From complex function theory to non-archimedean spaces

A number theoretical thread

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Abstract

Diophantine geometry and complex function theory have a long and well known history of mutual friendship, attested, for example, by the fruitful interactions between height functions and potential theory.

In the last 50 years, interactions even deepened with the invention of Arakelov geometry (Arakelov, Gillet/Soulé, Faltings) and its application by Szpiro/Ullmo/Zhang to equidistribution theorems and the Bogomolov conjecture.

Roughly at the same time, Berkovich invented a new kind of non-archimedean analytic spaces which possess a rich and well behaved geometric structure. This opened the way to non-archimedean potential theory (Baker/Rumely, Favre/Rivera-Letelier, Thuillier...), or to arithmetic/geometric equidistribution theorems in this case.

More recently, Ducros and myself introduced basic ideas from tropical geometry and a construction of Lagerberg to construct a calculus of (p,q)-forms on Berkovich spaces, which is an analogue of the corresponding calculus on complex manifolds, and seems to be an attractive candidate for being the p-adic side of height function theory.

Heights

Arakelov geometry

Non-archimedean geometries

Real forms and currents in non-archimedean geometry

Metrized line bundles

Heights

Naïve heights

The **height** of a point $\mathrm{x} \in \mathbf{P}_n(\overline{\mathbf{Q}})$ is a quantitative measure of its complexity, roughly the number of bits we need to write it down: if $\mathrm{x} = [x_0 : \cdots : x_n]$, with $x_0, \ldots, x_n$ integer and coprime, then

$$h(\mathrm{x}) = \log \sup(|x_0|, \ldots, |x_n|).$$

Northcott property

For every integer $d \geqslant 1$ and every real number B, the set of points $\mathrm{x} \in \mathbf{P}_n(\overline{\mathbf{Q}})$ such that $[\mathbf{Q}(\mathrm{x}) : \mathbf{Q}] \leqslant d$ and $h(x) \leqslant B$ is finite.

Naïve heights — functoriality

The height has good functorial properties: let $F\colon \mathbf{P}_n \dashrightarrow \mathbf{P}_m$ be defined by coprime homogeneous polynomials $F_0, \ldots, F_m$ of the same degree d, and let $B_F = V(F_0, \ldots, F_m)$ be the indetermination locus.

- There exists $c_F \in \mathbf{R}$ such that $h(F(\mathrm{x})) \leqslant dh(\mathrm{x}) + c_F$ for all $\mathrm{x} \in \mathbf{P}_n(\overline{\mathbf{Q}})$ such that $\mathrm{x} \notin B_F$;
- For every closed subvariety X of $\mathbf{P}_n$ such that $X \cap B_F = \emptyset$, there exists $c_X \in \mathbf{R}$ such that $h(F(\mathrm{x})) \geqslant dh(\mathrm{x}) - c_X$ for all $x \in X(\overline{\mathbf{Q}})$.

Weil heights

WEIL transformed this functoriality properties into a **height machine**:

- For every projective variety X and every line bundle L on X, a height function h_L on $X(\overline{\mathbf{Q}})$;
- For every morphism $f\colon X \to Y$, an equality $h_L \circ f = h_{f^*L}$.

However: because of the unspecified constants c_F, c_X, the height function h_L is only defined up to a bounded function, and the functorial equality only holds up to a bounded error term.

This is relatively innoccuous when one looks at points of large height, but forbids of talking of **the** height of a point in general.

Normalizing the height: Algebraic dynamics

In various contexts, it is important to exhibit a specific height function.

TATE, SILVERMAN...

Given $f\colon X \to X$, a line bundle L and an integer $q \geqslant 2$ such that $f^*L \simeq L^q$, there is a unique height function $\widehat{h}_L$ such that $\widehat{h}_L(f(x)) = q h_L(x)$ for all $x \in X(\overline{\mathbf{Q}})$.

- If L is ample, preperiodic points are characterized by the formula $\widehat{h}_L(x) = 0$.
- If X is an abelian variety, $f = [2]$
  L is ample and symmetric, then $[2]^*L \simeq L^4$
  and $\widehat{h}_L$ induces a positive quadratic form on $X(\overline{\mathbf{Q}}) \otimes \mathbf{R}$ (NÉRON, TATE).

Normalizing the height: the Mahler measure

Let $t \in \overline{\mathbf{Q}}$, let $F \in \mathbf{Z}[T]$ be its minimal polynomial, written as $F = c(T - a_1)\dots(T - a_d)$, with $a_1, \dots, a_d \in \mathbf{C}$, one has

$$
\begin{aligned}
h([1:t]) &= \frac{1}{\deg(F)} \log(\mathrm{M}(F)) \\
&= \frac{1}{\deg(F)} \left(\log(|c|) + \sum_{j=1}^{d} \log(\sup(\left|a_j\right|, 1)) \right).
\end{aligned}
$$

Here $\mathrm{M}(F)$ is the **Mahler measure** of F.

Height zero: t is a unit, all conjugates are inside the unit disk.
Kronecker: $t = 0$ or t is a root of unity.

Recall the Lehmer conjecture: there should exist $c_{\mathrm{L}} > 0$ such that $h([1:t]) > c_{\mathrm{L}} / \deg(t)$ unless $h([1:t]) = 0$.

Normalizing the height: capacity theory

Fekete/Szegő: replace the unit disk by a (reasonable) compact K set and $\log(\sup(|\cdot|, 1))$ by the **potential** of K:

- g_K is zero on K, harmonic outside of K;
- $g_K(z) = \log(|z|/c_K) + o(1)$ for $z \to \infty$.

The constant c_K is the **capacity** of K, aka its **transfinite diameter.**

This gives rise to a height h_K and there are theorems (Fekete/Szegő, Serre, Rumely...) that assert the finiteness of the set of points of height zero if $c_K < 1$, or the existence of many points of arbitrarily small height if $c_K \geqslant 1$.

Arakelov geometry

The setup

Arakelov geometry is a slightly hi-tech machinery that furnishes a **geometric** framework to specify the height functions, functorially.

Instead of a projective $\mathbf{Q}$-variety X and a line bundle L on X, one considers:

- A projective and flat scheme $\mathscr{X}$ over $\operatorname{Spec}(\mathbf{Z})$;
- A line bundle $\mathscr{L}$ on $\mathscr{X}$;
- Hermitian metrics on the complex line bundle $\mathscr{L}_{\mathbf{C}}$ over the complex manifold $X(\mathbf{C})$.

Arakelov heights

An inductive definition associates to such triples $(\mathscr{X}, \mathscr{L}, \|\cdot\|)$ specific functions $h_{\overline{\mathscr{L}}}$ that apply to all closed subschemes of $\mathscr{X}$.

For integral subschemes which are vertical, it is essentially a geometric degree: if $Z \subset \mathscr{X}_{\mathbf{F}_p}$, one obtains

$$\deg_{\mathscr{L}_{\mathbf{F}_p}}(Z) \log(p).$$

For integral subschemes $\mathscr{Z}$ which are horizontal, one obtains the height of their generic fiber Z (multiplied by its geometric degree):

$$(1 + \dim(Z)) h_{\overline{\mathscr{L}}}(\mathscr{Z}_{\mathbf{Q}}) \deg_{\mathscr{L}_{\mathbf{Q}}}(Z).$$

Normalizing Arakelov heights

What perspective do we get with respect to normalized heights when X is a projective $\mathbf{Q}$-variety and L is a line bundle on X? Finding appropriate models and appropriate metrics:

- Néron–Tate height: Néron models, metric with translation invariant first Chern form;
- Rumely's potential theory on curves: p-adic capacities via appropriate models;
- Algebraic dynamics: good reduction is rare, and one still needs an approximation process (Zhang's adelic metrics).

Arakelov heights: applications

Analogues of the Hilbert–Samuel theorem lead to an important theorem by Zhang that compares the height of a variety with the essential infimum of the heights of its points: if $\mathscr{L}$ is relatively ample and $c_1(\overline{\mathscr{L}})$ is a positive $(1,1)$-form, then

$$\sup_{Y \subsetneq X} \inf_{x \in X(\overline{\mathbf{Q}}) \setminus Y} h_{\overline{\mathscr{L}}}(x) \geqslant h_{\overline{\mathscr{L}}}(X).$$

Szpiro/Ullmo/Zhang observed that this lower bound leads to an **equidistribution theorem** when it is sharp.

Ullmo/Zhang used this equidistribution property to establish the Bogomolov conjecture for curves/subvarieties of abelian varieties.

Non-archimedean geometry: motivations

Arakelov geometry

- Put non-archimedean places on the same footing as archimedean ones;
- Phenomena of bad reductions, absence of good models;
- Equidistribution theorems of points of small height, resp. in algebraic dynamics.

Asymptotic aspects of archimedean geometry

- Conjectures of Kontsevich–Soibelman;
- Degenerations of archimedean dynamics towards non-archimedean dynamics;
- Asymptotic expansions of integrals.

Non-archimedean geometries

Algebraic geometry vs. analytic geometry

Building blocks:

	alg. geometry	non-arch. geometry
algebra	finitely generated over a field	affinoid algebras
space	spectrum $\mathrm{Spec}(A)$	analytic spectrum $\mathscr{M}(A)$
points	prime ideals	multiplicative seminorms
	eq. classes of morphisms to fields	eq. classes of morphisms to complete valued fields

Non-archimedean geometry

Affinoid algebras: quotients of the algebra $k\{T_1, \dots, T_r\}$ of power series whose coefficients converge to 0.

The analytic **spectrum** $\mathscr{M}(A)$ is the set of multiplicative seminorms p on A, functions $p \colon A \to \mathbf{R}_+$ such that

- $p(f+g) \leqslant p(f) + p(g)$ for $f, g \in A$;
- $p(fg) = p(f)p(g)$ for $f, g \in A$;
- $p(\lambda) = |\lambda|$ for $\lambda \in k$.

Topology: the coarsest such that all maps $f \mapsto p(f)$ are continuous

Notation: p is viewed as a point of $\mathscr{M}(A)$, hence $p(f)$ is written $|f(p)|$.

Analytic spaces in the sense of Berkovich (continued)

Interests of Berkovich's theory:

- Good topology (locally contractible, locally compact);
- Interesting/fruitful interaction with real numbers;
- Possess both a topology and a Grothendieck topology.

Other theories:

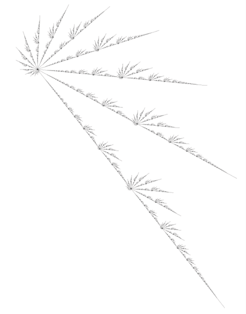
- Naïve: has not enough local compactness for our work;
- Tate: has not enough points;
- Raynaud: lacks a "visualization framework"
- Huber: has too many points.

Berkovich spaces: the affine line

Let's picture $\mathbf{A}_k^{1,\mathrm{an}} = \mathscr{M}(k[T])$ (k algebraically closed)

There are four sorts of points:

1. (rational points) $a \in k$,
   evaluation semi-norm $p_a : f \mapsto |f(a)|$;
2. (rational disks) $a \in k$, $r \in |k^\times|^{\mathbf{Q}}$,
   Gauss norm $p_{D(a,r)}$ on the disk $D(a,r)$;
3. (irrational disks) $a \in k$, $r \notin |k^\times|^{\mathbf{Q}}$,
   Gauss norm $p_{D(a,r)}$ on the disk $D(a,r)$;
4. (the rest) infimums of Gauss norms associated with decreasing chains of disks with "empty" intersection.



(from Hrushovski, Loeser, Poonen, *Ens. Math., 2014)*

Berkovich spaces: skeletons

Berkovich's framework fits very well with Raynaud's approach via formal models:

Formal schemes $\mathscr{X}$ over k° have a **generic fiber** $\mathscr{X}_\eta$.

There is a **specialization map**: $\mathrm{sp}\colon \mathscr{X}_\eta \to \mathscr{X}_s$.

If $\mathscr{X}$ is, say, semi-stable, then:

- Composants points of the special fiber $\mathscr{X}_s$ have a unique preimage in $\mathscr{X}_\eta$ ("**Gauss points**");
- More generally: there is a canonical embedding of the dual complex $S(\mathscr{X}_s)$ of $\mathscr{X}_s$ into $\mathscr{X}_\eta$ — the **skeleton**;
- There is a canonical **retraction deformation** from $\mathscr{X}_\eta$ onto this skeleton.

Real forms and currents in non-archimedean geometry

Moments and polyhedra

A **moment** on an analytic space X is a morphism $f = (f_1, \ldots, f_n) \colon X \to \mathbf{G}_{\mathrm{m}}^n$.

This gives rise to a **tropicalization**, a continuous map:

$$f_{\mathrm{trop}} \colon X \to \mathbf{R}^n, \qquad x \mapsto (\log(|f_1(x)|), \ldots, \log(|f_n(x)|)).$$

Proposition (Bieri–Groves, Kapranov...)

If X is compact, then $f_{\mathrm{trop}}(X)$ is a compact polyhedral subspace of $\mathbf{R}^n$, of dimension $\leqslant \dim(X)$.

In a non-Berkovich setup, one would have to consider the *closure* of this image.

Smooth functions

Definition

*A function φ on X is **(G-)smooth** if (G-)locally, there exist:*

- *A moment $f : X \to \mathbf{G}_{\mathrm{m}}^n$,*
- *A smooth function u on $\mathbf{R}^n$*

such that $\varphi = u \circ f_{\mathrm{trop}}$.

If X is top. separated and locally holomorphically separated:

- **Stone–Weierstrass**: $\mathscr{C}^\infty_{(c)}(X)$ is dense in $\mathscr{C}_{(c)}(X)$ for the compact open topology;
- If X is paracompact, then every open covering admits a **smooth partition of unity.**

Supercalculus

Analogies:

$\mathbf{R}^n$	$\mathbf{C}^n$
convex functions	plurisubharmonic functions
Monge–Ampère operator	
$\det \frac{\partial^2}{\partial x_i \partial x_j}$	$\det \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$
"supercalculus"	differential calculus wrt $dz, d\bar{z}$.

Berezin–Lagerberg supercalculus on $\mathbf{R}^n$:
take the tensor product of two copies of the De Rham complex to obtain a theory of (p, q)-forms.

Lagerberg differential forms

On an open subset U of $\mathbf{R}^n$, objects of the form

$$\sum_{\substack{I=(i_1<\cdots<i_p)\\ J=(j_1<\cdots<j_q)}} \alpha_{IJ}\, \mathrm{d}' x_{i_1} \wedge \cdots \wedge \mathrm{d}' x_{i_p} \otimes \mathrm{d}'' x_{j_1} \wedge \cdots \wedge \mathrm{d}'' x_{j_q}$$

where α_{IJ} are smooth functions on U.

Differential operators: d', d''

Anti-involution: $\mathrm{J}\,\mathrm{d}' x = \mathrm{d}'' x$, $\mathrm{J}\,\mathrm{d}'' x = -\,\mathrm{d}' x$.

(Super)differential forms on analytic spaces

The idea is simply to "pull-back" superforms by local tropicalizations.

Local objects $f^*\alpha$ in **tropical charts**: open $U \subset X$, moment $f : U \to \mathbf{G}_{\mathrm{m}}^n$, superform α around $f_{\mathrm{trop}}(U)$

Sheafification process: Sheaves $\mathscr{A}_X^{p,q}$ on X, for $0 \leqslant p, q \leqslant \dim(X)$.

Proposition

- *If $f_{\mathrm{trop}} = g_{\mathrm{trop}}$, then $f^*\alpha = g^*\alpha$.*
- *If $f^*\alpha = 0$ and U is compact, then $\alpha = 0$ on $f_{\mathrm{trop}}(U)$.*

Differential operators: $\mathrm{d}', \mathrm{d}''$; anti-involution J.

Integration

Let $d = \dim(X)$ and consider a local (d,d)-form $\omega = f^*\alpha$, where $f\colon V \to \mathbf{R}^n$, U compact.

To define $\int_V f^*\alpha$, integrate α on $f_{\text{trop}}(V)$.

The tropicalization $f_{\text{trop}}(V)$ is piecewise d-dimensional in $\mathbf{R}^n$.

If $n = d$, one could say

$$\int h\, \mathrm{d}' x_1 \wedge \mathrm{d}'' x_1 \wedge \cdots \wedge \mathrm{d}' x_n \wedge \mathrm{d}'' x_n = \int h \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

but this depends on the choice of coordinates.

One needs to define an additional structure on $f_{\text{trop}}(V)$: **calibration**. (In tropical geometry: **weights**.)

There is a **balancing condition.**

Currents

Currents are defined as the dual of compactly supported forms.

Bigraded sheaf.

Differential operators d', d''; antiinvolution J.

Functorial push-forward for compact morphisms.

Integration current δ_X.
More generally $u_*\delta_Y$ for compact $u\colon Y \to X$.

Boundary integration currents: $\delta_{\partial(X)}$, $u_*\delta_{\partial(Y)}$.

Stokes formula: $\int_X \mathrm{d}'\,\omega = \int_{\partial(X)} \omega$
More generally $u_*\delta_{\partial(Y)} = \mathrm{d}'\,u_*\delta_Y$.

Poincaré–Lelong formula

Functions $u\colon X \to \mathbf{R} \cup \{-\infty\}$ which are integrable on every compact skeleton define currents, e.g., continuous outside of a Zariski closed subset of empty interior.

Theorem ("Poincaré–Lelong")

For $f \in \mathscr{M}(X)^\times$, regular meromorphic function on X,

$$\mathrm{d}'\mathrm{d}''[\log|f|] = \delta_{\mathrm{div}(f)}.$$

$\mathrm{div}(f)$ is the divisor of f
the current $\delta_{\mathrm{div}(f)}$ is defined by additivity.

Another formula

In complex analysis, $\mathrm{d}'\mathrm{d}'' \log \sup(|z|, r)$ is the integration current on the circle of radius r.

Theorem

For $f : X \to \mathbf{A}^1$, $r \in \mathbf{R}$,

$$\mathrm{d}'\mathrm{d}'' \left[\sup \left(\log |f|, r \right) \right] = \delta_{f^{-1}(\eta_r)},$$

integration current over the $\mathscr{H}(\eta_r)$-analytic space $f^{-1}(\eta_r)$.

Here, η_r is the Gauss point corresponding to the disk $D(0, r)$.

Metrized line bundles

Metrized line bundles

Let $\overline{L} = (L, \|\cdot\|)$ be a line bundle on X with a smooth metric.

Curvature form, locally defined by $c_1(\overline{L}) = \mathrm{d}'\mathrm{d}'' \log \|s\|^{-1}$, for any local non-vanishing section s of L.

Proposition

Let $\mathscr{X}$ be a proper k-scheme, $\mathscr{L}$ a line bundle on $\mathscr{X}$. Take $(X, L) = (\mathscr{X}^{\mathrm{an}}, \mathscr{L}^{\mathrm{an}})$. Then

$$\int_X c_1(\overline{L})^n = (c_1(\mathscr{L})^n | \mathscr{X}).$$

Remark (Y. Liu): There are cycle classes in d''-cohomology, defined using the Gersten complex.

Formal metrics and their curvature

Let $\mathscr{X}$ be a proper formal k°-scheme (normal, say),
let $\mathscr{L}$ be a line bundle on $\mathscr{X}$
take $(X, L) = (\mathscr{X}_\eta, \mathscr{L}_\eta)$.

Then L has a natural "formal" metric which is continuous,
but not smooth in general, so that $c_1(\overline{L})$ is a $(1,1)$-current.

Formal metrics and their volume

Translating Bedford-Taylor theory from complex analysis to the current framework, we can consider **products** of these currents $c_1(\overline{L})$, using smooth approximations.

Theorem

Assume that k is endowed with a nontrivial discrete absolute value. Then

$$c_1(\overline{L})^n = \sum_{\xi \in X} m_\xi (c_1(\mathscr{L})^n | V_\xi) \delta_\xi .$$

Here, ξ runs over points of X which reduce to the generic point of a component V_ξ of the special fiber $\widetilde{\mathscr{X}}$, and m_ξ is its multiplicity.

Curvature forms of formal metrics (cont'd)

The currents $c_1(\overline{L})^p$ can be described in two different ways:

- By integration on suitable polyhedral subspaces of X — theory of **PL currents**, which enjoys similar properties to tropical intersection theory;
- On a large open subset (which carries all of its mass), as a sum of integration currents on fibers;

The positivity of this current can be characterized in terms of the numerical positivity of the cycle class $c_1(\mathscr{L}_s)^p$ on the special fiber $\mathscr{X}_s$.

Perspectives

- Non-archimedean Arakelov geometry (Gubler–Künnemann);
- Local heights via analytic geometry rather than formal models;
- Relation with Chow groups and K-theory (Liu, Mikami);
- Study of psh functions (Thuillier, Maculan, Jell, Wanner, Boucksom–Favre–Jonsson,...);
- Monge–Ampère problem (Kontsevich–Tschinkel, B-F-J, Jell–Martin–G–K...);
- Non-archimedean limits of archimedean integrals (Ducros, Hrushovski, Loeser).