Differential forms and currents on Berkovich spaces

Antoine Chambert-Loir (**Université de Paris**)
Joint work with Antoine Ducros (**Sorbonne Université**)
Institut de mathématiques de Jussieu–Paris-Rive-Gauche
arXiv:1204.6277

Valuation theory Seminar
Decidability, definability and computability in number theory
Internet, 2nd december, 2020

Introduction

Motivations:

- Non-archimedean aspects of Arakelov geometry
- Asymptotic aspects of archimedean geometry

Introduction

Motivations:

- Non-archimedean aspects of Arakelov geometry
  - Put non-archimedean places on the same footing as archimedean ones;
  - Phenomena of bad reductions, absence of good models;
  - Equidistribution theorems of points of small height, resp. in algebraic dynamics.
- Asymptotic aspects of archimedean geometry

Introduction

Motivations:

- Non-archimedean aspects of Arakelov geometry
  - Put non-archimedean places on the same footing as archimedean ones;
  - Phenomena of bad reductions, absence of good models;
  - Equidistribution theorems of points of small height, resp. in algebraic dynamics.
- Asymptotic aspects of archimedean geometry
  - Conjectures of Kontsevich–Soibelman;
  - Degenerations of archimedean dynamics towards non-archimedean dynamics;
  - Asymptotic expansions of integrals.

First step

We build a theory of **real valued differential forms** and **currents** on analytic spaces in the sense of Berkovich giving rise to:

- (p, q) forms for $p, q \leqslant n$ (dimension of space);
- integration of (n, n)-forms;
- by duality, currents;
- classical formulas, such as the Poincaré–Lelong formula;
- theory of metrized line bundles, their curvature forms,...

Further developments

- Study of the tropical Dolbeauly cohomology of Berkovich spaces, relation with Chow groups (Jell, Wanner; Liu, Mikami);
- Tropical intersection theory (Gubler, Künnemann);
- Asymptotic expansions of archimedean integrals (Ducros, Hrushovski, Loeser);
- Non-archimedean degenerations of archimedean algebraic dynamics (Boucksom, Favre, Jonsson)...

Contents

Contents

Analogies

$\mathbf{R}^n$	$\mathbf{C}^n$
convex functions	plurisubharmonic functions
Monge–Ampère operator	
$\det \frac{\partial^2}{\partial x_i \partial x_j}$	$\det \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$
“supercalculus”	differential calcul wrt $dz, d\bar{z}$.

Superforms (after Lagerberg)

Let U be an open subset of $\mathbf{R}^n$.

A **(super)form** of type (p, q) on U is an element of

$$\mathscr{A}^{p,q}(U) = \mathscr{C}^\infty(U) \otimes \bigwedge^p (\mathbf{R}^n)^* \otimes \bigwedge^q (\mathbf{R}^n)^*.$$

In coordinates:

$$\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ}(x)\, \mathrm{d}' x_{i_1} \wedge \cdots \wedge \mathrm{d}' x_{i_p} \otimes \mathrm{d}'' x_{j_1} \wedge \cdots \wedge \mathrm{d}'' x_{j_q}.$$

Bigraded algebra: $\mathscr{A}(U) = \bigoplus_{p,q} \mathscr{A}^{p,q}(U)$, exterior product $\wedge$

Involution J defined by $\mathrm{J}\,\mathrm{d}' x = \mathrm{d}'' x, \quad \mathrm{J}\,\mathrm{d}'' x = -\mathrm{d}' x$.

Notion of **symmetric form**: $\mathrm{J}\alpha = \alpha$.

Differential operators:

$$\mathrm{d}'\colon \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p+1,q}(U), \qquad \mathrm{d}''\colon \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p,q+1}(U).$$

Examples: for $f \in \mathscr{A}^{0,0}(U) = \mathscr{C}^\infty(U)$,

$$\mathrm{d}'\mathrm{d}'' f = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\, \mathrm{d}' x_i \otimes \mathrm{d}'' x_j$$

$$(\mathrm{d}'\mathrm{d}'' f)^n = n!\,\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \mathrm{d}' x_1 \wedge \mathrm{d}'' x_1 \wedge \cdots \wedge \mathrm{d}' x_n \wedge \mathrm{d}'' x_n.$$

Supercurrents

Integral: For $\alpha = f\, \mathrm{d}' x_1 \wedge \mathrm{d}'' x_1 \wedge \cdots \wedge \mathrm{d}' x_n \wedge \mathrm{d}'' x_n \in \mathscr{A}^{n,n}(U)$, set

$$\int_U \alpha = \int_U f(x) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

— *depends* on the choice of affine coordinates.

Currents:

- **currents** = (continuous) linear forms on superforms;
- **differential calculus** defined by duality;
- (p, q)-forms define $(n - p, n - q)$-currents;
- positive forms, positive currents and their products (à la Bedford–Taylor), and **formulas**, such as:

$$(\mathrm{d}'\mathrm{d}'' \log \max(0, x_1, \ldots, x_n))^n = \delta_0.$$

Contents

Analytic spaces in the sense of Berkovich

Let k be a field, complete for a non-archimedean absolute value.

With a k-algebra A (either of finite type, or affinoid), Berkovich associates an analytic **spectrum** $\mathscr{M}(A)$:

$\mathscr{M}(A)$ is a set of multiplicative seminorms p on A, functions $p\colon A \to \mathbf{R}_+$ such that

- $p(f+g) \leqslant p(f) + p(g)$ for $f, g \in A$;
- $p(fg) = p(f)p(g)$ for $f, g \in A$;
- $p(\lambda) = |\lambda|$ for $\lambda \in k$.

Topology: the coarsest such that all maps $f \mapsto p(f)$ are continuous

Notation: p is viewed as a point of $\mathscr{M}(A)$, hence $p(f)$ is written $|f(p)|$.

Analytic spaces in the sense of Berkovich (continued)

Interests of Berkovich's theory:

- Good topology (locally contractible, locally compact);
- Interesting/fruitful interaction with real numbers;
- Possess both a topology and a Grothendieck topology.

Analytic spaces in the sense of Berkovich (continued)

Interests of Berkovich's theory:

- Good topology (locally contractible, locally compact);
- Interesting/fruitful interaction with real numbers;
- Possess both a topology and a Grothendieck topology.

Other theories:

- Naïve: has not enough local compactness for our work;
- Tate: has not enough points;
- Raynaud: lacks a "visualization framework"
- Huber: has too many points.

Tropicalizations

Let k be a field, complete for a non-archimedean absolute value.

Torus: $\mathbf{G}_{\mathrm{m}} = \mathscr{M}(k[T, T^{-1}])$;

tropicalization: continuous map $\mathbf{G}_{\mathrm{m}} \to \mathbf{R}$, $x \mapsto \log |T(x)|$

Tropicalizations

Let k be a field, complete for a non-archimedean absolute value.

Torus: $\mathbf{G}_\mathrm{m} = \mathscr{M}(k[T, T^{-1}])$;

tropicalization: continuous map $\mathbf{G}_\mathrm{m} \to \mathbf{R}$, $x \mapsto \log |T(x)|$

Definition

*Let X be a k-analytic space. A **moment** on X is a morphism $f : X \to \mathbf{G}_\mathrm{m}^d$.*

Tropicalization: $f_\mathrm{trop} : X \to \mathbf{R}^d$.

If X is compact, then $f_\mathrm{trop}(X)$ is a compact polyhedral subspace of $\mathbf{R}^d$.

Smooth functions

Let X be a k-analytic space.

Definition

A function φ on X is **(G-)smooth** *if (G-)locally, there exist*
– a moment $f: X \to \mathbf{G}_{\mathrm{m}}^d$,
– a smooth function u on an open neighborhood of $f_{\mathrm{trop}}(X)$ in $\mathbf{R}^d$
such that $\varphi = u \circ f_{\mathrm{trop}}$.

If X is locally holomorphically separated, topologically separated, then there are plenty of smooth functions:

1. Stone–Weierstrass: $\mathscr{C}^\infty_{(\mathrm{c})}(X)$ is dense in $\mathscr{C}_{(\mathrm{c})}(X)$ for the compact open topology;
2. If X is paracompact, then every open covering admits a smooth partition of unity.

Differential forms

Formal construction of a sheaf $\mathscr{A}_X^{p,q}$ and a G-sheaf $\mathscr{A}_{X_G}^{p,q}$ from **tropical charts**: from

$$X \supset U \xrightarrow{f} \mathbf{G}_{\mathrm{m}}^d, \quad P \supset f_{\mathrm{trop}}(U), \quad \alpha \in \mathscr{A}^{p,q}(P)$$

get $f^*\alpha$.

Lemma

$f_{\mathrm{trop}} = g_{\mathrm{trop}}$ *implies* $f^*\alpha = g^*\alpha$.

More or less formally, one obtains a $\mathrm{d}', \mathrm{d}''$ differential calculus, a notion of currents...

Theorem (Jell)

$(\mathscr{A}_X^{0,*}, \mathrm{d}'')$ *is a resolution of* $\mathbf{R}_X$.

Contents

Skeletons

Gauss point $\eta_r \in \mathbf{G}_{\mathrm{m}}^d$, associated with $r = (r_1, \ldots, r_n) \in \mathbf{R}^d$:

it is the (semi)norm given by $\sum a_m T^m \mapsto \sup_m |a_m| \, e^{mr}$

Skeleton of $\mathbf{G}_{\mathrm{m}}^n$: $S(\mathbf{G}_{\mathrm{m}}^d) = \{\eta_r \, ; \, r \in \mathbf{R}^d\} \simeq \mathbf{R}^d$ (as a top. space)

Skeletons

Gauss point $\eta_r \in \mathbf{G}_\mathrm{m}^d$, associated with $r = (r_1, \ldots, r_n) \in \mathbf{R}^d$:

it is the (semi)norm given by $\sum a_m T^m \mapsto \sup_m |a_m| e^{mr}$

Skeleton of $\mathbf{G}_\mathrm{m}^n$: $S(\mathbf{G}_\mathrm{m}^d) = \{\eta_r \,;\, r \in \mathbf{R}^d\} \simeq \mathbf{R}^d$ (as a top. space)

Let $f \colon X \to \mathbf{G}_\mathrm{m}^d$ be a moment, $n = \dim(X)$ (X equidimensional)

Characteristic polyedron of f, $\Sigma_f = \bigcup_{p \colon \mathbf{G}_\mathrm{m}^d \to \mathbf{G}_\mathrm{m}^n} (p \circ f)^{-1}(S(\mathbf{G}_\mathrm{m}^n))$

It has a canonical polyhedral structure (Ducros).

$f_{\mathrm{trop}} \colon \Sigma_f \to \mathbf{R}^d$ is a polyhedral immersion.

Skeletons

Gauss point $\eta_r \in \mathbf{G}_{\mathrm{m}}^d$, associated with $r = (r_1, \ldots, r_n) \in \mathbf{R}^d$:

it is the (semi)norm given by $\sum a_m T^m \mapsto \sup_m |a_m| e^{mr}$

Skeleton of $\mathbf{G}_{\mathrm{m}}^n$: $S(\mathbf{G}_{\mathrm{m}}^d) = \{\eta_r \,;\, r \in \mathbf{R}^d\} \simeq \mathbf{R}^d$ (as a top. space)

Let $f\colon X \to \mathbf{G}_{\mathrm{m}}^d$ be a moment, $n = \dim(X)$ (X equidimensional)

Characteristic polyedron of f, $\Sigma_f = \displaystyle\bigcup_{p\colon \mathbf{G}_{\mathrm{m}}^d \to \mathbf{G}_{\mathrm{m}}^n} (p \circ f)^{-1}(S(\mathbf{G}_{\mathrm{m}}^n))$

It has a canonical polyhedral structure (Ducros).

$f_{\mathrm{trop}}\colon \Sigma_f \to \mathbf{R}^d$ is a polyhedral immersion.

$$x \in \Sigma_f \iff \mathrm{trop.\ dim}_x(f) = n \qquad \text{(tropical dim. of } f \text{ at } x\text{)}$$

Skeletons

Gauss point $\eta_r \in \mathbf{G}_m^d$, associated with $r = (r_1, \ldots, r_n) \in \mathbf{R}^d$:
it is the (semi)norm given by $\sum a_m T^m \mapsto \sup_m |a_m| e^{mr}$

Skeleton of $\mathbf{G}_m^n$: $S(\mathbf{G}_m^d) = \{\eta_r \,;\, r \in \mathbf{R}^d\} \simeq \mathbf{R}^d$ (as a top. space)

Let $f \colon X \to \mathbf{G}_m^d$ be a moment, $n = \dim(X)$ (X equidimensional)

Characteristic polyedron of f, $\Sigma_f = \bigcup_{p \colon \mathbf{G}_m^d \to \mathbf{G}_m^n} (p \circ f)^{-1}(S(\mathbf{G}_m^n))$

It has a canonical polyhedral structure (Ducros).

$f_{\text{trop}} \colon \Sigma_f \to \mathbf{R}^d$ is a polyhedral immersion.

$x \in \Sigma_f \Longleftarrow \text{trop. } \dim_x(f) = n$ (tropical dim. of f at x)

(converse $\Longrightarrow$ holds if $x \notin \partial(X)$)

Support of (n, n)-forms

Let $\alpha \in \mathscr{A}^{n,n}(X)$.

Locally, the support of α is contained in a polyhedral subspace of X, built from local skeletons.

One wants to integrate α on this polyhedral subspace, cell by cell.

Support of (n, n)-forms

Let $\alpha \in \mathscr{A}^{n,n}(X)$.

Locally, the support of α is contained in a polyhedral subspace of X, built from local skeletons.

One wants to integrate α on this polyhedral subspace, cell by cell.

Difficulty: Lagerberg's definition of the integral of an (n, n)-superform on $\mathbf{R}^n$ depends on the choice of a basis.

Support of (n, n)-forms

Let $\alpha \in \mathscr{A}^{n,n}(X)$.

Locally, the support of α is contained in a polyhedral subspace of X, built from local skeletons.

One wants to integrate α on this polyhedral subspace, cell by cell.

Difficulty: Lagerberg's definition of the integral of an (n, n)-superform on $\mathbf{R}^n$ depends on the choice of a basis.

Solution: define "calibrations" of the cells.

Cellular decomposition

$f\colon X \to \mathbf{G}_{\mathrm{m}}^{d}$, Σ_f

$\mathscr{C}$, convenient cellular decomposition of Σ_f:

- the intersection of two cells, the boundary of a cell, are unions of cells;
- for every cell C, $f_{\mathrm{trop}}\colon C \xrightarrow{\sim} f_{\mathrm{trop}}(C)$, convex polyhedron of $\mathbf{R}^d$;
- $\partial(X)$ does not meet open n-cells;
- each open n-cell is open in Σ_f.

Basic diagram for an n-cell C:

$$\begin{array}{ccccc} C & \hookrightarrow & X & \xrightarrow{f} & \mathbf{G}_{\mathrm{m}}^{d} \\ & & & \searrow^{p\circ f} & \big\downarrow p \\ & & & & \mathbf{G}_{\mathrm{m}}^{n} \end{array} \qquad \begin{array}{ccc} f_{\mathrm{trop}}(C) & \hookrightarrow & \mathbf{R}^{d} \\ \big\downarrow \simeq & & p_{\mathrm{trop}} \big\downarrow \big\uparrow \sigma_p \\ (p\circ f)_{\mathrm{trop}}(C) & \hookrightarrow & \mathbf{R}^{n} \end{array}$$

where σ_p is the unique affine section of $p \circ f$ with image $\langle f_{\mathrm{trop}}(C)\rangle$.

Calibration

$$\begin{array}{ccccc} C & \hookrightarrow & X & \xrightarrow{f} & \mathbf{G}_{\mathrm{m}}^{d} \\ & & & \underset{p\circ f}{\searrow} & \downarrow p \\ & & & & \mathbf{G}_{\mathrm{m}}^{n} \end{array} \qquad \begin{array}{ccc} f_{\mathrm{trop}}(C) & \hookrightarrow & \mathbf{R}^{d} \\ \downarrow \simeq & & p_{\mathrm{trop}} \downarrow \uparrow \sigma_p \\ (p\circ f)_{\mathrm{trop}}(C) & \hookrightarrow & \mathbf{R}^{n} \end{array}$$

Theorem

- *$p \circ f$ is finite and flat of some degree $d_C(p \circ f)$ at each point of $\mathring{C}$.*
- *Up to sign, the n-vector $d_C(p \circ f) \cdot \sigma_{p,*}(e_1 \wedge \cdots \wedge e_n) \in \bigwedge^n \langle f_{\mathrm{trop}}(C) \rangle$ does not depend on the choice of p.*

$\rightarrow$ canonical **calibration** of the cell C.

Calibration and integration

The canonical calibration of a cell C allows to integrate:

- any (n, n)-form on C
- any $(n-1, n)$-form on $\partial(C)$ (sign convention: outer normal)

By summing the contributions of n-cells of a convenient cellular decomposition, one gets:

- an integration map, $\mathscr{A}_c^{n,n}(X) \to \mathbf{R}$, $\omega \mapsto \int_X \omega = \sum_C \int_C \omega$
- a boundary integration map, $\mathscr{A}_c^{n-1,n}(X) \to \mathbf{R}$, $\omega \mapsto \int_{\partial(X)} \omega = \sum_C \int_{\partial(C)} \omega$

Integration

Theorem

The support of boundary integration is contained in $\partial(X)$.

More or less equivalent to the balancing condition in **tropical geometry.**

Theorem (Stokes formula)

For $\omega \in \mathscr{A}_c^{n-1,n}(X)$, one has $\displaystyle\int_X \mathrm{d}' \omega = \int_{\partial(X)} \omega$.

Contents

Definition

Let X be equidimensional, $\dim(X) = n$, without boundary

As in differential geometry, **currents** are defined as (continuous) linear forms on differential forms:

$$\mathscr{D}^{p,q}(X) = (\mathscr{A}_c^{p,q}(X))^* = \mathscr{D}_{n-p,n-q}(X).$$

$\mathrm{d}', \mathrm{d}''$-differential calculus, involution J — by duality (with a sign)

$\alpha \in \mathscr{A}(X)$, $T \in \mathscr{D}(X)$, $T \wedge \alpha \colon \omega \mapsto \langle T, \alpha \wedge \omega \rangle$.

Sheaf property

Integration currents

Integration currents: $\varphi\colon Y \to X$ topologically proper map, $\dim(Y) = m$,

$$\varphi_* \delta_Y \left(: \alpha \mapsto \int_Y \varphi^* \alpha \right) \in \mathscr{D}^{m,m}(X)$$

$$\varphi_* \delta_{\partial(Y)} \left(: \alpha \mapsto \int_{\partial(Y)} \varphi^* \alpha \right) \in \mathscr{D}^{m-1,m}(X)$$

Poincaré–Lelong equation

Functions $u\colon X \to \mathbf{R} \cup \{-\infty\}$ which are integrable on every compact skeleton define currents, e.g., continuous outside of a Zariski closed subset of empty interior.

Theorem

For $f \in \mathscr{M}(X)^\times$, regular meromorphic function on X,

$$\mathrm{d}'\mathrm{d}''[\log|f|] = \delta_{\mathrm{div}(f)}.$$

In complex analysis, $\mathrm{d}'\mathrm{d}'' \log \sup(|z|, r)$ is the integration current on the circle of radius r.

Theorem

For $f : X \to \mathbf{A}^1$, $r \in \mathbf{R}$,

$$\mathrm{d}'\mathrm{d}'' \left[\sup \left(\log |f|, r \right) \right] = \delta_{f^{-1}(\eta_r)},$$

integration current over the $\mathscr{H}(\eta_r)$-analytic space $f^{-1}(\eta_r)$.

Metrized line bundles

Let $\overline{L} = (L, \|\cdot\|)$ be a line bundle on X with a smooth metric.

Curvature form, locally defined by $c_1(\overline{L}) = \mathrm{d}'\mathrm{d}'' \log \|s\|^{-1}$, for any local non-vanishing section s of L.

Proposition

Let $\mathscr{X}$ be a proper k-scheme, $\mathscr{L}$ a line bundle on $\mathscr{X}$; take $(X, L) = (\mathscr{X}^{\mathrm{an}}, \mathscr{L}^{\mathrm{an}})$. Then

$$\int_X c_1(\overline{L})^n = (c_1(\mathscr{L})^n | \mathscr{X}).$$

Remark (Y. Liu): There are cycle classes in d''-cohomology, defined using the Gersten complex.

Formal metrics and their curvature

Let $\mathscr{X}$ be a proper formal k°-scheme (normal, say),

let $\mathscr{L}$ be a line bundle on $\mathscr{X}$

take $(X, L) = (\mathscr{X}_\eta, \mathscr{L}_\eta)$.

Then L has a natural "formal" metric which is continuous,

but not smooth in general, so that $c_1(\overline{L})$ is a $(1,1)$-current.

Formal metrics and their volume

Translating Bedford-Taylor theory from complex analysis to the current framework, we can consider **products** of these currents $c_1(\overline{L})$, using smooth approximations.

Theorem

Assume that k is endowed with a nontrivial discrete absolute value. Then

$$c_1(\overline{L})^n = \sum_{\xi \in X} m_\xi (c_1(\mathscr{L})^n | V_\xi) \delta_\xi .$$

Here, ξ runs over points of X which reduce to the generic point of a component V_ξ of the special fiber $\widetilde{\mathscr{X}}$, and m_ξ is its multiplicity.

Curvature forms of formal metrics (cont'd)

The currents $c_1(\overline{L})^p$ can be described in two different ways:

- By integration on suitable polyhedral subspaces of X — theory of **PL currents**, which enjoys similar properties to tropical intersection theory;
- On a large open subset (which carries all of its mass), as a sum of integration currents on fibers;

Perspectives

- Non-archimedean Arakelov geometry (Gubler–Künnemann);
- Local heights via analytic geometry rather than formal models;
- Relation with Chow groups and K-theory (Liu, Mikami);
- Study of psh functions (Thuillier, Maculan, Jell, Wanner, Boucksom–Favre–Jonsson,...);
- Monge–Ampère problem (Kontsevich–Tschinkel, B-F-J, Jell–Martin–G–K...);
- Non-archimedean limits of archimedean integrals (Ducros, Hrushovski, Loeser).