an exact sequence if and only if u is a monomorphism and $v : Y \to Z$ is its cokernel. The description of the cokernel of v shows that there exists a morphism $w : Z \to \Sigma X$ such that the diagram $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle. Conversely, given such a distinguished triangle with vertices in $\boldsymbol{C}$, the construction of the kernel and cokernel of u proves that $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is an exact sequence.

c) Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle. Let us show that the induced complex $H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z)$ is exact at $H^0(Y)$.

1) Let us first assume that X belongs to $\boldsymbol{D}^{\leqslant 0}$ and prove that $H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \to 0$ is exact.

In fact, we first begin by treating the particular case where all of X, Y, Z belong to $\boldsymbol{D}^{\leqslant 0}$.

Let $T \in \boldsymbol{C}$. Applying the (contravariant) cohomological functor $\boldsymbol{D}(\cdot, T)$ to the given triangle, we obtain an exact sequence of abelian groups:

$$\boldsymbol{D}(\Sigma X, T) \xrightarrow{w^*} \boldsymbol{D}(Z, T) \xrightarrow{v^*} \boldsymbol{D}(Y, T) \xrightarrow{u^*} \boldsymbol{D}(X, T).$$

Since $\Sigma X \in \boldsymbol{D}^{\leqslant -1}$ and $T \in \boldsymbol{D}^{\geqslant 0}$, one has $\boldsymbol{D}(\Sigma X, T) = 0$. On the other hand, since $T \in \boldsymbol{D}^{\geqslant 0}$, the morphism $\varepsilon_Z^0 : Z \to \tau_{\geqslant 0} Z$ induces an isomorphism $\boldsymbol{D}(\tau_{\geqslant 0} Z, T) \xrightarrow{\sim} \boldsymbol{D}(Z, T)$. Since $T \in \boldsymbol{D}^{\leqslant 0}$, the morphism $\eta_{\tau_{\geqslant 0} Z} : \tau_{\leqslant 0}\tau_{\geqslant 0} Z \to \tau_{\geqslant 0} Z$ induces an isomorphism $\boldsymbol{D}(\tau_{\geqslant 0} Z, T) \xrightarrow{\sim} \boldsymbol{D}(H^0(Z), T)$. In this way, the previous exact sequence rewrites as the exact sequence

$$0 \to \boldsymbol{C}(H^0(Z), T) \xrightarrow{H^0(v)^*} \boldsymbol{C}(H^0(Y), T) \xrightarrow{H^0(u)^*} \boldsymbol{C}(H^0(X), T),$$

since $H^0(X), H^0(Y), H^0(Z)$ belong to $\boldsymbol{C}$. Since this holds for every object T of $\boldsymbol{C}$, the initial diagram is exact.

We now return to the general case where only X is assumed to belong to $\boldsymbol{D}^{\leqslant 0}$.

Let us first prove that the morphism $\tau_{\geqslant 1} v \in \boldsymbol{D}(\tau_{\geqslant 1} Y, \tau_{\geqslant 1} Z)$ is an isomorphism. Let $T \in \boldsymbol{D}^{\geqslant 1}$; applying the contravariant cohomological functor $\boldsymbol{D}(\cdot, T)$ to the initial distinguished triangle furnishes an exact sequence

$$\boldsymbol{D}(\Sigma X, T) \xrightarrow{w^*} \boldsymbol{D}(Z, T) \xrightarrow{v^*} \boldsymbol{D}(Y, T) \xrightarrow{u^*} \boldsymbol{D}(X, T).$$

Since $X \in \boldsymbol{D}^{\leqslant 0}$, one has $\Sigma X \in \boldsymbol{D}^{\leqslant 0}$ as well, and $\boldsymbol{D}(\Sigma X, T) = \boldsymbol{D}(X, T) = 0$, because $T \in \boldsymbol{D}^{\geqslant 1}$. Consequently, the morphism $v^* : \boldsymbol{D}(Z, T) \to \boldsymbol{D}(Y, T)$ is an isomorphism. Making use of the adjunction $(\tau_{\geqslant 1}, \cdot)$, we obtain that $\tau_{\geqslant 1}(v)^* : \boldsymbol{D}(\tau_{\geqslant 1} Z, T) \to \boldsymbol{D}(\tau_{\geqslant 1} Y, T)$ is an isomorphism. Since this holds for every object of $\boldsymbol{D}^{\geqslant 1}$, we finally deduce that $\tau_{\geqslant 1} v$ is an isomorphism, as claimed.

Let us now build an octahedron

$$\begin{array}{ccccccc}
X & \longrightarrow & \tau_{\leqslant 0} Y & \longrightarrow & U & \longrightarrow & \Sigma X \\
\| & & \downarrow{\scriptstyle \eta_Y} & & \vdots & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & \Sigma X \\
 & & \downarrow{\scriptstyle \varepsilon_Y} & & \vdots & & \\
 & & \tau_{\geqslant 1} Y & = & \tau_{\geqslant 1} Y & & \\
 & & \downarrow & & \vdots & & \\
\Sigma X & \longrightarrow & \Sigma \tau_{\leqslant 0} Y & \longrightarrow & \Sigma U & \longrightarrow & \Sigma^2 X
\end{array}$$

where the left vertical distinguished triangle is the truncation triangle associated with Y. Let us consider the right vertical distinguished triangle $U \to Z \to \tau_{\geqslant 1}Y \to \Sigma U$. The morphism $Z \to \tau_{\geqslant 1}Y$ is precisely the composition of the unit $Z \to \tau_{\geqslant 1}Z$ and of the inverse of the isomorphism $\tau_{\geqslant 1}v$. Consequently, the object U is isomorphic to $\tau_{\leqslant 0}Z$, and this distinguished triangle is isomorphic to the canonical truncation triangle of Z. The top horizontal distinguished triangle then identifies with the triangle

$$X \xrightarrow{u'} \tau_{\leqslant 0}Y \xrightarrow{\tau_{\leqslant 0}v} \tau_{\leqslant 0}Z \to \Sigma X,$$

where u' is the unique morphism such that $\varepsilon_Y \circ u' = u$. The three objects $X, \tau_{\leqslant 0}Y, \tau_{\leqslant 0}Z$ belong to $\boldsymbol{D}^{\leqslant 0}$; applying the case already established, we obtain the desired exact sequence.

2) One proves similarly (or by passing to the opposite category), that if $Z \in \boldsymbol{D}^{\geqslant 0}$, then the diagram $0 \to H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z)$ is an exact sequence.

3) Let us finally establish the general case. We begin with an octahedron

$$\begin{array}{ccccccc}
\tau_{\leqslant 0}X & \longrightarrow & X & \longrightarrow & \tau_{\geqslant 1}X & \longrightarrow & \Sigma\tau_{\leqslant 0}X \\
\| & & \downarrow{\scriptstyle u} & & \downarrow & & \| \\
\tau_{\leqslant 0}X & \xrightarrow{u'} & Y & \longrightarrow & U & \longrightarrow & \Sigma\tau_{\leqslant 0}X \\
 & & \downarrow{\scriptstyle v} & & \downarrow & & \\
 & & Z & = & Z & & \\
 & & \downarrow{\scriptstyle w} & & \downarrow & & \\
\Sigma\tau_{\leqslant 0}X & \longrightarrow & \Sigma X & \longrightarrow & \Sigma\tau_{\geqslant 1}X & \longrightarrow & \Sigma^2\tau_{\leqslant 0}X.
\end{array}$$

in which the left vertical triangle is the initially given distinguished triangle. Since $\tau_{\leqslant 0}X \in \boldsymbol{D}^{\leqslant 0}$, the second horizontal distinguished triangle furnishes an exact sequence

$$H^0(X) \xrightarrow{H^0(u)} H^0(Y) \to H^0(U) \to 0.$$

Since $\tau_{\geqslant 1}X \in \boldsymbol{D}^{\geqslant 1}$, one has $\Sigma\tau_{\geqslant 1}X \in \boldsymbol{D}^{\geqslant 0}$ and the the second vertical distinguished triangle (shifted once) furnishes an exact sequence

$$0 \to H^0(U) \to H^0(Z) \to H^0(\Sigma\tau_{\geqslant 1}X).$$

The composition of the epimorphism $H^0(Y) \to H^0(U)$ and of the monomorphism $H^0(U) \to H^0(Z)$ is the morphism $H^0(v)$; we thus have established the exactness of $H^0(X) \to H^0(Y) \to H^0(Z)$ at the middle object.

This concludes the proof of *c*), hence of the theorem. □

4.2.3. — If $X \in \mathbf{D}^{<0}$, then $H^0(X) = \tau_{\geqslant 0}\tau_{\leqslant 0}X = \tau_{\geqslant 0}X = 0$. Similarly, if $X \in \mathbf{D}^{>0}$, then $\tau_{\leqslant 0}X = 0$ hence $H^0(X) = 0$.

For every integer n and every object X of $\mathbf{D}$, we set $H^n(X) = H^0(\Sigma^n X)$. With this notation, any distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\mathbf{D}$ gives rise to a long exact sequence

$$\cdots \to H^{n-1}(Z) \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \dots$$

in $\mathbf{C}$.

Observe that $H^n(X) = \Sigma^n \tau_{\geqslant n}\tau_{\leqslant n}X$. If $X \in \mathbf{D}^{<n}$ or $X \in \mathbf{D}^{>n}$, then $H^n(X) = 0$.

Let m be an integer and let $\eta_X : \tau_{\leqslant m}X \to X$ be the canonical morphism. For every integer $n > m$, one has $H^n(\tau_{\leqslant m}X) = 0$, since $\tau_{\leqslant m}X \in \mathbf{D}^{\leqslant m}X \subset \mathbf{D}^{\leqslant n}X$. On the other hand, if $n \leqslant m$, then $H^n(\eta_X) : H^n(\tau_{\leqslant m}X) \to H^n(X)$ is an isomorphism; indeed,

$$H^n(\tau_{\leqslant m}X) = \tau_{\geqslant n}\tau_{\leqslant n}\tau_{\leqslant m}X \xrightarrow{H^n(\eta_X)} \tau_{\geqslant n}\tau_{\leqslant n}X = H^n(X).$$

Similarly, one has $H^n(\tau_{\geqslant m}X) = 0$ for $n < m$, while the canonical morphism $\varepsilon_X : X \to \tau_{\geqslant m}X$ induces an isomorphism $H^n(X) \xrightarrow{\sim} H^n(\tau_{\geqslant m}X)$ for $n \geqslant m$.

Definition (**4.2.4**). — *A truncation structure* $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 1})$ *on* $\mathbf{D}$ *is said to be* non-degenerate *if* $\bigcap_n \mathbf{D}^{\geqslant n}$ *and* $\bigcap_n \mathbf{D}^{\leqslant n}$ *are reduced to zero objects.*

The canonical truncation structure on the derived category $\mathbf{D}(\mathbf{A})$ of an abelian category (example 4.1.6) is nondegenerate. Indeed if $X \in \bigcap_n \mathbf{D}(\mathbf{A})^{\geqslant n}$, then $H^j(X) = 0$ for every $j \in \mathbf{Z}$, so that the zero morphism $0 \to X$ is a homologism.

However, the "degenerate" truncation structures of example 4.1.5 are not non-degenerate.

Proposition (**4.2.5**). — *Let* $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 1})$ *be a nondegenerate truncation structure on* $\mathbf{D}$. *Then the following properties hold:*

a) *An object* $X \in \mathbf{D}$ *is zero if and only if* $H^j(X) = 0$ *for every integer* j;

b) *An object* $X \in \mathbf{D}$ *belongs to* $\mathbf{D}^{\leqslant n}$ *if and only if* $H^j(X) = 0$ *for every integer* j *such that* $j > n$;

c) *An object* $X \in \mathbf{D}$ *belongs to* $\mathbf{D}^{\geqslant n}$ *if and only if* $H^j(X) = 0$ *for every integer* j *such that* $j < n$;

d) *A morphism* $u : X \to Y$ *in* $\mathbf{D}$ *is an isomorphism if and only if* $H^j(u) : H^j(X) \to H^j(Y)$ *is an isomorphism for every integer* j.

Proof. — a) If $X = 0$, then $H^j(X) = 0$ for every j, because H^j is an additive functor. Conversely let X be an object of $\mathbf{D}$ such that $H^j(X) = 0$ for every integer j.

First assume that there exists an integer n such that $X \in \mathbf{D}^{\leqslant n}$. Then $0 = H^n(X) = \tau_{\geqslant n}(X)$, so that $X \in \mathbf{D}^{\leqslant n-1}$. By induction, one has $X \in \bigcap_m \mathbf{D}^{\leqslant m} X = \{0\}$. Similarly, if there exists an integer n such that $X \in \mathbf{D}^{\geqslant n}$, then one has $0 = H^n(X) = \tau_{\leqslant n}(X)$, hence $X \in \mathbf{D}^{\geqslant n+1}$ and, by induction, $X \in \bigcap_m \mathbf{D}^{\geqslant m} = \{0\}$.

In the general case, let us consider the canonical triangle $\tau_{\leqslant 0}X \to X \to \tau_{\geqslant 1}X \to \Sigma\tau_{\leqslant 0}X$. Applying the functor H^0, it induces a long exact sequence

$$\dots H^{n-1}(X) \to H^{n-1}(\tau_{\geqslant 1}X) \to H^n(\tau_{\leqslant 0}X) \to H^n(X) \to \dots$$

so that $H^{n-1}(\tau_{\geqslant 1}X) \simeq H^n(\tau_{\leqslant 0}X)$ for every integer n. Let $n \in \mathbf{Z}$; if $n > 0$, we have $H^n(\tau_{\leqslant 0}X) = 0$; otherwise, one has $n \leqslant 0$, then $H^n(\tau_{\leqslant 0}X) \simeq H^{n-1}(\tau_{\geqslant 1}X) = 0$ since $n-1 \leqslant 0$. Since $\tau_{\leqslant 0}X \in \mathbf{D}^{\leqslant 0}$, the particular case already treated shows that $\tau_{\leqslant 0}X = 0$. One proves similarly that $\tau_{\geqslant 1}X = 0$. Then, the distinguished triangle $0 \to X \to 0 \to 0$ proves that $X = 0$.

b) We already know that if $X \in \mathbf{D}^{\leqslant n}$, then $H^j(X) = 0$ for every integer $j > n$. Conversely, let X be an object of $\mathbf{D}$ such that $H^j(X) = 0$ for every integer $j > n$. Then the object $\tau_{\geqslant n+1}X$ satisfies $H^j(X) = 0$ for every $j \in \mathbf{Z}$. By assertion *a*), one has $\tau_{\geqslant n+1}X = 0$, hence $X \in \mathbf{D}^{\leqslant n}$, by corollary 4.1.9.

c) This is analogous: one proves that $H^j(\tau_{\leqslant n-1}X) = 0$ for every integer j, hence $\tau_{\leqslant n-1}X = 0$, hence $X \in \mathbf{D}^{\geqslant n}$.

d) If u is an isomorphism, then so is $H^n(u)$ for every integer n. Let us assume, conversely, that $H^n(u)$ is an isomorphism for every integer n. Let us complete u into a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$. Applying the functor H^0, we obtain a long exact sequence:

$$\dots \to H^{n-1}(Z) \xrightarrow{H^{n-1}(w)} H^n(X) \xrightarrow{H^n(u)} H^n(Y) \xrightarrow{H^n(v)} H^n(Z) \xrightarrow{H^n(w)} H^{n+1}(X) \to \dots$$

in $\mathbf{C}$. Using that $H^n(u)$ is an isomorphism for every n, one deduces $H^n(Z) = 0$ for every n. By *a*), this implies that $Z = 0$. Consquently, u is an isomorphism. □

4.3. t-exact functors

Definition (4.3.1). — *Let* $\mathbf{D}$ *and* $\mathbf{D}'$ *be triangulated categories endowed with truncation structures and let* $F\colon \mathbf{D} \to \mathbf{D}'$ *be a triangulated functor. One says that*

F *is* right t-exact *if* $F(\mathbf{D}^{\leqslant 0}) \subset \mathbf{D}'^{\leqslant 0}$, *and that it is* left t-exact *if* $F(\mathbf{D}^{\geqslant 1}) \subset \mathbf{D}'^{\geqslant 1}$. *One says that* F *is* t-exact *if it is both left t-exact and right t-exact.*

4.3.2. — Assume that F is left t-exact. By translation, one observes that $F(\mathbf{D}^{\geqslant n+1}) \subset \mathbf{D}'^{\geqslant n+1}$ for every integer n.

Let moreover $X \in \mathbf{D}$, let $\tau_{\leqslant 0}X \to X \xrightarrow{\varepsilon} \tau_{>0}X \to \Sigma\tau_{\leqslant 0}X$ be the canonical triangle. Applying F, we obtain a distinguished triangle $F(\tau_{\leqslant 0}X) \to F(X) \xrightarrow{F(\varepsilon)} F(\tau_{>0}X) \to \Sigma F(\tau_{\leqslant 0}X)$ in $\mathbf{D}'$. By assumption, $F(\tau_{>0}X) \in \mathbf{D}'^{>0}$; consequently, the morphism $F(\varepsilon) : F(X) \to F(\tau_{>0}X)$ factors through a unique morphism $\tau_{>0}F(X) \to F(\tau_{>0}X)$.

4.3.3. — Assume that F is right t-exact. By translation, one observes similarly that $F(\mathbf{D}^{\leqslant n}) \subset \mathbf{D}'^{\leqslant n}$ for every integer n.

Moreover, for every object, the morphism $F(\eta) : F(\tau_{\leqslant 0}X) \to F(X)$ factors through a unique morphism $F(\tau_{\leqslant 0}X) \to \tau_{\leqslant 0}F(X)$, where $\eta : \tau_{\leqslant 0}X \to X$ is the canonical morphism.

4.3.4. — Let $F : \mathbf{D} \to \mathbf{D}'$ be a triangulated functor between triangulated categories endowed with truncation structures. Let $\mathbf{C}$ and $\mathbf{C}'$ be their hearts, and let $\tilde{F} = H^0 \circ F : \mathbf{C} \to \mathbf{C}'$; it is an additive functor.

Proposition (**4.3.5**). — a) *If* F *is left t-exact, then* $\tilde{F}$ *is left exact.*

Moreover, for every object $X \in \mathbf{D}^{\geqslant 0}$, *the canonical morphism* $\eta_X : \tau_{\leqslant 0}X \to X$ *induces an isomorphism* $H^0 \circ F(\eta_X) : \tilde{F}(H^0(X)) \simeq H^0(F(X))$.

b) *If* F *is right t-exact, then* $\tilde{F}$ *is right exact.*

Moreover, for every object $X \in \mathbf{D}^{\leqslant 0}$, *the canonical morphism* $\varepsilon_X : X \to \tau_{\geqslant 0}X$ *induces an isomorphism* $H^0 \circ F(\varepsilon_X) : H^0(F(X)) \to \tilde{F}(H^0(X))$.

Proof. — a) Let us assume that F is left t-exact. Let $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ be an exact sequence in $\mathbf{C}$. By theorem 4.2.2, there exists a morphism $w : Z \to \Sigma X$ such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle. Applying the triangulated functor F, we obtain a distinguished triangle $F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{F(w)} \Sigma F(X)$ in $\mathbf{D}'$. Since F is left t-exact and X, Y, Z belong to $\mathbf{D}^{\geqslant 0}$, their images F(X), F(Y), F(Z) belong to $\mathbf{D}'^{\geqslant 0}$; in particular, $H^{-1}(F(Z)) = 0$. The long exact sequence associated with the previous triangle and the cohomological functor H^0 give rise to an exact sequence

$$0 \to H^0(F(X)) \xrightarrow{H^0F(u)} H^0(F(Y)) \xrightarrow{H^0F(v)} H^0(F(Z)).$$

This proves that the functor $H^0 \circ F$ is left exact.

Let then $X \in \boldsymbol{D}^{\geqslant 0}$, so that $\tau_{\leqslant 0}X = H^0(X)$. Applying as above triangulated functor F and the cohomological functor H^0, the canonical triangle $\tau_{\leqslant 0}X \xrightarrow{\eta_X} X \to \tau_{>0}X \to \Sigma\tau_{\leqslant 0}X$ leads to an exact sequence

$$0 \to H^0F(\tau_{\leqslant 0}X) \xrightarrow{H^0F(\eta)} H^0F(X) \to H^0F(\tau_{>0}X).$$

Since $\tau_{>0}X \in \boldsymbol{D}^{\geqslant 1}$, the left t-exactness of F implies that $F(\tau_{>0}X) \in \boldsymbol{D}'^{\geqslant 1}$, hence $H^0(F(\tau_{>0}X)) = 0$. Consequently, the morphism $H^0 \circ F(\eta_X) \colon \tilde{F}(H^0(X)) = H^0F(H^0(X)) \to H^0F(X)$ is an isomorphism, as claimed.

b) The case of a right t-exact functor is treated similarly. □

Corollary **(4.3.6)**. — *Let $\boldsymbol{D}, \boldsymbol{D}', \boldsymbol{D}''$ be triangulated categories endowed with truncation structures. Let $F \colon \boldsymbol{D} \to \boldsymbol{D}'$ and $G \colon \boldsymbol{D}' \to \boldsymbol{D}''$ be triangulated functors.*

a) *If F and G are left t-exact, then $G \circ F$ is left t-exact and the morphism of functors $H^0G(\eta) \colon \widetilde{G} \circ \widetilde{F} \to \widetilde{G \circ F}$ is an isomorphism, where $\eta \colon \tau_{\leqslant 0} \to \mathrm{id}$ is the counit of the adjunction $(\cdot, \tau_{\leqslant 0})$ in $\boldsymbol{D}$.*

b) *If F and G are right t-exact, then $G \circ F$ is right t-exact and the morphism of functors $H^0G(\varepsilon) \colon \widetilde{G \circ F} \to \widetilde{G} \circ \widetilde{F}$ is an isomorphism, where $\varepsilon \colon \mathrm{id} \to \tau_{\geqslant 0}$ is the unit of the adjunction $(\tau_{\geqslant 0}, \cdot)$ in $\boldsymbol{D}$.*

Proof. — a) Assume that F and G are left t-exact. Then $G \circ F(\boldsymbol{D}^{\geqslant 1}) \subset G(\boldsymbol{D}'^{\geqslant 1}) \subset \boldsymbol{D}''^{\geqslant 1}$, so that $G \circ F$ is left t-exact as well.

Let then $X \in \boldsymbol{C}$. By definition, $\widetilde{G \circ F}(X) = H^0(G(F(X)))$. Let $Y = F(X)$; since F is left t-exact, $Y \in \boldsymbol{D}'^{\geqslant 0}$, and $\tilde{F}(X) = H^0(Y)$. Consequently, the morphism $H^0G(\eta_X) \colon \tilde{G}(\tilde{F}(X)) \to H^0G(Y) = \widetilde{G \circ F}(X)$ is an isomorphism.

b) The case of right t-exact functors is analogous. □

Proposition **(4.3.7)**. — *Let $\boldsymbol{D}$ and $\boldsymbol{D}'$ be triangulated categories endowed with truncation structures. Let $G \colon \boldsymbol{D} \to \boldsymbol{D}'$ and $F \colon \boldsymbol{D}' \to \boldsymbol{D}$ be triangulated functors. Assume that F is right adjoint to G.*

Then, F is left t-exact if and only if G is right t-exact. If these properties hold, then $\tilde{F}$ is right adjoint to $\tilde{G}$.

Proof. — Let us assume that F is left t-exact and let us prove that G is right t-exact. Let X be an object of $\boldsymbol{D}^{\leqslant 0}$ and let us prove that $\mathrm{G(X)} \in \boldsymbol{D}'^{\leqslant 0}$. We use the criterion of corollary 4.1.9; let $\mathrm{Y} \in \boldsymbol{D}'^{\geqslant 1}$; since F is left t-exact, one has $\mathrm{F(Y)} \in \boldsymbol{D}^{\geqslant 1}$; consequently, $\boldsymbol{D}'(\mathrm{G(X),Y}) \simeq \boldsymbol{D}(\mathrm{X,F(Y)}) = 0$; this proves that $\mathrm{G(X)} \in \boldsymbol{D}^{\leqslant 0}$.

Conversely, these arguments prove that if G is right t-exact, then F is left t-exact.

Let X be an object of $\boldsymbol{C}'$ and Y be an object of $\boldsymbol{C}$. Since $\mathrm{F(Y)} \in \boldsymbol{D}^{\geqslant 0}$, one has $\tilde{\mathrm{F}}(\mathrm{Y}) = \mathrm{H}^0(\mathrm{F(Y)}) = \tau_{\leqslant 0}\mathrm{F(Y)}$; similarly, $\tilde{\mathrm{G}}(\mathrm{X}) = \mathrm{H}^0(\mathrm{G(X)}) = \tau_{\geqslant 0}\mathrm{G(X)}$.

The adjunction (G, F) furnishes a bifunctorial isomorphism $\boldsymbol{D}(\mathrm{X,F(Y)}) \simeq \boldsymbol{D}'(\mathrm{G(X),Y})$. Since $\mathrm{X} \in \boldsymbol{D}^{\leqslant 0}$, the adjunction $(\cdot, \tau_{\leqslant 0})$ furnishes a bifunctorial isomorphism $\boldsymbol{C}(\mathrm{X},\tilde{\mathrm{F}}(\mathrm{Y})) \simeq \boldsymbol{D}(\mathrm{X,F(Y)})$. Similarly, the adjunction $(\tau_{\geqslant 0}, \cdot)$ furnishes a bifunctorial isomorphism $\boldsymbol{C}'(\tilde{\mathrm{G}}(\mathrm{X}),\mathrm{Y}) \simeq \boldsymbol{D}'(\mathrm{G(X),Y})$.

The composition of these isomorphisms is a bifunctorial isomorphism $\boldsymbol{C}(\mathrm{X},\tilde{\mathrm{F}}(\mathrm{Y})) \simeq \boldsymbol{C}'(\tilde{\mathrm{G}}(\mathrm{X}),\mathrm{Y})$. In particular, $\tilde{\mathrm{F}}$ is right adjoint to $\tilde{\mathrm{G}}$. □

4.4. Glueing truncation structures

Proposition **(4.4.1)**. — *Let $\boldsymbol{D}$ be a triangulated category, let $\boldsymbol{M}$, $\boldsymbol{N}$ be two full triangulated subcategories such that every object isomorphic to an object of $\boldsymbol{M}$ (resp. $\boldsymbol{N}$) belongs to $\boldsymbol{M}$ (resp. $\boldsymbol{N}$). We make the following hypotheses:*

(i) *For every* $\mathrm{A} \in \boldsymbol{M}$ *and every* $\mathrm{B} \in \boldsymbol{N}$, *one has* $\boldsymbol{D}(\mathrm{A,B}) = 0$;

(ii) *For every object* $\mathrm{X} \in \boldsymbol{D}$, *there exists a distinguished triangle* $\mathrm{A} \to \mathrm{X} \to \mathrm{B} \to \Sigma\mathrm{A}$, *where* $\mathrm{A} \in \boldsymbol{M}$ *and* $\mathrm{B} \in \boldsymbol{N}$.

Let $\mathrm{Q}_{\boldsymbol{M}} : \boldsymbol{D} \to \boldsymbol{D}/\boldsymbol{M}$ *and* $\mathrm{Q}_{\boldsymbol{N}} : \boldsymbol{D} \to \boldsymbol{D}/\boldsymbol{N}$ *be the localization functor from $\boldsymbol{D}$ to its quotients by the subcategories $\boldsymbol{M}$ and $\boldsymbol{N}$ respectively.*

a) *The functor* $\mathrm{Q}_{\boldsymbol{M}}|_{\boldsymbol{N}} : \boldsymbol{N} \to \boldsymbol{D}/\boldsymbol{M}$ *is an equivalence of categories. It admits a quasi-inverse whose composition with* $\mathrm{Q}_{\boldsymbol{M}}$ *is a left adjoint* $\tau_{\boldsymbol{N}}$ *of the inclusion* $\boldsymbol{N} \to \boldsymbol{D}$.

b) *The functor* $\mathrm{Q}_{\boldsymbol{N}}|_{\boldsymbol{M}} : \boldsymbol{M} \to \boldsymbol{D}/\boldsymbol{N}$ *is an equivalence of categories. It admits a quasi-inverse whose composition with* $\mathrm{Q}_{\boldsymbol{N}}$ *is a right adjoint* $\tau_{\boldsymbol{M}}$ *of the inclusion* $\boldsymbol{M} \to \boldsymbol{D}$.

We shall sum up the conclusion of the proposition by saying that the diagrams

$$0 \to \boldsymbol{M} \to \boldsymbol{D} \xrightarrow{\tau_{\boldsymbol{N}}} \boldsymbol{N} \to 0 \qquad \text{and} \qquad 0 \to \boldsymbol{N} \to \boldsymbol{D} \xrightarrow{\tau_{\boldsymbol{M}}} \boldsymbol{M} \to 0$$

are exact sequences of triangulated categories.

Proof. — The two statements are obtained one from another by passing to the opposite category, so that we only prove the first one.

Let us observe that $(\boldsymbol{M}, \boldsymbol{N})$ is a truncation structure on $\boldsymbol{D}$. By corollaries 4.1.8 and 4.1.9, an object X of $\boldsymbol{D}$ belongs to $\boldsymbol{N}$ if and only if $\boldsymbol{D}(\mathrm{A}, \mathrm{X}) = 0$ for every object A of $\boldsymbol{M}$, and an object X of $\boldsymbol{D}$ belongs to $\boldsymbol{M}$ if and only if $\boldsymbol{D}(\mathrm{X}, \mathrm{B}) = 0$ for every object B of $\boldsymbol{N}$. Moreover, the subcategories $\boldsymbol{M}$ and $\boldsymbol{N}$ are thick.

Let X, Y be objects of $\boldsymbol{N}$. By proposition 2.5.17, the localization functor $\mathrm{Q}_{\boldsymbol{M}} : \boldsymbol{D} \to \boldsymbol{D}/\boldsymbol{M}$ induces an isomorphism

$$\boldsymbol{N}(\mathrm{X}, \mathrm{Y}) = \boldsymbol{D}(\mathrm{X}, \mathrm{Y}) \xrightarrow{\sim} (\boldsymbol{D}/\boldsymbol{M})(\mathrm{Q}(\mathrm{X}), \mathrm{Q}(\mathrm{Y})),$$

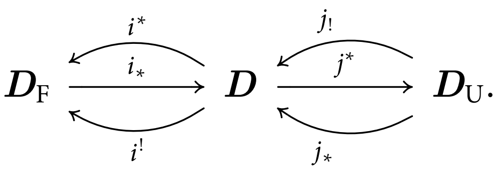
so that the functor $\mathrm{Q}_{\boldsymbol{M}}$ is fully faithful.

Let $\tau_{\boldsymbol{M}} : \boldsymbol{D} \to \boldsymbol{M}$ and $\tau_{\boldsymbol{N}} : \boldsymbol{D} \to \boldsymbol{N}$ be the truncation functors associated with this truncation structure. Let X be an object of $\boldsymbol{D}$; in the canonical distinguished triangle $\tau_{\boldsymbol{M}}\mathrm{X} \to \mathrm{X} \to \tau_{\boldsymbol{N}}\mathrm{X} \to \Sigma\tau_{\boldsymbol{M}}\mathrm{X}$, the morphism $\mathrm{X} \to \tau_{\boldsymbol{N}}\mathrm{X}$ induces an isomorphism in $\boldsymbol{D}/\boldsymbol{N}$, by construction of the quotient category, because $\tau_{\boldsymbol{M}}\mathrm{X} \in \boldsymbol{M}$. Consequently, $\mathrm{Q}_{\boldsymbol{N}}(\mathrm{X}) \xrightarrow{\to} \mathrm{Q}_{\boldsymbol{N}}(\tau_{\boldsymbol{N}}\mathrm{X})$. This proves that the functor $\mathrm{Q}_{\boldsymbol{N}}$ is essentially surjective.

We then observe that the functor $\tau_{\boldsymbol{M}} : \boldsymbol{D} \to \boldsymbol{M}$ is a right adjoint of the inclusion functor $\boldsymbol{M} \hookrightarrow \boldsymbol{D}$. By what precedes, it induces a quasi-inverse of the functor $\mathrm{Q}_{\boldsymbol{N}}$. □

4.4.2. The general context of glueing. — We fix some notation which will remain in force in the following sections; the motivation for this situation will be explained in example 4.4.3.

We assume given three triangulated categories $\boldsymbol{D}$, $\boldsymbol{D}_{\mathrm{U}}$, $\boldsymbol{D}_{\mathrm{F}}$ and six triangulated functors:



We also make the following hypotheses:

(i) The two pairs (i^*, i_*) and $(i_*, i^!)$ are adjoint.

(ii) The two pairs $(j_!, j^*)$ and (j^*, j_*) are adjoint. In each case, the corresponding units will be denoted by a letter ε, the corresponding counits by a letter η.

(iii) The functors i_*, $j_!$ and j_* are fully faithful. Equivalently, the counits $i^*i_* \to \mathrm{id}$ and $j^*j_* \to \mathrm{id}$ are isomorphisms, as well as the units $\mathrm{id} \to i^!i_*$ and $\mathrm{id} \to j^*j_!$.

(iv) One has $j^*i_* = 0$. As a consequence, its left adjoint $i^*j_! = 0$, and its right adjoint $i^!j_* = 0$. Moreover, for $X \in \boldsymbol{D}_F$ and $Y \in \boldsymbol{D}_U$, one has $\boldsymbol{D}(j_!Y, i_*X) = \boldsymbol{D}_U(Y, j^*i_*X) = 0$ and $\boldsymbol{D}(i_*X, j_*Y) = \boldsymbol{D}_U(j^*i_*X, Y) = 0$.

(v) For every object $X \in \boldsymbol{D}$, there exists distinguished triangles

$$j_!j^*X \xrightarrow{\eta} X \xrightarrow{\varepsilon} i_*i^*X \to \Sigma j_!j^*X$$

and

$$i_*i^!X \xrightarrow{\eta} X \xrightarrow{\varepsilon} j_*j^*X \to \Sigma i_*i^!X.$$

By corollary 2.2.7, the unlabeled arrows of these triangles are uniquely determined and these triangles are functorial in X.

Observe the symmetry: passing to the opposite categories interchanges $i^!$ and i^* on the one hand, and $j_!$ and j_* on the other hand.

As a consequence of these hypotheses, we note the following functorial isomorphism, for every $Y \in \boldsymbol{D}_U$:

$$\boldsymbol{D}_U(j_!Y, j_*Y) \simeq \boldsymbol{D}_U(Y, j^*j_*Y) \simeq \boldsymbol{D}_U(Y, Y).$$

Example **(4.4.3)**. — The important example of such a glueing context, and the motivation for the notation, comes from topology.[2] Then, $\boldsymbol{D}$, $\boldsymbol{D}_U$ and $\boldsymbol{D}_F$ are the derived categories $\mathbf{D}(\boldsymbol{Ab}(X))$, $\mathbf{D}(\boldsymbol{Ab}(U))$ and $\mathbf{D}(\boldsymbol{Ab}(F))$ of the categories of abelian sheaves on a topological space X, an open subset U, and the closed complement subset $F = X - U$. Let $i\colon F \to X$ and $j\colon U \to X$ be the inclusions.

(i) The extension by zero functor $i_! = i_*\colon \boldsymbol{Ab}(F) \to \boldsymbol{Ab}(X)$ is exact, and induces a functor, still denoted i_*, from $\mathbf{D}_F$ to $\mathbf{D}$. The functor of restriction to F, $i^*\colon \boldsymbol{Ab}(X) \to \boldsymbol{Ab}(F)$, is also exact, and induces a triangulated functor $i^*\boldsymbol{D} \to \boldsymbol{D}_F$. The functor i_* does not admit a right adjoint at the level of categories of sheaves, but Verdier duality provides a right adjoint $i^!\colon \boldsymbol{D} \to \boldsymbol{D}_F$ at the level of derived categories.

[2] What follows is not strictly true; one should rather assume that X, U, F are moderate topological spaces and restrict to the subcategories of the indicated derived categories consisting of complexes with constructible cohomology. This will hopefully be cleaned up once the sections on Verdier duality and constructible sheaves are written.

(ii) Similarly, the extension by zero functor $j_! : \boldsymbol{Ab}(\mathrm{U}) \to \boldsymbol{Ab}(\mathrm{X})$ is exact and fully faithful and induces a functor $j_! : \boldsymbol{D}_{\mathrm{U}} \to \boldsymbol{D}$.

The functor of restriction to U, $j^* : \boldsymbol{Ab}(\mathrm{X}) \to \boldsymbol{Ab}(\mathrm{U})$, is exact as well, and induces a functor $j^* \boldsymbol{D} \to \boldsymbol{D}_{\mathrm{U}}$.

On the other hand, the functor $j_* : \boldsymbol{Ab}(\mathrm{U}) \to \boldsymbol{Ab}(\mathrm{X})$ is only left exact and we denote by $j_* : \boldsymbol{D}_{\mathrm{U}} \to \boldsymbol{D}$ its right derived functor.

(iii) The full faithfulness of i_*, $j_!$ and j_* holds at the level of categories of sheaves, it remains true at the level of derived categories.

(iv) The relation $j^* i_* = 0$ holds at the level of categories sheaves, and remains true at the level of derived categories.

(v) Every abelian sheaf $\mathscr{F}$ on X gives rise to an exact sequence

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0.$$

Applied to every term of complex C of abelian sheaves on X, these exact sequences furnish an exact sequences of complexes of abelian sheaves, hence, passing to the homotopy category $\boldsymbol{K}(\boldsymbol{Ab}(\mathrm{X}))$, a distinguished triangle

$$j_! j^* \mathrm{C} \to \mathrm{C} \to i_* i^* \mathrm{C} \to \Sigma j_! j^* \mathrm{C}.$$

The second required distinguished triangle is deduced from this one by applying the duality functor.

For every $\mathrm{Y} \in \boldsymbol{D}_{\mathrm{U}}$, the identity morphism $\mathrm{id}_{\mathrm{Y}} \in \boldsymbol{D}_{\mathrm{U}}(\mathrm{Y}, \mathrm{Y})$ corresponds, via the isomorphism $\boldsymbol{D}(j_! \mathrm{Y}, j_* \mathrm{Y}) \simeq \boldsymbol{D}_{\mathrm{U}}(\mathrm{Y}, \mathrm{Y})$ to a morphism $j_! \mathrm{Y} \to j_* \mathrm{Y}$. At the level of sheaves, this morphism is nothing but the fact that the sheaf $j_! \mathscr{F}$ is a subsheaf of $j_* \mathscr{F}$.

Proposition **(4.4.4)**. — *With the hypotheses of §4.4.2, one has the following three exact sequences of triangulated categories:*

$$0 \to \boldsymbol{D}_{\mathrm{F}} \xleftarrow{i^*} \boldsymbol{D} \xleftarrow{j_!} \boldsymbol{D}_{\mathrm{U}} \to 0$$

$$0 \to \boldsymbol{D}_{\mathrm{F}} \xrightarrow{i_*} \boldsymbol{D} \xrightarrow{j^*} \boldsymbol{D}_{\mathrm{U}} \to 0$$

$$0 \to \boldsymbol{D}_{\mathrm{F}} \xleftarrow{i^!} \boldsymbol{D} \xleftarrow{j_*} \boldsymbol{D}_{\mathrm{U}} \to 0.$$

Proof. — Since the triangulated functors $i_* : \boldsymbol{D}_{\mathrm{F}} \to \boldsymbol{D}$ and $j_! : \boldsymbol{D}_{\mathrm{U}} \to \boldsymbol{D}$ are fully faithful, they induce equivalences of triangulated categories from $\boldsymbol{D}_{\mathrm{F}}$ and $\boldsymbol{D}_{\mathrm{U}}$ to their images. The hypothesis (iv) and the first distinguished triangle (v) of §4.4.2 allow us to apply proposition 4.4.1 to the triangulated category $\boldsymbol{D}$ and to its pair $(i_* D_F, j_! D_U)$ of triangulated subcategories. It furnishes two exact sequences of triangulated categories

$$0 \to D_F \xrightarrow{i_*} D \xrightarrow{j^*} D_U \to 0 \qquad \text{and} \qquad 0 \to D_U \xrightarrow{j_!} D \xrightarrow{i^*} D_F \to 0.$$

Indeed, with the notation of that proposition, the two functors τ_{D_U} and τ_{D_F} are induced by the distinguished triangle (v), hence are given by $\tau_{D_U} X = j_! j^* X$ and $\tau_{D_F} X = i_* i^* X$; finally, we have identified D_F and D_U as a subcategory of D via the functors $j_!$ and i_* respectively.

The same argument applied to the second distinguished triangle of §4.4.2 (v) furnishes two exact sequences of triangulated categories:

$$0 \to D_F \xrightarrow{i_*} D \xrightarrow{j^*} D_U \to 0 \qquad \text{and} \qquad 0 \to D_U \xrightarrow{j_*} D \xrightarrow{i^!} D_F \to 0.$$

The second exact sequence is the one that was missing. □

Definition **(4.4.5)**. — *Let* $(D_U^{\leqslant 0}, D_U^{\geqslant 1})$ *and* $(D_F^{\leqslant 0}, D_F^{\geqslant 1})$ *be truncation structures on* D_U *and* D_F *respectively. Let* $D^{\leqslant 0}$ *and* $D^{\geqslant 1}$ *be the full subcategories of* D *whose objects are given by*

(4.4.5.1) $$\operatorname{ob}(D^{\leqslant 0}) = \{X \in \operatorname{ob}(D)\,;\ j^*X \in D_U^{\leqslant 0} \text{ and } i^*X \in D_F^{\leqslant 0}\}$$

(4.4.5.2) $$\operatorname{ob}(D^{\geqslant 1}) = \{X \in \operatorname{ob}(D)\,;\ j^*X \in D_U^{\geqslant 1} \text{ and } i^!X \in D_F^{\geqslant 1}\}.$$

Theorem **(4.4.6)**. — *The pair* $(D^{\leqslant 0}, D^{\geqslant 1})$ *given by definition 4.4.5 is a truncation structure on* D.

We say that this truncation structure of D is obtained by *glueing* the given truncation structures on D_U and D_F.

Proof. — We check the axioms of a truncation structure.

a) By construction of the categories $D^{\leqslant 0}$ and $D^{\geqslant 0}$, they contain any object of D which is isomorphic to one of their objects.

b) Let $X \in \operatorname{ob}(D^{\leqslant 0})$ and let $Y \in \operatorname{ob}(D^{\geqslant 1})$. Applying the contravariant cohomological functor $D(\cdot, Y)$ to the distinguished triangle $j_! j^* X \to X \to i_* i^* X \to \Sigma j_! j^* X$, we obtain an exact sequence

$$D(i_* i^* X, Y) \to D(X, Y) \to D(j_! j^* X, Y).$$

By adjunction of the pair $(i_*, i^!)$, one has $D(i_* i^* X, Y) \simeq D_F(i^* X, i^! Y) = 0$, since $i^* X \in D_F^{\leqslant 0}$ and $i^! Y \in D_F^{\geqslant 1}$. Similarly, the pair $(j_!, j^*)$ is adjoint, hence

$\boldsymbol{D}(j_! j^* \mathrm{X}, \mathrm{Y}) \simeq \boldsymbol{D}_{\mathrm{U}}(j^*\mathrm{X}, j^*\mathrm{Y}) \simeq 0$, since $j^*\mathrm{X} \in \boldsymbol{D}_{\mathrm{U}}^{\leqslant 0}$ and $j^*\mathrm{Y} \in \boldsymbol{D}_{\mathrm{U}}^{\geqslant 1}$. Consequently, $\boldsymbol{D}(\mathrm{X}, \mathrm{Y}) = 0$.

c) The inclusions $\Sigma \boldsymbol{D}^{\leqslant 0} \subset \boldsymbol{D}^{\leqslant 0}$ and $\Sigma^{-1} \boldsymbol{D}^{\geqslant 1} \subset \boldsymbol{D}^{\geqslant 1}$ follow from the fact that the functors i^*, j^* and $i^!$ are triangulated.

d) Let X be an object of $\boldsymbol{D}$; let us construct a distinguished triangle $\mathrm{A} \to \mathrm{X} \to \mathrm{B} \to \Sigma \mathrm{A}$, where $\mathrm{A} \in \boldsymbol{D}^{\leqslant 0}$ and $\mathrm{B} \in \boldsymbol{D}^{\geqslant 1}$.

Let $g : \mathrm{X} \to j_* \tau_{>0} j^* \mathrm{X}$ be the unique morphism whose image under the adjunction isomorphism $\boldsymbol{D}(\mathrm{X}, j_* \tau_{>0} j^* \mathrm{X}) \simeq \boldsymbol{D}_{\mathrm{U}}(j^*\mathrm{X}, \tau_{>0} j^*\mathrm{X})$ is the canonical morphism $\varepsilon_{j^*\mathrm{X}} : j^*\mathrm{X} \to \tau_{>0} j^*\mathrm{X}$. We complete g into a distinguished triangle $\mathrm{Y} \xrightarrow{f} \mathrm{X} \xrightarrow{g} j_* \tau_{>0} j^* \mathrm{X} \to \Sigma \mathrm{Y}$.

Similarly, let $v : \mathrm{Y} \to i_* \tau_{>0} i^* \mathrm{Y}$ be the unique morphism whose image under the adjunction isomorphism $\boldsymbol{D}(\mathrm{Y}, i_* \tau_{>0} i^* \mathrm{Y}) \simeq \boldsymbol{D}_{\mathrm{F}}(i^*\mathrm{Y}, \tau_{>0} i^*\mathrm{Y})$ is the canonical morphism $\varepsilon_{i^*\mathrm{Y}} : i^*\mathrm{Y} \to \tau_{>0} i^*\mathrm{Y}$. Let us complete v to a distinguished triangle $\mathrm{A} \xrightarrow{u} \mathrm{Y} \xrightarrow{v} i_* \tau_{>0} i^* \mathrm{Y} \to \Sigma \mathrm{A}$.

Let us then build an octahedron:

$$
\begin{array}{ccccccc}
\mathrm{A} & \xrightarrow{u} & \mathrm{Y} & \xrightarrow{v} & i_* \tau_{>0} i^* \mathrm{Y} & \longrightarrow & \Sigma \mathrm{A} \\
\| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \| \\
\mathrm{A} & \xrightarrow{f \circ u} & \mathrm{X} & \xrightarrow{w} & \mathrm{B} & \longrightarrow & \Sigma \mathrm{A} \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \\
 & & j_* \tau_{>0} i^* \mathrm{X} & = & j_* \tau_{>0} i^* \mathrm{X} & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma \mathrm{A} & \xrightarrow{u} & \Sigma \mathrm{Y} & \xrightarrow{v} & \Sigma i_* \tau_{>0} i^* \mathrm{Y} & \longrightarrow & \Sigma^2 \mathrm{A}.
\end{array}
$$

It suffices to prove that $\mathrm{A} \in \boldsymbol{D}^{\leqslant 0}$ and $\mathrm{B} \in \boldsymbol{D}^{\geqslant 1}$.

Applying the functor j^* to the second vertical triangle, we obtain a distinguished triangle

$$j^* i_* \tau_{>0} i^* \mathrm{Y} \xrightarrow{j^*(h)} j^* \mathrm{B} \xrightarrow{j^*(k)} j^* j_* \tau_{>0} j^* \mathrm{X} \to \Sigma j^* i_* \tau_{>0} i^* \mathrm{Y}.$$

Since $j^* i_* = 0$, the morphism $j^*(k)$ is an isomorphism. Composed with the counit $\eta : j^* j_* \xrightarrow{\sim} \mathrm{id}$ (which is an isomorphism, because j_* is fully faithful), we obtain an isomorphism $\eta_{\tau_{>0} j^*\mathrm{X}} \circ j^*(k) : j^*\mathrm{B} \xrightarrow{\sim} \tau_{>0} j^* \mathrm{X}$. In particular, $j^*\mathrm{B} \in \boldsymbol{D}_{\mathrm{U}}^{\geqslant 1}$.

Similarly, applying the functor $i^!$ to this second vertical triangle, we obtain the distinguished triangle

$$i^! i_* \tau_{>0} i^* Y \xrightarrow{i^!(h)} i^! B \xrightarrow{i^!(k)} i^! j_* \tau_{>0} i^! X \to \Sigma i^! i_* \tau_{>0} i^* Y.$$

Since $i^! j_* = 0$, the morphism $i^!(h)$ is an isomorphism. Composed with the unit $\varepsilon\colon \mathrm{id} \xrightarrow{\sim} i^! i_*$ (which is an isomorphism, because i_* is fully faithful), we obtain an isomorphism $i^!(h) \circ \varepsilon_{\tau_{>0} i^* Y} \colon \tau_{>0} i^* Y \xrightarrow{\sim} i^! B$. Consequently, $i^! B \in \boldsymbol{D}_F^{\geqslant 1}$.

Let us now apply j^* to the second horizontal triangle; we obtain a distinguished triangle

$$j^* A \xrightarrow{j^*(f \circ u)} j^* X \xrightarrow{j^*(w)} j^* B \to \Sigma j^* A.$$

Observe that $\varepsilon_{\tau_{>0} j^* X} \circ j^*(k) \circ j^*(w) = \varepsilon_{\tau_{>0} j^* X} \circ j^*(k \circ w) = \varepsilon_{\tau_{>0} j^* X} \circ j^*(g) = \varepsilon_{j^* X}$. Consequently, we have a distinguished triangle

$$j^* A \xrightarrow{j^*(f \circ u)} j^* X \xrightarrow{\varepsilon_{j^* X}} \tau_{>0} j^* X \to \Sigma j^* A.$$

Equivalently, the morphism $j^*(f \circ u)$ factors uniquely through an isomorphism $j^* A \xrightarrow{\sim} \tau_{\leqslant 0} j^* X$. In particular, $j^* A \in \boldsymbol{D}_U^{\leqslant 0}$.

Let us apply i^* to the first horizontal triangle; this furnishes a distinguished triangle

$$i^* A \xrightarrow{i^*(u)} i^* Y \xrightarrow{i^*(v)} i^* i_* \tau_{>0} i^* Y \to \Sigma i^* A.$$

The counit $\eta\colon i^* i_* \xrightarrow{\sim} \mathrm{id}$ is an isomorphism, because i_* is fully faithful, and one has $\eta_{\tau_{>0} i^* Y} \circ i^*(v) = \eta i^* Y$. Consequently, there exists a distinguished triangle of the form

$$i^* A \xrightarrow{i^*(u)} i^* Y \xrightarrow{\eta i^* Y} \tau_{>0} i^* Y \to \Sigma i^* A.$$

This implies that the morphism $i^*(u)$ factors uniquely through an isomorphism $i^* A \xrightarrow{\sim} \tau_{\leqslant 0} i^* Y$. In particular, $i^* A \in \boldsymbol{D}_F^{\leqslant 0}$.

We thus have proved that $A \in \boldsymbol{D}^{\leqslant 0}$ and $B \in \boldsymbol{D}^{\geqslant 1}$, as claimed.

Consequently, $(\boldsymbol{D}^{\leqslant 0}, \boldsymbol{D}^{\geqslant 1})$ is a truncation structure on $\boldsymbol{D}$. □

Proposition **(4.4.7)**. — a) *The functors $j_!$ and i^* are right t-exact;*

b) *The functors j^* and i_* are t-exact;*

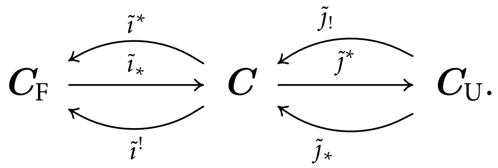
c) *The functors j_* and $i^!$ are left t-exact.*

Proof. — If $X \in \boldsymbol{D}^{\leqslant 0}$, then $j^* X \in \boldsymbol{D}_U^{\leqslant 0}$ and $i^* X \in \boldsymbol{D}_F^{\leqslant 0}$; consequently, j^* and i^* are right t-exact. If $X \in \boldsymbol{D}^{\geqslant 1}$, then $j^* X \in \boldsymbol{D}_U^{\geqslant 1}$ and $i^! X \in \boldsymbol{D}_F^{\geqslant 1}$; consequently, j^* and $i^!$ are left t-exact.

By proposition 4.3.7, the functor $j_!$ is right t-exact and the functor j_* is left t-exact, because $j_!$ is a left adjoint of j^*, and j_* is a right adjoint of j^*.

Since it is a right adjoint of i^* and a left adjoint of $i^!$, proposition 4.3.7 implies that the functor i_* is t-exact. □

Corollary **(4.4.8)**. — *Let* $\boldsymbol{C}_{\mathrm{U}}$ *and* $\boldsymbol{C}_{\mathrm{F}}$ *denote the hearts of the given truncation structures on* $\boldsymbol{D}_{\mathrm{U}}$ *and* $\boldsymbol{D}_{\mathrm{F}}$*, and let* $\boldsymbol{C}$ *be the heart of the truncation structure on* $\boldsymbol{D}$ *which is obtained by glueing. For each of the six functors* $\mathrm{F} \in \{i_*, i^!, i^*, j^*, j_!, j_*\}$*, we let* $\tilde{\mathrm{F}} = \mathrm{H}^0 \circ \mathrm{F}$ *the corresponding functors between the hearts:*



a) *The adjoint pairs* $(j_!, j^*)$*,* (j^*, j_*)*,* (i^*, i_*) *and* $(i_*, i^!)$ *give rise to adjoint pairs* $(\tilde{\jmath}_!, \tilde{\jmath}^*)$*,* $(\tilde{\jmath}^*, \tilde{\jmath}_*)$*,* $(\tilde{\imath}^*, \tilde{\imath}_*)$ *and* $(\tilde{\imath}_*, \tilde{\imath}^!)$*. Moreover, the functors* $\tilde{\imath}_*$*,* $\tilde{\jmath}_!$ *and* $\tilde{\jmath}_*$ *are fully faithful.*

b) *One has* $\tilde{\jmath}^* \circ \tilde{\imath}_* = 0$*,* $\tilde{\imath}^* \circ \tilde{\jmath}_! = 0$ *and* $\tilde{\imath}^! \circ \tilde{\jmath}_* = 0$*. For every object* $\mathrm{X} \in \boldsymbol{C}_{\mathrm{F}}$ *and every object* $\mathrm{Y} \in \boldsymbol{C}_{\mathrm{U}}$*, one has* $\boldsymbol{C}(\tilde{\imath}_*\mathrm{X}, \tilde{\jmath}_*\mathrm{Y}) = \boldsymbol{C}(\tilde{\jmath}_!\mathrm{Y}, \tilde{\imath}_*\mathrm{X}) = 0$.

c) *For every object* $\mathrm{X} \in \boldsymbol{C}$*, there are exact sequences*

$$0 \to \tilde{\imath}_*\mathrm{H}^{-1}(i^*\mathrm{X}) \to \tilde{\jmath}_!\tilde{\jmath}^*\mathrm{X} \to \mathrm{X} \to \tilde{\imath}_*\tilde{\imath}^*\mathrm{X} \to 0$$

and

$$0 \to \tilde{\imath}_*\tilde{\imath}^!\mathrm{X} \to \mathrm{X} \to \tilde{\jmath}_*\tilde{\jmath}^*\mathrm{X} \to \tilde{\imath}_*\mathrm{H}^1(i^!\mathrm{X}) \to 0.$$

Proof. — a) The first part follows from proposition 4.3.7.

Let $\mathrm{X}, \mathrm{Y} \in \boldsymbol{C}_{\mathrm{U}}$. One has bifunctorial isomorphisms

$$\begin{aligned}
\boldsymbol{C}(\tilde{\jmath}_!\mathrm{X}, \tilde{\jmath}_!\mathrm{Y}) &\simeq \boldsymbol{D}(\tau_{\geqslant 0} j_!\mathrm{X}, \tau_{\geqslant 0} j_!\mathrm{Y}) && (j_! \text{ is right t-exact}) \\
&\simeq \boldsymbol{D}(j_!\mathrm{X}, \tau_{\geqslant 0} j_!\mathrm{Y}) && (\text{by adjunction of } (\tau_{\geqslant 0}, \cdot)) \\
&\simeq \boldsymbol{D}_{\mathrm{U}}(\mathrm{X}, j^*\tau_{\geqslant 0} j_!\mathrm{Y}) && (\text{by adjunction of } (j_!, j^*)) \\
&\simeq \boldsymbol{D}_{\mathrm{U}}(\mathrm{X}, \tau_{\geqslant 0} j^* j_!\mathrm{Y}) && (\text{by t-exactness of } j^*) \\
&\simeq \boldsymbol{D}_{\mathrm{U}}(\mathrm{X}, \tau_{\geqslant 0}\mathrm{Y}) && (\text{because } \mathrm{id} \simeq j^* j_!) \\
&\simeq \boldsymbol{D}(\mathrm{X}, \mathrm{Y}) \simeq \boldsymbol{C}(\mathrm{X}, \mathrm{Y}),
\end{aligned}$$

under which a morphism $f \in \boldsymbol{C}(\mathrm{X}, \mathrm{Y})$ corresponds with $\tilde{\jmath}_! f \in \boldsymbol{C}(\tilde{\jmath}_!\mathrm{X}, \tilde{\jmath}_!\mathrm{Y})$. This proves that the functor $\tilde{\jmath}_!$ is fully faithful.

One proves similarly that the functor $\tilde{\jmath}_*$ is fully faithful.

Let then $X, Y \in \mathcal{C}_F$. Since the functor i_* is t-exact, its restriction to $\mathcal{C}_F$ coincides with the functor $\tilde{\imath}_*$. Since i_* is fully faithful, the functor $\tilde{\imath}_*$ is fully faithful as well.

b) The three equalities follow from corollary 4.3.6 and the fact that j^* and i_* are both left (or right) t-exact, that i^* and $j_!$ are both right t-exact, and that j_* and $i^!$ are both left t-exact.

c) Let $X \in \mathcal{C}$ and let us apply the functor H^0 to the canonical distinguished triangle

$$j_!j^*X \to X \to i_*i^*X \to \Sigma j_!j^*X.$$

One obtains an exact sequence

$$H^{-1}(X) \to H^{-1}(i_*i^*X) \to H^0(j_!j^*X) \to H^0(X) \to H^0(i_*i^*X) \to H^1(j_!j^*X).$$

Since $X \in \mathcal{C}$, one has $H^0(X) = X$ and $H^{-1}(X) = 0$. Since j^* and $j_!$ are both right t-exact, one has $H^0(j_!j^*X) = \tilde{\jmath}_!\tilde{\jmath}^*X$. Since i_* and i^* are both right t-exact, one has $H^0(i_*i^*X) = \tilde{\imath}_*\tilde{\imath}^*X$ and $H^{-1}(i_*i^*X) = i_*H^{-1}(i^*X) = \tilde{\imath}_*H^{-1}(i^*X)$. Finally, $\Sigma X \in \mathcal{D}^{\leqslant -1}$; since $j_!$ and j^* are both right t-exact, this implies $j_!j^*X \in \mathcal{D}^{\leqslant 0}$, hence $\Sigma j_!j^*X \in \mathcal{D}^{\leqslant -1}$ and $H^0(\Sigma j_!j^*X) = 0$. This furnishes the desired exact sequence

$$\tilde{\imath}_*H^{-1}(i^*X) \to \tilde{\jmath}_!\tilde{\jmath}^*X \to X \to \tilde{\imath}_*\tilde{\imath}^*X \to 0.$$

The second exact sequence is established similarly, by applying the functor H^0 to the distinguished triangle $i_*i^!X \to X \to j_*j^*X \to \Sigma i_*i^!X$. □

Proposition (**4.4.9**). — *The truncation structure on $\mathcal{D}$ is nondegenerate if and only if the given truncation structures on $\mathcal{D}_U$ and $\mathcal{D}_F$ are nondegenerate.*

Proof. — Let us assume that the given truncation structures on $\mathcal{D}_U$ and $\mathcal{D}_F$ are nondegenerate, and let us prove that the truncation structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ is nondegenerate as well.

Let $X \in \bigcap \mathcal{D}^{\leqslant n}$. Consequently, $j^*X \in \mathcal{D}_U^{\leqslant n}$ and $i^*X \in \mathcal{D}_F^{\leqslant n}$ for every integer n, so that $j^*X = 0$ and $i^*X = 0$. The distinguished triangle $j_!j^*X \to X \to i_*i^*X \to \Sigma j_!j^*X$ then proves that $X = 0$.

Similarly, let $X \in \bigcap_n \mathcal{D}^{\geqslant n}$. This implies that $j^*X \in \bigcap_n \mathcal{D}_U^{\geqslant n}$ and $i^!X \in \bigcap_n \mathcal{D}_F^{\geqslant n}$, so that $j^*X = 0$ and $i^!X = 0$. The distinguished triangle $i_*i^!X \to X \to j_*j^*X \to \Sigma i_*i^!X$ then proves that $X = 0$.

Conversely, let us assume that the truncation structure on $\mathcal{D}$ is nondegenerate.

It follows from the fact that the functor i_* is t-exact and fully faithful that the truncation structure on $\mathcal{D}_F$ is nondegenerate. Let indeed $X \in \bigcap \mathcal{D}_F^{\leqslant n}$. Since i_* is t-exact, one has $i_*X \in \boldsymbol{D}_F^{\leqslant n}$, hence $i_*X = 0$. Since $X \simeq i^*i_*X$, one has $X = 0$. Let then $X \in \bigcap \boldsymbol{D}_F^{\geqslant n}$. Since i_* is t-exact, one has $i_*X \in \boldsymbol{D}_F^{\geqslant n}$, hence $i_*X = 0$. Since $X \simeq i^*i_*X$, one has $X = 0$. This proves that the truncation structure on $\boldsymbol{D}_F$ is nondegenerate, as claimed.

Using that the functor $j_!$ is right t-exact and fully faithful, one proves that $\bigcap \boldsymbol{D}_U^{\leqslant n} = 0$. Using that the functor j_* is left t-exact and fully faithful, one proves that $\bigcap \boldsymbol{D}_U^{\geqslant n} = 0$. This shows that the truncation structure on $\boldsymbol{D}_F$ is nondegenerate. □

Example **(4.4.10)**. — Let us start with a given truncation structure $(\boldsymbol{D}_F, \boldsymbol{D}_F^{\geqslant 1})$ on $\boldsymbol{D}_F$ but with the degenerate truncation structure $(\boldsymbol{D}_U, 0)$ on $\boldsymbol{D}_U$ and let us consider the resulting truncation structure on $\boldsymbol{D}$.

Let $\tau^F_{\leqslant 0} \colon \boldsymbol{D} \to \boldsymbol{D}^{\leqslant 0}$ be the corresponding truncation functor, right adjoint to the inclusion of the subcategory $\boldsymbol{D}^{\leqslant 0}$ whose objects X are characterized by the condition $i^*X \in \boldsymbol{D}_F^{\leqslant 0}$.

Let us go back to the proof of theorem 4.4.6, especially part *d*). With the notation of that proof, we have $\tau_{>0} j^*X = 0$ (because of the choice of the truncation structure on $\boldsymbol{D}_U$), hence $Y = X$, $f = \mathrm{id}$; moreover, $A = \tau^F_{\leqslant 0}X$. Also, h is an isomorphism (two out of three arrows defining the morphism of horizontal triangles are isomorphisms), so that $B \simeq i_*\tau_{>0}i^*Y$. Then, the second horizontal distinguished triangle of the octahedron furnishes the canonical truncation triangle of X:

$$\tau^F_{\leqslant 0}X \to X \to i_*\tau_{>0}i^*X \to \Sigma\tau^F_{\leqslant 0}X.$$

Finally, for every $X \in \boldsymbol{D}$, one has

$$H^0(X) = \tau^F_{\leqslant 0}\tau_{\geqslant 0}X = \tau^F_{\leqslant 0}i_*\tau_{\geqslant 0}i^*X = i_*H^0(i^*X),$$

since i_* is t-exact.

Example **(4.4.11)**. — Still starting with the truncation structure $(\boldsymbol{D}_F^{\leqslant 0}, \boldsymbol{D}_F^{\geqslant 1})$ on $\boldsymbol{D}_F$, let us consider the degenerate truncation structure $(0, \boldsymbol{D}_U)$ on $\boldsymbol{D}_U$.

In that case, an object X of $\boldsymbol{D}$ belongs to $\boldsymbol{D}^{\geqslant 1}$ if and only if $i^!X \in \boldsymbol{D}_F^{\geqslant 0}$. Let us denote by $\tau^F_{\geqslant 0}$ the left adjoint of the inclusion functor $\boldsymbol{D}^{\geqslant 0} \to \boldsymbol{D}$. For every object X, the canonical truncation triangle of X relative to this truncation structure writes

$$i_*\tau_{<0}i^!X \to X \to \tau^F_{\geqslant 0}X \to \Sigma i_*\tau_{<0}i^!X,$$

and the cohomological functor is computed as $H^0(X) = i_*H^0(i^!X)$.

Example **(4.4.12)**. — Let us now start with the degenerate truncation structure $(\boldsymbol{D}_{\mathrm{F}}, 0)$ on $\boldsymbol{D}_{\mathrm{F}}$ and with the given truncation structure $(\boldsymbol{D}_{\mathrm{U}}^{\leqslant 0}, \boldsymbol{D}_{\mathrm{U}}^{\geqslant 0})$ on $\boldsymbol{D}_{\mathrm{U}}$. An object X of $\boldsymbol{D}$ belongs to $\boldsymbol{D}^{\leqslant 0}$ if and only if $j^*\mathrm{X} \in \boldsymbol{D}_{\mathrm{U}}^{\leqslant 0}$. We denote by $\tau_{\leqslant 0}^{\mathrm{U}}$ be the associated truncation functor, right adjoint of the inclusion $\boldsymbol{D}^{\leqslant 0} \to \boldsymbol{D}$. The canonical truncation triangle associated with an object X writes

$$\tau_{\leqslant 0}^{\mathrm{U}}\mathrm{X} \to \mathrm{X} \to j_*\tau_{>0}j^*\mathrm{X} \to \Sigma\tau_{\leqslant 0}^{\mathrm{U}}\mathrm{X},$$

and the cohomological functor is given by $\mathrm{H}^0\mathrm{X} = j_*\mathrm{H}^0 j^*\mathrm{X}$.

Example **(4.4.13)**. — Finally, we start with the degenerate truncation structure $(0, \boldsymbol{D}_{\mathrm{F}})$ on $\boldsymbol{D}_{\mathrm{F}}$ and with the given truncation structure $(\boldsymbol{D}_{\mathrm{U}}^{\leqslant 0}, \boldsymbol{D}_{\mathrm{U}}^{\geqslant 0})$ on $\boldsymbol{D}_{\mathrm{U}}$. An object X of $\boldsymbol{D}$ belongs to $\boldsymbol{D}^{\geqslant 0}$ if and only if $j^*\mathrm{X} \in \boldsymbol{D}_{\mathrm{U}}^{\geqslant 0}$. Let us denote by $\tau_{\geqslant 0}^{\mathrm{U}}$ the associated truncation functor, left adjoint of the inclusion $\boldsymbol{D}^{\geqslant 0} \to \boldsymbol{D}$. For every object X of $\boldsymbol{D}$, the canonical truncation functor writes

$$j_!\tau_{<0}j^*\mathrm{X} \to \mathrm{X} \to \tau_{\geqslant 0}^{\mathrm{U}}\mathrm{X} \to \Sigma j_!\tau_{<0}j^*\mathrm{X},$$

and the cohomological functor is given by $\mathrm{H}^0\mathrm{X} = j_!\mathrm{H}^0(j^*\mathrm{X})$.

Remark **(4.4.14)**. — The four truncation structures on $\boldsymbol{D}$ described in examples 4.4.10, 4.4.11, 4.4.12 and 4.4.13 can be used to describe the truncation structure given by theorem 4.4.6. Indeed, one has the formulas

$$\tau_{\leqslant 0} = \tau_{\leqslant 0}^{\mathrm{F}}\tau_{\leqslant 0}^{\mathrm{U}} \qquad \text{and} \qquad \tau_{\geqslant 0} = \tau_{\geqslant 0}^{\mathrm{F}}\tau_{\geqslant 0}^{\mathrm{U}}.$$

To prove the first formula, let us go back to the octahedron drawn in the course of the proof of theorem 4.4.6. In that diagram, the first vertical distinguished triangle identifies with the canonical truncation triangle associated with the truncation structure of example 4.4.12, so that $\mathrm{Y} = \tau_{\leqslant 0}^{\mathrm{U}}\mathrm{X}$. Then $\tau_{\leqslant 0}\mathrm{X} = \mathrm{A} = \tau_{\leqslant 0}^{\mathrm{F}}\mathrm{Y} = \tau_{\leqslant 0}^{\mathrm{F}}\tau_{\leqslant 0}^{\mathrm{U}}\mathrm{X}$.

The other formula is deduced from that one by passing to the opposite categories (and exchanging $i^!$ and i^* on the one hand, and j_* and $j_!$ on the other hand).

4.5. Extensions

We retain the notation of the previous section, as given in §4.4.2.

Definition **(4.5.1)**. — *Let* $\mathrm{Y} \in \boldsymbol{D}_{\mathrm{U}}$. *An* extension *of* Y *is an object* X *of* $\boldsymbol{D}$ *endowed with an isomorphism* $u\colon j^*\mathrm{X} \xrightarrow{\sim} \mathrm{Y}$.

Let (X, u) be an extension of Y. By the adjunction (j^*, j_*), the datum of the morphism u is equivalent with that of a morphism $u^\sharp \in \boldsymbol{D}(\mathrm{X}, j_*\mathrm{Y})$. On the other hand, the morphism $u^{-1} : \mathrm{Y} \to j^*\mathrm{X}$ corresponds with a morphism $u^\flat \in \boldsymbol{D}(j_!\mathrm{Y}, \mathrm{X})$ under the adjunction $(j_!, j^*)$. This furnishes a diagram

$$j_!\mathrm{Y} \xrightarrow{u^\flat} \mathrm{X} \xrightarrow{u^\sharp} j_*\mathrm{Y}$$

whose composition is the element of $\boldsymbol{D}(j_!\mathrm{Y}, j_*\mathrm{Y}) \simeq \boldsymbol{D}_{\mathrm{U}}(\mathrm{Y}, j^* j_*\mathrm{Y}) \simeq \boldsymbol{D}_{\mathrm{U}}(\mathrm{Y}, \mathrm{Y})$ corresponding to id_{Y}.

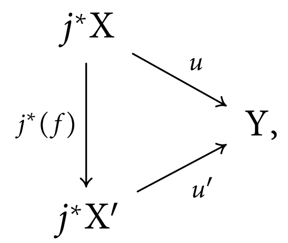
After translation and identifying $j^*\mathrm{X}$ with Y, the canonical distinguished triangle $i_* i^! \mathrm{X} \to \mathrm{X} \to j_* j^* \mathrm{X} \to \Sigma i_* i^! \mathrm{X}$ furnishes a distinguished triangle:

(4.5.1.1)
$$\mathrm{X} \xrightarrow{u^\sharp} j_*\mathrm{Y} \to i_* i^! \Sigma\mathrm{X} \to \Sigma\mathrm{X}.$$

Applying the functor i^* to this triangle and using the isomorphism of functors $i^* i_* \xrightarrow{\sim} \mathrm{id}$, we obtain another distinguished triangle:

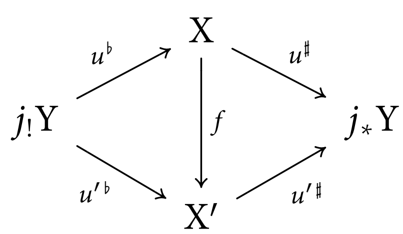
(4.5.1.2)
$$i^*\mathrm{X} \xrightarrow{i^*(u^\sharp)} i^* j_*\mathrm{Y} \to i^! \Sigma\mathrm{X} \to \Sigma i^*\mathrm{X}.$$

4.5.2. — If (X, u) and (X', u') are two extensions of Y, a morphism of extensions from X to X′ is an element $f \in \boldsymbol{D}(\mathrm{X}, \mathrm{X}')$ that makes the following diagram commutative



in other words, such that $u' \circ j^*(f) = u$.

This furnishes the commutative diagram



Conversely, a morphism $f : \mathrm{X} \to \mathrm{X}'$ is a morphism of extensions if and only if $f \circ u^\flat = u'^\flat$, if and only if $u'^\sharp \circ f = u^\sharp$.

Extensions of a given object $\mathrm{Y} \in \boldsymbol{D}_{\mathrm{U}}$ form a category.

Example (**4.5.3**). — Let Y be an object of $\boldsymbol{D}_{\mathrm{U}}$. The object $j_!\mathrm{Y}$, endowed with the isomorphism $\varepsilon_{\mathrm{Y}}\colon \mathrm{Y} \xrightarrow{\sim} j^* j_! \mathrm{Y}$, unit of the adjunction $(j_!, j^*)$, is an extension of Y. The object $j_*\mathrm{Y}$, endowed with the isomorphism $\eta_{\mathrm{Y}}\colon j^* j_* \mathrm{Y} \xrightarrow{\sim} \mathrm{Y}$, counit of the adjunction (j^*, j_*), is an extension of Y. Moreover, the canonical morphism $j_!\mathrm{Y} \to j_*\mathrm{Y}$ is a morphism of extensions.

Example (**4.5.4**). — Let Y be an object of $\boldsymbol{D}_{\mathrm{U}}$, let p be an integer and let $\mathrm{X} = \tau^{\mathrm{F}}_{\geqslant p} j_! \mathrm{Y}$.

Let us apply the functor j^* to the canonical distinguished triangle $i_*\tau_{<p} i^! j_! \mathrm{Y} \to j_!\mathrm{Y} \xrightarrow{v} \tau^{\mathrm{F}}_{\geqslant p} j_! \mathrm{Y} \to \Sigma i_* \tau_{<p} i^! j_! \mathrm{Y}$; since $j^* i_* = 0$, we get a distinguished triangle $0 \to j^* j_! \mathrm{Y} \xrightarrow{j^*(v)} j^* \tau^{\mathrm{F}}_{\geqslant p} j_! \mathrm{Y} \to 0$, so that $j^*(v)$ is an isomorphism. Letting $u = j^*(v) \circ \varepsilon_{\mathrm{Y}}$ be its composition with the unit $\varepsilon_{\mathrm{Y}}\colon \mathrm{Y} \xrightarrow{\sim} j^* j_! \mathrm{Y}$, this furnishes an extension (X, u) of Y.

More precisely, let (X', u') be another extension of Y, where X′ is isomorphic to X as an object of $\boldsymbol{D}$. Let us show that there exists *at most one* morphism $f\colon \mathrm{X} \to \mathrm{X}'$ such that $u = u' \circ j^*(f)$, in other words, at most one an morphism of extensions from (X, u) to (X', u'), and that, in this case, f is an isomorphism. Indeed, let us complete the morphism $u'^\flat\colon j_!\mathrm{Y} \to \mathrm{X}$ to a distinguished triangle $\mathrm{Z} \to j_!\mathrm{Y} \xrightarrow{u'^\flat} \mathrm{X} \to \Sigma \mathrm{Z}$ and consider a partial morphism of distinguished triangles:

$$\begin{array}{ccccccc}
i_*\tau_{<p} i^! j_! \mathrm{Y} & \longrightarrow & j_!\mathrm{Y} & \xrightarrow{u^\flat} & \mathrm{X} & \longrightarrow & \Sigma i_* \tau_{<p} i^! j_! \mathrm{Y} \\
{\scriptstyle g}\downarrow & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Sigma g} \\
\mathrm{Z} & \longrightarrow & j_!\mathrm{Y} & \xrightarrow{u'^\flat} & \mathrm{X}' & \longrightarrow & \Sigma \mathrm{Z}.
\end{array}$$

Relative to the truncation structure on $\boldsymbol{D}$ of example 4.4.13, the object $i_*\tau_{<p} i^! j_! \mathrm{Y}$ is $< p$, and $\mathrm{X}' \simeq \tau^{\mathrm{F}}_{\geqslant p} j_! \mathrm{Y}$ is $\geqslant p$, hence $\Sigma^{-1}\mathrm{X}'$ is $\geqslant p$ as well. Consequently, $\boldsymbol{D}(i_*\tau_{<p} i^! j_! \mathrm{Y}, \Sigma^{-1}\mathrm{X}') = 0$. It then follows from corollary 2.2.6 that there exists at most one isomorphism of extensions.

Consequently, we will allow ourselves to say that an extension of Y “is” isomorphic to $\tau^{\mathrm{F}}_{\geqslant p} j_! \mathrm{Y}$.

Example (**4.5.5**). — Let Y be an object of $\boldsymbol{D}_{\mathrm{U}}$, let p be an integer, and let $\mathrm{X} = \tau^{\mathrm{F}}_{\leqslant p} j_* \mathrm{Y}$. Applying j^* to the canonical distinguished triangle $\tau^{\mathrm{F}}_{\leqslant p} j_! \mathrm{Y} \xrightarrow{v} j_! \mathrm{Y} \to i_* \tau_{>0} i^* \mathrm{Y} \to \Sigma \tau^{\mathrm{F}}_{\leqslant p} j_! \mathrm{Y}$, we obtain that $j^*(v)$ is an isomorphism. Then $u = \eta_{\mathrm{Y}} \circ j^*(v)\colon j^*\mathrm{X} \to \mathrm{Y}$ is an isomorphism, so that (X, u) is an extension of Y.

By duality, one deduces from the preceding example that if (X', u') is an extension of Y such that X′ is isomorphic to X, as an object of $\boldsymbol{D}$, there exists a unique morphism of extensions f from (X, u) to (X', u'), and it is an isomorphism.

Theorem **(4.5.6)**. — *Let* Y *be an object of* $\boldsymbol{D}_{U}$. *Let* p *be an integer. Then* $X = \tau^{F}_{<p} j_* Y = \tau^{F}_{>p} j_! Y$ *is the unique extension of* Y *such that* $i^*X \in \boldsymbol{D}_F^{<p}$ *and* $i^!X \in \boldsymbol{D}_F^{>p}$.

Proof. — Let (X, u) be an extension of Y and let $u^\flat : j_!Y \to X$ and $u^\sharp : X \to j_*Y$. be the morphisms deduced by adjunction. Let us prove that the following properties are equivalent:

(i) $i^*X \in \boldsymbol{D}_F^{<p}$ and $i^!X \in \boldsymbol{D}_F^{>p}$;
(ii) $i^!X \simeq \tau_{>p}(i^* j_* Y)$;
(iii) $i^*X \simeq \tau_{<p}(i^* j_* Y)$;
(iv) $X \simeq \tau^{F}_{<p} j_* Y$;
(v) $X \simeq \tau^{F}_{>p} j_! Y$.

Precisely, in (ii), we mean that the morphism $i^* j_* Y \to i^! \Sigma X$ appearing in the distinguished triangle (4.5.1.2) factors through an isomorphism $\tau_{\geqslant p}(i^* j_* Y) \to i^! \Sigma X$. Similarly, in (iii), we mean that the morphism $i^*(u^\sharp)$ factors through an isomorphism $i^*Y \to \tau_{<p}(i^* j_* Y)$.

If condition (i) holds, then the distinguished triangle (4.5.1.2) writes $i^* j_* Y$ as an extension of the object $i^*X \in \boldsymbol{D}_F^{<p}$ by the object $i^!X \in \boldsymbol{D}_F^{>p}$. By uniqueness of the truncation triangles associated with a truncation structure (proposition 4.1.7, *d*)), one has $i^*X \simeq \tau_{<p} i^* j_* Y$ and $i^!X \simeq \tau_{>p} i^* j_* Y$. This shows the implications (i)⇒(ii) and (i)⇒(iii).

(ii) implies that $i^!X \in \boldsymbol{D}_F^{>p}$ and that $\tau_{\geqslant p} i^* X = 0$, hence (i). Similarly, (iii) implies that $i^*X \in \boldsymbol{D}_F^{<p}$ and that $\tau_{\geqslant p} i^! \Sigma X = 0$, hence (i).

Assume (iv). By assumption, one has $i^*X \in \boldsymbol{D}_F^{<p}$. On the other hand, the uniqueness of a morphism of extensions $X \simeq \tau^{F}_{<p} j_* Y$ implies that $i_* i^! \Sigma X$ belongs to the $\geqslant p$-part of the truncation structure of example 4.4.10, that is, $i^! i_* i^! \Sigma X \in \boldsymbol{D}_F^{\geqslant p}$ and $j^* i_* i^! \Sigma X = 0$. This implies (i).

Conversely, if (i) holds, then $j^* i_* i^! \Sigma X = 0$, so that in the distinguished triangle (4.5.1.1), the objet X belongs to the $< p$-part and the object $i_* i^! \Sigma X$ belongs to the $\geqslant p$-part of the truncation structure of example 4.4.10. This implies that $X = \tau^{F}_{<p} j_* Y$, hence (iv).

The proof of the equivalence (i)⇔(v) is analogous.

This concludes the proof of the theorem. □

Proposition **(4.5.7).** — a) *The functor* $\tilde{j}^* : \boldsymbol{C} \to \boldsymbol{C}_U$ *identifies the abelian category* $\boldsymbol{C}_U$ *with the quotient of the abelian category* $\boldsymbol{C}$ *by the essential image* $\overline{\boldsymbol{C}}_F$ *of the functor* $\tilde{\imath}_*$.

b) *For every object* X *of* $\boldsymbol{C}$, $\tilde{\imath}_*\tilde{\imath}^*X$ *is the largest quotient of* X *that belongs to* $\overline{\boldsymbol{C}}_F$, *and* $\tilde{\imath}_*\tilde{\imath}^!X$ *is the largest subobject of* X *that belongs to* $\overline{\boldsymbol{C}}_F$.

Proof. — a) Let us first show that the category $\overline{\boldsymbol{C}}_F$ is the kernel of the functor $\tilde{j}^* : \boldsymbol{C} \to \boldsymbol{C}_U$. The relation $\tilde{j}^*\tilde{\imath}_* = 0$ implies that $\overline{\boldsymbol{C}}_F \subset \operatorname{Ker}(\tilde{j}^*)$. Conversely, let $X \in \boldsymbol{C}$ be such that $\tilde{j}^*X = 0$. The exact sequences of corollary 4.4.8, *c*), show that $\tilde{\imath}_*\tilde{\imath}^*X \simeq X \simeq \tilde{\imath}_*\tilde{\imath}^!X$; in particular X belongs to $\overline{\boldsymbol{C}}_F$. Consequently, the functor $\tilde{j}^*$ factors uniquely through a functor $T : \boldsymbol{C}/\overline{\boldsymbol{C}}_F \to \boldsymbol{C}_U$. Let $S : \boldsymbol{C}_U \to \boldsymbol{C}/\overline{\boldsymbol{C}}_F$ be the composition of the functor $\tilde{j}_!$ with the canonical functor $\boldsymbol{C} \to \boldsymbol{C}/\overline{\boldsymbol{C}}_F$. The isomorphism $\tilde{j}^* \circ \tilde{j}_! \simeq \mathrm{id}$ implies that $T \circ S \simeq \mathrm{id}$. On the other hand, the the first exact sequence of corollary 4.4.8,*c*), implies that $S \circ T \simeq \mathrm{id}$. Consequently, T is an equivalence of categories, as claimed.

b) Let X be an object of $\boldsymbol{C}$. By corollary 4.4.8, *c*), the canonical morphism $X \to \tilde{\imath}_*\tilde{\imath}^*X$ is an epimorphism. Conversely, let $v : X \to \tilde{\imath}_*Y$ be an epimorphism from X to an object of $\overline{\boldsymbol{C}}_F$. Since the pair $(\tilde{\imath}^*, \tilde{\imath}_*)$ is adjoint, the morphism v corresponds to a morphism $v^\flat : \tilde{\imath}^*X \to Y$, and $v^\flat = \eta_Y \circ \imath^*(v)$, where $\eta : \tilde{\imath}^* \circ \tilde{\imath}_* \xrightarrow{\sim} \mathrm{id}$ is the counit (it is an isomorphism because $\tilde{\imath}_*$ is fully faithful). Then $v = \tilde{\imath}_*(v^\flat) \circ \varepsilon_X$ factorizes uniquely through $\tilde{\imath}_*\tilde{\imath}^*X$.

Similarly, the canonical morphism $\tilde{\imath}_*\tilde{\imath}^!X \to X$ is a monomorphism. Let then $w : \tilde{\imath}_*Z \to X$ be a monomorphism from an object of $\overline{\boldsymbol{C}}_F$ to X. Let $w^\sharp : Z \to \tilde{\imath}^!X$ be the morphism associated with w by the adjunction $(\tilde{\imath}_*, \tilde{\imath}^!)$. If $\eta : \tilde{\imath}_* \circ \tilde{\imath}^! \xrightarrow{\sim} \mathrm{id}$ is its counit, then $w = \eta_X \circ \tilde{\imath}_*(w^\sharp)$ is the unique factorization of w through $\tilde{\imath}_*\tilde{\imath}^!X$, as claimed. ☐

4.5.8. — Let $Y \in \boldsymbol{C}_U$. Since the functor $j_!$ is right t-exact, one has $j_!Y \in \boldsymbol{D}^{\leqslant 0}$, and $\tilde{j}_!Y = \tau_{\geqslant 0} j_!Y$. Since the functor j_* is left t-exact, one has $\tilde{j}_*Y = \tau_{\leqslant 0} j_*Y$. Moreover, there exists a unique morphism $\tilde{u} : \tilde{j}_!Y \to \tilde{j}_*Y$ such that $\tilde{j}^*(\tilde{u})$ is the composition of the counit $\tilde{j}^*\tilde{j}_* \to \mathrm{id}$ and the unit $\mathrm{id} \to \tilde{j}^*\tilde{j}_!$ associated with the adjoint pairs $(\tilde{j}^*, \tilde{j}_*)$ and $(\tilde{j}_!, \tilde{j}^*)$. This leads to a canonical diagram

$$\begin{tikzcd} j_!Y \arrow[r] \arrow[rrr, bend right, "u"'] & \tilde{j}_!Y \arrow[r, "\tilde{u}"] & \tilde{j}_*Y \arrow[r] & j_*Y, \end{tikzcd}$$

where $u : j_!Y \to j_*Y$ is the canonical morphism.

Definition **(4.5.9)**. — *Let* $Y \in C_U$. *One defines* $\tilde{j}_{!*} = \mathrm{Im}(\tilde{j}_! Y \to \tilde{j}_* Y)$. *It is called the* middle extension *of* Y.

The middle extension fits naturally in a diagram

$$j_! Y \longrightarrow \tilde{j}_! Y \longrightarrow \tilde{j}_{!*} Y \longrightarrow \tilde{j}_* Y \longrightarrow j_* Y, \qquad u \colon j_! Y \to j_* Y.$$

When one applies the functor j^* to this diagram, the first term $j^* j_! Y$ and the last term $j^* j_* Y$ are isomorphic to Y via the unit of the adjunction $(j_!, j^*)$ and the counit of the adjunction (j^*, j_*) respectively, and all morphisms are isomorphisms. In this way, $\tilde{j}_{!*}$ is naturally an extension of Y.

Proposition **(4.5.10)**. — *Let* $Y \in C_U$. *One has the following relations:*

a) $X = \tilde{j}_! Y = \tau^F_{>-1} j_! Y = \tau^F_{<-1} j_* Y$ *is the only extension of* Y *in* $\boldsymbol{D}$ *such that* $i^* X \in \boldsymbol{D}_F^{<-1}$ *and* $i^! X \in \boldsymbol{D}_F^{>-1}$;

b) $X = \tilde{j}_{!*} Y = \tau^F_{>0} j_! Y = \tau^F_{<0} j_* Y$ *is the only extension of* Y *in* $\boldsymbol{D}$ *such that* $i^* X \in \boldsymbol{D}_F^{<0}$ *and* $i^! X \in \boldsymbol{D}_F^{>0}$;

c) $X = \tilde{j}_* Y = \tau^F_{>1} j_! Y = \tau^F_{<1} j_* Y$ *is the only extension of* Y *in* $\boldsymbol{D}$ *such that* $i^* X \in \boldsymbol{D}_F^{<1}$ *and* $i^! X \in \boldsymbol{D}_F^{>1}$.

Proof. — All these assertions follow from theorem 4.5.6, except for the identification of $\tilde{j}_! Y$, $\tilde{j}_{!*} Y$ and $\tilde{j}_* Y$ with the indicated extensions.

a) One has $j^* j_! Y \simeq Y \in C_U$; consequently, $\tilde{j}_! Y = \tau_{\geqslant 0} j_! Y = \tau^F_{\geqslant 0} \tau^U_{\geqslant 0} j_! Y = \tau^F_{\geqslant 0} j_! Y$.

c) One has $j^* j_* Y \simeq Y \in C_U$; consequently, $\tilde{j}_* Y = \tau_{\leqslant 0} j_* Y = \tau^F_{\leqslant 0} \tau^U_{\leqslant 0} j_* Y = \tau^F_{\leqslant 0} j_* Y$.

b) Let $X = \tau^F_{>0} j_! Y = \tau^F_{<0} j_* Y$; let us first show that $X \in \boldsymbol{C}$. Since $i^* X \in \boldsymbol{D}_F^{<0}$ and $j^* X \simeq Y \in C_U$, one has $X \in \boldsymbol{D}^{\leqslant 0}$; since $i^! X \in \boldsymbol{D}_F^{>0}$ and $j^* X \simeq Y \in C_U$, one has $X \in \boldsymbol{D}^{\geqslant 0}$; consequently, $X \in \boldsymbol{C}$, as claimed.

The cohomology functor associated with the truncation structure of example 4.4.10 is $i_* H^0 i^*$. Consequently, the canonical morphism $\tau^F_{<-1} j_* Y \to \tau^F_{<0} j_* Y$ can be completed to a distinguished triangle

$$\tau^F_{<-1} j_* Y \to \tau^F_{<0} j_* Y \to \Sigma i_* H^{-1} i^* j_* Y \to \Sigma \tau^F_{<-1} j_* Y.$$

By *a*), we have $\tau^F_{<-1} j_* Y = \tilde{j}_! Y$. By rotation, the preceding triangle gives the following distinguished triangle:

$$i_* H^{-1} i^* j_* Y \to \tilde{j}_! Y \to X \to \Sigma i_* H^{-1} i^* j_* Y.$$

Since the three vertices of this triangle belong to $\boldsymbol{C}$, the diagram

$$0 \to i_* \mathrm{H}^{-1} i^* j_* \mathrm{Y} \to \tilde{j}_! \mathrm{Y} \to \mathrm{X} \to 0$$

is an exact sequence, as claimed.

By duality, the canonical morphism $\tau^{\mathrm{F}}_{\geqslant -1} j_! \mathrm{Y} \to \tau^{\mathrm{F}}_{\geqslant 1} j_! \mathrm{Y}$ furnishes an exact sequence

$$0 \to \mathrm{X} \to \tilde{j}_* \mathrm{Y} \to i_* \mathrm{H}^1 i^! j_! \mathrm{Y} \to 0.$$

This proves that X is the image of the canonical morphism $\tilde{j}_! \mathrm{Y} \to \tilde{j}_* \mathrm{Y}$, hence $\mathrm{X} = \tilde{j}_{!*} \mathrm{Y}$, as was to be shown. □

Corollary (**4.5.11**). — *Let* $\mathrm{Y} \in \boldsymbol{C}_{\mathrm{U}}$. *Then* $\mathrm{X} = \tilde{j}_{!*} \mathrm{Y}$ *is the unique extension of* Y *in* $\boldsymbol{C}$ *which has no non-trivial subobject and no non-trivial quotient in* $\overline{\boldsymbol{C}}_{\mathrm{F}}$.

Proof. — Let $\mathrm{X} \in \boldsymbol{C}$ be an extension of Y. By proposition 4.5.7, *b*), the largest quotient of X that belongs to $\overline{\mathcal{C}}_{\mathrm{F}}$ is $\tilde{\imath}_* \tilde{\imath}^* \mathrm{X}$. One has $i^* \mathrm{X} \in \boldsymbol{D}^{\leqslant 0}_{\mathrm{F}}$, because i^* is right t-exact, hence $\tilde{\imath}^* \mathrm{X} = \mathrm{H}^0 i^* \mathrm{X}$. Since $\tilde{\imath}_*$ is exact and fully faithful, this quotient vanishes if and only if $i^* \mathrm{X} \in \boldsymbol{D}^{< 0}_{\mathrm{F}}$.

Similarly, the largest subobject of X that belongs to $\overline{\mathcal{C}}_{\mathrm{F}}$ is $\tilde{\imath}_* \tilde{\imath}^! \mathrm{X}$. Since $i^!$ is left t-exact, one has $i^! \mathrm{X} \in \boldsymbol{D}^{\geqslant 0}_{\mathrm{F}}$, hence $\tilde{\imath}^! \mathrm{X} = \mathrm{H}^0 i^! \mathrm{X}$. It vanishes if and only if $i^! \mathrm{X} \in \boldsymbol{D}^{> 0}_{\mathrm{F}}$.

The corollary thus follows from proposition 4.5.10. □

Corollary (**4.5.12**). — *The functor* $\tilde{j}_{!*} \colon \boldsymbol{C}_{\mathrm{U}} \to \boldsymbol{C}$ *if fully faithful and respects epimorphisms and monomorphisms. It induces an equivalence of categories from* $\boldsymbol{C}_{\mathrm{U}}$ *to the full subcategory of* $\boldsymbol{C}$ *consisting of objects* X *such that* $\tilde{\imath}^* \mathrm{X} = \tilde{\imath}^! \mathrm{X} = 0$.

However, the functor $\tilde{j}_{!*}$ is not exact in the middle in general, see (de Cataldo & Migliorini, 2009, p. 562).

Proof. — Let $f \colon \mathrm{Y} \to \mathrm{Z}$ be a morphism in $\boldsymbol{C}_{\mathrm{U}}$. The morphisms $\tilde{j}_!(f)$ and $\tilde{j}_*(f)$ fit in a diagram

$$\begin{array}{ccccc}
\tilde{j}_! \mathrm{Y} & \twoheadrightarrow & \tilde{j}_{!*} \mathrm{Y} & \hookrightarrow & \tilde{j}_* \mathrm{Y} \\
\downarrow {\scriptstyle \tilde{j}_!(f)} & & \downarrow {\scriptstyle \tilde{j}_{!*}(f)} & & \downarrow {\scriptstyle \tilde{j}_*(f)} \\
\tilde{j}_! \mathrm{Z} & \twoheadrightarrow & \tilde{j}_{!*} \mathrm{Z} & \hookrightarrow & \tilde{j}_* \mathrm{Z}.
\end{array}$$

Assume that f is a monomorphism. Since $\tilde{j}_*$ is left exact, the morphism $\tilde{j}_*(f)$ is a monomorphism, and one reads on the preceding diagram that $\tilde{j}_{!*}(f)$ is a monomorphism as well.

Similarly, assume that f is an epimorphism. Since $\tilde{j}_!$ is right exact, the morphism $\tilde{j}_!(f)$ is then an epimorphism. This implies that $\tilde{j}_{!*}(f)$ is an epimorphism as well.

Since $\tilde{j}^* \circ \tilde{j}_{!*} = \mathrm{id}$, the functor $\tilde{j}_{!*}$ is faithful. On the other hand, let $f\colon \tilde{j}_{!*}Y \to \tilde{j}_{!*}Z$ be a morphism in $\boldsymbol{C}$, and let

$$g = f - \tilde{j}_{!*} \circ \tilde{j}^*(f) : \tilde{j}_{!*}Y \to \tilde{j}_{!*}Z.$$

One has $\tilde{j}^*(g) = 0$; consequently, $\mathrm{Ker}(g)$ is a subobject of $\tilde{j}_{!*}Y$ such that $\tilde{j}^*(\mathrm{Ker}(g)) = Y$, because $\tilde{j}^*$ is exact. This implies that $\tilde{j}_{!*}Y/\mathrm{Ker}(g)$ belongs to the subcategory $\overline{\boldsymbol{C}}_F$, hence is zero, by corollary 4.5.11. Consequently, $\mathrm{Ker}(g) = \tilde{j}_{!*}Y$, hence $g = 0$ and $f = \tilde{j}_{!*} \circ \tilde{j}^*(f)$. Consequently, the functor $\tilde{j}_{!*}$ is fully faithful.

Let Y be an object of $\boldsymbol{C}_U$ and let $X = \tilde{j}_{!*}Y$. By proposition 4.5.10, one has $i^*X \in \boldsymbol{D}_F^{<0}$, hence $\tilde{\imath}^*X = H^0(i^*X) = 0$. Similarly, one has $\tilde{\imath}^!X = 0$. Conversely, let X be an object of $\boldsymbol{C}$ such that $\tilde{\imath}^*X = \tilde{\imath}^!X = 0$. Let $Y = \tilde{j}^*X$. By construction, X is an extension of Y in $\boldsymbol{C}$. By proposition 4.5.7, *b*), $\tilde{\imath}_*\tilde{\imath}^*X = 0$ is the largest quotient of X that belongs to $\overline{\boldsymbol{C}}_F$, and $\tilde{\imath}_*\tilde{\imath}^!X = 0$ is the largest subobject of X that belongs to $\overline{\boldsymbol{C}}_F$. By corollary 4.5.11, X is isomorphic to $\tilde{j}_{!*}Y$. This concludes the proof. □

Corollary **(4.5.13)**. — *The simple objects of the category* $\boldsymbol{C}$ *are the objects* $\tilde{j}_{!*}S$, *for* $S \in \boldsymbol{C}_U$ *simple, and the objects* $\tilde{\imath}_*T$, *for* $T \in \boldsymbol{C}_F$ *simple.*

Proof. — a) Let us first prove that for every object $S \in \boldsymbol{C}_U$, $\tilde{j}_{!*}S$ is simple if and only if S is simple.

Assume that $\tilde{j}_{!*}S$ is simple. Necessarily, S is nonzero; let $S' \to S$ be a nonzero subobject. Then $\tilde{j}_{!*}S' \to \tilde{j}_{!*}S$ is a subobject as well, because $\tilde{j}_{!*}$ preserves monomorphisms, and $\tilde{j}_{!*}S'$ is nonzero, since its image under $\tilde{j}^*$ is S'. Consequently, $\tilde{j}_{!*}S' \to \tilde{j}_{!*}S$ is an isomorphism, and applying $\tilde{j}^*$, we conclude that $S' \to S$ is an isomorphism. This shows that S is simple.

Conversely, let us assume that $\tilde{j}_{!*}S$ is not simple and let us prove that S is not simple. If $\tilde{j}_{!*}S \simeq 0$, then $S \simeq \tilde{j}^*\tilde{j}_{!*}S \simeq 0$. Let us thus assume that $\tilde{j}_{!*}S \not\simeq 0$ and let $T \to \tilde{j}_{!*}S$ be a subobject which is neither 0, nor an isomorphism; let $\tilde{j}_{!*}S \to T'$ be its cokernel, so that we have an exact sequence $0 \to T \to \tilde{j}_{!*}S \to T' \to 0$. Since $\tilde{j}^*$ is exact and $\tilde{j}^*\tilde{j}_{!*} \simeq \mathrm{id}$, we have an exact sequence $0 \to \tilde{j}^*T \to S \to \tilde{j}^*T' \to 0$. Since $\tilde{j}_{!*}S$ has no nonzero subboject in $\overline{\boldsymbol{C}}_F$, one has $\tilde{j}^*T \neq 0$; similarly, one has $\tilde{j}^*T' \neq 0$. This proves that S is not simple.

b) Since $\tilde{j}^*\tilde{j}_{!*} \simeq \mathrm{id}$, if, for two objects S and S′ of $\boldsymbol{C}_U$, the objects $\tilde{j}_{!*}S$ and $\tilde{j}_{!*}S'$ of $\boldsymbol{C}$ are isomorphic, then S and S′ are isomorphic.

c) Since the functor $\tilde{\imath}_*$ is exact and fully faithful, for every object T in $\mathcal{C}_F$, the object $\tilde{\imath}_*T$ of $\mathcal{C}$ is simple if and only if T is simple. Moreover, if T and T′ are two objects of $\mathcal{C}_F$ such that $\tilde{\jmath}_{!*}T$ and $\tilde{\jmath}_{!*}T'$ are isomorphic in $\mathcal{C}$, then T and T′ are isomorphic.

d) To conclude the proof, it suffices to prove that a simple object X of $\mathcal{C}$ is either of the form $\tilde{\jmath}_{!*}S$ for some object $S \in \mathcal{C}_U$, or of the form $\tilde{\imath}_*T$ for some object $T \in \mathcal{C}_F$. There are two cases. If X has a nonzero subobject, or a nonzero quotient, in $\overline{\mathcal{C}}_F$, then X is isomorphic to that object since it is simple. Otherwise, the relation $\tilde{\jmath}^* \circ \tilde{\jmath}_{!*} \simeq \mathrm{id}$ in $\mathcal{C}_U$ shows that X is an extension of $\tilde{\jmath}^*X$; since this extension has neither a nonzero subobject, nor a nonzero quotient in $\overline{\mathcal{C}}_F$, corollary 4.5.11 implies that $X \simeq \tilde{\jmath}_{!*}\tilde{\jmath}^*X$. □

CHAPTER 5

PERVERSE SHEAVES

5.0.1. — In this chapter, we only consider topological spaces which are locally compact and finite dimensional. If X is such a space, we write $\boldsymbol{D}(\mathrm{X})$ for its derived category of sheaves of abelian groups.

We recall that every continuous map $f : \mathrm{Y} \to \mathrm{X}$ of such topological spaces induces functors $f_!, f_* : \boldsymbol{D}(\mathrm{X}) \to \boldsymbol{D}(\mathrm{Y})$ and $f^*, f^! : \boldsymbol{D}(\mathrm{Y}) \to \boldsymbol{D}(\mathrm{X})$, related by adjunctions (f^*, f_*) and $(f_!, f^!)$.

5.1. Stratified spaces

Definition **(5.1.1)**. — *Let* X *be a topological space. A* stratification $\mathscr{S}$ *of* X *is a finite partition of* X *into nonempty locally closed subsets, called* strata*, such that the closure of a stratum is a union of strata.*

Example **(5.1.2)**. — Let n be an integer. The projective space $\mathbf{P}^n$ (considered as a complex manifold) admits a standard stratification $(\mathrm{S}_0, \dots, \mathrm{S}_n)$ such that for every i, the stratum S_i is an affine space $\mathbf{C}^i$ of (complex) dimension i, and its closure $\overline{\mathrm{S}_i} = \mathrm{S}_0 \cup \dots \cup \mathrm{S}_i$ is a projective subspace $\mathbf{P}^i$.

Example **(5.1.3)**. — Let G be a complex reductive algebraic group, let B be a Borel subgroup of G and let W be a Weyl group associated to the maximal torus of B. For example, one may take for G the linear group $\mathrm{GL}(n, \mathbf{C})$, for B be the subgroup of upper triangular matrices and for W be the subgroup of permutation matrices. The Bruhat decomposition $\mathrm{G} = \mathrm{BWB}$ induces a stratification $\mathscr{B} = (\mathrm{B}w\mathrm{B})_{w \in \mathrm{W}}$ of G.

Example **(5.1.4)**. — Let p and n be integers such that $1 \leqslant p \leqslant n$ and let $\mathrm{X} = \mathrm{Gr}(p, n)$ be the Grassmann varieties of p-dimensional subspaces of $\mathbf{C}^n$. By linear algebra, every such subspace V can be represented by a unique $p \times n$ matrix A_V in row reduced echelon form, the p row vectors of which form a basis of V. The pivot indices of this matrix A_V form a strictly increasing sequence $\mathbf{i} = (i_1, \dots, i_p)$ of integers, characterized by the relations

$$\dim(V \cap \mathbf{C}^m \times \{(0, \dots, 0)\}) \geqslant d \quad \Leftrightarrow \quad i_d \geqslant m.$$

The row reduced matrices associated with such a sequence $\mathbf{i}$ form an affine space $S_\mathbf{i}$ of dimension

$$(i_2 - i_1 - 1) + 2(i_3 - i_2 - 1) + \cdots + (p-1)(i_p - i_{p-1} - 1) + p(n - i_p)) \\ = \frac{1}{2}(2n - p + 1)p - (i_1 + \cdots + i_p).$$

This furnishes a stratification of the Grassmann variety in (open) "Schubert cells" which are complex affine spaces.

When $p = 1$, the Grassmann manifold $\mathrm{Gr}(1, n)$ is the projective space of dimension $n - 1$, and for $i \in \{1, \dots, n\}$, the affine space S_i has dimension $n - i$. One recovers (up to the indexing) the stratification of $\mathbf{P}^{n-1}$.

5.1.5. — Let X be a topological space and let $\mathscr{S}$ be a stratification of X.

Let $S \in \mathscr{S}$. By definition of a locally closed subset, S is open and dense in $\overline{S}$. By definition of a stratification, $\overline{S} - S$ is a union of strata, each of them has empty interior in $\overline{S}$.

The relation "$S \subset \overline{T}$" is an ordering on $\mathscr{S}$. Since it is equivalent to "$\overline{S} \subset \overline{T}$", it is reflexive and transitive. By the remark above, if $\overline{S} = \overline{T}$, then S and T are both dense in $\overline{S}$, hence $S = T$.

Lemma **(5.1.6)**. — *Let* X *be a topological space and let* $\mathscr{S}$ *be a stratification of* X. *Let* $S \in \mathscr{S}$ *and let* U *be the union of all strata* T *such that* $S \subset \overline{T}$. *Then* U *is a neighborhood of* S *in which* S *is closed.*

Proof. — Let us first show that U is a neighborhood of S. Assume otherwise and let $\mathscr{F}$ be an ultrafilter containing $X - U$ that converges to a point $s \in S$. Since $\mathscr{S}$ is finite, there exists a stratum $T \in \mathscr{S}$ such that $T \in \mathscr{F}$. By the definition of U, S is not contained in $\overline{T}$, a contradiction.

Let us then show that S is closed in U. Let $\mathscr{F}$ be an ultrafilter containing S that converges to a point s of U, and let us show that $s \in S$. Assume otherwise and let $T \neq S$ be the stratum of $\mathscr{S}$ that contains s; by the definition of U, we have $S \subset \overline{T}$. Then $S \subset \overline{T} - T$, which is closed in T. Consequently, $s \in \overline{T} - T$, a contradiction. □

Lemma **(5.1.7)**. — *Let* X *be a topological space and let* $\mathscr{S}$ *be a stratification of* X. *Assume that* $\mathrm{Card}(\mathscr{S}) \geqslant 2$. *Then there exists a closed subset* F *of* X *such that* $F \neq \varnothing$ *and* $F \neq X$, *which is a union of strata.*

Proof. — Let $S \in \mathscr{S}$ be a stratum such that $S \neq X$. By definition of a stratificatino, $\overline{S}$ is a union of strata, as well as $\overline{S} - S$; moreover, $\overline{S} - S$ is closed in $\overline{S}$. If $\overline{S} = X$, then we take $F = \overline{S} - S$. Otherwise, we take $F = \overline{S}$. □

5.2. Perverse sheaves

Definition **(5.2.1)**. — *Let* X *be a topological space and let* $\mathscr{S}$ *be a stratification of* X. *A function* $p \colon \mathscr{S} \to \mathbf{Z}$ *is called a* perversity *on* X *relative to the stratification* $\mathscr{S}$.

Definition **(5.2.2)**. — *Let* X *be a topological space and* p *be a perversity on* X *relative to a stratification* $\mathscr{S}$.

Let ${}^p\boldsymbol{D}(X)^{\leqslant 0}$ *be the full subcategory of* $\boldsymbol{D}(X)$ *whose objects* A *are characterized by the property*

(5.2.2.1) $$H^n(i_S^*A) = 0 \quad \text{for all } S \in \mathscr{S} \text{ and all } n > p(S).$$

Similarly, let ${}^p\boldsymbol{D}(X)^{\geqslant 0}$ *be the full subcategory of* $\boldsymbol{D}(X)$ *whose objects* A *are characterized by the property*

(5.2.2.2) $$H^n(i_S^!A) = 0 \quad \text{for all } S \in \mathscr{S} \text{ and all } n < p(S).$$

For every integer n, we also set ${}^p\boldsymbol{D}(X)^{\leqslant n} = \Sigma^{-n}\,{}^p\boldsymbol{D}(X)^{\leqslant n}$ and ${}^p\boldsymbol{D}(X)^{\geqslant n} = \Sigma^{-n}\,{}^p\boldsymbol{D}(X)^{\geqslant n}$.

Example **(5.2.3)** (Constant perversity). — Assume that p is constant with value $a \in \mathbf{Z}$; let us prove that ${}^p\boldsymbol{D}(X)^{\leqslant 0} = \boldsymbol{D}(X)^{\leqslant a}$ and ${}^p\boldsymbol{D}(X)^{\geqslant 0} = \boldsymbol{D}(X)^{\geqslant a}$.

Since the functor i_S^* on sheaves is exact, for every $S \in \mathscr{S}$, one has $\boldsymbol{D}(X)^{\leqslant a} \subset {}^p\boldsymbol{D}(X)^{\leqslant 0}$. Conversely, let $A \in {}^p\boldsymbol{D}(X)^{\leqslant 0}$ and let us prove that $A \in \boldsymbol{D}(X)^{\leqslant a}$. Since the standard truncation structure on $\boldsymbol{D}(X)$ is nondegenerate, it suffices to prove that $H^n(A) = 0$ for every integer n such that $n > a$. Let n be such an integer. By exactness of i_S^*, one has $i_S^*H^n(A) = H^n(i_S^*A) = 0$ for every $S \in \mathscr{S}$. Since the subspaces S, for $S \in \mathscr{S}$, cover X, this implies that all stalks of $H^n(A)$ are zero, hence $H^n(A) = 0$.

Let $A \in D(X)^{\geqslant a}$. Let $S \in \mathscr{S}$ be a stratum. Since i_S is an immersion, the functor $(i_S)_!$ on sheaves admits a right adjoint $(i_S)^!$, which is thus left exact, and of which $i_S^!$ is the right derived functor. Consequently, $i_S^! A \in D(S)^{\geqslant a}$ and $H^n i_S^! A = 0$ for every integer n such that $n < a$. This proves that A is an object of ${}^pD(X)^{\geqslant 0}$.

Conversely, let $A \in {}^pD(X)^{\geqslant 0}$ and let us prove that $A \in D(X)^{\geqslant a}$. Let us apply the triangulated functor $i_S^!$, composed with the cohomological functor H^0, to the (distinguished) truncation triangle $\tau_{<a}A \to A \to \tau_{\geqslant a}A \to \Sigma\tau_{<a}A$. We obtain an exact sequence

$$H^{n-1} i_S^! \tau_{\geqslant a} A \to H^n i_S^! \tau_{<a} A \to H^n i_S^! A \to H^n i_S^! \tau_{\geqslant a} A.$$

Since $\tau_{\geqslant a} A \in D(X)^{\geqslant a}$, we have $H^n i_S^! \tau_{\geqslant A} = 0$ for $n < a$, hence the previous exact sequence gives an isomorphism $H^n i_S^! \tau_{<a} A \xrightarrow{\sim} H^n i_S^! A$. Consequently, $\tau_{<a} A \in {}^pD(X)^{\geqslant 0}$. Replacing A by $\tau_{<a}A$, we may moreover assume that $A \in D(X)^{<a}$; let us then prove that $A = 0$. For every stratum $S \in \mathscr{S}$, we have $H^n(i_S^! A) = 0$ if $n < a$, because $A \in {}^pD(X)^{\geqslant 0}$, and $H^n(i_S^! A) = 0$ if $n \geqslant a$, because $A \in D(X)^{<a}$; consequently, $H^n(i_S^! A) = 0$ for every integer n, hence, $i_S^! A = 0$.

Let us prove that $i_S^* A = 0$ for every stratum $S \in \mathscr{S}$. We argue by induction, assuming the result true for every stratum T such that $S \subset \overline{T}$ and $S \neq T$. Let U be the union of all strata T such that $S \subset \overline{T}$; by lemma 5.1.6, it is a neighborhood of S in which S is closed. By the induction hypothesis, the support of $A|_U$ is contained in S; consequently $i_S^* A = i_S^! A = 0$, as claimed.

Since $\mathscr{S}$ covers X, we have $A = 0$, as was to be shown.

Example (**5.2.4**). — Let p and q be two perversities relative to the stratification $\mathscr{S}$ such that $p \leqslant q$. It follows from the definitions that ${}^pD(X)^{\leqslant 0} \subset {}^qD(X)^{\leqslant 0}$ and ${}^pD(X)^{\geqslant 0} \supset {}^qD(X)^{\geqslant 0}$.

In particular, if a is an integer such that $p \geqslant a$, then $D(X)^{\leqslant a} \subset {}^pD(X)^{\leqslant 0}$ and ${}^pD(X)^{\geqslant 0} \subset D(X)^{\geqslant a}$. Similarly, if b is an integer such that $p \leqslant b$, then ${}^pD(X)^{\leqslant 0} \subset D(X)^{\leqslant b}$ and $D(X)^{\geqslant b} \subset {}^pD(X)^{\geqslant 0}$.

Theorem (**5.2.5**). — *Let* X *be a topological space and* p *be a perversity on* X *relative to a stratification* $\mathscr{S}$. *The pair* $(D(X)^{\leqslant p}, D(X)^{\geqslant p+1})$ *is a truncation structure on* $D(X)$.

Proof. — We prove the result by induction on the cardinality of $\mathscr{S}$. If $\mathrm{Card}(\mathscr{S}) = 0$, then $X = \varnothing$, $D(X) = 0$, and the result is obvious. Assume that

$\mathrm{Card}(\mathscr{S}) = 1$, so that $\mathscr{S} = \{X\}$ and p is constant. It then follows from example 5.2.3 that the pair $(\boldsymbol{D}(X)^{\leqslant p}, \boldsymbol{D}(X)^{\geqslant p+1})$ is the standard truncation structure, shifted by $-p(X)$. Let us assume that $\mathrm{Card}(\mathscr{S}) \geqslant 2$; let F be a closed subset of X which is a union of strata, and such that $F \neq \varnothing$ and $F \neq X$ (lemma 5.1.7); let $U = X - F$. By induction, $(\boldsymbol{D}(U)^{\leqslant p}, \boldsymbol{D}(U)^{\geqslant p+1})$ and $(\boldsymbol{D}(F)^{\leqslant p}, \boldsymbol{D}(F)^{\geqslant p+1})$ are truncation structures on $\boldsymbol{D}(U)$ and $\boldsymbol{D}(F)$ respectively. It remains to observe that $(\boldsymbol{D}(X)^{\leqslant p}, \boldsymbol{D}(X)^{\geqslant p+1})$ is the truncation structure on $\boldsymbol{D}(X)$ which is deduced by glueing from these two truncation structures. □

Definition **(5.2.6)**. — *Let X be a topological space, let $\mathscr{S}$ be a stratification of X and let p be a perversity relative to $\mathscr{S}$. The truncation structure $({}^p\boldsymbol{D}(X)^{\leqslant 0}, {}^p\boldsymbol{D}(X)^{\geqslant 1})$ on $\boldsymbol{D}(X)$ is called the* p-perverse truncation structure. *Its heart is denoted by $\boldsymbol{M}(X)^p$; objects of $\boldsymbol{M}(X)^p$ are called* p-perverse sheaves.

Let us introduce the following notation: $\boldsymbol{D}(X)^{\leqslant p} = {}^p\boldsymbol{D}(X)^{\leqslant 0}$ and $\boldsymbol{D}(X)^{\geqslant p} = {}^p\boldsymbol{D}(X)^{\geqslant p}$. By example 5.2.3, it is consistent with the case of a constant perversity. It is also consistent with the intuitive understanding When p and q are two perversities such that $p \leqslant q$, it gives an intuitive explanation to the inclusions of example 5.2.4.

Similarly, the truncation functors associated with the p-perverse truncation structures will be denoted by $\tau_{\leqslant p}$ and $\tau_{\geqslant p}$, and the p-perverse cohomology functor will be denoted by H^p.

5.2.7. — Let X be a topological space, let $\mathscr{S}$ be a stratification of X and let p be a perversity relative to $\mathscr{S}$. Let $U \subset X$ be an open subset which is a union of strata, let $j\colon U \to X$ be the inclusion; let $F = X - U$ be the complementary subset and let $i\colon F \to X$ be the inclusion. We have adjoint triples of functors $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ relating the triangulated categories $\boldsymbol{D}(U)$, $\boldsymbol{D}(X)$ and $\boldsymbol{D}(F)$, giving rise to a glueing context: the p-perverse truncation structure on $\boldsymbol{D}(X)$ is obtained by glueing the p-perverse truncation structures on $\boldsymbol{D}(U)$ and $\boldsymbol{D}(F)$. We also have their variants on hearts $(\tilde{\imath}^*, \tilde{\imath}_*, \tilde{\imath}^!)$ and $(\tilde{\jmath}_!, \tilde{\jmath}^*, \tilde{\jmath}_*)$. We also have the intermediate extension functor $\tilde{\jmath}_{!*}$.

The functors $j_!$, i^* are right t-exact; the functors j^* and i_* are t-exact; the functors j_* and $i^!$ are left t-exact. The functors $\tilde{\jmath}_!$, $\tilde{\imath}^*$ are right exact; the functors $\tilde{\jmath}^*$ and $\tilde{\imath}_*$ are exact; the functors $\tilde{\jmath}_*$ and $\tilde{\imath}^!$ are left exact.

More generally, there are such functors associated with the immersion $j\colon \mathrm{U} \to \mathrm{U}'$, where U and U′ are two open subsets of X which are union of strata, etc. The various functors satisfy the expected transitivity properties.

Proposition **(5.2.8)**. — *Let* $\mathrm{A} \in \boldsymbol{M}(\mathrm{U})^p$.

a) *The object* $\tilde{j}_!\mathrm{A}$ *of* $\boldsymbol{M}(\mathrm{X})^p$ *is the unique extension* B *of* A *in* $\boldsymbol{D}(\mathrm{X})$ *such that for every stratum* $\mathrm{S} \subset \mathrm{F}$, *we have* $\mathrm{H}^n i_{\mathrm{S}}^*\mathrm{B} = 0$ *for* $n \geqslant p(\mathrm{S}) - 1$ *and* $\mathrm{H}^n i_{\mathrm{S}}^!\mathrm{B} = 0$ *for* $n \leqslant p(\mathrm{S}) - 1$.

b) *The object* $\tilde{j}_{!*}\mathrm{A}$ *of* $\boldsymbol{M}(\mathrm{X})^p$ *is the unique extension* B *of* A *in* $\boldsymbol{D}(\mathrm{X})$ *such that for every stratum* $\mathrm{S} \subset \mathrm{F}$, *we have* $\mathrm{H}^n i_{\mathrm{S}}^*\mathrm{A} = 0$ *for* $n \geqslant p(\mathrm{S})$ *and* $\mathrm{H}^n i_{\mathrm{S}}^!\mathrm{A} = 0$ *for* $n \leqslant p(\mathrm{S})$.

c) *The object* $\tilde{j}_*\mathrm{A}$ *of* $\boldsymbol{M}(\mathrm{X})^p$ *is the unique extension* B *of* A *in* $\boldsymbol{D}(\mathrm{X})$ *such that for every stratum* $\mathrm{S} \subset \mathrm{F}$, *we have* $\mathrm{H}^n i_{\mathrm{S}}^*\mathrm{A} = 0$ *for* $n \geqslant p(\mathrm{S}) + 1$ *and* $\mathrm{H}^n i_{\mathrm{S}}^!\mathrm{A} = 0$ *for* $n \leqslant p(\mathrm{S}) + 1$.

Proposition **(5.2.9)**. — *Let us assume that* $p(\mathrm{S}) \geqslant p(\mathrm{T})$ *for any two strata* S *and* T *such that* $\mathrm{S} \subset \overline{\mathrm{T}}$. *For every* $n \in \mathbf{N}$, *let* F_n *be the union of all strata* S *such that* $p(\mathrm{S}) < n$ *and let* U_n *be the union of all strata* S *such that* $p(\mathrm{S}) \leqslant n$; *let* j_n *be the inclusion of* U_n *into* U_{n+1}. *Then* F_n *is closed and* U_n *is open. Moreover, for every* p-*perverse sheaf* $\mathrm{A} \in \mathscr{M}(\mathrm{U}_n)$ *and every integer* a *such that* $p \leqslant a$ *and* $a \geqslant n$, *one has*

$$\tilde{j}_{!*}\mathrm{A} = \tau_{\leqslant a} j_{a,*} \dots \tau_{\leqslant n} j_{n,*}\mathrm{A}.$$

Proof. — The condition on p implies that for every stratum T such that $\mathrm{T} \subset \mathrm{F}_n$ and every stratum S such that $\mathrm{S} \subset \overline{\mathrm{T}}$, one has $p(\mathrm{S}) \geqslant p(\mathrm{T}) \geqslant n$, hence $\mathrm{S} \subset \mathrm{F}_n$. This implies that F_n is closed. Consequently, $\mathrm{U}_n = \mathrm{X} - \mathrm{F}_n$ is open.

To prove the desired formula by induction, it suffices to check that $\tilde{j}_{n,!*}\mathrm{A} = \tau_{\leqslant n} j_{n,*}\mathrm{A}$. Let $\mathrm{F} = \mathrm{U}_{n+1} - \mathrm{U}_n$. For every stratum $\mathrm{S} \in \mathscr{S}$ such that $\mathrm{S} \subset \mathrm{F}$, we have $p(\mathrm{S}) = n+1$, so that the p-perverse truncation structure of $\boldsymbol{D}(\mathrm{F})$ is the standard one shifted by $-(n+1)$. We have

$$\tilde{j}_{n,!*}\mathrm{A} = \tau^{\mathrm{F}}_{<0} j_{n,*}\mathrm{A} = \tau^{\mathrm{F,st}}_{\leqslant n} j_{n,*},$$

where $\tau^{\mathrm{F,st}}_{\leqslant n}$ is the partial truncation functor relative to the standard truncation structure on $\boldsymbol{D}(\mathrm{F})$. On U_n, we have $p \leqslant n$, so that $\mathrm{A} \in \boldsymbol{D}^{\leqslant n}(\mathrm{U}_n)$. Consequently, the canonical morphism

$$\tau^{\mathrm{F,st}}_{\leqslant n} j_{n,*}\mathrm{A} \to \tau_{\leqslant n} j_{n,*}\mathrm{A}$$

is mapped to an isomorphism after applying j_n^*; it is also mapped to an isomorphism after applying i_F^*, where i_F is the inclusion of F in U_{n+1}. Consequently, it is an isomorphism, and this concludes the proof of the proposition. □

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