Tame topology in number theory and geometry

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Abstract. — Summoned by Grothendieck in his Esquisse d'un programme (1985), tame topology is supposed to offer the flexibility of general topology without allowing its "pathological" constructions. Inspired by mathematical logic and real algebraic geometry, o-minimality is one solution to this program, proposed by van den Dries. The works of Peterzil and Starchenko showed that Serre's GAGA principle extends: if it is definable in an o-minimal structure, a complex analytic subset of $\mathbf{C}^n$ is necessarily algebraic.

In the last 10 years, these ideas have been made fruitful in number theory, where Zannier, Pila, then Tsimerman, Klingler, Ullmo and Yafaev proved the André-Oort conjecture concerning the geometry of subvarieties of Shimura varieties. An important tool is a counting theorem by Pila and Wilkie for points of $\mathbf{R}^n$ with rational coordinates with bounded numerator and denominator lying on a subset which is definable in an o-minimal structure.

Recently, Klingler, Bakker, Tsimerman, Brunebarbe used these ideas in Hodge theory, reproving for example a theorem of Cattani, Deligne and Kaplan regarding the algebraicity of the Hodge loci, or by proving a conjecture of Griffiths about the quasi-projectivity of the images of period maps. The aim of the lectures is to present these notions of diverse origins and, as far as possible, to describe how they interact.

In the preparation of these notes, I made extensive use of other surveys on various aspects of this topic, in particular Wilkie (2010); Peterzil & Starchenko (2011); Scanlon (2012); Bakker (2019); Fresán (2020).

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1. Tame topology

In his 1983 *Esquisse d'un programme*, published as (Grothendieck, 1997), Alexandre Grothendieck was advocating for some "tame topology" that would be flexible enough to "express with ease the topological intuition of shapes", while would be freed from the "spurious difficulties related to wild phenomena". For Grothendieck, the difficulty of proving Brouwer's theorem of invariance of domain, that is, of establishing a good dimension theory for topological manifolds, is already an indication that the topological theory is inadequate.

As a possible adequate framework, he mentions Hironaka's theory of semianalytic spaces, as well as real semialgebraic sets. In fact, he even suggests that there should exist a whole spectrum of tame topologies, for which semialgebraic sets would possibly form the coarsest such example, and semianalytic sets the finest one. These various topologies would be characterized, Grothendieck expects, by a list of properties, the most delicate of them being a *triangulability axiom*.

Such a framework has been proposed by van den Dries (1998), in the context of "mathematical logic", which proved extremly fruitful in the two following decades, that of *o-minimal geometry*.

Definition 1.1. — *A* geometry[(1)] *is the datum, for every integer n, of a set $\mathscr{D}_n$ of subsets of $\mathbf{R}^n$, satisfying the following properties:*

(1) *For every n, $\mathscr{D}_n$ is a boolean algebra: it contains the empty set, is stable under union, intersection and complement;*

(2) *For every n and every $A \in \mathscr{D}_n$, the subsets $A \times \mathbf{R}$ and $\mathbf{R} \times A$ of $\mathbf{R}^{n+1}$ belong to $\mathscr{D}_n$;*

(3) *For every n and every $A \in \mathscr{D}_{n+1}$, the image $p(A)$ of A under the projection $p\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ belongs to $\mathscr{D}_n$;*

(1) The standard word for this concept is *structure*, but that word has already too many meanings, even in model theory. Actually, the definition admits slight variations in the litterature; for example, van den Dries (1998) doesn't impose at the onset that the graphs of addition and multiplication belong to $\mathscr{D}_3$, allowing "piecewise linear" geometries, but the hypothesis appears soon after.

(4) *The set* $\mathscr{D}_1$ *contains all singletons* $\{a\}$, *for* $a \in \mathbf{R}$;

(5) *The set* $\mathscr{D}_2$ *contains the set* $\{(x, y) ; x < y\}$;

(6) *The graphs of addition and of multiplication belong to* $\mathscr{D}_3$;

(7) *For every* n *and every* i, j *such that* $1 \le i < j \le n$, *the set* $\{x \in \mathbf{R}^n ; x_i = x_j\}$ *belongs to* $\mathscr{D}_n$.

Thus, a geometry is the datum of the sets we are interested in, subject to a list of compatibilities which allow us to construct new sets from previous ones.

Using projections and intersections, one sees that whenever the graph of a function $f: \mathbf{R}^n \to \mathbf{R}^m$ belongs to $\mathscr{D}_{n+m}$, the image $f(\mathrm{A})$ of a set $\mathrm{A} \in \mathscr{D}_n$ belongs to $\mathscr{D}_m$. Similarly, if the graphs of functions $f: \mathbf{R}^n \to \mathbf{R}^m$ and $g: \mathbf{R}^p \to \mathbf{R}^n$ belong to $\mathscr{D}_{m+n}$ and $\mathscr{D}_{n+p}$, then the graph of their composition $f \circ g$ belongs to $\mathscr{D}_{m+p}$. Since the graphs of addition and multiplication belong to $\mathscr{D}_3$, this implies that $\mathscr{D}_{n+1}$ contains the graph of every polynomial in n variable.

1.2. — Given a geometry $(\mathscr{D}_n)$, the subsets of $\mathbf{R}^n$ that belong to $\mathscr{D}_n$ are called *definable* and the functions $f: \mathbf{R}^m \to \mathbf{R}^n$ whose graph belongs to $\mathscr{D}_{m+n}$ are called definable.

First order logic interprets boolean operations in $\mathbf{R}^n$ by logic connectors, intersection corresponds to conjunction (*and*), union to disjunction (*or*), and complement to negation (*not*). It also interprets projection by existential quantifiers: if a set $\mathrm{A} \subset \mathbf{R}^{n+1}$ is defined in $\mathbf{R}^{n+1}$ by a formula $\varphi(x_1, \dots, x_n, x_{n+1})$ in $(n+1)$ free variables $x_1, \dots, x_{n+1}$, and $p: \mathbf{R}^{n+1} \to \mathbf{R}^n$ is the projection given by $p(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$, then $p(\mathrm{A})$ is defined by the formula

$$\exists x_{n+1} \varphi(x_1, \dots, x_n, x_{n+1})$$

in the n free variables $x_1, \dots, x_n$.

This gives another point of view on definable sets, that explains the use of this adjective — these are those sets which can be defined using well formed formulas using logical connectors, quantifiers, and a given set of functions comprising all polynomials.

A consequence of this correspondence between defining formulas and definable subsets is that the closure, the interior, the boundary of a definable subset are again definable. Indeed, the closure can be defined by the classic ε-δ formula.

1.3. — At this stage, we could take for the definable sets of a geometry the family of all possible sets, setting $\mathscr{D}_n = \mathfrak{P}(\mathbf{R}^n)$, but this trivial solution does not give any insight.

It is a fundamental theorem of Tarski that one obtains a geometry in taking for $\mathscr{D}_n$ the set of all semialgebraic subsets of $\mathbf{R}^n$, those sets which are defined by polynomial equalities and inequalities. Indeed, Tarski proved that the image of a semialgebraic subset still is semialgebraic. This geometry is denoted by $\mathbf{R}_{\text{alg}}$.

By theorems of Łojasiewicz, Gabrielov and Hironaka, one also obtains a geometry in taking for $\mathscr{D}_n$ all so-called "finitely subanalytic" sets. These sets are defined in three steps: first, semianalytic sets are subsets of $\mathbf{R}^n$ locally defined by equalities and inequalities involving analytic functions; then, subanalytic sets are subsets of $\mathbf{R}^n$ which, locally, can be defined as the projection of a bounded semianalytic set; finally, finitely subanalytic sets are preimages of subanalytic subsets by the semialgebraic map

$$(x_1, \dots, x_n) \mapsto (x_1/\sqrt{1+x_1^2}, \dots, x_n/\sqrt{1+x_n^2})$$

(a semialgebraic bijection from $\mathbf{R}^n$ to $]-1;1[^n$). Equivalently, one adds to the semialgebraic sets all graphs of restrictions to the hypercube $[0;1]^n$ of real analytic functions on an open neighborhood of that hypercube. This geometry is denoted by $\mathbf{R}_{\text{an}}$.

Those two theories are also called *o-minimal* as they satisfy the following axiom — the letter "o" is for *order*.

Definition 1.4. — *A geometry is* o-minimal *if any definable set in* $\mathbf{R}$ *is a finite union of intervals.*

There are two alternative ways to reformulate this axiom:
- Definable sets in $\mathbf{R}$ coincide with semialgebraic sets;
- Definable sets in $\mathbf{R}$ have finitely many connected components.

It is a remarkable discovery of VAN DEN DRIES (1998) that this elementary property implies strong tameness properties in any dimension. Here is a small sample.

It follows from the definition that if a real-valued function f defined on a real interval $]a;b[$ is definable in an o-minimal geometry, its set of zeroes definable ($f^{-1}(0)$) is a finite union of intervals. More generally:

Proposition 1.5. — *Let* $f\colon]a;b[\to \mathbf{R}$ *be a function which is definable in an o-minimal geometry. Let* p *be an integer. Then there is a finite strictly increasing sequence* $(a_0, \dots, a_n)$, *with* $a = a_0$ *and* $a_n = b$ *such that for each* k, f *is either constant, or* $\mathscr{C}^p$ *and strictly monotone, on* $]a_{k-1}; a_k[$.

Theorem 1.6 **(Cylindric decomposition theorem).** — *Let* $\mathrm{A} \subset \mathbf{R}^{n+1}$ *be a subset which is definable in an o-minimal geometry. There exists a finite family* (A_j) *of definable subsets of* $\mathbf{R}^n$, *and for each* j, *a finite family* $(f_{j,k})$ *of definable functions from* A_j *to* $\mathbf{R}$, *such that* A *is the union of sets of the form*

- $\{(x,t)\ ;\ x \in \mathrm{A}_j$ *and* $t = f_{j,k}(x)\}$;
- $\{(x,t)\ ;\ x \in \mathrm{A}_j$ *and* $f_{j,k-1}(x) < t < f_{j,k}(x)\}$;
- $\{(x,t)\ ;\ x \in \mathrm{A}_j$ *and* $t < f_{j,k}(x)\}$;
- $\{(x,t)\ ;\ x \in \mathrm{A}_j$ *and* $t > f_{j,k}(x)\}$.

In particular, a function which is definable in an o-minimal geometry is "piecewise" of class $\mathscr{C}^p$, though it may not be piecewise $\mathscr{C}^\infty$.

Theorem 1.7 **(Triangulation theorem).** — *Let* $\mathrm{A} \subset \mathbf{R}^n$ *be a subset which is definable in an o-minimal geometry. Then there exists a definable homeomorphism from* A *to a simplicial subspace of* $\mathbf{R}^n$.

I recall that a simplicial subspace is a union of a finite set of disjoint "open" simplices in $\mathbf{R}^n$ such that the set of corresponding closed simplices satisfy the following property: the intersection of any two of them is a face of each of them and belongs to the set.

In particular, if a subset of $\mathbf{R}^n$, definable in an o-minimal geometry, is discrete, or even countable, then it is finite.

1.8. — For these statements to be practically useful, one needs to prove that the relevant sets be definable in some o-minimal geometry. As we will see in the rest of the course, this is often a major result in itself. We have seen that semialgebraic sets and semianalytic sets give rise to o-minimal geometry.

The previous results imply that not everything is o-minimal. For example, no o-minimal geometry can contain the graph of the sine function, since its set of zeroes (which would be definable) has infinitely many connected components.

The fundamental results are:

– There exists an o-minimal geometry $\mathbf{R}_{\text{exp}}$ that contains the graph of the exponential function (Wilkie, 1996) (proved in 1991);

– There exists an o-minimal geometry $\mathbf{R}_{\text{an,exp}}$ that contains all compact semianalytic sets and the graph of the exponential function (van den Dries & Miller, 1994; van den Dries *et al*, 1994).

Regarding their proofs, let us just quote Wilkie (1996):

> It is dicult to see how conventional analytic or differential geometric methods could be used to establish this result because of the essential singularity of the exponential function at infinity. The proof given here uses model-theoretic methods to analyse large zeros of systems of exponential-algebraic equations.

By model-theoretic methods, it is implied that one has to study not only definable subsets of $\mathbf{R}^n$, but also sets defined by the same formulas in real closed fields that extend $\mathbf{R}$. Since such fields are nonarchimedean, one tool at disposal is asymptotic analysis via valuation theory.

It is thus necessary to develop model-theoretic analogues of all important topological concepts (connectedness, compacity, etc.), with the potential gain that classic theorems which are false for arbitrary subsets of real closed fields (with the order topology) now hold true. For example, one proves that any two points of a definably connected subset are connected by a continuous definable map from $[0;1]$, or that a definable subset is definably compact if and only if it is bounded and closed.

On the other hand, the universe of all o-minimal geometries is complicated, perhaps more than what Grothendieck had envisioned. For example, there are pairs of o-minimal geometries which are not contained in a common o-minimal geometry (Le Gal, 2010; Rolin *et al*, 2003).

2. Complex analysis in an o-minimal setting

2.1. — When we identify the field $\mathbf{C}$ of complex numbers as $\mathbf{R}^2$ (real part, imaginairy part), addition and multiplication are expressed by polynomials, and modulus by a semialgebraic formula ($\sqrt{x^2+y^2}$). In a series of papers, Peterzil & Starchenko (2001, 2003, 2008, 2009) developed an o-minimal theory of complex analysis: studying fonctions on definable open subsets of $\mathbf{C}$ which are complex analytic and definable in an o-minimal geometry when viewed as functions from (a subset of) $\mathbf{R}^2$ to itself, and, more generally, studying analytic subspaces of $\mathbf{C}^n$ and functions on them which are definable in an o-minimal geometry.

Actually, such a theory had been developed earlier by KNEBUSCH (1982) and HUBER (1984), see (HUBER & KNEBUSCH, 1990), albeit in the more restrictive context of semialgebraic geometry.

Their discovery is that not only such a study is possible at all, which means that there is a model-theoretic approach to complex function theory over real closed fields, avoiding various topological and analytic tools which are unavailable in this general context, but also the theory is rich and fruitful — there are many nontrivial examples, and the consequences are often fantastic. This will be the main source of tension of the later chapters,

Without specific mention of the opposite, we assume given some o-minimal geometry is given, to which the adjective "definable" will refer. We insist that when a subspace of $\mathbf{C}^n$ is called definable, it means that it is definable as a subspace of $\mathbf{R}^{2n}$, with respect to the standard identification of $\mathbf{C}$ with $\mathbf{R}^2$, although any semialgebraic identification might do.

***Example 2.2.* —** These elementary examples show that the theory is both flexible and restrictive.

– Polynomials on $\mathbf{C}^n$, more generally on a definable subset of $\mathbf{C}^n$, are definable.

– Let U be an open subset of $\mathbf{C}^n$, and let $f\colon \mathrm{U} \to \mathbf{C}$ be a holomorphic function. For every open subspace V of U which is relatively compact in U, then $f|_{\mathrm{V}}$ is definable in $\mathbf{R}_{\mathrm{an}}$.

– The complex exponential function on $\mathbf{C}$ is not definable. Otherwise, the discrete subspace $2i\pi\mathbf{Z}$ of $\mathbf{R}$, which is defined by $e^z = 1$, would be definable, while every discrete subset which is definable in an o-minimal geometry is finite.

***Example 2.3* (Weierstrass's j-function). —** This is a a holomorphic function on the upper half-plane $\mathbf{h}$ which is invariant under the action of group $\Gamma = \mathrm{SL}(2,\mathbf{Z})$, inducing a bijection from $\mathbf{h}/\Gamma$ to $\mathbf{C}$.

It is not definable since it takes $j^{-1}(0)$ is discrete and infinite.

On the other hand, the action of $\mathrm{SL}(2,\mathbf{Z})$ admits a well known fundamental domain D, defined by the semialgebraic relations $|z| \geq 1$ and $-\frac{1}{2} \leq \Re(z) <$

$\frac{1}{2}$. Let us prove, following (Pila, 2011; Scanlon, 2012), that the restriction to D of j is definable in $\mathbf{R}_{\text{an,exp}}$.

Since $j(z+1) = j(z)$, the j function admits a Fourier expansion in terms of $q = e^{2i\pi z}$, which takes the form

$$j(z) = \mathrm{J}(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

The radius of convergence of $q\mathrm{J}(q)$ is equal to 1; when $z = x + iy \in \overline{\mathrm{D}}$, one has $y \geq \sqrt{3}/2$, hence $|q| = e^{-2\pi y} \leq e^{-\pi\sqrt{3}} < 1$. In other words, the restriction of J to the closed disk of radius $e^{-\pi\sqrt{3}}$ is definable in $\mathbf{R}_{\text{an}}$. Consider the relation $q = e^{-2\pi y}e^{2i\pi x}$ over $\overline{\mathrm{D}}$. We observe that x remains bounded, so that the second factor $e^{2i\pi x}$ is definable in $\mathbf{R}_{\text{an}}$; however, the first factor $e^{-2\pi y}$ is definable in $\mathbf{R}_{\text{exp}}$, but not less. Consequently, over $\overline{\mathrm{D}}$, q is a definable function of (x, y) in the o-minimal geometry $\mathbf{R}_{\text{an,exp}}$.

Consequently, $j|_{\overline{\mathrm{D}}}$ is definable in $\mathbf{R}_{\text{an,exp}}$.

2.4. — A main feature of the theory of definable analytic function theory is its interaction with algebraicity.

By Bianconi (2005), a germ of an analytic function which is definable in $\mathbf{R}_{\text{exp}}$ is semialgebraic.

Theorem 2.5. — (1) *An analytic function on* $\mathbf{C}^n$ *which is definable is a polynomial (Peterzil & Starchenko, 2003, theorem 2.17).*

(2) *An analytic subspace of* $\mathbf{C}^n$ *which is definable is algebraic (Peterzil & Starchenko, 2008, theorem 5.1).*

The second statement is an analogue of the theorem of Chow that claimes that closed analytic subspaces of $\mathbf{P}_n(\mathbf{C})$ are algebraic.

It is important to stress that these results are given independent proofs, valid in arbitrary real closed fields. However, as explained by Bakker (2019), they can often be proved by using advanced properties of complex function theory.

For example, an analytic function on $\mathbf{C}$ which is definable cannot have an essential singularity at infinity (otherwise, by Picard's Great theorem, it would take almost all values infinitely many often), hence it is a polynomial by the Casorati–Weierstrass theorem.

Similarly, if we view a definable analytic subspace of dimension d of $\mathbf{C}^n$ as a subspace of $\mathbf{P}_n(\mathbf{C})$, the cylindric decomposition theorem implies that it has bounded d-dimensional Hausdorff measure; then a theorem of Bishop implies that its closure is an analytic subspace of $\mathbf{P}_n(\mathbf{C})$, which is algebraic by the standard form of Chow's theorem.

2.6. — The scope of these results has been enlarged by Bakker, Brunebarbe & Tsimerman (2019) to a "definable GAGA" theorem that compares *definable analytic spaces* and their sheaf cohomology with their natural one.

Definable open subsets of $\mathbf{C}^n$, together with finite coverings, induce a Grothendieck topology on $\mathbf{C}^n$, giving rise to the definable site $\mathbf{C}^n_{\mathrm{def}}$ of $\mathbf{C}^n$. If U is a definable open subset of $\mathbf{C}^n$, one can consider the subspace $\mathscr{O}^{\mathrm{def}}_{\mathbf{C}^n}(\mathrm{U})$ of $\mathscr{O}_{\mathrm{C}^n}(\mathrm{U})$ consisting of those analytic functions which are definable. By restriction, these rings give rise to a sheaf $\mathscr{O}^{\mathrm{def}}_{\mathbf{C}^n}$ on the definable set $\mathbf{C}^n_{\mathrm{def}}$.

The basic definable analytic spaces are the closed subspaces of a definable open subset U of $\mathbf{C}^n$ defined by finitely many definable analytic functions on U, in other words, a finitely generated ideal I of $\mathscr{O}_{\mathrm{C}^n}(\mathrm{U})$. We then form the locally ringed site $(|\mathrm{V}(\mathrm{I})|, \mathscr{O}_{\mathrm{U}}/\mathrm{I})$.

More generally, a definable analytic space is a locally ringed site which has a *finite* covering by basic definable analytic space.

There is a natural notion of a coherent sheaf on a definable analytic space. In fact, the definable analogue of Oka's theorem (Bakker *et al*, 2019, theorem 2.16) implies that if X is a definable analytic space, then the structure ring $\mathscr{O}_{\mathrm{X}}$ is coherent, and finitely presented $\mathscr{O}_{\mathrm{X}}$-modules are coherent.

Quotients of definable analytic spaces by a closed étale equivalence relations exist in the category of definable analytic spaces (Bakker *et al*, 2019, corollary 2.19).

2.7. — A complex algebraic variety X, more generally, a complex algebraic spaces, gives rise to a definable analytic space $\mathrm{X}^{\mathrm{def}}$, and a coherent sheaf $\mathscr{F}$ on X gives rise to a coherent sheaf $\mathscr{F}^{\mathrm{def}}$ on $\mathrm{X}^{\mathrm{def}}$.

A definable analytic space X gives rise to an analytic space X^{an}, and a coherent sheaf $\mathscr{F}$ on X gives rise to a coherent sheaf $\mathscr{F}^{\mathrm{an}}$ on X^{an}.

These assignments are functorial.

2.8. — If X is a complex algebraic variety, then $(\mathrm{X}^{\mathrm{def}})^{\mathrm{an}}$ is nothing but the analytification X^{an} of X, as defined in (Serre, 1956) and analogously for coherent sheaves. If X is projective, Serre's GAGA theorems (Serre, 1956, théorèmes 2 and 3) assert that the analytification functor induces an equivalence of abelian categories $\mathbf{Coh}(X) \to \mathbf{Coh}(X^{an})$, and the theorem of Chow follows as an immediate consequence. As showed by Grothendieck, it suffices to assume that X is proper.

It may interesting to refine this comparison and study what happens in the two steps, first from X to X^{def}, and then from X^{def} to $(X^{def})^{an}$.

***Theorem 2.9* (Bakker, Brunebarbe & Tsimerman, 2019, theorem 2.22)** *Let* X *be a definable analytic space. The analytification functor* $\mathscr{F} \mapsto \mathscr{F}^{an}$ *from* $\mathbf{Coh}(X)$ *to* $\mathbf{Coh}(X^{an})$ *is faithful and exact.*

This theorem relates definable coherent analytic sheaves with general coherent analytic sheaves. If X is compact, definability in $\mathbf{R}_{an}$ is automatic, so that this functor is an equivalence of categories when the chosen o-minimal geometry contains $\mathbf{R}_{an}$. One cannot expect that this holds true in general, since there may be analytic functions which are not definable (violating full faithfulness for the structure sheaf), and there may be coherent analytic sheaves which are not definable (violating essential surjectivity).

The following theorems relate algebraic geometry with definable analytic geometry, and the picture is more complete.

***Theorem 2.10.* —** (1) *Let* X *be a complex algebraic space. The definabilization functor* $\mathscr{F} \mapsto \mathscr{F}^{def}$ *from* $\mathbf{Coh}(X)$ *to* $\mathbf{Coh}(X^{def})$ *is fully faithful and exact; its essential image is closed under taking subobjects and quotients (Bakker, Brunebarbe & Tsimerman, 2019, theorem 3.1).*

(2) *Let* M *be an algebraic space and let* $X \subset M^{def}$ *be a closed definable analytic subspace. Then there is a unique closed algebraic subspace of* M *of which* X *is the definabilization (Bakker et al, 2019, corollary 3.8).*

(3) *Let* X, Y *be algebraic spaces and let* $f\colon X^{def} \to Y^{def}$ *be a definable analytic map; there exists a unique algebraic morphism* $\varphi\colon X \to Y$ *such that* $f = \varphi^{def}$ *(Bakker et al, 2019, corollary 3.9).*

Note the necessity of introducing algebraic spaces: there are many examples of quotients of étale equivalence relation on a complex algebraic variety which do not exist as a scheme but do exist as an algebraic space; on the other hand, these quotients exist in the context of definable analytic spaces (see §2.6).

***Theorem 2.11* (Bakker, Brunebarbe & Tsimerman, 2019, theorem 4.2)** *Let* X *be an algebraic space, let* M *be a definable analytic space and let* $f\colon X^{\mathrm{def}} \to M$ *be a definable analytic map. If* f *is proper, then there exists a unique factorization* $f = j \circ \varphi^{\mathrm{def}}$, *where* $\varphi\colon X \to Y$ *is a proper dominant*[2] *morphism of algebraic spaces and* $j\colon Y^{\mathrm{def}} \to M$ *a closed immersion. Moreover,* $f^{\mathrm{an}}(X^{\mathrm{an}}) = j^{\mathrm{an}}(Y^{\mathrm{an}})$.

Being the image of a definable analytic map f, the set $f^{\mathrm{an}}(X^{\mathrm{an}})$ is of course definable on M. Via the identification with with the image of the closed immersion j^{an}, this theorem induces on that definable set the structure of an algebraic space coming from that of Y.

3. Bi-algebraicity

3.1. — Many conjectures or theorems in diophantine geometry can be phrased in terms of subvarieties of abelian varieties. For example, the Mordell conjecture, a theorem of Faltings (1983), is a particular case of a conjecture by Lang (1974) about the rational points of subvarieties of abelian varieties, the Manin–Mumford conjecture, proved by Raynaud (1983*a*,*b*), is about the torsion points of abelian varieties which belong to a given subvariety.

Pila & Zannier (2008) proposed a new strategy for proving such results that builds on properties of tame geometry, together with a number theoretical theorem of Pila & Wilkie (2006), see theorem 4.6 below.

3.2. — Another input of this strategy is a fresh look at theorems of Ax (1971, 1972) that provide a functional analogue to the Schanuel conjecture according to which, if $\alpha_1, \dots, \alpha_n$ are $\mathbf{Q}$-linearly independent complex numbers, then the transcendence degree over $\mathbf{Q}$ of the field generated by

$$\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n}$$

is at least n. Note that this conjectures implies many important transcendence results as particular cases; the Lindemann–Weierstrass theorem corresponds to the case where $\alpha_1, \dots, \alpha_n$ are algebraic. Already for $n = 1$, it implies that if α is algebraic and nonzero, then e^α is transcendental, hence e is transcendental (taking $\alpha = 1$) as well as π (taking $\alpha = 2\pi i$).

[2] A morphism φ of algebraic space is called *dominant* if the canonical morphism of sheaves $\mathscr{O}_Y \to \varphi_* \mathscr{O}_X$ is injective.

***Theorem 3.3* (Ax (1971)).** — *Let* $f\colon \mathbf{C}^n \to (\mathbf{C}^*)^n$ *be the map that sends* $(z_1,\dots,z_n)$ *to* $(e^{z_1},\dots,e^{z_n})$ *and let* D_n *be its graph; it is a closed irreducible analytic subspace of* $\mathbf{C}^n \times (\mathbf{C}^*)^n$*. Let* $p\colon \mathbf{C}^n \times (\mathbf{C}^*)^n \to (\mathbf{C}^*)^n$ *be the projection onto the second factor. Let* $\mathrm{M} \subset \mathbf{C}^n \times (\mathbf{C}^*)^n$ *be an algebraic subvariety and let* V *be a connected component of* $\mathrm{M} \cap \mathrm{D}_n$*, as a closed analytic subspace. If* $p(\mathrm{V})$ *is not contained in a translate of a strict subtorus of* $(\mathbf{C}^*)^n$*, then*

$$\dim(\mathrm{M}) \geq \dim(\mathrm{V}) + n.$$

Note that $(\mathbf{C}^*)^n$ is an algebraic group; its closed algebraic subgroups are defined by families monomial equations $z_1^{a_1}\dots z_n^{a_n} = 1$; subtori are connected algebraic subgroups. Consequently, an irreducible analytic subspace U of $(\mathbf{C}^*)^n$ is contained in a translate of a strict subtorus if and only if there exists a nonzero $(a_1,\dots,a_n) \in \mathbf{Z}^n$ such that the monomial $z_1^{a_1}\dots z_n^{a_n}$ is constant on U.

When $p(\mathrm{V})$ is contained in a translate of a strict subtorus, a change of coordinates reduces us to the case where that subtorus is $(\mathbf{C}^*)^m \times \{1\}$. Then the theorem implies the weaker inequality $\dim(\mathrm{M}) \geq \dim(\mathrm{V}) + m$, and this is optimal in general.

The relation between Schanuel's conjecture and this theorem is as follows: if $d = \dim(\mathrm{V})$, then around a smooth point, the analytic space V can be locally parameterized as $(f_1,\dots,f_n,e^{f_1},\dots,e^{f_n})$, where $f_1,\dots,f_n$ are n converging power series in d variables. The transcendence degree of the field generated by $(f_1,\dots,f_n,e^{f_1},\dots,e^{f_n})$ is the dimension of the smallest algebraic subvariety M' of $\mathbf{C}^n \times (\mathbf{C}^*)^n$ that contains V; since $\mathrm{M}' \subset \mathrm{M}$, it is thus smaller than $\dim(\mathrm{M})$. If we replace M by M', which does not change V, the inequality of theorem 3.3 implies that the transcendence degree over $\mathbf{C}$ of the field generated by $(f_1,\dots,f_n,e^{f_1},\dots,e^{f_n})$ is at least $n+d$. This strengthens Schanuel's conjecture — under the stronger hypothesis that $(f_1,\dots,f_n)$ are linearly independent modulo constants.

3.4. — In the same way as the Lindemann–Weierstrass theorem is a consequence of the Schanuel conjecture, applied to algebraic numbers and their exponentials, Ax's theorem 3.3 admits as corollary the following statement, known as the Ax–Lindemann theorem: *Let* U *be an algebraic subvariety of* $(\mathbf{C}^*)^n$ *and let* V *be a maximal irreducible algebraic subvariety of* $\mathbf{C}^n$ *such that* $\exp(\mathrm{V}) \subset \mathrm{U}$*. Then* V *is a translate of a* $\mathbf{Q}$*-rational vector subspace of* $\mathbf{C}^n$*,* *and* $\exp(V)$ *is a translate of an algebraic subgroup of* $(\mathbf{C}^*)^n$. (Here, exp is the exponential map from $\mathbf{C}^n$ to $(\mathbf{C}^*)^n$).

Indeed, we may assume that $\exp(V)$ is not contained in any translate of a strict subtorus of $(\mathbf{C}^*)^n$; the goal is then to show that $U = (\mathbf{C}^*)^n$ and $V = \mathbf{C}^n$. If one applies theorem 3.3 to $M = V \times U$, one obtains that $M \cap D_n$ is isomorphic to V; then $\dim(V) + \dim(U) = \dim(M) \geq \dim(V) + n$, so that $\dim(U) \geq n$. Necessarily $U = (\mathbf{C}^*)^n$, hence $V = \mathbf{C}^n$, as was to be shown.

This can be generalized slightly further: *Let* V *be an irreducible algebraic subvariety of* $\mathbf{C}^n$ *and let* M *be the Zariski closure of its image under the map* $z \mapsto (z, \exp(z))$ *from* $\mathbf{C}^n$ *to* $(\mathbf{C}^*)^n$*; then there exists a translate* T *of a subtorus of* $(\mathbf{C}^*)^n$ *such that* $M = V \times T$. Explicitly, the subtorus T has for Lie algebra the the smallest $\mathbf{Q}$-rational subspace of $\mathbf{C}^n$ such that V is contained in a translate of L.

3.5. — As already noted by Ax (1972), this paradigm holds in more general contexts than that of the exponential function. A general framework, would consider complex algebraic varieties S and T, an analytic map $f\colon S \to T$, and the question would ask: if M is an algebraic subvariety of $S \times T$, V an irreducible algebraic variety contained in the intersection $M \cap \Gamma_f$ of M with the graph of Γ, can one prove a pertinent lower bound for $\dim(M) - \dim(V)$ provided V is "generic"? Conversely, can one describe the "special" varieties V for which $\dim(M) < \dim(V) + \dim(T)$?

Actually, S and T need not be algebraic varieties themselves, it is sufficient for the question to make sense that they be semialgebraic open subsets of algebraic varieties S', T'. Then, an irreducible algebraic subvariety of $S \times T$ can be defined as an irreducible component, in the sense of analytic geometry, of the the trace on $S \times T$ of an algebraic subvariety of $S' \times T'$.

And we can broaden the picture still a little bit by replacing the pair $(S \times T, \Gamma_f)$ by a pair (U, A), where U is a semialgebraic open subset of an algebraic variety and A is an irreducible analytic subvariety of U; given an irreducible algebraic subvariety of U, and an irreducible algebraic variety V contained in $U \cap A$, are there pertinent lower bounds for $\dim(M) - \dim(V)$?

And, at this point, one could even dream about nonarchimedean companions.

3.6. — Ax (1972) treated the case of the complex exponential map, $\exp_G : L \to G$ of an algebraic group G, where L is its Lie algebra. Special varieties of G are those which are contained in a translate of a strict algebraic subgroup. His techniques belong to differential algebra.

In their new proof of the Manin–Mumford conjecture, Pila & Zannier (2008) gave a new proof of Ax's theorem for abelian varieties which builds on o-minimal arguments (the definability of $\exp_G$ in the o-minimal geometry $\mathbf{R}_{\mathrm{an,exp}}$) and the number theoretical result of Pila & Wilkie (2006). Note also the purely o-minimal proof of Peterzil & Starchenko (2018).

In his proof of the André–Oort conjecture for a product of modular curves, Pila (2011) gave a new statement of Ax–Lindemann–Weierstrass type, where f is the uniformization of a product of modular curves, that is, S is a power $\mathbf{h}^n$ of the upper half-plane and $f(z_1, \dots, z_n) = (j(z_1), \dots, j(z_n))$, where j is Weierstrass's j-function.

This result has been extended by Ullmo & Yafaev (2014) for the uniformization of compact Shimura varieties, Pila & Tsimerman (2014) for the uniformization of the Siegel modular variety, Klingler *et al* (2016) for the uniformization of general Shimura varieties, and Gao (2017) for "mixed" Shimura varieties.

The general Ax–Schanuel theorem for Shimura varieties is due to Mok, Pila & Tsimerman (2019). Previously, Pila & Tsimerman (2016) had treated the case of the j-function. See also Tsimerman (2015) for an o-minimal approach of Ax's theorem about the classical exponential function.

A significant part of these papers consists in proving that the relevant analytic maps, restricted to adequate fundamental domains, are definable in the o-minimal geometry $\mathbf{R}_{\mathrm{an,exp}}$.

3.7. — In the rest of this section, we explain the proof of the Ax–Lindemann theorem for the exponential map of abelian varieties, following Peterzil & Starchenko (2018). The word "definable" will refer to the o-minimal geometry $\mathbf{R}_{\mathrm{an}}$.

Let A be an abelian variety, let T be its Lie algebra and let $e : T \to A$ be its exponential map. The kernel of e is a lattice Λ in T. Let F be a fundamental parallelogram for this lattice; it is compact and the restriction of e to F is definable.

Let X be an irreducible algebraic subvariety of T and let Z be the Zariski closure of $e(\mathrm{X})$ in A; we wish to prove that Z is a translate of an abelian subvariety of A. Let B be the neutral component of the stabilizer of Z, it is an abelian subvariety of A and let L be its Lie algebra.

Let Z′ be the preimage of Z in T; it is a Λ-invariant complex analytic subspace of V. Its intersection with F, $\mathrm{Z}' \cap \mathrm{F}$, is definable.

All proofs in this kind of business introduce at some point some set like the following one: let Σ be the set of all $v \in \mathrm{T}$ such that $(v + \mathrm{X}) \cap \mathrm{F}$ is a nonempty subset of Z′; it is definable.

(1) *One has* $\mathrm{X} \subset \mathrm{F} - (\Sigma \cap \Lambda)$.

Let $x \in \mathrm{X}$; there exists $\lambda \in \Lambda$ such that $x + \lambda \in \mathrm{F}$, because F is a fundamental domain; consequently, $(\lambda+\mathrm{X}) \cap \mathrm{F}$ is nonempty. Moreover, $e(\lambda+\mathrm{X}) = e(\mathrm{X}) \subset \mathrm{Z}$, hence $\lambda + \mathrm{X} \subset \mathrm{Z}'$. This shows that $\lambda \in \Sigma \cap \Lambda$. Then $x = (x + \lambda) - \lambda \in \mathrm{F} - (\Sigma \cap \Lambda)$.

(2) *One has* $e(\Sigma) + \mathrm{Z} = \mathrm{Z}$.

Let $v \in \Sigma$. The analytic space Z′ contains an open subset of the irreducible algebraic set $v + \mathrm{X}$, hence it contains all of it, so that $\mathrm{X} \subset -v + \mathrm{Z}'$. Then $e(\mathrm{X}) \subset -e(v) + e(\mathrm{Z}') = -e(v) + \mathrm{Z}$. Since Z is the Zariski closure of $e(\mathrm{X})$, this implies that $\mathrm{Z} \subset -e(\mathrm{V}) + \mathrm{Z}$, hence $e(v) + \mathrm{Z} \subset \mathrm{Z}$. Since Z is irreducible and closed, one has the equality $e(\mathrm{V}) + \mathrm{Z} = \mathrm{Z}$.

(3) *If the stabilizer of* Z *is finite, then* X *is a point.*

Assume that stabilizer of Z is finite; by the previous step, $e(\Sigma)$ is finite. Then Σ is contained in a finite union of translates of Σ, hence it is discrete. Since Σ is definable in an o-minimal geometry, it is finite. Then, step 2 implies that X is contained in a finite union of translates of F by elements Λ; since F is bounded, this implies that X is bounded. Since it is an irreducible algebraic subvariety of an affine space, it is reduced to a point.

(4) *Conclusion of the proof.*

The previous step was the case $\mathrm{B} = 0$. The stabilizer of Z is of the form $\mathrm{S} + \mathrm{B}$, where S is a finite subset of A. Since the exponential map e is surjective, there exists a finite subset Δ of T such that $\mathrm{S} = e(\Delta)$. Then

$$e(\Sigma) \subset \mathrm{S} + \mathrm{B} = e(\Delta + \mathrm{L}),$$

so that

$$\Sigma \subset \Delta + \Lambda + \mathrm{L}.$$

The set $\Delta + \Lambda$ is discrete; in particular, it is countable. The image of Σ in T/L is definable and contained in the image of $\Delta + \Lambda$, which is countable. Consequently, there exists a finite subset Δ' of Δ such that $\Sigma \subset \Delta' + L$. Then $X \subset F - (\Sigma \cap \Lambda) \subset F - \Delta' + L$.

Consider the image of X modulo L; it is an irreducible constructible subset of T/L, which is contained in the image of $F - \Delta'$, hence is relatively compact. As above, this implies that the image of X modulo L is a point, so that there exists $\xi \in T$ such that $X \subset \xi + L$. Then $Z \subset e(\xi) + B$, and, finally, $Z = e(\xi) + B$.

Corollary 3.8. — *Let* V *be an algebraic subvariety of an abelian variety* A, *let* X *be a maximal irreducible subvariety of* T *such that* $e(X) \subset V$. *Then* $e(X)$ *is a translate of an abelian subvariety of* A.

Proof. — Let W be the Zariski closure of $e(X)$; by the Ax–Lindemann theorem, it is a translate of an abelian subvariety of A. It is also contained in V. Consequently, $e^{-1}(W)$ is an irreducible algebraic subvariety of T contained in $e^{-1}(V)$. By maximality, one has $e^{-1}(W) = X$, hence $e(X) = W$. □

4. Points of bounded height in definable sets

We now turn to interaction of o-minimal geometries with number theory.

4.1. — The *height* of a rational number α is defined by $H(\alpha) = \sup(|p|, |q|)$ is $\alpha = p/q$ is written as a fraction in lowest terms. It measures the arithmetic complexity of that number, at least in the sense that for every $T > 0$, there are finitely many rational numbers α such that $H(\alpha) \leq T$.

The notion extends naturally to heights of points in $\mathbf{Q}^n$ by setting

$$H(\alpha_1, \dots, \alpha_n) = \sup(H(\alpha_1), \dots, H(\alpha_n))$$

if $\alpha_1, \dots, \alpha_n$ are rational numbers. Again, for every $T > 0$, there are finitely many rational points $\alpha \in \mathbf{Q}^n$ such that $H(\alpha) \leq T$.

It also extends to rational points projective varieties, and satisfies approximate functorial properties.

This notion, close variants of it, as well as technically involved elaborations of it, is a standard tool of diophantine theory, from Fermat's proofs by infinite descent, to Mordell–Weil's theorem, to the proofs of Mordell's conjecture by Faltings (1983); Vojta (1991); Bombieri (1990), and many other important results.

4.2. — A classic theorem of Dirichlet asserts that when T goes to infinity, the number of $\alpha \in \mathbf{Q}$ such that $\mathrm{H}(\alpha) \leq \mathrm{T}$ is asymptotically equivalent to $12/\pi^2 \cdot \mathrm{T}^2$. The idea is that there roughly $2\mathrm{T}^2$ pairs of integers (p, q) such that $|p|, |q| \leq \mathrm{T}$ and $q \geq 1$, but one should sieve out 1/4th of it (those pairs consisting of even integers), and also 1/9th of the remaining lot (those pairs consisting of integers divisible by 3), and so on for all primes, leading to the expansion

$$2 \cdot \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) \cdot \mathrm{T}^2 = 2/\zeta(2) \cdot \mathrm{T}^2 = 12/\pi^2 \cdot \mathrm{T}^2.$$

In $\mathbf{Q}^n$, one gets a similar result, growing as T^{2n}…

4.3. — When it is impossible to prove that a diophantine equation has finitely many solutions, more generally that a subset $\mathrm{X} \subset \mathbf{R}^n$ contains finitely many rational points, one can at least try to prove that when T grows to infinity, the cardinality of the set

$$\mathrm{X}(\mathbf{Q}; \mathrm{T}) = \{\alpha \in \mathrm{X} \cap \mathbf{Q}^n \; ; \; \mathrm{H}(\alpha) \leq \mathrm{T}\}$$

grows much smaller than what is a priori possible.

If X is the affine line, we have seen that the cardinality of $\mathrm{X}(\mathbf{Q}; \mathrm{T})$ grows like T^2. If X is an elliptic curve, the cardinality of $\mathrm{X}(\mathbf{Q}; \mathrm{T})$ grows like $\log(\mathrm{T})^r$, where r is the rank of the finitely generated abelian group $\mathrm{X}(\mathbf{Q})$. Before it was known that $\mathrm{X}(\mathbf{Q})$ is finite, Mumford (1965) had proven that on a curve X of genus $g \geq 2$, the cardinality of $\mathrm{X}(\mathbf{Q}; \mathrm{T})$ has a log-logarithmic upper bound.

Starting with Jarník (1926), there have also been investigations when X is a more general than an algebraic variety. In this section, we describe a theorem of Pila & Wilkie (2006) that says in essence that if X is definable in some o-minimal geometry, then $\mathrm{X}(\mathbf{Q}; \mathrm{T})$ is small unless it is explained by semialgebraic subsets of X.

4.4. — Let X be a subset of $\mathbf{R}^n$. The *algebraic part* $\mathrm{X}^{\mathrm{alg}}$ of X is the union of all its connected infinite semialgebraic subsets. The complement $\mathrm{X} - \mathrm{X}^{\mathrm{alg}}$ is called its transcendental part and is denoted by X^{tr}.

The algebraic part X^{alg} contains the topological interior of X. Its precise determination may be delicate; theorems of Ax–Schanuel type may be useful. Here is an elementary and classical example, given in Pila & Wilkie (2006), see also Bhardwaj & van den Dries (2021) for a complete and detailed exposition.

Example 4.5. — Let X be the set of all $(x, y, z) \in \mathbf{R}^3$ such that $1 < x, y < 2$ and $z = x^y$. It is definable in $\mathbf{R}_{\exp}$.

For every rational number $b = p/q \in]1; 2[$, the set X_b of all $(x, y, z) \in X$ such that $y = b$ can be defined by $1 < x < 2$, $y = b$ and $z^q = x^p$, hence X_b is a semialgebraic curve contained in X.

Conversely, let us consider a semialgebraic curve contained in X. Cellular decomposition shows that it can be parameterized under the form $(t, y(t), z(t))$, or $(a, t, z(t))$, or (a, b, t) where t varies in some open real interval, and $y(t)$, $z(t)$ represent $\mathscr{C}^1$-semialgebraic functions, while a, b are real numbers. Let us analyse all three cases:

(1) Case (a, b, t): it is incompatible with the equation $z = x^y = a^b$;

(2) Case $(a, t, z(t))$: it implies $z(t) = a^t$, but the exponential function is not semialgebraic, hence $t \mapsto a^t$ is not semialgebraic unless $a = 1$;

(3) Case $(t, y(t), z(t))$: then $z(t) = t^{y(t)} = e^{y(t)\log(t)}$ is semialgebraic, hence its logarithmic derivative $z'(t)/z(t) = y'(t)\log(t) + y(t)/t$ is semialgebraic as well. Since $y(t)$ and $y'(t)$ are semialgebraic but $\log(t)$ is not semialgebraic, we get that $y'(t) = 0$. The considered cell takes the form $(t, b, z(t))$ and $z(t) = t^b$ is semialgebraic, which implies that b is a rational number.

Indeed, if $z(t) = t^b$ is algebraic on some interval, there is a nontrivial polynomial $P \in \mathbf{R}[X, Y]$ such that $P(t, t^b)$ vanishes an open, possibly smaller, interval. By analytic continuation, this implies that $P(t, t^b)$ vanishes identically on $\mathbf{R}_+$. If $P = \sum c_{m,n} X^m Y^n$, we get $\sum c_{m,n} t^{m+nb} \equiv 0$. If b is irrational, then all exponents $m + nb$ are distinct, so that $P(t, t^b)$ behaves like $c_{m,n} t^{m+nb}$, where (m, n) is the unique pair such that $c_{m,n} \neq 0$ that makes $m + nb$ maximal. This concludes the proof.

We thus have proved that X^{alg} is the union of all these semialgebraic curves X_b, for $b \in]1; 2[$.

Theorem 4.6 **(Pila & Wilkie, 2006, theorem 1.9)**
Let $X \subset \mathbf{R}^n$ *be a set definable in an o-minimal geometry. For every* $\varepsilon > 0$, *there exists a real number c such that*

$$\mathrm{Card}(X^{\mathrm{tr}}(\mathbf{Q}; T)) \leq c \cdot T^{\varepsilon}$$

for all $T > 1$.

Here, the constant c depends on X and ε.

Actually, and this is very important for applications, Pila & Wilkie (2006) established various refinements of this theorem in two directions. Firstly, they can allow the definable set X to vary in a definable family, and the constant c is uniform in this family. Secondly, they can define a an exceptional set Z which is definable and contained in X^{alg}, so that $\mathrm{Card}(X - Z(\mathbf{Q}; T)) \leq c \cdot T^{\varepsilon}$; the interest is that the set X^{alg} is not definable in general, as the previous example shows. They can also combine these two generalizations.

The final statement, due to Pila (2009), makes use of the notion of "block families", and is probably too technical to be worth being quoted here.

4.7. — From the number theoretical side, the proof of theorem 4.6 relies on the "determinant method" that is sometimes attributed to the paper of Bombieri & Pila (1989), but is in fact much older since underlies all the transcendental number theory of the 20th century, from the work of Thue (1909) on diophantine approximation to elaborate versions in Arakelov geometry. In fact, very close arguments can already be found in the elementary proof of Hilbert's irreducibility theorem by Dörge (1927)(3).

The initialization step, simplified from (Pila, 2004, prop. 4.2), is as follows.

Lemma 4.8. — *Consider n smooth functions* $f_1, \dots, f_n$ *from the hypercube* $[0; 1]^m$ *to* $\mathbf{R}$ *and let X be the image of* $f = (f_1, \dots, f_n)$. *Let* $\varepsilon > 0$. *There exists an integer* $d \geq 1$ *and a real number c such that for every* $T > 1$, *the set* $X(\mathbf{Q}; T)$ *is contained in at most* $c \cdot T^{\varepsilon}$ *hypersurfaces of degree d.*

Proof. — Let $N = \binom{d+n}{n}$ be the number of monomials of degree $\leq d$ in n indeterminantes. Fix a small hypercube Q of size r contained in $[0; 1]^m$ and consider N points $p_1, \dots, p_N$ in Q with $f(p_j) \in X(\mathbf{Q}; T)$. We consider

(3) That paper is in German. See also Lang (1960) for a nice and concise exposition in French, and Chambert-Loir (2005) for a textbook presentation in English.

the determinant

$$\Delta = \det(f(p_j)^{\mu})_{\mu,j}$$

indexed by monomials μ of degree $\leq d$ and $j \in \{1,\dots,\mathrm{N}\}$. The equality $\Delta = 0$ means that the points $f(p_j)$ belong to a common hypersurface of degree d; so assume that $\Delta \neq 0$. Let $b_j \in \mathbf{N}^*$ be the smallest strictly positive integer such that $b_j f(p_j) \in \mathbf{Z}^n$; by definition of the height of $f(p_j)$, b_j and $b_j f_i(p_j)$ are integers of absolute value $\leq \mathrm{T}$. In particular, $\Delta \cdot \prod_{j=1}^{\mathrm{N}} b_j^d$ is a rational integer; since it is nonzero, its absolute value is at least 1, so that

$$|\Delta| \geq \mathrm{T}^{-\mathrm{N}d}.$$

On the other hand, since the points p_j all belong to the hypercube Q, they are at distance at most $r\sqrt{m}$ one from the other. To evaluate the determinant Δ, one can perform column operations on its matrix so as to eliminate more and more terms of the Taylor expansion. This gives rise to an upper bound of the form $|\Delta| \leq c \cdot r^{\mathrm{B}}$, for some integer B which only depends on N, hence on d, and some real number c which depends on the size of the derivatives of f on the hypercube Q. According to Pila (2004), the number of derivatives of the f_j that need to be considered grows as $d^{n/m}$, and B grows as $d^{n(m+1)/m}$.

For $r < \mathrm{T}^{-\mathrm{N}d/\mathrm{B}} c^{-1/\mathrm{B}}$, we get a contradiction; that is, the points $p_1,\dots,p_{\mathrm{N}}$ belong to a common hypersurface of degree d.

Taking $r = \mathrm{T}^{-\mathrm{N}d/\mathrm{B}} c^{-1/\mathrm{B}}/2$, say, we cover the hypercube $[0;1]^m$ by $1/r^m = \mathrm{T}^{\mathrm{N}dm/\mathrm{B}} c^{m/\mathrm{B}} 2^m$ small hypercubes Q. Finally, the set $\mathrm{X}(\mathbf{Q};\mathrm{T})$ is covered by $\mathrm{T}^{\mathrm{N}dm/\mathrm{B}} c^{m/\mathrm{B}} 2^m$ hypersurfaces of degree d.

It remains to pay attention to the numerology and to observe that when $d \to +\infty$, the exponent $\mathrm{N}d/\mathrm{B}$ tends to 0. By the expressions given above for N and B, one has $\mathrm{N}d/\mathrm{B} \approx d^{\alpha}$, with

$$\alpha = n + 1 - n\frac{m+1}{m} = 1 - \frac{n}{m} < 0.$$

That concludes the proof. □

4.9. — This lemma, rather a more precise version of it, allows to start an induction process, replacing the initial definable set X by the finitely many intersections $\mathrm{X} \cap \mathrm{H}$ with hypersurfaces of degree d that come out of the lemma. For those hypersurfaces H such that $\mathrm{X} \cap \mathrm{H}$ has dimension 0, the intersection is finite, and uniformly bounded when H varies, leading to a $O(\mathrm{T}^{\varepsilon})$ bound.

To go on when $\dim(X \cap H) \geq 1$, it is necessary to parameterize these sets further, while having some control on the derivatives of the parameterization.

A second input of PILA & WILKIE (2006) is thus an o-minimal version of a "reparametrization lemma" due to GROMOV (1987) and YOMDIN (1987*a*,*b*). One difficulty is that it needs to be applied to all possible $X \cap H$, uniformly when H varies among all relevant hypersurfaces of degree $\leq d$, consequently requiring a version with parameters.

It is in this inductive step that the o-minimality assumption shows its strength, in the uniformity that it allows.

4.10. — For the applications, a more general version of the counting theorem is necessary, where one takes into account not only rational points of X, but also points with coordinates in algebraic fields of given degree. The height machine extends naturally to algebraic numbers: the height of an algebraic number is essentially the maximum of the heights of the coefficients of its minimal polynomial. Then Northcott's theorem guarantees that for any integer d and any real number T, the set of algebraic numbers α such that $[\mathbf{Q}(\alpha):\mathbf{Q}] \leq d$ and $H(\alpha) \leq T$ is finite. Extending the methods of PILA & WILKIE (2006) to this framework, PILA (2009) showed that for any $\varepsilon > 0$, there is a real number c such that the number of points $x = (x_1, \dots, x_n) \in X^{\mathrm{tr}}$ such that $[\mathbf{Q}(x_j):\mathbf{Q}] \leq d$ and $H(x_j) \leq T$ for all j is smaller than cT^{ε}.

4.11. — It is a natural question to strengthen the counting theorem 4.6 to an upper bound of the form

$$\mathrm{Card}(X^{\mathrm{tr}}(\mathbf{Q};T)) \leq c(\log(T))^{N}.$$

This is in fact conjectured by PILA & WILKIE (2006) when X is definable in $\mathbf{R}_{\exp}$.

As already remarked by PILA & WILKIE (2006), for every strictly decreasing function $\varepsilon\colon \mathbf{R}_+ \to \mathbf{R}$ which tends to 0 at infinity, there exists a transcendental analytic function f on $[0;1]$ and a sequence (T_j) of going to infinity such that

$$\mathrm{Card}(X^{\mathrm{tr}}(\mathbf{Q};T_j)) \geq T_j^{\varepsilon(T_j)}.$$

Taking $\varepsilon(t) = 1/\sqrt{\log(t)}$, one has $t^{\varepsilon(t)} = e^{\sqrt{(\log(t)))}}$ is not bounded from above by any power of $\log(t)$, so that the above question would have a negative answer without any restriction on the considered o-minimal geometry.

More recently, Binyamini & Novikov (2017) have proved such a result when X is definable in the o-minimal geometry generated by the restrictions $\exp|_{[0;1]}$ and $\sin|_{[0;\pi]}$.

5. Special points and special subvarieties

5.1. — To this day, the most stunning application of o-minimal geometry to diophantine geometry is probably the proof of the André–Oort conjecture for Shimura varieties, following a strategy of Pila & Zannier (2008), that started with a paper of Pila (2011) for products of modular curves, and was rapidly generalized by two main teams of mathematicians, Pila & Tsimerman (2013, 2014); Tsimerman (2018) and Ullmo & Yafaev (2014); Gao (2017) that lead to a proof of the André–Oort conjecture for arbitrary (mixed) Shimura under the Generalized Riemann hypothesis[(4)], and unconditionally for Shimura varieties of Siegel type. Their work builds on the proof of the already evoked "hyperbolic Ax–Lindemann theorem". They also need height estimates for cm-points which follow from the Generalized Riemann hypothesis, but could be deduced from of an averaged version of a conjecture of Colmez regarding the height of abelian varieties with complex multiplication. A proof of these height estimates has just been announced by Pila, Shankar & Tsimerman (2021).

In this section, I describe this André–Oort conjecture in the particular case of modular curves and present a sketch of its proof.

5.2. — To motivate the arguments, I start with a sketch of the proof of the Manin–Mumford conjecture following Pila & Zannier (2008).

Proposition 5.3. — *Let* A *be a complex abelian variety and let* V *be a subvariety of* A. *If* V *does not contain any translate of an abelian subvariety of* A *of strictly positive dimension, then* $V \cap A_{\mathrm{tors}}$ *is finite.*

Proof. — We assume that A is defined over a number field K, the general case can be deduced from it by some spreading-out argument; see for example (Raynaud, 1983*a*, beginning of §7).

(4) Earlier, a proof of the conjecture conditional under GRH had already been proposed by Edixhoven & Yafaev (2003).

Let $e\colon T \to A$ be the exponential map of A and let $F \subset T$ be a bounded open subset which contains a fundamental domain for the action of the lattice $\Lambda = \ker(e)$; since e is analytic and F is bounded, the map $e|_F$ is definable in $\mathbf{R}_{an}$, and $X = e^{-1}(V) \cap F$ is an analytic set contained in F, definable in $\mathbf{R}_{an}$.

It follows from the Ax–Lindemann theorem for abelian varieties that X^{alg} is empty. Indeed, if $e^{-1}(V)$ contains an infinite connected semialgebraic set, it contains its Zariski closure Y, because X is an analytic subset. Then the Zariski closure of $e(Y)$ is translate $b + B$ of an abelian subvariety of A, contained in V, hence $B = 0$ and Y is a point; contradiction.

Let us identify T with $\mathbf{R}^{2g}$ in such a way that Λ is mapped to $\mathbf{Z}^{2g}$, and F to some open neighborhood of $[0;1]^{2g}$. The points of T whose image in A are torsion points correspond to the points of $\mathbf{Q}^{2g}$.

On the other hand, arguing by contradiction, let us assume that V contains infinitely many torsion points. It then contains points a of arbitrarily high order N. A theorem of Masser (1984) gives a lower bound for the degree of the field of definition of a, of the form $[K(a) : K] \gg N^{\rho}$, for some real number $\rho > 0$. The Galois orbit of a thus contains that many points, which are all contained in V, giving rise to the same number of points in $X \cap [0;1[^{2g}$. These points have coordinates of the form m/N, with $0 \leq m < N$, hence their heights are bounded by N. In particular, $\operatorname{Card}(X(\mathbf{Q};N)) \gg N^{\rho}$.

Since $X = X^{tr}$ and N can be taken arbitrarily large, this contradicts the counting theorem 4.6. □

5.4. — From that statement and various reductions, Pila & Zannier (2008) deduce the full form of the Manin–Mumford conjecture. Namely, if A is a complex abelian variety and V an irreducible algebraic subvariety of A, then the Zariski closure of $V \cap A_{tors}$ is a finite union of translates of abelian subvarieties of A by torsion points.

One first proves that the set of translates of abelian subvarieties of A which are contained in V has only finitely many maximal elements. Their union is an algebraic subvariety of V, which we call its *special locus* and denote by V^{spec}. As in the previous proposition, one then proves that $V - V^{spec}$ has only finitely many torsion points. In fact, with the notation of the proof, the Ax–Lindemann theorem implies that the algebraic part of X is equal to $e^{-1}(V^{spec}) \cap F$.

On the other hand, if $V = \bigcup(b_j + B_j)$, where B_j is an abelian subvariety of A and $b_j \in A$, then either b_j is a torsion point in A/B_j, in which case $b_j + B_j$ contains a Zariski dense set of torsion points, or b_j is not a torsion point in A/B_j and $b_j + B_j$ does not contain any torsion point.

The conclusion is that up to a finite number of torsion points, the Zariski closure of $V \cap A_{\mathrm{tors}}$ is the union of those $b_j + (B_j)_{\mathrm{tors}}$ for those j such that $b_j \in (A/B_j)_{\mathrm{tors}}$.

5.5. — We now pass to the André–Oort conjecture. Let E be a complex elliptic curve, presented as a one-dimensional complex torus $\mathbf{C}/\Lambda$, where Λ is a lattice in $\mathbf{C}$. Endomorphisms of E are induced by homotheties of $\mathbf{C}$ that stabilize the lattice Λ; usually, there are no other endomorphisms than the integer multiplications $z \mapsto nz$, for $n \in \mathbf{Z}$, but some exceptional elliptic curves have more endomorphisms.

When Λ is normalized so that it has a basis of the form $(1, \tau)$, where $\tau \in \mathbf{h}$ belongs to the Poincaré upper half-plane, they correspond to the case where τ generates an imaginary quadratic field. Then, the ring End(E) is an order in that quadratic field $\mathbf{Q}(\tau)$, the curve E is said to admit complex multiplication, or to be a CM-elliptic curve, and the j-invariant $j(\tau)$ is an algebraic number. Also, any elliptic curve which is isogeneous to a CM-elliptic curve is still CM, with the same imaginary quadratic field.

By the way, a theorem in transcendental number theory independently proved by Gel'fond and Schneider characterizes CM-elliptic curves as those curves $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$, for which both $\tau \in \mathbf{h}$ and $j(\tau)$ is an algebraic number.

5.6. — Let $N \geq 1$ be an integer. The holomorphic functions $j(z)$ and $j(Nz)$ are not algebraically independent, and there exists an irreducible polynomial $\Phi_N \in \mathbf{Z}[X, Y]$ such that $\Phi_N(j(z), j(Nz)) = 0$. This polynomial is symmetric in X and Y, homogeneous of degree $\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$. The plane curve that it defines in $\mathbf{C}^2$ is birational to the modular curve $X_0(N)$ that parameterizes isogenies between elliptic curves with a cyclic kernel of degree N, the two projections to $\mathbf{C}$ corresponding to the two natural morphisms from $X_0(N)$ to $X_0(1)$.

Say that an algebraic subvariety V of $\mathbf{C}^n$ is *special* if it is an irreducible component of an algebraic subvariety defined by equations of the form $z_a = c$, for c some j-invariant of a CM-elliptic curve, or $\Phi_N(z_a, z_b) = 0$, for $N \geq 1$, and $a, b \in \{1, \dots, n\}$.

Assume that V is a special subvariety of $\mathbf{C}^n$. Using the fact that CM j-invariants are dense in $\mathbf{C}$, and that the j-invariants of two isogeneous elliptic curves are simultaneously CM or not-CM, one observes that V contains a Zariski dense set of points of the form $(z_1, \dots, z_n)$, all of whose coordinates of which are CM j-invariants.

Conversely, when restricted to products of modular curves, the André–Oort conjecture characterizes special subvarieties of $\mathbf{C}^n$ as those algebraic subvarieties which satisfy this density property.

5.7. — Let V be an irreducible algebraic subvariety of $\mathbf{C}^n$ containing a Zariski-dense set of CM-points. Since CM-points are algebraic numbers, the variety V is defined over a number field, say F.

Consider the map induced coordinatewise by Weierstrass's j-function, from $\mathbf{h}^n$ to $\mathbf{C}^n$ and that we still denote by j. Let D be the standard semialgebraic fundamental domain for the action of $SL_2(\mathbf{Z})$ on $\mathbf{h}$; as we have seen in example 2.3, the map $j|_D$ is definable in the o-minimal geometry $\mathbf{R}_{\text{an,exp}}$, so that the map $j\colon D^n \to \mathbf{C}^n$ is definable in $\mathbf{R}_{\text{an,exp}}$. The inverse image $X = j^{-1}(V) \cap D^n$ is thus definable.

Every CM-point z in V is the image by j of exactly one point $\tau \in X$, and each coordinate of τ is imaginary quadratic. For any automorphism σ of $\overline{\mathbf{Q}}$ that is the identity on the definition field K of V, one has $\sigma(z) \in K$, so that each such point z gives rise to $[K(z) : K]$-distinct points on V, and $[K(z) : K]$-distinct points on X. If τ is imaginary quadratic, then the the theory of complex multiplication and Siegel's theorem on class numbers imply that $[\mathbf{Q}(j(\tau)) : \mathbf{Q}] \gg H(\tau)^{1/2-\varepsilon}$.

The counting theorem 4.6 then implies that up to finitely exceptions, all of these points τ belong to X^{alg}.

5.8. — On the other hand, the "hyperbolic Ax–Lindemann–Weierstrass" theorem furnishes a description of X^{alg} as traces on D^n of a finite union of *geodesic* subvarieties of $\mathbf{h}^n$, meaning irreducible components of (the trace on $\mathbf{h}^n$ of) algebraic subvarieties of $\mathbf{C}^n$ defined by equations of the form $z_b = \gamma \cdot z_a$ for $1 \leq a, b \leq n$ and $\gamma \in SL_2(\mathbf{Q})$, or $z_b = c$, for $c \in \mathbf{h}$.

If a geodesic subvariety contains CM-points, the points $c \in \mathbf{h}$ that appears in equations of the form $z_b = c$ are necessarily themselves CM-points. Ultimately, this implies that this geodesic subvariety gives rise to a special subvariety of V.

This concludes the sketch of Pila's proof of the André–Oort conjecture.

6. Definability of period mappings

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