Flatness in non-Archimedean analytic geometry

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Abstract. This text is devoted to the systematic study of flatness in the context of Berkovich analytic spaces. After having shown through a counter-example that naive flatness in that context is not stable under base change, we introduce the notion of *universal* flatness and we study a first important class of universally flat morphisms, that of quasi-smooth ones.

We then show the existence of local *dévissages* (in the spirit of Raynaud and Gruson) for coherent sheaves, which we use, together with a study of the local rings of 'generic fibers' of morphisms, to prove that a flat, boundaryless morphism is universally flat.

After that we prove that the image of a compact analytic space by a universally flat morphism can be covered by a compact, relatively Cohen-Macaulay and zero-dimensional multisection, and the image of the latter is shown to be a compact analytic domain of the target. It follows that the image of a compact analytic space by a universally flat morphism is a compact analytic domain of the target. This was first proved in the rigid-analytic context by Raynaud, but our proof is completely different: it is based upon Temkin's theory of the reduction of analytic germs and quantifiers elimination in the theory of non-trivially valued algebraically closed fields, and it uses neither formal models nor flattening techniques.

We end the paper by showing, using Kiehl's method, Zariski-openness of the universal flatness and of the quasi-smoothness loci of a given morphism.

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Contents

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Introduction

In scheme theory, flatness was used in a significant way for the first time by Serre in his celebrated paper GAGA ([23]), where the following plays a crucial role: if X is a complex algebraic variety and if X^{an} denotes the corresponding analytic space, then for every $x \in X(\mathbb{C})$ the ring $\mathscr{O}_{X^{an},x}$ is flat over $\mathscr{O}_{X,x}$. After that, flatness became in a few years, under the influence of Grothendieck and his school, a central concept in algebraic geometry, for at least two reasons.

• A geometrical one: it has turned out that the notion of a flat morphism is exactly the right one to translate in rigorous language the (intuitive, but quite vague) idea of a *family*.

• A technical one: (faithfully) flat morphisms have very good descent properties.

What we do in this paper is starting a systematic investigation of flatness in *non-Archimedean analytic geometry*, and more precisely in the *Berkovich* setting ([1], [2]). We have chosen to give 'purely Berkovich' proofs, without using formal or algebraic models, and the related subtle, sophisticated results (flattening, semi-stable reduction, reduced fiber theorem...). Therefore we sometimes give new proofs of facts which were originally established using some of those highly non-trivial theorems about models (see below some examples of such facts in the summary of our results). The only 'reduction arguments' we will use are based upon Temkin's theory of the reduction of analytic germs, which involves general valuation theory, and especially Riemann-Zariski spaces.

Flatness in rigid geometry

In fact, flatness has already been considered in non-Archimedean analytic geometry, but in the *rigid-analytic* context ([7] and [8]), where its definition is as simple as one may hope: a morphism $Y \to X$ between rigid spaces is rig-flat at a point $y \in Y$ if it is flat as a morphism of locally ringed spaces, that is to say, if $\mathscr{O}_{Y,y}$ is a flat $\mathscr{O}_{X,x}$-module. Flatness in that sense behaves well – it is stable under base change and ground field extension. But contrary to what happens in scheme theory, this is by no way obvious, because base change and ground field extension are defined using *complete* tensor products. Roughly speaking, the proofs proceed as follows:

• the study of *rigid* flatness is reduced to that of *formal* flatness, thanks to formal avatars of Raynaud-Gruson flattening techniques, which are used to build a flat formal model of any given rig-flat morphism;

• the study of *formal* flatness is reduced to that of *algebraic* flatness, in a more standard way (dividing by various ideals of definition and using flatness criteria in the spirit of [18], §5).

Let us mention that this general strategy (formal flattening and reduction modulo an ideal of definition to replace an analytic problem with an algebraic one) was also used by Raynaud to prove the following fact: if $\varphi : Y \to X$ is a flat morphism between affinoid rigid spaces, then $\varphi(Y)$ is a finite union of affinoid domains of X (*cf.* [8], cor. 5.11).

Flatness in Berkovich geometry: the first problems

We fix from now on and for the whole introduction a complete, non-Archimedean field k, and we will only consider *Berkovich* analytic spaces. An analytic space without any mention of the ground field will be an analytic space defined over *any* complete extension of k, and stability under base change will always implicitly refer to base change by any analytic space with this convention – hence stability under ground field extension will be a particular case of stability under base change.

Let φ be a morphism of *good* k-analytic spaces (the class of good analytic spaces, which contains affinoid spaces, analytifications of algebraic varieties, and generic fibers of affine and proper formal schemes, is, to the knowledge of the author, the only one in which the local rings are relevant). Let us say that φ is flat at a point of its source if it is flat at that point as a morphism of locally ringed spaces, exactly like in the rigid setting.

We immediately face a big problem. Indeed, in this context, the use of complete tensor products does not just make proofs of stability under base change more complicated, as it does in the rigid case: stability under base change is actually *wrong*.

Let us give a counter-example. Roughly speaking, it is due to a boundary phenomenon: it consists in the embedding into the affine plane of a curve which is drawn on some bi-disc and can not be extended; the problem occurs at the unique boundary point of the curve. We are now going describe it more precisely; the reader will find detailed proofs of what follows in section 2(2.19-2.23). Let $r > 0$, let $f = \sum a_i T^i \in k[[T]]$ be a power series whose radius of convergence is r, and let Y be the closed one-dimensional k-disc of radius r. The Shilov boundary of Y consists in one point y (the one that corresponds to the semi-norm $\sum b_i T^i \mapsto \max |b_i| r^i$). Denote by φ the morphism $(\mathsf{Id}, f) : Y \to \mathbb{A}_k^{2,\mathrm{an}}$ and by X the closed analytic domain of $\mathbb{A}_k^{2,\mathrm{an}}$ defined by the inequality $|T_1| \leqslant r$; note that $\varphi(Y) \subset X$; more precisely, $\varphi(Y)$ is the Zariski-closed subset of X defined by the equation $T_2 = f(T_1)$.

One can show that $\mathscr{O}_{\mathbb{A}_k^{2,\mathrm{an}},\varphi(y)}$ is a field: this is due to the fact that $\varphi(Y)$ can not be extended to a curve defined *around* $\varphi(y)$, because the radius of convergence of f is exactly r. As a consequence, φ is flat at the point y. Now:

• $Y = \varphi^{-1}(X) \to X$ is a closed immersion of a one-dimensional space in a purely two-dimensional space, hence is not flat at y;

• if L is any complete extension of k such that Y_L has a L-rational point y' lying above y, then $\varphi(y')$ belongs to the topological interior of X_L in $\mathbb{A}_L^{2,\mathrm{an}}$ (because $\varphi(y')$ is a rigid point); therefore $\varphi_L : Y_L \to \mathbb{A}_L^{2,\mathrm{an}}$ is, around y', a closed immersion of a one-dimensional space in a purely two-dimensional space, hence is not flat at y'.

Some comments. The above counter-example is, in some sense, archetypal, because boundary phenomena are actually the only obstruction for flatness to be stable under base change: if $Y \to X$ is a morphism between good k-analytic spaces and if $y \in Y$ is such that $Y \to X$ is inner at y (or, more generally, extendable – see 5.11), then if $Y \to X$ is flat at y, it remains so after any (good) base change (th. 5.12.3). This had already been proved by Berkovich (in a completely different way) in some non-published notes about flatness.

It is then clear why such problems can not occur in the rigid setting: this is because boundary points are *never* rigid.

The notion of universal flatness. Because of the aforementioned problems, the right notion to consider, for a morphism between two good k-analytic spaces, seems to be that of *universal* flatness at a point (2.7) – one simply *requires* stability under base change.

Description of our results

We are now going to describe with some details the content of every section (except section 0, devoted to remindings about analytic geometry and Temkin's theory); but for the sake of simplicity, we will sometimes present here only a simplified version of our results – essentially by weakening the assumptions to focus on a significant particular case (*e.g.* the space is compact, or the valuation is non-trivial, etc.). Before beginning the description, let us fix some notations.

We fix a subgroup Γ of $\mathbb{R}_+^*$ such that $|k^*|.\Gamma \neq \{1\}$; there is a natural notion of a Γ-*strict* k-analytic space (0.23 *et sq.*), which will often appear in our claims, in the assumptions as well as in the conclusions; we have introduced this notion because it doesn't cause any complication in the proofs, and helps keeping track of the parameters involved in the definitions of our objects.

If $Y \to X$ is a morphism between k-analytic spaces and if $x \in X$, the fiber of $Y \to X$ at x will be denoted by Y_x.

If L/K is an extension of graded fields (that is, of $\mathbb{R}_+^*$-graded fields ; see 0.1 for more explanations) we denote by $\mathbb{P}_{L/K}$ the space of graded valuations on L which are trivial on K. For the basic definitions, properties and notations of those objects, see 0.2 and 0.5; let us simply mention here that if Δ is a subgroup of $\mathbb{R}_+^*$, a quasi-compact open subset of $\mathbb{P}_{L/K}$ is said to be Δ-*strict* if it can be defined using only homogeneous elements whose degree belongs to Δ.

Section 1. In this section one establishes the following result (th. 1.8): let K be a graded field, let F be a graded extension of K, let L be a graded extension of F, let Δ be a subgroup of $\mathbb{R}_+^*$ and let V be a Δ-strict quasi-compact open subset of $\mathbb{P}_{L/K}$; the image of V on $\mathbb{P}_{F/K}$ is a Δ-strict quasi-compact open subset of the latter.

The proof consists in reducing to the case where the fields involved are trivially graded (and where Δ is consequently trivial), and where V is affine, say $\mathsf{V} = \mathbb{P}_{L/K}\{f_1, \ldots, f_n\}$. This particular situation is covered by th. 1.4; the latter not only tells that the image of V is a quasi-compact open subset of $\mathbb{P}_{F/K}$, it ensures the existence of finitely many closed points $y_1, \ldots, y_m$ on $\mathsf{Spec}\ F[f_1, \ldots, f_n]$ so that the image of V is the union of the images of the $\mathbb{P}_{\kappa(y_j)/K}\{f_1(y_j), \ldots, f_n(y_j)\}$'s on $\mathbb{P}_{F/K}$; the latter fact will play a crucial role in the proof of th. 6.2.

The proof of thm. 1.4 itself is based upon two results: quantifiers elimination for non-trivially valued algebraically closed fields, on one hand (*cf.* [21], 4.17, or [10], th. 2.14); and an *explicit* description of the image of V in the case where L is algebraic over F on the other one (lemmas 1.1 and 1.2, prop. 1.3).

After having proved those purely valuation-theoretic results, we come to analytic geometry. We use th. 1.8 together with Temkin's theory to prove that if $(Y, y) \to (X, x)$ is a morphism of k-analytic germs, then there exists a smallest analytic domain (Z, x) of (X, x) through which $(Y, y) \to (X, x)$ goes; if moreover (Y, y) is Γ-strict and (X, x) is separated, the germ (X, x) is Γ-strict (1.9.2).

This section ends with a geometrical interpretation of the Krull dimension of $\mathscr{O}_{X,x}$ for (X, x) a good k-analytic germ (cor. 1.12).

Section 2. In this section, we first generalize somme well-known GAGA-type results about flatness (prop. 2.5 and th. 2.6; see also comments at 2.3 *et sq.*). Then we come to the definition of universal flatness in the good case (2.7), give some basic examples and give a proof of some (quite classical) facts about locally finite, flat morphisms (prop. 2.16). After that, we study in full detail the aforementioned counter-example showing that flatness is not automatically universal.

Then we come to the definition of universal flatness in the general, that is, non necessarily good, case; this definition uses good analytic domains and the definition in the good case, but not the local rings of non-good spaces, about which almost nothing is known.

At the end of the section we prove that as far as exact sequences are concerned, universally flat coherent sheaves behave as expected, that is, like flat coherent sheaves in algebraic geometry (lemma 2.29, prop. 2.30, and prop. 2.31); though proofs are quite straightforward, they require a little bit more arguments than the classical ones.

Section 3. The goal of that section is to introduce a fundamental class of universally flat morphisms, namely, that of *quasi-smooth* morphisms (def. 3.5); they are the Berkovich counterpart of the so-called *rig-smooth* morphisms of rigid geometry; we have chosen to call them *quasi-smooth* to be coherent with the terminology *quasi-étale* which was introduced by Berkovich in [3]. The theorems we prove here are by no way surprising. They are certainly well known for most of them, at least in the rigid setting; but for the sake of completeness, and also to handle the non-strict case as well as what happens at a non-rigid point of a strictly k-analytic space, we have chosen to write all the proofs.

The most important result we establish is the following: if $Y \to X$ is a morphism between k-analytic spaces, if $y \in Y$ and if x is its image on X, then Y is quasi-smooth over X at y if and only if Y is universally X-flat at y and Y_x is geometrically regular at y (prop. 3.14). We then show that if $Y \to X$ is quasi-smooth at y, the coherent sheaf $\Omega_{Y/X}$ is free at y of rank equal to the relative dimension of Y over X at y (prop. 3.16; this fact is actually *equivalent* to quasi-smoothness at y when $Y = \mathscr{M}(k)$ by cor. 3.15 and 3.2.2).

After that we come to the links between quasi-smoothness and smoothness. We prove that if Y and X are good, then $Y \to X$ is quasi-smooth at y if and only if y has an affinoid neighborhood in Y which is X-isomorphic to an affinoid domain of a smooth X-space (prop. 3.18); it easily follows, *always under our goodness assumption*, that Y is smooth at y if and only if it is quasi-smooth and inner at y. Surprisingly, we don't know whether this is true without goodness assumption (see 3.21); if it were not, it would mean that Berkovich's definition of smoothness is not G-local on the target, contrary to boundaryless quasi-smoothness.

At the end of the section we prove that if Y and X are good and if Y is quasi-smooth over X at y then $\mathsf{Spec}\ \mathscr{O}_{Y,y} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ is flat with geometrically regular fibers (prop. 3.23); it follows that the usual commutative algebra properties are preserved by quasi-smooth morphisms (cor. 3.24).

Section 4. Let us begin with a remark. Let $Y \to X$ is a morphism between noetherian *schemes*, let $y \in Y$ and let x denote its image on X. If $\mathscr{O}_{X,x}$ is a field, x is the generic point of an irreducible component of X, which implies that $\mathscr{O}_{Y_x,y}$ is equal to $\mathscr{O}_{Y,y}$.

The goal of that section is to prove a Berkovich avatar of this result (when the local rings are relevant, that is, when Y and X are good). So, let $Y \to X$ be a morphism of good k-analytic spaces, let $y \in Y$ and let x be its image on X; assume that $\mathscr{O}_{X,x}$ is a field. One can not expect $\mathsf{Spec}\ \mathscr{O}_{Y_x,y} \to \mathsf{Spec}\ \mathscr{O}_{Y,y}$ to be an isomorphism in general, because the definition of Y_x involves two completion operations, one for defining the complete residue field $\mathscr{H}(x)$, and another one for defining the fiber itself (through complete tensor products with $\mathscr{H}(x)$).

But this maps has nevertheless nice properties, *as soon as y belongs to* $\mathsf{Int}\ Y/X$: it is then flat with CI (that is, complete intersection) fibers, and even with regular fibers if char. $k = 0$ (th. 4.15).

A first counter-example (4.16.1 – it again involves the non-extendable curve already considered in section 2 to provide a counter-example to universality of flatness) shows that the innerness assumption can not be removed. A second one (4.16.2), due to Temkin, shows that one can not expect in general the fibers to be regular in positive characteristic, even when $Y \to X$ is finite and flat.

First remark. For proving our theorem 4.15 on 'generic fibers' we need a preliminary result on smooth morphisms which also has, to our opinion, its own interest (th. 4.9). Though it could be proved using the decomposition (after a suitable base change) of a smooth morphism into a sequence of smooth relative curves of a certain kind ([2], §3.7), we have chosen another method, which we think is simpler. Indeed, the existence of the aforementioned decomposition follows from the (highly non-trivial) semi-stable reduction for curves, but we proceed using only elementary arguments: the explicit description of some open subsets of the relative affine space over a given space, and the fact that an étale morphisms of k-analytic germs $(T,t) \to (Z,z)$ inducing an isomorphism $\mathscr{H}(t) \simeq \mathscr{H}(z)$ is itself an isomorphism.

Let us say a few words about this result (th. 4.9) on smooth morphisms. In complex analytic geometry, a morphism is smooth if and only if it is locally on the source and the target a product by an open polydisc. This is definitely false in Berkovich's setting (in any positive genus projective curve over an algebraically closed non-Archimedean field, those are points which don't admit any neighborhood isomorphic to an open disc), and th. 4.9 provides, in some sense, the best 'approximation' of such a result one can expect in our context. In the case where $|k^*| \neq \{1\}$, it says the following (the trivially valued case is a little bit more complicated): if $Y \to X$ is a smooth morphism between good k-analytic spaces and if x belongs to the image of Y, there exists an étale morphism $X' \to X$ whose image contains x, and an open subset of $Y \times_X X'$

which is isomorphic the product of X' and an open polydisc. This implies the openness of $Y \to X$ and the fact that $Y \to X$ has locally sections on its image for the étale topology (cor. 4.10, cor. 4.12). Note that the openness of smooth morphisms had already been proved by Berkovich using the aformentionned decomposition of smooth morphisms, see [2], cor. 3.7.4.

Second remark. Our main motivation to write theorem 4.15 was to use it in our proof, based upon dévissages, of the fact that a morphism which is flat and boundaryless at a given point is *universally* flat at that point (th. 5.12.3). This in fact only uses the flatness of the map $\mathscr{O}_{Y_x,y} \to \mathscr{O}_{Y,y}$; but we have chosen to also prove that its fibers are CI (and regular in char. 0) because it only requires a few extra-lines in the proof, and it seems to us that it is intrinsically interesting.

Section 5. This section is inspired by the first part of Raynaud and Gruson's work on flatness ([22]); our proofs essentially follow those of *loc. cit.*, but with some extra-work due to specific analytic problems. In particular, th. 1.1.1. of *loc. cit.* had to be replaced with cor. 4.7 of [14].

We begin with some technical results involving universal flatness and universal injectivity (def. 5.1); prop. 5.6 and prop. 5.7 are the respective analogs of lemma 2.2 and th. 2.1 of *loc. cit.* But there is something new: prop. 5.7 also provides an assertion which is specific to the boundaryless case, and is established using our theorem 4.15 on the local rings of 'generic fibers' (more precisely, its flatness claim).

Let $Y \to X$ be a morphism between good k-analytic spaces and assume that Y is Γ-strict. Let $\mathcal{F}$ be a coherent sheaf on Y and let $y \in \mathsf{Supp}\, \mathcal{F}$. After having introduced the notion of a *Γ-strict X-dévissage* of $\mathcal{F}$ at y (def. 5.9), we show that there always exists such a dévissage with length related to the dimension and the codepth of $\mathcal{F}$ at y (th. 5.10); this is the analytic counterpart of prop. 1.2.3 of *loc. cit.* Using our preliminary results already mentioned (prop. 5.7), we show that once given a dévissage, universal flatness of $\mathcal{F}$ is equivalent to the universal injectivity of some arrows which are part of the dévissage. Thanks to the part of prop. 5.7 which is specific to the inner case, we also obtain that if $Y \to X$ is inner at y, or more generally if $\mathcal{F}$ is extendable at y (5.11), then X-flatness of $\mathcal{F}$ over X is automatically universal (this is part of th. 5.12.3).

After that, we apply our results to the particular case where $\mathcal{F}$ is universally X-flat at y and of codepth zero at y, that is, CM at y. What we get (th. 5.15) is the following: if $\mathcal{F}$ is CM at y then there exists a Γ-strict affinoid neighborhood V of $y \in Y$, a Γ-strict affinoid domain T of a smooth X-space, and a finite morphism $V \to T$ with respect to which $\mathcal{F}$ is (universally) flat; note that the converse implication is easy (lemma 5.14).

Let us make two remarks.

First remark. In that paper, the main applications of the results we prove in this section are the universality of flat, boundaryless morphisms and the local description of relatively CM coherent sheaves, both of which are mentioned above. But we hope that they will have many other ones, especially concerning the image of a morphism between compact analytic spaces, and the problem of the definition, and of the existence, of non-Archimedean flatifiers (both questions are very likely related to each other).

Second remark. The formal version of Raynaud-Gruson methods also provides dévissages ([8], §3); what it precisely gives can be translated the following way in Berkovich's language: assume that $|k^*| \neq \{0\}$ and let $Y \to X$ be a morphism between strictly k-analytic spaces; let $\mathcal{F}$ be a coherent sheaf on Y with support Y. Then $\mathcal{F}$ admits *G-locally on* Y a strict X-dévissage.

So we have brought to improvements (and a purely Berkovich proof, without formal models): our results also hold for non-strict spaces, and in the good case, they provide *local* (and not only G-local) dévissages.

Section 6. Using formal flattening and openness of the flat, finitely presented morphism of schemes, Raynaud proved that if $|k^*| \neq \{1\}$ and if $\varphi : Y \to X$ is a flat morphism between two affinoid rigid spaces, then $\varphi(Y)$ is a finite union of affinoid domains of X (*cf.*[8], cor. 5.11). What we do in this section is essentially giving a purely Berkovich-theoretic proof of this fact, avoiding the use formal models and flattening; we in fact extend a little bit Raynaud's result, by removing the strictness assumption, and by handling not only the case where $Y \to X$ is universally flat, but more generally the case where Y is the support of a universally X-flat coherent sheaf $\mathcal{F}$ (th. 6.4). But to make the presentation simpler, we will now describe our results and methods only in the case where $\mathcal{F} = \mathscr{O}_Y$.

Our first result (thm. 6.1) tells the following: if Y is a compact, Γ-strict k-analytic space and if φ is a zero-dimensional, CM morphism (*i.e.* φ is universally flat with CM fibers) from Y to a separated analytic space X, then $\varphi(Y)$ is a Γ-strict compact analytic domain of X. The proofs consists in two steps:

- one first reduces, essentially thanks to our local description of CM maps (th. 5.15) to the case where both Y and X are affinoid and Γ-strict and where Y is a Galois-invariant affinoid domain of a finite Galois cover of X;
- the assertion is then proven by using the existence, for every point y of Y, of a smallest analytic domain of $(X, \varphi(y))$ through which $(Y, y) \to (X, \varphi(y))$ goes (1.9.2), in order to reduce to the case where $Y \to X$ is étale, hence open.

After that we prove (th. 6.3) that if the valuation of k is non-trivial, if Y is a compact *strictly* k-analytic space, if X is a separated k-analytic space and if $\varphi : Y \to X$ is a universally flat morphism, then there exists a compact strictly k-analytic space X', a CM zero-dimensional map $\psi : X' \to X$ and an X-morphism $\sigma : X' \to Y$, such that $\psi(X') = \varphi(Y)$. In other words, the image of φ is covered by a zero-dimensional, CM multisection. Note that strictness of Y can not be avoided here: if $X = \mathscr{M}(k)$ and $Y \neq \emptyset$, our theorem exactly says that Y has a rigid point – this is nothing but the *Nullstellensatz*, which is not true in general in the non-strict case[1].

The main step in proving th. 6.3 consists in establishing kind of a local version of it, and more precisely in building, for every $y \in Y$, finitely many zero-dimensional CM multisections which cover the image of the germ (Y, y); we have in fact given that step the status of an independant theorem (th. 6.2). Let us mention some essential ingredients of the proof:

- Temkin's reduction and our theorem 1.4 about the image of a map between Riemann-Zariski-spaces; the finitely many closed points involved in the latter

[1] We don't pretend to have given a new proof of the *Nullstellensatz*, because the latter is *used* in our proof – see 6.2.7.

play a crucial role by indicating, in some sense, in which 'directions' around y the multisections have to be drawn;

• the fact that in the non-trivially valued case, a smooth morphism between good k-analytic spaces admits an étale multisection over every point of its image (cor. 4.12).

The equality $\varphi(Y) = \psi(X')$ implies, in view of the aforementioned result about the image of a zero-dimensional CM morphism (th. 6.1), that $\varphi(Y)$ is a compact strictly analytic domain of X. Now by using in a straightforward way the Shilov section associated with a suitable k-free polyray, we generalize this latter as follows (th. 6.4): if Y is a Γ-strict, compact k-analytic space and φ a universally flat morphism from Y to a separated k-analytic space X, then $\varphi(Y)$ is a compact, Γ-strict analytic domain of X.

One straightforwardly deduces from this that a *boundaryless*, universally flat morphism is open (th. 6.6); this had already been proved by Berkovich, in another way, in his unpublished notes.

We eventually remark something which, to our knowledge, had not been noticed till now: in both results about the image of universally flat morphisms (th. 6.4 and th. 6.6), the universal flatness assumption for the map $\varphi : Y \to X$ involved can be replaced by the following: X is normal, Y and X are purely dimensional, and φ is purely of relative dimension $\mathsf{dim}\ Y - \mathsf{dim}\ X$. The proof proceeds by reducing to the universally flat case thanks to cor. 4.7 of [14].

Section 7. In this last section of the paper, we apply Kiehl's methods (which he introduced in [20] for the analog question in complex analytic geometry) to prove that the universal flatness locus of a morphism $\varphi : Y \to X$ is a Zariski-open subset of Y (th. 7.1); we then deduce that the set of points of Y at which Y is quasi-smooth of given relative dimension is a Zariski-open subset of Y (th. 7.4); the proof uses the openness (but not *a priori* Zariski-openness) of this locus, which is established in section 3 (3.10.8).

Acknowledgements

I started to think about the topics this paper is devoted to when I was asked some questions about flatness by Brian Conrad and Michael Temkin, for their work [12]. I realized quite quickly that answering them would take much more time that I had originally expected... and it eventually gave rise to the present work. I thus would like to thank warmly Conrad and Temkin for having given me the initial inspiration – and also for a lot of fruitful discussions since then.

0 Some remindings and notations about analytic geometry

Graded fields and graded valuations

(0.1) In [25], Temkin introduced powerful tools for the local study of analytic spaces and analytic morphisms. Those tools are based upon *graded* commutative algebra. It turns out that most classical notions of classical commutative algebra have graded counterparts, and that the usual theorems often remain *mutatis* *mutandis* true in the graded context; one only has essentially to add the words 'graded' or 'homogeneous' at suitable places. We will therefore most of the time make a free use of the graded translations of well-known facts. The justifications are left to the reader, who can also fruitfully report to the first paragraph of [25].

By a *graded ring*, we will always mean an $\mathbb{R}_+^*$-*graded ring*; the notation relative to the graduation will then be *multiplicative.* Any classical ring can be considered as a trivially graded ring, that is, a graded ring in which any element is homogeneous of degree 1. Therefore, all what we are going to do now also apply while working with classical rings; in this situation, we will often omit the word 'graded'.

If K is a graded ring and if r is an element of $\mathbb{R}_+^*$, we will denote by K^r the set of homogeneous elements of degree r of K. If Δ is a subgroup of $\mathbb{R}_+^*$, we will denote by K^Δ the graded subring $\bigoplus\limits_{\delta\in\Delta} K^\delta$ of K.

(0.2) Let K be a graded field (that is, a graded ring in which any non-zero *homogeneous* element is invertible) and let Γ be an ordered group with multiplicative notation. A Γ-*graded valuation* on K is a map $|.|$ defined on the set of homogeneous elements of K with values in $\Gamma\cup\{0\}$ which satisfies the following conditions:

i) $|1|=1, |0|=0$, and $|ab|=|a|.|b|$ for every couple (a,b) of homogeneous elements;

ii) for every couple (a,b) of homogeneous elements *of the same degree* we have $|a+b|\leqslant \max(|a|,|b|)$.

If we don't need to focus on the group Γ, or if the latter is clear from the context, we will simply talk about a *graded valuation* on K; if K is a field (viewed as a trivially graded field), a graded valuation on K is nothing but a classical Krull valuation on K; there is a natural notion of equivalence of graded valuations on K.

(0.3) Let K be a graded field. If $|.|$ is a graded valuation on K, the subset

$$\bigoplus_r \{\lambda\in K^r,\ |\lambda|\leqslant 1\}$$

of K is a graded subring of K which is called the *graded ring of* $|.|$. It is a local graded ring, whose unique maximal graded ideal is $\bigoplus_r\{\lambda\in K^r,\ |\lambda|<1\}$, and whose residue graded field is called the *residue graded field* of $|.|$ .Two graded valuations on K are equivalent if and only if they have the same graded ring.

A graded subring A of K is a graded valuation ring, that is, the graded ring of a graded valuation, if and only if for every non-zero homogeneous element λ of K, one has $\lambda\in A$ or $\lambda^{-1}\in A$; or, what amounts to the same, if and only if A is a graded local subring of K which is maximal for the domination relation; hence by Zorn's lemma, every graded local subring of K is dominated by a graded valuation ring.

Let $|.|$ be a graded valuation on K, let A be its graded ring and let k be its graded residue field. If $|.|'$ is a graded valuation on k, the pre-image of the graded ring of $|.|'$ inside A is a graded valuation ring of K. The corresponding graded valuation is called the *composition* of $|.|$ and $|.|'$ and has the same residue graded field as $|.|'$.

(0.4) Let K be a graded field and let Γ be an ordered group. Let $(s_1, \ldots, s_n)$ be positive real numbers. We denote by $K(s_1^{-1}S_1, \ldots, s_n^{-1}S_n)$ the graded field of fractions of the graded domain

$$K[s_1^{-1}S_1, \ldots, s_n^{-1}S_n] := \bigoplus_r \left(\bigoplus_I K^{r\mathbf{s}^{-I}} \mathbf{S}^I \right).$$

If $|.|$ is any graded Γ-valuation on K, we will denote by $|.|_{\text{Gauß}}$ the graded Γ valuation on $K(s_1^{-1}S_1, \ldots, s_n^{-1}S_n)$ that sends any homogeneous element $\sum a_I \mathbf{S}^I$ to $\max |a_I|$.

(0.5) Let K be a graded field. If L is any graded extension of K, we will denote by $\mathbb{P}_{L/K}$ the 'graded Riemann-Zariski space of L over K ', that is, the set of equivalence classes of valuations on L whose restriction to K is trivial (or, in other words, whose graded ring contains K). For any finite set E of homogeneous elements of L, one denotes by $\mathbb{P}_{L/K}\{E\}$ the subset of $\mathbb{P}_{L/K}$ that consists in graded valuations $|.|$ such that $|f| \leqslant 1$ for every $f \in E$. We endow $\mathbb{P}_{L/K}$ with the topology generated by the $\mathbb{P}_{L/K}\{f_1, \ldots, f_n\}$'s, where $f_1, \ldots, f_n$ are homogeneous elements of L. An open subset of $\mathbb{P}_{L/K}$ will be said to be *affine* it is equal to $\mathbb{P}_{L/K}\{f_1, \ldots, f_n\}$ for suitable f_i's; note that $\mathbb{P}_{L/K}$ is affine (indeed, one has $\mathbb{P}_{L/K} = \mathbb{P}_{L/K}\{\emptyset\}$). Any affine open subset of $\mathbb{P}_{L/K}$ (especially, $\mathbb{P}_{L/K}$ itself) is quasi-compact ([11],5.3.6).

If F is any graded extension of L, and if E is graded subfield of F such that $E \cap L = K$, the restriction induces a continuous map $\mathbb{P}_{F/E} \to \mathbb{P}_{L/K}$; if $E = K$, this map is surjective. If Δ is a subgroup of $\mathbb{R}_+^*$, a quasi-compact open subset U of $\mathbb{P}_{L/K}$ will be said to be Δ-*strict* if U is the pre-image of some quasi-compact open subset of $\mathbb{P}_{L^\Delta/K^\Delta}$.

By definition, U is Δ-strict if and only if it is a finite union of affine open subsets whose definition only involve homogeneous elements with degrees belonging to Δ. If U is affine, say $\mathsf{U} = \mathbb{P}_{L/K}\{f_1, \ldots, f_n\}$, then U is Δ-strict if and only if the degree of every f_i belongs to $\sqrt{\mathfrak{D}.\Delta}$, where $\mathfrak{D}$ is the group of degrees of homogeneous elements of K (this follows from [25], prop. 2.5 *i*)).

If $\Delta = \{1\}$, we will simply say *strict* instead of $\{1\}$-strict.

(0.6) Remark. Our definition of Δ-strictness is very closed to that of Temkin in [25](which is also used in [12]), the only difference being that we don't require Δ to contain $\mathfrak{D}$; actually, Δ-strictness in our sense is equivalent to $\mathfrak{D}.\Delta$-strictness in Temkin's one.

Analytic geometry: general facts

(0.7) A *non-Archimedean absolute value* on a field k is an $\mathbb{R}_+^*$-valuation on k. An *analytic field* is a field endowed with a non-Archimedean absolute value for which it is complete; note that any field endowed with the *trivial* absolute value is an analytic field. Let k be an analytic field.

(0.7.1) An *analytic extension* of k is an analytic field L together with an isometric embedding $k \hookrightarrow L$.

(0.7.2) In this text, we will denote by $\widetilde{k}$ the *graded* reduction of k. It is a graded ring which was was defined by Temkin in [25]; for every $r > 0$, its subgroup $\widetilde{k}^r$ of homogeneous elements of degree r is the quotient

$$\{\lambda \in k,\ |\lambda| \leqslant r\}/\{\lambda \in k,\ |\lambda| < r\}.$$

Note that the field $\widetilde{k}^1$ is the residue field of the valuation $|.|$ in the classical sense. If λ is any element of k and if r is a positive real number such that $|\lambda| \leqslant r$, we will denote by $\widetilde{\lambda}^r$ the image of λ in $\widetilde{k}^r$. If $\lambda \neq 0$ and if $r = |\lambda|$ we will write $\widetilde{\lambda}$ instead of $\widetilde{\lambda}^r$; if $\lambda = 0$ we set $\widetilde{\lambda} = 0$.

(0.7.3) Remark. If $|k| = \{1\}$, then $\widetilde{k} = \widetilde{k}^1$; if not, it doesn't seem that $\widetilde{k}$ can be interestingly interpreted as a residue graded field in the sense of 0.3.

(0.8) If k is an analytic field, the notion of a *k-analytic space* will be that of Berkovich ([2]). If X is such a space, and if L is an analytic extension of k, we will denote by X_L the L-analytic space deduced from X by extending the ground field to L. An *analytic space* (without mention of any ground field) is a pair (X, k) where k is an analytic field and X a k-analytic space; a morphism between two analytic spaces (Y, L) and (X, k) consists in an isometric embedding $k \hookrightarrow L$ and a morphism $Y \to X_L$ of L-analytic spaces. While speaking about analytic spaces and morphisms between them, we will of course most of the time omit to mention the fields and the isometric embeddings involved.

Similarly we define an affinoid algebra (resp. space) as a pair $(\mathscr{A}, k)$ (resp. (X, k)) where k is an analytic field and $\mathscr{A}$ a k-affinoid algebra (resp. and X a k-analytic space).

We fix an analytic field k.

(0.9) A *good* analytic space is an analytic space whose every point has an affinoid neighborhood, and hence a basis of affinoid neighborhoods.

If X is an analytic space (resp. a scheme) and if $x \in X$, then $\mathscr{H}(x)$ (resp. $\kappa(x)$) will denote the complete residue field (resp. the residue field) of x.

If $f : Y \to X$ is a morphism of k-analytic spaces and if $x \in X$, the fiber of f at x will be denoted either by Y_x or by $f^{-1}(x)$; it is an $\mathscr{H}(x)$-analytic space.

(0.10) General convention. While working with a given affinoid space denoted by a capital letter (eg. Y, X, S, T, U, V, W...) we will usually denote the spectrum of its algebra of functions by the corresponding calligraphic letter (eg. $\mathscr{Y}$, $\mathscr{X}$, $\mathscr{S}$, $\mathscr{T}$, $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$...). If, say, $T \to X$ (or $Y \to X$, or...) is a morphism between k-affinoid spaces, and if $x \in X$, the spectrum of the algebra of functions on T_x (or Y_x, or...) will be denoted by $\mathscr{T}_x$ (or $\mathscr{Y}_x$, or...). If Y, or X, or... is a k-affinoid space and if L is an analytic extension of k, the spectrum of the algebra of functions on Y_L, or X_L, or... will be denoted by $\mathscr{Y}_L$, or $\mathscr{X}_L$, or....

(0.11) Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a k-free polyray, that is, a finite family of positive real numbers which is free when one views it as a family of elements of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{R}_+^*/|k^*|)$. Let $\mathbf{T}$ be a family of n indeterminates. The set of power series $\sum_{I \in \mathbb{Z}^n} a_I \mathbf{T}^I$ with coefficients in k such that $|a_I| r^I \to 0$ as $|I| \to \infty$ is a field; once equipped with the norm $\sum a_I \mathbf{T}^I \mapsto \max |a_I| r^I$, it becomes an analytic extension of k we denote by $k_{\mathbf{r}}$. Now, let X be a k-affinoid space, let $\mathscr{A}$ be its algebra of functions. We write $X_{\mathbf{r}}$ instead of $X_{k_{\mathbf{r}}}$. For any $x \in X$, the map $\sum a_I \mathbf{T}^I \mapsto \max |a_I(x)| \mathbf{r}^I$ is a bounded multiplicative semi-norm on $k_{\mathbf{r}} \widehat{\otimes} \mathscr{A}$; the corresponding point of $X_{\mathbf{r}}$ will be denoted by $\mathfrak{s}(x)$. The map $\mathfrak{s}$ is a continuous section of $X_{\mathbf{r}} \to X$ we will call the *Shilov section.*

(0.12) Any analytic space X is endowed with a set-theoretic Grothendieck topology, which is called the G-*topology*, and which is finer than its usual topology. The corresponding site is denoted by X_{G}; it inherits a sheaf of rings $\mathscr{O}_{X_{\mathrm{G}}}$. If X is good, the restriction $\mathscr{O}_X$ of $\mathscr{O}_{X_{\mathrm{G}}}$ to the category of open subsets of X makes X a locally ringed space. Both sheaves of rings $\mathscr{O}_{X_{\mathrm{G}}}$ and $\mathscr{O}_X$ are coherent (for proofs[2] see [15], lemma 0.1).

(0.12.1) If X is good, the forgetful functor induces an equivalence between the categories of coherent $\mathscr{O}_{X_{\mathrm{G}}}$-modules and that of coherent $\mathscr{O}_X$-modules which preserves the cohomology groups ([2], prop. 1.3.4 and 1.3.6)

(0.12.2) Convention. In the sequel, it will be sufficient for us to work with sheaves on X_{G}, and we won't need to pay a special attention, in the good case, to the restriction of such a sheaf to the category of open subsets of X. For that reason, and for sake of simplicity, a coherent $\mathcal{O}_{X_{\mathrm{G}}}$-module will simply be called a *coherent sheaf on* X, and we will write $\mathscr{O}_X$ instead of $\mathscr{O}_{X_{\mathrm{G}}}$.

Nevertheless, be aware that if X is good and if $x \in X$, the notation $\mathscr{O}_{X,x}$ will still denote the stalk at x of the classical, i.e. *non G-topological, sheaf $\mathscr{O}_X$. In other words,*

$$\mathscr{O}_{X,x} := \varinjlim_{U \text{ } \mathbf{open} \text{ neighborhood of } x} \mathscr{O}_X(U).$$

(0.12.3) A subset of an analytic space X that can be defined as the zero locus of a coherent sheaf of ideals is called a *Zariski-closed* subset of X; as suggested by the terminology, the Zariski-closed subsets are exactly the closed subsets of a topology, the so-called *Zariski-topology*; the property of beeing a Zariski-closed subset is of G-local nature (*cf.* [15], §0.2.3 and prop. 4.2). If $\mathcal{I}$ is a coherent sheaf of ideals on X and if Y is the corresponding Zariski-closed subset, then Y can be given a natural structure of an analytic space, so that there is a morphism $\iota : Y \hookrightarrow X$ whose underlying continous map is the inclusion and which is such

[2] It was pointed out to the author by Jérôme Poineau that there is a mistake in both proofs. Indeed, in every of them one starts with a *surjection* $\mathscr{O}^n \to \mathscr{O}$ and proves that its kernel is locally finitely generated, though in order to get the coherence one should establish such a finiteness result for any, *i.e.* non necessarily surjective, map $\mathscr{O}^n \to \mathscr{O}$; but it turns out that the proofs don't make any use of those inaccurate surjectivity assumptions.

that $\iota_* \mathscr{O}_Y \simeq \mathscr{O}X/\mathcal{I}$; once it is equipped with this structure, we will call Y the *closed analytic subspace* of X defined by $\mathcal{I}$.

(0.12.4) If $\mathcal{F}$ is a coherent sheaf on an analytic space X, its *support* $\mathsf{Supp}\,\mathcal{F}$ is the closed analytic subspace of X defined by the annihilator of $\mathcal{F}$.

(0.12.5) If $\mathscr{A}$ is an affinoid algebra the global section functor induces an equivalence between the category of coherent sheaves on $\mathscr{M}(\mathscr{A})$ and that of finitely generated $\mathscr{A}$-modules, and the latter is itself equivalent to that of coherent sheaves on $\mathsf{Spec}\,\mathscr{A}$ (this theorem is essentially due to Kiehl, *cf.* [2], §1.2); we thus get an equivalence between the category of coherent sheaves on $\mathscr{M}(\mathscr{A})$ and that of coherent sheaves on $\mathsf{Spec}\,\mathscr{A}$. If $\mathcal{F}$ is a coherent sheaf on $\mathscr{M}(\mathscr{A})$ we will also denote by $\mathcal{F}$ the corresponding coherent sheaf on $\mathsf{Spec}\,\mathscr{A}$.

(0.12.6) Let X be a scheme (resp. a good analytic space). If $\mathcal{F}$ is a coherent sheaf on X and if $x \in X$ we will denote by $\mathcal{F} \otimes \mathscr{O}_{X,x}$ the stalk of $\mathcal{F}$ at x (resp. the stalk of *the restriction of* $\mathcal{F}$ *to the category of open subsets of* X at x). If A is any $\mathscr{O}_{X,x}$-algebra, we will write $\mathscr{F} \otimes \mathsf{A}$ instead of $(\mathcal{F} \otimes \mathscr{O}_{X,x}) \otimes_{\mathscr{O}_{X,x}} \mathsf{A}$. In the case of a good analytic space we have $\mathcal{F} \otimes \mathscr{O}_{X,x} = \mathcal{F}(V) \otimes_{\mathscr{O}_X(V)} \mathscr{O}_{X,x}$ for any affinoid neighborhood V of x in X.

Remark that if X is affinoid and if $\mathbf{x} \in \mathscr{X}$, then according to our conventions $\mathcal{F} \otimes \mathscr{O}_{\mathscr{X},\mathbf{x}}$ will denote the stalk at $\mathbf{x}$ of the coherent sheaf on $\mathscr{X}$ that corresponds to $\mathcal{F}$.

(0.12.7) Let X be an analytic space, let $\mathcal{F}$ be a coherent sheaf on X and let $x \in X$. If V is a good analytic domain of X containing x, then $\mathcal{F}_{|V} \otimes \mathscr{O}_{V,x}$ and $\mathcal{F}_{|V} \otimes \mathscr{H}(x)$ are well-defined (see above).

For the sake of simplicity, we will simply write $\mathcal{F} \otimes \mathscr{O}_{V,x}$ instead of $\mathcal{F}_{|V} \otimes \mathscr{O}_{V,x}$.

As far as $\mathcal{F}_{|V} \otimes \mathscr{H}(x)$ is concerned, it follows from its construction that it is equal to $\mathcal{F}(V) \otimes_{\mathscr{O}_X(V)} \mathscr{H}(x)$, and hence *doesn't depend on* V; we will therefore simply denote it by $\mathcal{F} \otimes \mathscr{H}(x)$. If A is any algebra over $\mathscr{O}_{V,x}$ (resp. $\mathscr{H}(x)$) we will write $\mathcal{F} \otimes \mathsf{A}$ instead of $(\mathcal{F} \otimes \mathscr{O}_{V,x}) \otimes_{\mathscr{O}_{V,x}} \mathsf{A}$ (resp. $(\mathcal{F} \otimes \mathscr{H}(x)) \otimes_{\mathscr{H}(x)} \mathsf{A}$).

(0.12.8) Let $\pi : Y \to X$ be a morphism of schemes (resp. analytic spaces) and let $\mathcal{E}$ be a coherent sheaf on X. When the $\otimes$ symbol is used with one of the meanings mentioned at 0.12.6 and 0.12.7 above, we will often simply write $\mathcal{E} \otimes$ rather than $\pi^* \mathcal{E} \otimes$.

(0.12.9) Let X be a good analytic space, let $\mathcal{F}$ be a coherent sheaf on X and let $\mathcal{I}$ be its annihilator. If $x \in X$ then $x \in \mathsf{Supp}\,\mathcal{F}$ if and only if $\mathcal{F} \otimes \mathscr{O}_{X,x}$ is non-zero. Indeed, the annihilator of $\mathcal{F} \otimes \mathscr{O}_{X,x}$ is equal to $\mathcal{I} \otimes \mathscr{O}_{X,x}$; therefore $\mathcal{F} \otimes \mathscr{O}_{X,x} = 0$ if and only if $\mathcal{I} \otimes \mathscr{O}_{X,x}$ is contained in the maximal ideal of $\mathscr{O}_{X,x}$, which exactly means that x is lying on the closed analytic subspace defined by $\mathcal{I}$, that is, on $\mathsf{Supp}\,\mathcal{F}$.

(0.12.10) Let X be an affinoid space. To any $\mathscr{X}$-scheme of finite type $\mathscr{Y}$ is associated functorialy a good k-analytic space $\mathscr{Y}^{an}$; it comes equipped with a map to X and with a morphism of locally ringed spaces to $\mathscr{Y}$. To any coherent sheaf $\mathcal{E}$ on $\mathscr{Y}$ is associated functorially a coherent sheaf $\mathcal{E}^{an}$ on $\mathscr{Y}^{an}$. If $\mathscr{Y} = \mathscr{X}$

then $\mathscr{Y}^{an} = X$ and $\mathcal{E} \mapsto \mathcal{E}^{an}$ is a quasi-inverse of the equivalence of categories we mentioned above (hence in this case we will sometimes write $\mathcal{E}$ instead of $\mathcal{E}^{an}$). If y is a point of $\mathscr{Y}^{an}$ and if $\mathbf{y}$ denotes its image on $\mathscr{Y}$, we have for any coherent sheaf $\mathscr{E}$ on $\mathscr{Y}$ a natural isomorphism between $\mathcal{E}^{an} \otimes \mathscr{O}_{\mathscr{Y}^{an},y}$ and $\mathcal{E} \otimes \mathscr{O}_{\mathscr{Y}^{an},y}$.

(0.13) Let us now list various properties we will often use in the sequel.

(0.13.1) If X is a good analytic space and if $x \in X$, the residue field of the local ring $\mathscr{O}_{X,x}$ is a dense subfield of $\mathscr{H}(x)$: this follows immediatly from the definition of the latter.

(0.13.2) If X is a good analytic space and if $x \in X$, then the local ring $\mathscr{O}_{X,x}$ is noetherian, henselian ([2], th. 2.1.4 and th. 2.1.5), and excellent ([15], th. 2.13).

(0.13.3) If X is a good k-analytic space and if L is an analytic extension of k, then $X_L \to X$ is a faithfully flat map of locally ringed spaces (for flatness, see [2], cor. 2.1.3; for surjectivity, see [14], 0.5).

(0.13.4) If X is a good analytic space and if V is a good analytic domain of X then for any $x \in V$ the map $\mathsf{Spec}\ \mathscr{O}_{V,x} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ is flat with geometrically reduced fibers ([15], th. 3.3).

(0.13.5) If X is an affinoid space and if $\mathscr{Y}$ is an $\mathscr{X}$-scheme of finite type, then $\mathscr{Y}^{an} \to \mathscr{Y}$ is surjective, and if y is a point on $\mathscr{Y}^{an}$ whose image on $\mathscr{Y}$ is denoted by $\mathbf{y}$, then $\mathsf{Spec}\ \mathscr{O}_{\mathscr{Y}^{an},y} \to \mathsf{Spec}\ \mathscr{O}_{\mathscr{Y},\mathbf{y}}$ is flat with geometrically reduced fibers ([15], th. 3.3).

(0.14) There is a good notion of dimension for an analytic spaces, which is due to Berkovich ([1], §2); some basic results about it were proven by the author ([14], §1). If X is a k-analytic space we will denote by $\mathsf{dim}_k\ X$ the k-analytic dimension of X; if x is a point of X, we will denote by $\mathsf{dim}_{k,x}\ X$ the k-analytic dimension of X at x. In both cases if there is no ambiguity about the ground field, one will simply write $\mathsf{dim}\ X$ and $\mathsf{dim}_x\ X$. As an example, if $Y \to X$ is a morphism of k-analytic spaces, if $x \in X$ and if $y \in Y_x$, then $\mathsf{dim}_y\ Y_x$ will mean $\mathsf{dim}_{\mathscr{H}(x),y}\ Y_x$; this integer is also called the *relative dimension* of $Y \to X$ at x; if $\mathcal{F}$ is a coherent sheaf on Y and if $y \in \mathsf{Supp}\ \mathcal{F}$, we will call $\mathsf{dim}_y\ (\mathsf{Supp}\ \mathcal{F})_x$ the *relative dimension of $\mathcal{F}$ at y with respect to X.*

While speaking about an analytic space X, its dimension will always have to be understood as being taken over the field which is not mentioned but is part of the datum of X.

(0.15) If K is an analytic field and if L is a complete extension of K, we will set

$$\mathsf{d}(L/K) = \mathsf{tr.\ deg}(\widetilde{L}_1/\widetilde{K}_1) + \mathsf{dim}_{\mathbb{Q}}\ \mathbb{Q} \otimes_{\mathbb{Z}} (|L^*|/|K^*|)\ \in \mathbb{N} \cup \{+\infty\}.$$

One can also define directly $\mathsf{d}(L/K)$ as the transcendance degree of the graded field extension $\widetilde{K} \hookrightarrow \widetilde{L}$. It is related to the dimension: if X is a k-analytic space, then $\mathsf{dim}\ X = \sup\limits_{x \in X} \mathsf{d}\ (\mathscr{H}(x)/k)$ (*cf.* [14], 2.14).

(0.15.1) Let $d \in \mathbb{N}$ and let $Y \to X$ be a *surjective* morphism between k-analytic spaces whose relative dimension is equal to d at every point of Y. It follows immediatly from 0.15 that $\mathsf{dim}\ Y = \mathsf{dim}\ X + d$.

(0.15.2) Let $\varphi : Y \to X$ be a finite morphism between k-analytic spaces; the image $\varphi(Y)$ is a Zariski-closed subset of X. Let $y \in Y$ and let x be its image on X. If y is the only pre-image of x on Y, then $\mathsf{dim}_x\ \varphi(Y) = \mathsf{dim}_y\ Y$; indeed, one immediatly reduces to the case where Y, X and $\varphi(Y)$ are k-affinoid. It is now sufficient to show that if Z is an irreducible component of Y, then $\mathsf{dim}\ \varphi(Z) = \mathsf{dim}\ Z$; but this follows from 0.15.1.

(0.16) Remark. If X is an analytic space and if $x \in X$, then $\mathsf{dim}_x\ X$ is defined as the minimal dimension of an analytic domain of X containing x; in fact, it is coincides with the minimal dimension of an *open* analytic domain of X containing x. Indeed, let us denote the latter by d. We obviously have $\mathsf{dim}_x\ X \leqslant d$. Let us prove the converse inequality. Let $V_1, \ldots, V_n$ be affinoid domains of X containing x such that $\bigcup V_i$ is a neighborhood of x and let F be the union of all irreducible components of the V_i's which do *not* contain x. There exist an open neighborhood U of x in $(\bigcup V_i) - F$. For every i, this neighborhood intersects only the irreducible components of V_i that contain x; all those components are of dimension bounded by $\mathsf{dim}_x\ X$; we therefore have $\mathsf{dim}\ U = \max \mathsf{dim}\ (U \cap V_i) \leqslant \mathsf{dim}_x\ X$, whence the claim.

(0.17) If X is a k-analytic space and if Y is a subset of X, we will denote by $\overline{Y}^X$ (resp. $\overline{Y}^{X_{\text{Zar}}}$) the topological closure of Y inside X (resp. the closure of Y inside X for the *Zariski-topology* of the latter).

(0.18) If X is a k-affinoid space and if $x \in X$, the integer

$$\inf_{V \text{ aff. nghb. of } x} \mathsf{dim}_k\ \overline{\{x\}}^{V_{\text{Zar}}}$$

depends only on (X, x); it will be called the *k-analytic central dimension* of the germ (X, x) and will be denoted by $\mathsf{centdim}_k\ (X, x)$, or simply by $\mathsf{centdim}\ (X, x)$ if there is no ambiguity about the ground field.

Validity of a property at a point

(0.19) Let P be a property of noetherian local rings (resp. of finitely generated modules over a noetherian local ring, resp. of a diagram of linear maps between finite modules over such a ring). As an example, we may consider for P the property for a noetherian local ring to be regular, that for a finitely generated module over a noetherian local ring to be CM or to be flat (that is, free), that for a morphism between modules over such a ring to be injective, or that for a sequence of linear maps between modules over such a ring to be exact.

(0.19.1) The validity of P at a point: the good case. Let X be a good analytic space or a scheme, (resp. let X be a good analytic space or a scheme and let $\mathcal{E}$ be a coherent sheaf on X, resp. let X be a good analytic space or a scheme and let D be a diagram of linear maps between coherent sheaves on X).

Let $x \in X$. We will say that X (resp. $\mathcal{E}$, resp. D) *satisfies* P *at* x if $\mathscr{O}_{X,x}$ (resp. $\mathcal{E} \otimes \mathscr{O}_{X,x}$, resp. $\mathsf{D} \otimes \mathscr{O}_{X,x}$) satisfies P.

In order to extend the definition of the validity of P at a point to the case of a non-necessarily good analytic space, we introduce the following technical definition.

(0.19.2) Definition. The property P is said to be *strongly local* if for every good analytic space Y (resp. for every good analytic space Y and every coherent sheaf $\mathcal{G}$ on Y, resp. for every good analytic space Y and every diagram D of linear maps between coherent sheaves on Y), for every good analytic domain Z of Y and for every $y \in Z$, the following are equivalent:

i) Y (resp. $\mathcal{G}$, resp. D) satisfies P at y;

ii) Z (resp. $\mathcal{G}_{|Z}$, resp. $\mathsf{D}_{|Z}$) satisfies P at y.

(0.19.3) The validity of P at a point: the general case. *Assume that* P *is strongly local.* Let X be an analytic space (resp. let X be an analytic space and let $\mathcal{E}$ be a coherent sheaf on X, resp. let X be an analytic space and let D be a diagram of linear maps between coherent sheaves on X). Let $x \in X$. The following are easily seen to be equivalent:

α) for every good analytic domain U of X containing x, U (resp. $\mathcal{E}_{|U}$, resp. $\mathsf{D}_{|U}$) satisfies P at x;

β) there exists a good analytic domain U of X containing x such that U (resp. $\mathcal{E}_{|U}$, resp. D) satisfies P at x.

If both those equivalent assertions are true, we will say that X (resp. $\mathcal{E}$, resp. D) *satisfies* P *at* x. This definition is compatible with the preceeding one in the good case. If X (resp. $\mathcal{E}$, resp. D) satisfies P at every point of X, we will simply say that it satisfies P.

(0.19.4) Remark. If P is strongly local, if X is an analytic space (resp. if X is an analytic space and if $\mathcal{E}$ is a coherent sheaf on X, resp. if X is an analytic space and if D is a diagram of linear maps between coherent sheaves on X), if W is an analytic domain of X and if $x \in W$, it is obvious that X (resp. $\mathcal{E}$, resp. D) satisfies P at x if and only if W (resp. $\mathcal{E}_{|W}$, resp. $\mathsf{D}_{|W}$) satisfies P at x.

(0.19.5) Examples. Let us list some properties which are strongly local.

a) As far as noetherian local rings are concerned, the properties to be regular, R_m for some $m \in \mathbb{N}$, Gorenstein, or a complete intersection (*cf.* 0.13.4).

b) As far as finitely generated modules over noetherian local rings are concerned, the properties to be Cohen-Macaulay or S_m for some m, or that to be flat (*cf.* 0.13.4).

c) As far as diagram of linear maps are concerned, the exactness of a sequence (thanks to 0.13.4); in particular, injectivity, surjectivity and bijectivity of a single arrow are strongly local.

(0.19.6) Remark about surjectivity. If X is a k-analytic space, if $x \in X$ and if $\mathcal{G} \to \mathcal{F}$ is a linear map between coherent sheaves on X, then Nakayama's lemma ensures that $\mathcal{G} \to \mathcal{F}$ is surjective at the point x if and only if the arrow $\mathcal{G} \otimes \mathscr{H}(x) \to \mathcal{F} \otimes \mathscr{H}(x)$ is onto.

(0.20) GAGA-principles. Let P be any of the properties listed at 0.19.5 *a*) (resp. 0.19.5 *b*), resp. 0.19.5 *c*)) and let X be an affinoid space. Let $\mathscr{Y}$ be an $\mathscr{X}$-scheme of finite type (resp. let $\mathscr{Y}$ be an $\mathscr{X}$-scheme of finite type and let $\mathcal{E}$ be a coherent sheaf on $\mathscr{Y}$, resp. let $\mathscr{Y}$ be an $\mathscr{X}$-scheme of finite type and let D be a diagram of linear maps between coherent sheaves on $\mathscr{Y}$). Let y be a point of $\mathscr{Y}^{an}$ and let $\mathbf{y}$ be its image on $\mathscr{Y}$. Then it follows from 0.13.5 that $\mathscr{Y}$ (resp. $\mathcal{E}$, resp. D) satisfies P at $\mathbf{y}$ if and only if $\mathscr{Y}^{an}$ (resp. $\mathcal{E}^{an}$, resp. D^{an}) satisfies P at y.

(0.21) If X is an analytic space and if $\mathcal{E} \to \mathcal{F}$ is a map of coherent sheaves on X, then the *bijectivity locus* of $\mathcal{E} \to \mathcal{F}$ will be the set of points of X at which this map is bijective; we will denote it by $\mathsf{Bij}(\mathcal{E} \to \mathcal{F})$. It is a Zariski-open subset; to see that, one immediatly reduces to the affinoid case, and then one uses 0.20 and the well-known scheme-theoretic version of our claim.

(0.22) Let $Y \to X$ be a morphism between good analytic spaces or a morphism of schemes. Let $\mathcal{F}$ be a coherent sheaf on Y, let $y \in Y$ and let x be its image on X. The sheaf $\mathcal{F}$ will be said to be *X-flat at y* if $\mathcal{F} \otimes \mathscr{O}_{Y,y}$ is a flat $\mathscr{O}_{X,x}$-module. If $\mathscr{O}_Y$ is X-flat at y, we will simply say that Y is X-flat at y, or that $Y \to X$ is flat at y.

Γ-strictness

(0.23) We fix a subgroup Γ of $\mathbb{R}_+^*$ such that $|k^*|.\Gamma \neq \{1\}$ (that is, such that $|k^*| = 1 \Rightarrow \Gamma \neq \{1\}$). We are going to define the notions of a Γ-strict analytic space and of a Γ-strict germ, and we will relate them to that of a Γ-strict quasi-compact open subset of a graded Zariski-Riemann space (0.5). As for the latter, there is small difference between our convention and Temkin's one ([25], [12]): we don't require Γ to contain $|k^*|$, and what we call Γ-strictness would be $|k^*|$-strictness according to Temkin's conventions.

Most proofs of that section are essentially straightforward adaptations of those of Temkin in [25], hence won't be written in full detail; the interested reader may find complete ones in [12].

(0.24) Let $\mathscr{A}$ be a k-affinoid algebra and let Γ be a subgroup of $\mathbb{R}_+^*$ such that $|k^*|.\Gamma \neq \{1\}$ (that is, such that $|k^*| = 1 \Rightarrow \Gamma \neq \{1\}$). We will say that a $\mathscr{A}$ is *Γ-strict* if it is a quotient of $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ for suitables r_j's *belonging to* Γ.

(0.24.1) If $\mathscr{A}$ is a quotient of $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ for suitables r_j's *belonging to* $\sqrt{|k^*|.\Gamma}$, then $\mathscr{A}$ is Γ-strict. To see that, one first easily reduces to the case where the r_j's all belong to $\sqrt{\Gamma}$. There exists a k-free polyray $\mathbf{s} = (s_1, \ldots, s_m)$ such that every s_i belongs to Γ and such that every r_j belongs to $\sqrt{|k_{\mathbf{s}}^*|}$. Then $k_{\mathbf{s}}\widehat{\otimes}_k\mathscr{A}$ is strictly $k_{\mathbf{s}}$-affinoid; the proof of cor. 2.1.8 of [1] together with an easy induction on m shows then that $\mathscr{A}$ is Γ-strict.

(0.24.2) A k-affinoid space will be said to be Γ-strict if its algebra of analytic functions is Γ-strict.

If X is such a space, the spectral semi-norm of any analytic function f on X belongs to $\sqrt{|k^*|.\Gamma} \cup \{0\}$. Indeed, as we saw above, there exists a k-free polyray $\mathbf{s} = (s_1, \ldots, s_m)$ such that every s_i belongs to Γ and such that $X_{\mathbf{s}}$ is strictly $k_{\mathbf{s}}$-affinoid. The spectral semi-norm of f can be computed on $X_{\mathbf{s}}$; hence it follows from [9], 6.2.1/4 that it belongs to $\sqrt{|k_{\mathbf{s}}^*|} \subset \sqrt{|k^*|.\Gamma} \cup \{0\}$.

Conversely, let X be a k-affinoid space such that the spectral semi-norm of every analytic function on X belongs to $\sqrt{|k^*|.\Gamma} \cup \{0\}$; then X is Γ-strict. Indeed, let $\mathscr{A}$ be the algebra of analytic functions on X, and let us fix an admissible epimorphism $k\{T_1/r_1, \ldots, T_n/r_n\} \to \mathscr{A}$. For every i, let s_i be the spectral radius of the image of T_i in $\mathscr{A}$. By assumption, $s_i \in \sqrt{|k^*|.\Gamma} \cup \{0\}$. Now set $t_i = s_i$ if $s_i \neq 0$ and take for t_i any element of $\sqrt{|k^*|.\Gamma} \cap [0; r_i]$ if $s_i = 0$. The admissible epimorphism $k\{T_1/r_1, \ldots, T_n/r_n\} \to \mathscr{A}$ factors then through an admissible epimorphism $k\{T_1/t_1, \ldots, T_n/t_n\} \to \mathscr{A}$, whence the Γ-strictness of $\mathscr{A}$.

(0.24.3) The class of Γ-strict spaces is a *dense* class in the sense of [2], §1 (the assumption that $\Gamma.|k^*| \neq \{1\}$ has precisely been maid to ensure this property). It thus gives rise to a corresponding category of analytic spaces, that of the Γ-strict k-analytic spaces; we will see below (lemma 0.29 *iii*)) that it is a *full* subcategory of the category of all k-analytic spaces. We also have a natural notion of a Γ-strict k-analytic germ.

If U and V are two Γ-strict affinoid domains of a separated k-analytic space X, then $U \cap V$ is a Γ-strict affinoid domain of X; as a consequence, a separated k-analytic space is Γ-strict if and only if it admits a G-covering by Γ-strict affinoid domains.

(0.24.4) Any k-affinoid space (resp. k-analytic space, resp. k-analytic germ) is $\mathbb{R}_+^*$-strict. If $|k^*| \neq \{1\}$, a k-affinoid space (resp. k-analytic space, resp. k-analytic germ) is $\{1\}$-strict if and only if it is a strictly k-affinoid space (resp. a strictly k-analytic space, resp. a strictly k-analytic germ).

The reduction of germs

(0.25) If X is any topological space, an *admissible* X-*space* will be a connected, non-empty, quasi-compact topological space Y equipped with a continuous map $\mathsf{Y} \to \mathsf{X}$ which induces, for any $\mathsf{y} \in \mathsf{Y}$, an homeomorphism between an open neighborhood of y and an open subset of X.

(0.26) Let k be an analytic field and let (X,x) be a k-analytic germ. Temkin has defined in [25] the *graded reduction* of a germ (X,x); it is an admissible $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$-space $\widetilde{(X,x)}$; this construction is functorial in (X,x).

The germ (X,x) is separated if and only if $\widetilde{(X,x)}$ is an open subset of $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$.

We fix a subgroup Γ *of* $\mathbb{R}_+^*$ *such that* $|k^*|.\Gamma \neq \{1\}$.

(0.27) The notion of a Γ-strict quasi-compact open subset of $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ extends as follows. Let U be an admissible $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$-space. We will say that U is Γ-*strict* if there exists an admissible $\mathbb{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$-space V such that

$$\mathsf{U} \simeq \mathsf{V} \times_{\mathbb{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}} \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}} \quad .$$

If it is the case, it follows from prop. 2.6 of [25] that V is uniquely determined.

(0.28) Lemma. *Let* (X,x) *be a* k*-analytic germ. The following are equivalent:*

i) x *has a* Γ*-strict* k*-affinoid neighborhood in* X*;*

ii) there exist homogeneous elements $f_1,\ldots,f_n$ *of* $\widetilde{\mathscr{H}(x)}$ *whose degrees all belong to* Γ *and such that* $\widetilde{(X,x)} = \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{f_1,\ldots,f_n\}$.

Proof. Implication *i)* $\Rightarrow$ *ii)* follows from the fact that the spectral semi-norm of an analytic function on a Γ-strict k-affinoid space is an element of $\sqrt{|k^*|.\Gamma}$ (0.24.2). Assume that *ii)* is true; then the germ (X,x) is good by th. 5.1 of [25]; in other words, x has an affinoid neighborhood V in x. One can assume that all f_i's are non-zero; for every i, denote by r_i the degree of f_i. One can shrink V such that there exist analytic functions $h_1,\ldots,h_n$ on V satisfying for every i the equalities $|h_i(x)| = r_i$ and $\widetilde{h_i(x)}_{r_i} = f_i$. The n-uple $(h_1,\ldots,h_n)$ defines a morphism $\mathbf{h} : V \to \mathbb{A}_k^{n,an}$; set $t = \mathbf{h}(x)$. The quasi-compact open subset $\widetilde{(X,x)}$ of $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ is the pre-image of $\mathbb{P}_{\widetilde{\mathscr{H}(t)}/\widetilde{k}}\{\widetilde{T_1(t)}_{r_1},\ldots,\widetilde{T_n(t)}_{r_n}\}$, where the T_i's are the coordinate functions on the affine space. Let T be the affinoid domain of $\mathbb{A}_k^{n,an}$ defined by the inequalities $|T_i| \leqslant r_i$ for $i = 1,\ldots,n$.

Thanks to prop. 4.1 *iii)* of [25], the map $(X,x) \to (\mathbb{A}_k^{n,an},t)$ goes through (T,t). Hence one can shrink V so that there exist an affinoid neighborhood W of t in $\mathbb{A}_k^{n,an}$ satisfying $\mathbf{h}(V) \subset W \cap T$. Since $\mathbb{A}_k^{n,an}$ has no boundary, one can even choose W to be Γ-strict. As $\widetilde{(V,x)}$ is the pre-image of $\widetilde{(W\cap T,t)}$ inside $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, the morphism $V \to W \cap T$ is inner at v ([25], th. 5.2). As $W \cap T$ is Γ-strict, lemma 2.5.11 of [1] immediatly implies that x has a Γ-strict affinoid neighborhood in V, hence in X. $\square$

The lemma that follows is directly inspirated by prop. 2.6, lemma 4.9 and cor. 4.10 of [25]; once one knows lemma 0.28 above, their proofs can be straightforwardly adapted to give assertions *i, ii* and *iii*). Assertion *iv*) follows directly from the constructions.

(0.29) Lemma (after Temkin, see prop. 2.6, lemma 4.9 and cor. 4.10 of [25]). *Let (X,x) and (Y,y) be k-analytic germs.*

i) The germ (X,x) is Γ-strict if and only $\widetilde{(X,x)}$ is Γ-strict; if it is the case, the unique admissible $\mathbb{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$-space $\widetilde{(X,x)}$ comes from will be denoted by $\widetilde{(X,x)}^{\Gamma}$.

ii) If both (Y,y) and (X,x) are Γ-strict, any morphism $(Y,y)\to(X,x)$ induces a map $\widetilde{(Y,y)}^{\Gamma}\to\widetilde{(X,x)}^{\Gamma}$.

iii) The category of Γ-strict k-analytic germs (resp. Γ-strict k-analytic spaces) is a full subcategory of the category of all k-analytic germs (resp. k-analytic spaces);

iv) If $\Gamma=\{1\}$ and if (X,x) is $\{1\}$-strict, that is, strictly k-analytic, then $\widetilde{(X,x)}_{\{1\}}$ is nothing but the non-graded reduction of (X,x) defined by Temkin in [24]*; we will write $\widetilde{(X,x)}^{1}$ instead of $\widetilde{(X,x)}^{\{1\}}$.* $\square$

(0.30) Let (X,x) and (Y,y) be two Γ-strict k-analytic germs.

(0.30.1) The germ (X,x) is separated if and only if $\widetilde{(X,x)}^{\Gamma}\to\mathbb{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$ is an open immersion; it follows from lemma 0.29 and from prop. 4.8 *iii*) of [25].

(0.30.2) The germ (X,x) is good if and only if it fulfils the equivalent conditions of lemma 0.28; this follows from prop. 2.5 *i*) of [25].

(0.30.3) The assignment $(V,x)\mapsto\widetilde{(V,x)}^{\Gamma}$ induces a bijection between the set of Γ-strict analytic domains of (X,x) and that of quasi-compact, non-empty open subsets of $\widetilde{(X,x)}^{\Gamma}$; it follows from lemma 0.29 and from th. 4.5 of [25].

(0.30.4) If (V,x) is a Γ-strict analytic domain of (X,x), then a given morphism $(Y,y)\to(X,x)$ goes through (V,x) if and only if $\widetilde{(Y,y)}^{\Gamma}\to\widetilde{(X,x)}^{\Gamma}$ goes through $\widetilde{(V,x)}^{\Gamma}$; it follows from lemma 0.29 and from prop. 4.1 *iii*) of [25].

(0.30.5) A given morphism $(Y,y)\to(X,x)$ is boundaryless if and only if $\widetilde{(Y,y)}^{\Gamma}\simeq\widetilde{(X,x)}^{\Gamma}\times_{\mathbb{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}}\mathbb{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}}$; it follows from lemma 0.29 and from th. 5.2 of [25].

1 About analytic germs

We fix from now on and till the end of the paper an analytic field k and a subgroup Γ of $\mathbb{R}_+^*$ such that $|k^*|.\Gamma\neq\{1\}$. By an analytic (or affinoid) space without any mention of a ground field, we will always mean an analytic (or affinoid) space (X,L) with L an analytic extension of k; but as usual, L will be most of the time not mentioned at all.

Maps between Riemann-Zariski spaces: the trivially graded case

(1.1) Lemma. *Let K be a field, let $|.|$ be a valuation on it. Let L be an algebraic extension of K ; let $|.|'$ be a valuation on L extending $|.|$. Let $S_1, \ldots, S_n$ be indeterminates, and let $|.|''$ be an extension of $|.|'$ to $L(S_1, \ldots, S_n)$ whose restriction to $K(S_1, \ldots, S_n)$ is equal to $|.|_{\text{Gauß}}$. Then $|.|'' = |.|'_{\text{Gauß}}$.*

Proof. We denote by k (resp. ℓ) the residue field of $|.|$ (resp. $|.|'$). By construction, $|S_i|_{\text{Gauß}} = 1$ for any i, and the images of the S_i's in the residue field of $|.|_{\text{Gauß}}$ are algebraically independant over k. As L is algebraic over F, ℓ is algebraic over k. Therefore, the images of the S_i's in the residue field of $|.|''$ are algebraically independant over ℓ. It implies that if $P = \sum a_I \mathbf{S}^I$ is any element of the ring $L[S_1, \ldots, S_n]$ such that $|a_I|' \leqslant 1$ for any I, then $|P|'' < 1$ if and only if $|a_I|' < 1$ for any I. Now if $P = \sum a_I \mathbf{S}^I$ is any non-zero element of $L[S_1, \ldots, S_n]$, dividing P by a coefficient of maximal valuation and applying the preceeding results immediatly yields $|P|'' = \max |a_I|'$; therefore $|.|'' = |.|'_{\text{Gauß}}$. □

(1.2) Lemma. *Let F be a field and let $P = T^n + a_{n-1}T^{n-1} + \ldots + a_0$ be a monic polynomial belonging to $F[T]$; set $a_n = 1$. Assume that P is split in F. Let $|.|$ be a valuation on F. The following are equivalent:*

i) $|\lambda| > 1$ *for every root* λ *of* P *in* F;

ii) $|a_0| > |a_j|$ *for any* $j > 0$.

Proof. If *i)* is true then *ii)* follows immediatly from the usual relations between the coefficients and the roots of P. Suppose that *ii)* is true and let λ be a root of P. As $P(\lambda) = 0$ there exists $j > 0$ such that $|a_0| \leqslant |a_j \lambda^j|$. Since $|a_0| > |a_j|$, this implies that $|\lambda| > 1$. □

(1.3) Proposition. *Let k be a field, let K be an extension of k, and let L be an algebraic extension of K. Let U be a quasi-compact open subset of $\mathbb{P}_{L/k}$. Its image V on $\mathbb{P}_{K/k}$ is a quasi-compact open subset of the latter.*

Proof. We can assume that U is equal to $\mathbb{P}_{K/k}\{f_1, \ldots, f_l\}$ for suitable elements f_i's of L. Set $f = f_1S_1 + \ldots f_lS_l \in L(\mathbf{S}) := L(S_1, \ldots, S_l)$. Let $P = T^n + a_{n-1}T^{n-1} + \ldots + a_0$ be the minimal polynomial of f over $K(\mathbf{S})$. Let $\mathbb{K}$ be a finite extension of $K(\mathbf{S})$ in which P splits. Let $|.|$ be a valuation on K whose restriction to k is trivial. We fix an extension $|.|_0$ of $|.|_{\text{Gauß}}$ to $\mathbb{K}$.

(1.3.1) *The valuation $|.|$ belongs to V if and only if $|.|_{\text{Gauß}}$ extends to a valuation $|.|''$ on $L(\mathbf{S})$ such that $|f|'' \leqslant 1$.* Indeed, let us first assume that $|.| \in \mathsf{V}$. This means that it extends to a valuation $|.|'$ on L such that $|f_i|' \leqslant 1$ for every i, and $|'|_{\text{Gauß}}$ is then an extension of $|_{\text{Gauß}}|$ to $L(\mathbf{S})$ satisfying the required properties.

Conversely, assume that $|.|_{\text{Gauß}}$ extends to a valuation $|.|''$ on $L(\mathbf{S})$ such that $|f|'' \leqslant 1$ and let $|.|'$ denotes the restriction of $|.|''$ to L. Thanks to lemma 1.1, one has $|.|'' = |'|_{\text{Gauß}}$; therefore, the inequality $|f|'' \leqslant 1$ simply means that $|f_i|' \leqslant 1$ for every i, and we are done.

(1.3.2) The valuation $|_{\text{Gauß}}|$ admits an extension $|'|'$ to $L(\mathbf{S})$ such that $|f|'' \leqslant 1$ if and only if there exists a root λ of P in $\mathbb{K}$ such that $|\lambda|_0 \leqslant 1$; according to lemma 1.2, the latter condition is equivalent to the existence of a positive j such that $|a_0|_{\text{Gauß}} \leqslant |a_j|_{\text{Gauß}}$. Hence V is eventually equal, in view of 1.3.1 above, to the preimage under $|.| \mapsto |.|_{\text{Gauß}}$ of $\bigcup\limits_{a_j \neq 0} \mathbb{P}_{K(\mathbf{S})/k}\{a_0/a_j\}$; this preimage is easily seen, by the very definition of $|.|_{\text{Gauß}}$ for a given $|.|$, to be a quasi-compact open subset of $\mathbb{P}_{K/k}$. $\square$

(1.4) Theorem. *Let K be a field, let F be an extension of K and let L be an extension of F. Let $f_1, \ldots, f_n$ be finitely many elements of L and let A be the F-subalgebra of L generated by the f_i's. For any $y \in \mathsf{Spec}\, A$, denote by p_y the map $\mathbb{P}_{\kappa(y)/K} \to \mathbb{P}_{F/K}$.*

i) The image V of $\mathbb{P}_{L/K}\{f_1, \ldots, f_n\})$ on $\mathbb{P}_{F/K}$ is a quasi-compact open subset of the latter.

ii) There exist finitely many closed *points $y_1, \ldots, y_m$ of $\mathsf{Spec}\, A$ such that*

$$\mathsf{V} = \bigcup_j p_{y_j}(\mathbb{P}_{\kappa(y_j)/K}\{f_1(y_j), \ldots, f_n(y_j)\}\ ).$$

Proof. If y is any closed point of $\mathsf{Spec}\, A$, its residue field $\kappa(y)$ is finite over F. Proposition 1.3 then ensures that $p_y(\ \mathbb{P}_{\kappa(y)/K}\{f_1(y), \ldots, f_n(y)\}\ )$ is a quasi-compact open subset of $\mathbb{P}_{F/K}$. As V is a quasi-compact topological space, it is therefore enough, to establish both *i*) and *ii*), to prove that

$$\mathsf{V} = \bigcup_{y \in \mathsf{C}} p_y(\mathbb{P}_{\kappa(y)/K}\{f_1(y), \ldots, f_n(y)\}\ )$$

where C is the set of *all* closed points of $\mathsf{Spec}\, A$.

(1.4.1) Let us first prove that

$$\bigcup_{y \in \mathsf{C}} p_y(\ \mathbb{P}_{\kappa(y)/K}\{f_1(y), \ldots, f_n(y)\}\ ) \subset \mathsf{V}.$$

We will even show that

$$\bigcup_{y \in \mathsf{Spec}\, A} p_y(\ \mathbb{P}_{\kappa(y)/K}\{f_1(y), \ldots, f_n(y)\}\ ) \subset \mathsf{V}.$$

Let y be any point of $\mathsf{Spec}\, A$ and let $|.|$ be a valuation on F which is trivial on K and which belongs to $p_y(\ \mathbb{P}_{\kappa(y)/K}\{f_1(y), \ldots, f_n(y)\}\ )$, that is, which extends to a valuation $|.|'$ on $\kappa(y)$ which satisfies the inequality $|f_j(y)|' \leqslant 1$ for any j. Let $|.|''$ be a valuation on L whose ring dominates $\mathscr{O}_{\mathsf{Spec}\, A, y}$. The residue field $\mathbb{K}$ of $|.|''$ is an extension of $\kappa(y)$; we choose an extension $|.|'''$ of $|.|'$ to $\mathbb{K}$. The composition of $|.|''$ and $|.|'''$ is a valuation on L whose restriction to F is equal to $|.|$ and whose ring contains the f_j's. Hence $|.| \in p_{L/F}(\mathbb{P}_{L/K}\{f_1, \ldots, f_n\})$.

(1.4.2) Let us now prove that

$$\mathsf{V} \subset \bigcup_{y \in \mathsf{C}} p_y(\ \mathbb{P}_{\kappa(y)/K}\{f_1(y), \ldots, f_n(y)\}\ ).$$

Let $|.|$ be a valuation on F which is trivial on K and which belongs to V, that is, which extends to a valuation $|.|'$ on L which satisfies the inequalities $|f_i|' \leqslant 1$

for all i's. If $|.|$ is trivial, we choose a closed point y on $\mathsf{Spec}\, A$ (this is possible since $A \neq \{0\}$); then $|.|$ is the restriction to F of the trivial valuation on $\kappa(y)$, whose ring contains evidently the $f_j(y)$'s, and we are done.

Suppose that $|.|$ is non trivial, let $\mathbb{L}$ be an algebraic closure of L and let $\mathbb{F}$ be the algebraic closure of F inside $\mathbb{L}$. We choose an extension $|.|''$ of $|.|'$ to $\mathbb{L}$. Let $(P_1, \ldots, P_m)$ be polynomials which generate the ideal of relations between the f_i's over the field F.

The system of equations and inequalities (in variables $x_1, \ldots, x_n$)

$$\{P_j(x_1, \ldots, x_n) = 0\}_{j=1,\ldots,m} \text{ and } \{|x_i|'' \leqslant 1\}_{i=1,\ldots,n}$$

has a solution in $\mathbb{L}$, provided by the f_i's. The quantifiers elimination for non-trivially valued, algebraically closed fields ensures then that it has a solution in $\mathbb{F}$; that is exactly what was needed. $\square$

Maps between Riemann-Zariski spaces: the general case

(1.5) Lemma. *Let*

$$\begin{array}{ccc} E & \longrightarrow & F \\ \uparrow & & \uparrow \\ K & \longrightarrow & L \end{array}$$

be a commutative diagram of graded fields such that $E \otimes_K L \to F$ is injective. Let A (resp. B, resp. C) be a graded valuation ring of K (resp. L, resp. E); assume that $B \cap K = C \cap K = A$. There exists a graded valuation ring D of F such that $D \cap E = C$ and $D \cap L = B$.

Proof. We denote by a, b and c the respective residue graded fields of A, B and C. We choose a maximal graded ideal of the non-zero graded ring $\mathsf{b} \otimes_{\mathsf{a}} \mathsf{c}$; let d be the corresponding quotient.

Since C has no A-torsion, it is A-flat (with respect to the graded tensor product). Therefore $C \otimes_A B \hookrightarrow C \otimes_A L = (C \otimes_A K) \otimes_K L$. As $C \otimes_A K$ is simply a graded localization of C by a homogeneous multiplicative subset which doesn't contain zero, $C \otimes_A K \hookrightarrow E$. It follows that $C \otimes_A L \hookrightarrow E \otimes_K L \hookrightarrow F$. The natural map $B \otimes_A C \to B.C \subset F$ is thus an isomorphism. Hence there exists a (unique) map $B.C \to \mathsf{d}$ extending both $B \to \mathsf{b} \to \mathsf{d}$ and $C \to \mathsf{c} \to \mathsf{d}$. The kernel of this map is a homogeneous prime ideal (because its target is a graded field); by Zorn's lemma the corresponding graded localization of $B.C$ is dominated by a graded valuation ring D of F. By construction, D satisfies the required property. $\square$

(1.6) Corollary. *Let K, L, E, F be as above and let $\mathfrak{l}$ be a graded subfield of L; set $\mathfrak{k} = K \cap \ell$. Define π, ρ, φ and ψ by the commutative diagram*

$$\begin{array}{ccc} \mathbb{P}_{F/\mathfrak{l}} & \xrightarrow{\pi} & \mathbb{P}_{E/k} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{P}_{L/\mathfrak{l}} & \xrightarrow{\rho} & \mathbb{P}_{K/\mathfrak{k}} \end{array} .$$

If U *is any subset of* $\mathbb{P}_{E/\mathfrak{k}}$*, then* $\varphi(\pi^{-1}(\mathsf{U})) = \rho^{-1}(\psi(\mathsf{U}))$.

Proof. The inclusion $\varphi(\pi^{-1}(\mathsf{U})) \subset \rho^{-1}(\psi(\mathsf{U}))$ is obvious (and follows only from the commutativity of the diagram). Now, let $|.| \in \rho^{-1}(\psi(\mathsf{U}))$. This means that $|.|$ is a graded valuation on L, trivial over $\mathfrak{l}$, and that there exists a graded valuation $|.|' \in \mathsf{U}$ such that $|.|'_{|K} = |.|_{|K}$. Thanks to lemma 1.5 above, there exists a graded valuation $|.|''$ on F whose restriction to E is equal to $|.|'$, and whose restriction to L is equal to $|.|$. The latter fact implies that the restriction of $|.|''$ to $\mathfrak{l}$ is trivial; therefore $|.|'' \in \pi^{-1}(\mathsf{U})$ and $|.| \in \varphi(\pi^{-1}(\mathsf{U}))$. □

(1.7) Corollary 1.6 above can be used every time one has a commutative diagram of graded fields

$$\begin{tikzcd} E \arrow[r] & F \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

such that $E \otimes_K L \to F$ is injective. Let us give two examples of such a diagram, which will play a role in the sequel.

(1.7.1) Let K be a graded field and let L be a graded extension of K. The natural map $L^1 \otimes_{K^1} K \to L$ is injective. Indeed, let Δ be the group of degrees of non-zero homogeneous elements of K; one has $K = \bigoplus_{\delta \in \Delta} K^\delta$. For every $\delta \in \Delta$, the K^1-vector space K^δ is one-dimensional, and the L^1-vector space L^δ is also one-dimensional and contains K^δ; it follows that $L^1 \otimes_{K^1} K^\delta \to L^\delta$ is an isomorphism. Therefore

$$L^1 \otimes_{K^1} K \simeq \bigoplus_{\delta \in \Delta} L^\delta \subset L,$$

whence the claim.

(1.7.2) Let K be a graded field, let $s_1, \ldots, s_n$ be positive real numbers, and let L be a graded extension of K. Let us write $K(\mathbf{s}^{-1}\mathbf{S})$ (resp. $L(\mathbf{s}^{-1}\mathbf{S})$) for $K(s_1^{-1}S_1, \ldots, s_m^{-1}S_m)$ (resp. ...). The natural arrow $L \otimes_K K(\mathbf{s}^{-1}\mathbf{S}) \to L(\mathbf{s}^{-1}\mathbf{S})$ is then injective. Indeed, it follows directly from the definition that

$$L \otimes_K K[\mathbf{s}^{-1}\mathbf{S}] \simeq L[\mathbf{s}^{-1}\mathbf{S}].$$

Therefore $L \otimes_K K(\mathbf{s}^{-1}\mathbf{S})$ appears as a localization of the graded domain $L[\mathbf{s}^{-1}\mathbf{S}]$ by a homogeneous multiplicative system which doesn't contain zero; hence it embeds in the fraction field $L(\mathbf{s}^{-1}\mathbf{S})$ of $L[\mathbf{s}^{-1}\mathbf{S}]$.

(1.8) Theorem. *Let* K *be a graded field, let* F *be a graded extension of* K *and let* L *be a graded extension of* F*. Let* Δ *be a subgroup of* $\mathbb{R}_+^*$ *and let* V *be a* Δ*-strict quasi-compact open subset of* $\mathbb{P}_{L/K}$*. Its image on* $\mathbb{P}_{F/K}$ *is a* Δ*-strict quasi-compact open subset of the latter.*

Proof. We will first treat a particular case.

(1.8.1) *Assume that* $\Delta = \{1\}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{P}_{L/K} & \xrightarrow{\pi} & \mathbb{P}_{L_1/K_1} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{P}_{F/K} & \xrightarrow{\rho} & \mathbb{P}_{F^1/K^1} \end{array} .$$

By assumption, there exists a quasi-compact open subset U of $\mathbb{P}_{L^1/K^1}$ such that $\mathsf{V} = \pi^{-1}(\mathsf{U})$. By cor. 1.6 and 1.7.1, one has $\varphi(\mathsf{V}) = \rho^{-1}(\psi(\mathsf{U}))$. By th. 1.4, $\psi(\mathsf{U})$ is a quasi-compact open subset of $\mathbb{P}_{F^1/K^1}$, whence the result.

(1.8.2) *Let us treat now the general case.* Let $s_1, \ldots, s_m$ be elements of Δ such that the degrees of the f_i's all belong to the subgroup of $\mathbb{R}^*_+$ generated by the s_j's. Write $K(\mathbf{s}^{-1}\mathbf{S})$ (resp. $L(\mathbf{s}^{-1}\mathbf{S})$) for $K(s_1^{-1}S_1, \ldots, s_m^{-1}S_m)$ (resp. ...). Let us consider the commutative diagram

$$\begin{array}{ccc} \mathbb{P}_{L(\mathbf{s}^{-1}\mathbf{S})/k} & \xrightarrow{\mu} & \mathbb{P}_{L/k} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathbb{P}_{K(\mathbf{s}^{-1}\mathbf{S})/k} & \xrightarrow{\nu} & \mathbb{P}_{K/k} \end{array} .$$

The quasi-compact open subset $\mu^{-1}(\mathsf{V})$ of $\mathbb{P}_{L(\mathbf{s}^{-1}\mathbf{S})/k}$ is strict by choice of the s_j's. It follows therefore from 1.8.1 that $\theta(\mu^{-1}(\mathsf{V}))$ is a strict quasi-compact open subset of $\mathbb{P}_{K(\mathbf{s}^{-1}\mathbf{S})/k}$. By cor. 1.6 and 1.7.2, $\theta(\mu^{-1}(\mathsf{V})) = \nu^{-1}(\varphi(\mathsf{V}))$. As a consequence, a graded valuation $|.|$ on K which is trivial on k belongs to $\varphi(\mathsf{V})$ if and only if $|.|_{\text{Gauß}}$ belongs to $\theta(\mu^{-1}(\mathsf{V}))$. The latter being a strict quasi-compact open subset, if follows from the very definition of $|.| \mapsto |.|_{\text{Gauß}}$ and from the choice of the s_j's that $\varphi(\mathsf{V})$ is a Δ-strict quasi-compact open subset of $\mathbb{P}_{K/k}$. $\square$

The smallest subdomain containing the image of a germ

(1.9) Let $(Y, y) \to (X, x)$ be a morphism of k-analytic germs and assume that (Y, y) is Γ-strict.

(1.9.1) Let V be an open subset of $\widetilde{(Y, y)}$ such that $\mathsf{V} \to \mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$ identifies V with a Γ-strict quasi-compact open subset of $\mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$, and let U be the image of V on $\widetilde{(X, x)}$. It follows from th. 1.8 that $\mathsf{U} \to \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ identifies U with a Γ-strict quasi-compact open subset of $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$.

(1.9.2) Let Z be the image of $\widetilde{(Y, y)}$ on $\widetilde{(X, x)}$. It follows from 1.9.1 above that Z admits a finite covering $\bigcup \mathsf{Z}_i$ where Z_i is for every i homeomorphic to a Γ-strict quasi-compact open subset of $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$; as a consequence, Z is a quasi-compact open subset of $\widetilde{(X, x)}$, and is Γ-strict if $\widetilde{(X, x)} \subset \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, that is, if (X, x) is separated.

The quasi-compact, non-empty open subset Z corresponds to an analytic domain (Z, x) of (X, x). By what we have just seen, the analytic domain (Z, x)

has a finite covering by separated, Γ-strict analytic domains, and is itself Γ-strict as soon as (X, x) is separated. In view of 0.30.4, (Z, x) *is the smallest analytic domain of* (X, x) *through which* $(Y, y) \to (X, x)$ *goes.*

It follows from the construction that if $(Y_1, y), \ldots, (Y_n, y)$ are analytic domains of (Y, y) such that $(Y, y) = \bigcup (Y_i, y)$ an if (Z_i, x) denotes for every i the smallest analytic domain of (X, x) through which $(Y_i, y) \to (X, x)$ goes, then $(Z, x) = \bigcup (Z_i, x)$.

(1.9.3) If $\widetilde{(Y, y)} \to \widetilde{(X, x)}$ is surjective, then $(Z, x) = (X, x)$; note that by 0.30.5 it happens in particular if $(Y, y) \to (X, x)$ is boundaryless.

(1.9.4) Let Δ be a subgroup of $\mathbb{R}_+^*$ such that $|k^*|.\Delta \neq \{1\}$ and such that $(Y, y), (X, x)$ and (Z, x) are Δ-strict. It follows from cor. 1.6 that the image of $\widetilde{(Y, y)}^{\Delta}$ on $\widetilde{(X, x)}^{\Delta}$ is equal to $\widetilde{(Z, x)}^{\Delta}$.

(1.10) Let us now mention some straightforward 'global' consequences of 1.9.2; let $\varphi : Y \to X$ be a morphism of k-analytic spaces.

(1.10.1) Let $y \in Y$ such that (Y, y) is Γ-strict; there exist a Γ-strict compact analytic neighborhood V of y in Y and a compact analytic domain U of X which is a finite union of compact, Γ-strict analytic domains, such that $\varphi(V) \subset U$. Note that if X is separated then U is itself Γ-strict.

(1.10.2) Assume that Y is compact and Γ-strict, and that X is separated. Using 1.10.1, one deduces the existence of a Γ-strict compact analytic domain W of X such that $\varphi(Y) \subset W$.

Germs and dimension

(1.11) Lemma. *Let X be an affinoid space and let $x \in X$. Let $\mathbf{x}$ be its image on $\mathscr{X}$ and let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}_{\mathscr{X},\mathbf{x}}$. The following are equivalent :*

- *i)* $\mathsf{centdim}(X, x) = \mathsf{dim}\ \overline{\{x\}}^{X_{Zar}}$*;*
- *ii)* $\mathfrak{m}\mathscr{O}_{X,x}$ *is the maximal ideal of* $\mathscr{O}_{X,x}$*;*
- *iii)* $\mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{X,x} = \mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{\mathscr{X},\mathbf{x}}$.

Proof. By flatness of the local map $\mathscr{O}_{\mathscr{X},\mathbf{x}} \to \mathscr{O}_{X,x}$, assertion *iii)* is equivalent to the fact that the codimension of the closed point ω of $\mathsf{Spec}\ \mathscr{O}_{X,x}$ inside the special fiber of $\mathsf{Spec}\ \mathscr{O}_{X,x} \to \mathscr{O}_{\mathscr{X},\mathbf{x}}$ is equal to zero; but this is the case if and only if this fiber is *set-theoretically* equal to $\{\omega\}$, that is, if and only if $\mathscr{O}_{X,x}/\mathfrak{m}$ is artinian; as the fibers of $\mathsf{Spec}\ \mathscr{O}_{X,x} \to \mathscr{O}_{\mathscr{X},\mathbf{x}}$ are (geometrically) reduced (0.13.4) the latter is equivalent to the fact that $\mathscr{O}_{X,x}/\mathfrak{m}$ is a field, which means exactly that $\mathfrak{m}\mathscr{O}_{X,x}$ is the maximal ideal of $\mathscr{O}_{X,x}$; hence *ii)* $\iff$ *iii)*.

On can prove that *i)* $\iff$ *ii)* after replacing X by any Zariski-closed subspace of X containing x; hence by taking for such a space the one that corresponds to the reduced closed subscheme $\overline{\{\mathbf{x}\}}$ of $\mathscr{X}$, one can assume that $\mathscr{X}$ is irreducible and reduced, and that $\mathfrak{m} = 0$; in this situation, X is also irreducible and reduced (*cf.* 0.13.5). Let d be the dimension of X; it coincides now with $\mathsf{dim}\ \overline{\{x\}}^{X_{\mathsf{Zar}}}$.

Assume that i) is true, that is, that $d = \mathsf{centdim}(X, x)$. Let f be a non-zero element of $\mathscr{O}_{X,x}$, and let V be an affinoid neighborhood of x on which f is defined. Let Y and Z be two irreducible components of V containing x. Both are of dimension d; if $Y \neq Z$, there intersection is a Zariski-closed subset of V containing x and of dimension strictly lower than d, which contradicts i). Then there is only one irreducible component Y of V containing x. The zero-locus of f on Y is a Zariski-closed subset of Y which is not equal to the whole Y, because if it where then f would vanish on a neighborhood of x, hence would vanish in the reduced local ring $\mathscr{O}_{X,x}$; but by assumption, this is not the case. Therefore the dimension of the zero-locus of f is strictly lower than d, and then this locus can not contain x because of i). As a consequence, f is invertible in $\mathscr{O}_{X,x}$. Thus the latter is a field, and we have proved ii).

Assume that ii) is true, that is, that $\mathscr{O}_{X,x}$ is a field. Let V be an affinoid neighborhood of x in X and let Z be a Zariski-closed subset of V containing x. Let $f_1, \ldots, f_n$ be analytic functions on V which generate the vanishing ideal of Z. For any i, we have $f_i(x) = 0$; the image of f_i in $\mathscr{O}_{X,x}$ is then not invertible, hence is zero. Therefore Z contains a neighborhood U of x in V. As X is irreducible of dimension d, it is purely d-dimensional and the dimension of U is equal to d. This implies that the dimension of Z (which is bounded by d) is also equal to d; therefore, i) is proved. □

(1.12) Corollary. *Let X be a good analytic space and let $x \in X$. One has the equality* $\mathsf{centdim}(X, x) + \mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{X,x} = \mathsf{dim}_x\ X$.

Proof. Set $d = \mathsf{centdim}(X, x)$. One can assume that X is k-affinoid and that $\overline{\{x\}}^{X_{\mathsf{Zar}}} = d$. Let $\mathbf{x}$ be the image of x on $\mathscr{X}$. Let $X_1, \ldots, X_n$ be the irreducible components of X that go through x. The Zariski-closure of x in X is an irreducible Zariski-closed subset Z of X of dimension d, which is included in every X_i. For any i, set $d_i = \mathsf{dim}\ X_i$ and $\delta_i = \mathsf{codim}_{\mathsf{Krull}}(Z, X_i)$; one has $d_i = \delta_i + d$ ([14], prop. 1.11).

Now $\mathsf{dim}_x\ X = \max d_i$; the preceeding lemma ensures that $\mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{X,x}$ coincides with $\mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{\mathscr{X},\mathbf{x}}$, that is, with $\mathsf{codim}_{\mathsf{Krull}}(Z, X)$ which is nothing but $\max \delta_i$. The corollary follows immediatly. □

(1.13) Lemma. *If $Y \to X$ is a morphism between good k-analytic spaces, if $y \in Y$ is a point at which this map is finite, and if x denotes the image of y on X, then* $\mathsf{centdim}(X, x) = \mathsf{centdim}(Y, y)$.

Proof. One can shrink Y and X so that both are affinoid, and so that $\mathsf{dim}\ \overline{\{x\}}^{X_{\mathsf{Zar}}} = \mathsf{centdim}(X, x)$ and $\mathsf{dim}\ \overline{\{y\}}^{Y_{\mathsf{Zar}}} = \mathsf{centdim}(Y, y)$. Now the image of $\overline{\{y\}}^{Y_{\mathsf{Zar}}}$ on X coincides with $\overline{\{x\}}^{X_{\mathsf{Zar}}}$; but it follows then from 0.15.1that $\mathsf{dim}\ \overline{\{x\}}^{X_{\mathsf{Zar}}} = \mathsf{dim}\ \overline{\{y\}}^{Y_{\mathsf{Zar}}}$, whence our claim. □

2 Flatness and universal flatness

Algebraic flatness *versus* analytic flatness

(2.1) Lemma. *Let* $\begin{array}{ccc} D & \longleftarrow & C \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$ *be a commutative diagramm of commutative rings such that C (resp. D) is flat (resp. faithfully flat) over A (resp. B). If M is a B-module such that $D \otimes_B M$ is C-flat, then M is A-flat.*

Proof. Let $N \hookrightarrow N'$ be an injective linear map between two A-modules. As C is A-flat, $C \otimes_A N \hookrightarrow C \otimes_A N'$. As $M \otimes_B D$ is C-flat,

$$\underbrace{(M \otimes_B D) \otimes_C (C \otimes_A N)}_{(M \otimes_B D) \otimes_A N} \hookrightarrow \underbrace{(M \otimes_B D) \otimes_C (C \otimes_A N')}_{(M \otimes_B D) \otimes_A N'}.$$

In other words, $(N \otimes_A M) \otimes_B D \hookrightarrow (N' \otimes_A M) \otimes_B D$. Faithfull flatness of the B-algebra D now implies that $N \otimes_A M \hookrightarrow N' \otimes_A M$. □

Fitting together 0.13.5 and lemma 2.1 above immediately gives sthe following result.

(2.2) Lemma. *Let $\mathscr{A} \to \mathscr{B}$ be a morphism between k-affinoid algebras; let $\mathscr{Y}$ (resp. $\mathscr{X}$) be a $\mathscr{B}$-scheme (resp. an $\mathscr{A}$-scheme) of finite type, and let $\mathscr{Y} \to \mathscr{X}$ be an $\mathscr{A}$-morphism. Let $y \in \mathscr{Y}^{an}$ and let $\mathbf{y}$ be its image on $\mathscr{Y}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathscr{Y}$. If $\mathcal{F}^{an}$ is $\mathscr{X}^{an}$-flat at y, then $\mathcal{F}$ is $\mathscr{X}$-flat at $\mathbf{y}$.* □

(2.3) We would like now prove the converse implication in some particular cases; let us first mention two cases in which it is more or less well-known.

(2.3.1) Let $Y \to X$ be a *finite* morphism between affinoid spaces, let $y \in Y$ and let $\mathbf{y}$ be its image on $\mathscr{Y}$. If $\mathcal{F}$ is a coherent sheaf on Y, then $\mathscr{F}$ is X-flat at y if and only if it is $\mathscr{X}$-flat at $\mathbf{y}$: this is essentially prop. 3.2.1 of [2] – the latter is written only for $\mathscr{F} = \mathscr{O}_Y$, but its proof works actually for any coherent sheaf.

(2.3.2) Let $Y \to X$ be a morphism between affinoid spaces, let $y \in Y$ and let x be its image on X ; denote by $\mathbf{y}$ and $\mathbf{x}$ their respective images on $\mathscr{Y}$ and $\mathscr{X}$. If x and y are rigid and if $\mathcal{F}$ is a coherent sheaf on Y then $\mathcal{F}$ is X-flat at y if and only if it is $\mathscr{X}$-flat at $\mathbf{y}$. If $|k^*| \neq \{1\}$ and if Y and X are strictly k-affinoid, this is a classical assertion of rigid-analytic geometry, but its proof is very simple and immediately extends to our situation: indeed, one knows from [18], cor. 5.8 that $\mathcal{F} \otimes \mathscr{O}_{Y,y}$ is flat over $\mathscr{O}_{X,x}$ if and only if $\mathcal{F} \otimes \widehat{\mathscr{O}_{Y,y}}$ is flat over $\widehat{\mathscr{O}_{X,x}}$, and that $\mathcal{F} \otimes \mathscr{O}_{\mathscr{Y},\mathbf{y}}$ is flat over $\mathscr{O}_{\mathscr{X},\mathbf{x}}$ if and only if $\mathcal{F} \otimes \widehat{\mathscr{O}_{\mathscr{Y},\mathbf{y}}}$ is flat over $\widehat{\mathscr{O}_{\mathscr{X},\mathbf{x}}}$; but as x and y are rigid, $\widehat{\mathscr{O}_{X,x}} = \widehat{\mathscr{O}_{\mathscr{X},\mathbf{x}}}$ and $\widehat{\mathscr{O}_{Y,y}} = \widehat{\mathscr{O}_{\mathscr{Y},\mathbf{y}}}$ ([2], lemma 2.6.3), whence our claim.

(2.4) Now we are going to prove two results. The first one is a generalization of 2.3.1 to any morphism between schemes of finite type over a given affinoid algebra; the second one will concern any morphism between affinoid spaces, and extend both 2.3.1 and 2.3.2.

(2.5) Proposition. *Let $\mathscr{Y} \to \mathscr{X}$ be a morphism between schemes of finite type over a given affinoid algebra. Let $\mathcal{F}$ be a coherent sheaf on $\mathscr{Y}$, let y be a point of $\mathscr{Y}^{an}$ and let $\mathbf{y}$ be its image on $\mathscr{Y}$. If $\mathcal{F}$ is $\mathscr{X}$-flat at $\mathbf{y}$, then $\mathcal{F}^{an}$ is $\mathscr{X}^{an}$-flat at y.*

Proof. We can assume that $\mathscr{X}$ is affine. Let x be the image of y on $\mathscr{X}^{an}$ and let U be an affinoid neighborhood of x in $\mathscr{X}^{an}$. There is a natural map from $\mathscr{O}_{\mathscr{X}}(\mathscr{X})$ to $\mathscr{O}_{\mathscr{U}}(\mathscr{U})$ which induces a morphism $\mathscr{U} \to \mathscr{X}$, and the space $\mathscr{Y}^{an} \times_{\mathscr{X}^{an}} U$ can be identified with $(\mathscr{Y} \times_{\mathscr{X}} \mathscr{U})^{an}$. Let us call $\mathbf{z}$ the image of y on $\mathscr{Y} \times_{\mathscr{X}} \mathscr{U}$. Since flatness is preserved by any *scheme-theoretic* base change, $\mathcal{F} \otimes \mathscr{O}_{\mathscr{Y} \times_{\mathscr{X}} \mathscr{U}, \mathbf{z}}$ is $\mathscr{O}_{\mathscr{U},\mathbf{x}}$-flat, where $\mathbf{x}$ is the image of x on $\mathscr{U}$. The ring $\mathscr{O}_{\mathscr{Y}^{an} \times_{\mathscr{X}^{an}} U, y}$ is $\mathscr{O}_{\mathscr{Y} \times_{\mathscr{X}} \mathscr{U}, \mathbf{z}}$-flat by 0.13.5; therefore, $\mathcal{F} \otimes \mathscr{O}_{\mathscr{Y}^{an} \times_{\mathscr{X}^{an}} U, y'}$ is $\mathscr{O}_{\mathscr{U},\mathbf{x}}$-flat. One concludes with a straightforward limit argument. □

(2.6) Theorem. *Let $\mathscr{A} \to \mathscr{B}$ be a morphism between k-affinoid algebras and let $Y \to X$ be the induced arrow between the corresponding affinoid spaces. Let M be a finitely generated $\mathscr{B}$-module, let $y \in Y$ and let x be its image on X. Let $\mathbf{y}$ (resp. $\mathbf{x}$) be the image of y on $\mathscr{Y}$ (resp. of x on $\mathscr{X}$). Assume that there exists a Zariski-closed subspace Z of Y containing y such that $Z \to X$ is finite. Suppose that $M \otimes_{\mathscr{B}} \mathscr{O}_{\mathscr{Y},\mathbf{y}}$ is $\mathscr{O}_{\mathscr{X},\mathbf{x}}$-flat; then $M \otimes_{\mathscr{B}} \mathscr{O}_{Y,y}$ is $\mathscr{O}_{X,x}$-flat.*

Proof. Let V be an affinoid neighborhood of x in X and let W be the pre-image of V on Y. We denote by $\mathscr{A}_V$ (resp. $\mathscr{B}_W$) the algebra of analytic functions on V (resp. W); we set $M_W = M \otimes_{\mathscr{B}} \mathscr{B}_W$. Let $\mathfrak{p}$ the prime ideal of $\mathscr{A}$ that corresponds to $\mathbf{x}$, and let $\mathscr{I}$ be the ideal of $\mathscr{B}$ that corresponds to Z. Let η (resp. ξ) be the image of y (resp. x) on $\mathscr{W}$ (resp. $\mathscr{V}$).

We are going to prove that $M \otimes_{\mathscr{B}} \mathscr{O}_{\mathscr{W},\eta}$ is a flat $\mathscr{O}_{\mathscr{V},\xi}$-module. By [18], th. 5.6 the latter is true if and only if the two following conditions are satisfied:

i) $M \otimes_{\mathscr{B}} \mathscr{O}_{\mathscr{W},\eta}/\mathfrak{p}$ is $\mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$-flat;

ii) for any $d > 0$, the natural map

$$M \otimes_{\mathscr{B}} (\mathfrak{p}^d \mathscr{O}_{\mathscr{W},\eta}/\mathfrak{p}^{d+1}\mathscr{O}_{\mathscr{W},\eta}) \to \mathfrak{p}^d(M \otimes_{\mathscr{B}} \mathscr{O}_{\mathscr{W},\eta})/\mathfrak{p}^{d+1}(M \otimes_{\mathscr{B}} \mathscr{O}_{\mathscr{W},\eta})$$

is an isomorphism.

(2.6.1) *Let us prove i).* The quotient $\mathscr{B}/\mathscr{I}$ is a finite $\mathscr{A}$-algebra; we therefore have $\mathscr{B}_W/\mathscr{I} = (\mathscr{B}/\mathscr{I}) \otimes_{\mathscr{A}} \mathscr{A}_V$. We set $N = M/\mathfrak{p}M$.

Let n be a non-negative integer. As $\mathscr{O}_{\mathscr{X},\mathbf{x}}/\mathfrak{p}$ is a field, the $\mathscr{O}_{\mathscr{X},\mathbf{x}}/\mathfrak{p}$-module $(\mathscr{I}^n N/\mathscr{I}^{n+1}N) \otimes_{\mathscr{A}} \mathscr{O}_{\mathscr{X},\mathbf{x}}/\mathfrak{p}$ is flat. It follows that $(\mathscr{I}^n N/\mathscr{I}^{n+1}N) \otimes_{\mathscr{A}} \mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$ is a flat $\mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$-module. It can be rewritten as

$$(\mathscr{I}^n N/\mathscr{I}^{n+1}N) \otimes_{\mathscr{B}/\mathscr{I}} (\mathscr{B}/\mathscr{I}) \otimes_{\mathscr{A}} \mathscr{A}_V \otimes_{\mathscr{A}_V} \mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$$

$$= (\mathscr{I}^n N/\mathscr{I}^{n+1}N) \otimes_{\mathscr{B}/\mathscr{I}} (\mathscr{B}_W/\mathscr{I}) \otimes_{\mathscr{A}_V} \mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}.$$

The ring $\mathscr{O}_{\mathscr{W},\eta}/(\mathfrak{p} + \mathscr{I})$ is a localization of $(\mathscr{B}_W/\mathscr{I}) \otimes_{\mathscr{A}_V} \mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$. For that reason, $(\mathscr{I}^n N/\mathscr{I}^{n+1}N) \otimes_{\mathscr{B}/\mathscr{I}} \mathscr{O}_{\mathscr{W},\eta}/(\mathfrak{p} + \mathscr{I})$ is $\mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$-flat.

We obviously have

$$(\mathscr{I}^n N/\mathscr{I}^{n+1}N)\otimes_{\mathscr{B}/\mathscr{I}}\mathscr{O}_{\mathscr{W},\eta}/(\mathfrak{p}+\mathscr{I})=(\mathscr{I}^n N/\mathscr{I}^{n+1}N)\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta}.$$

As $\mathscr{O}_{\mathscr{W},\eta}$ is a flat $\mathscr{B}$-algebra, the latter coincides with

$$\mathscr{I}^n(N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta})/\mathscr{I}^{n+1}(N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta}).$$

The $\mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$-module $\mathscr{I}^n(N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta})/\mathscr{I}^{n+1}(N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta})$ is thus flat for any non-negative n. It obviously implies that for any such n, the $\mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$-module

$$(N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta})/\mathscr{I}^{n+1}(N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta})$$

is flat. By [16], chapt. 0, §10.2.6, $M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta}/\mathfrak{p}=N\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{W},\eta}$ is then $\mathscr{O}_{\mathscr{V},\xi}/\mathfrak{p}$-flat; hence $i)$ is true.

(2.6.2) *Let us prove $ii)$.* Let d be a positive integer. By assumption, $M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{Y},\mathbf{y}}$ is $\mathscr{O}_{\mathscr{X},\mathbf{x}}$-flat. By [18], th. 5.6, the natural map

$$M\otimes_{\mathscr{B}}(\mathfrak{p}^d\mathscr{O}_{\mathscr{Y},\mathbf{y}}/\mathfrak{p}^{d+1}\mathscr{O}_{\mathscr{Y},\mathbf{y}})\to\mathfrak{p}^d(M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{Y},\mathbf{y}})/\mathfrak{p}^{d+1}(M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{Y},\mathbf{y}})$$

is then an isomorphism. As $\mathscr{O}_{\mathscr{W},\eta}$ is a flat $\mathscr{O}_{\mathscr{Y},\mathbf{y}}$-algebra, $ii)$ follows immediately.

(2.6.3) Remark. The existence of Z (resp. the $\mathscr{O}_{\mathscr{X},\mathbf{x}}$-flatness of $M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{Y},\mathbf{y}}$) was used only while proving $i)$ (resp. $ii)$).

(2.6.4) *Conclusion.* Let T be any affinoid neighborhood of y in W, and let τ be the image of y on $\mathscr{T}$. As $\mathscr{O}_{\mathscr{T},\tau}$ is a flat $\mathscr{O}_{\mathscr{W},\eta}$-algebra, the $\mathscr{O}_{\mathscr{V},\xi}$-module $M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{T},\tau}$ is flat.

We thus have shown the following: if V is any neighborhood of x in X, if T is any affinoid neighborhood of y in the pre-image of V inside Y, and if τ (resp. ξ) denotes the image of y (resp. x) on $\mathscr{T}$ (resp. $\mathscr{V}$), then $M\otimes_{\mathscr{B}}\mathscr{O}_{\mathscr{T},\tau}$ is $\mathscr{O}_{\mathscr{V},\xi}$-flat. A straightforward limit argument then ensures that $M\otimes_{\mathscr{B}}\mathscr{O}_{Y,y}$ is $\mathscr{O}_{X,x}$-flat. $\square$

Universal flatness

(2.7) Let $Y\to X$ be a morphism between good k-analytic spaces, and let $\mathcal{F}$ be a coherent sheaf on Y. Let $y\in Y$. We will say that $\mathcal{F}$ is *universally X-flat at y* if for any any good analytic space X', sfor any morphism $X'\to X$, and for any point y' lying above y on $Y':=Y\times_X X'$, the pull-back of $\mathcal{F}$ on Y' is X'-flat at y'. If $\mathcal{F}=\mathscr{O}_Y$, we will simply say that Y is universally X-flat at y, or that $Y\to X$ is universally flat at y. If $Y\to X$ is universally flat at any point of Y, we will simply say that $Y\to X$ is universally flat.

(2.8) Remark. If $\mathcal{F}$ is universally X-flat at y, then it is in particular X-flat at y.

Examples

(2.9) Let $Y \to X$ be a morphism between good k-analytic spaces and let $y \in Y$. Let X' be a good analytic space, let $X' \to X$ be a morphism, and let y' be a point of $Y' := Y \times_X X'$ lying over y. If $\mathcal{F}$ is a coherent sheaf on Y which is universally X-flat at y, then its inverse image on Y' is universally X'-flat at y'.

(2.10) Let $Z \to Y$ and $Y \to X$ be morphisms between good k-analytic spaces, let $z \in Z$ and let y be its image on Y. If $Z \to Y$ is universally flat at z, and if $Y \to X$ is universally flat at y, then $Z \to X$ is universally flat at z.

(2.11) If X is a good analytic space and if Y is a good analytic domain of X, then $Y \hookrightarrow X$ is universally flat (0.13.4).

(2.12) Let $\varphi : Y \to X$ be a morphism of good k-analytic spaces and let $y \in Y$. Let V be a good analytic domain of Y containing y and let U be a good analytic domain of X containing $\varphi(V)$. Let $\mathcal{F}$ be a coherent sheaf on Y. It follows straightforwardly from 2.11 above that the following are equivalent:

α) $\mathcal{F}$ is universally X-flat at y;

β) $\mathcal{F}_{|V}$ is universally U-flat at y.

(2.13) Let $\mathscr{A}$ be an affinoid algebra, and let $\mathscr{Y}$ and $\mathscr{X}$ be two $\mathscr{A}$-schemes of finite type. Let $\mathscr{Y} \to \mathscr{X}$ be an $\mathscr{A}$-morphism, let y be a point of $\mathscr{Y}^{an}$ and let $\mathbf{y}$ be the image of y on $\mathscr{Y}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathscr{Y}$ which is $\mathscr{X}$-flat at $\mathbf{y}$. Then $\mathcal{F}^{an}$ is universally $\mathscr{X}^{an}$-flat at y. Indeed, one can assume that $\mathscr{X}$ is affine. Let V be an affinoid space, let $V \to \mathscr{X}^{an}$ be a morphism, and let y' be a point of $\mathscr{Y}^{an} \times_{\mathscr{X}^{an}} V$ lying above y; there is a natural map $\mathscr{O}_{\mathscr{X}}(\mathscr{X}) \to \mathscr{O}_{\mathscr{V}}(\mathscr{V})$ which induces a morphism $\mathscr{V} \to \mathscr{X}$, and $\mathscr{Y}^{an} \times_{\mathscr{X}^{an}} V$ can be identified with $(\mathscr{Y} \times_{\mathscr{X}} \mathscr{V})^{an}$. Let us call $\mathbf{y}'$ the image of y' on $\mathscr{Y} \times_{\mathscr{X}} \mathscr{V}$. As flatness behaves well under *scheme-theoretic* base change, the pull-back of $\mathcal{F}$ on $\mathscr{Y} \times_{\mathscr{X}} \mathscr{V}$ is $\mathscr{V}$-flat at $\mathbf{y}'$; by prop. 2.5, the pull-back of $\mathcal{F}^{an}$ on $\mathscr{Y}^{an} \times_{\mathscr{X}^{an}} V$ is V-flat at $\mathbf{y}'$, whence the claim.

(2.14) Let Y be a good k-analytic space. The structure map $Y \to \mathscr{M}(k)$ is universally flat. To see that, one can assume that Y is k-affinoid. Let X be an affinoid space. Let $y \in Y \times_k X$, and let x be its image on X. Let U be an affinoid neighborhood of x in X, and let V be an affinoid neighborhood of y in $Y \times_X U$. Let $\mathscr{B}$ (resp. $\mathscr{A}$, resp. $\mathscr{C}$) be the respective algebras of analytic functions on Y, U and V. The $\mathscr{A}\widehat{\otimes}_k\mathscr{B}$-algebra $\mathscr{C}$ is flat ([1], prop. 2.2.4); the $\mathscr{A}$-algebra $\mathscr{A}\widehat{\otimes}_k\mathscr{B}$ is flat ([2], lemma 2.1.2, the fact that K is a field is not used in its proof); hence $\mathscr{C}$ is $\mathscr{A}$-flat. By a straightforward limit argument, $\mathscr{O}_{Y\times_k X,y}$ is a flat $\mathscr{O}_{X,x}$-algebra.

The flat, locally finite morphisms

We will use what we have just done to show some results which were already proven in [2], §3.2 when $\mathcal{F} = \mathscr{O}_Y$; we include the proofs (which may differ of those of [2]) for the convenience of the reader.

(2.15) Lemma. *Let $Y \to X$ be a finite morphism between good k-analytic spaces, let $x \in X$ and let $y_1, \ldots, y_r$ be the pre-images of x. Let $\mathcal{F}$ be a coherent sheaf on Y. The $\mathscr{O}_{X,x}$-module $\pi_*\mathcal{F} \otimes \mathscr{O}_{X,x}$ is isomorphic to $\prod \mathcal{F} \otimes \mathscr{O}_{Y,y_i}$. In particular, $\pi_*\mathcal{F}$ is flat at x if and only if $\mathcal{F}$ is X-flat at every y_i.*

Proof. It follows straightforwardly from the fact (due to properness of $Y \to X$) that for any neighborhood V of $\{y_1, \ldots, y_r\}$, there exist an affinoid neighborhood U of x in X whose pre-image is included in V and is a *disjoint* union $\coprod V_i$, where V_i is for every i an affinoid neighborhood of y_i in Y. □

(2.16) Proposition. *Let $Y \to X$ be a morphism between good k-analytic spaces and let y be a point of Y at which this morphism is finite; let x be the image of y on X. Let $\mathcal{F}$ be a coherent sheaf on Y. The following are equivalent:*

i) $\mathcal{F}$ is X-flat at y;

ii) there exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\pi_*\mathcal{F}_{|T}$ is flat at x;

iii) there exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\pi_*\mathcal{F}_{|T}$ is a free $\mathscr{O}_S$-module;

iv) there exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\mathcal{F}(T)$ is a flat $\mathscr{O}_S(S)$-module;

v) there exist an affinoid domain T of Y containing y and an affinoid domain S of X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\mathcal{F}(T)$ is a flat $\mathscr{O}_S(S)$-module;

vi) $\mathcal{F}$ is universally X-flat at y.

Proof. Suppose that *i*) is true. As $Y \to X$ is finite at y, there exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ for which y is the only pre-image of x; as $\mathcal{F}$ is X-flat at y, lemma 2.15 tells us that $\pi_*\mathcal{F}_{|T}$ is T-flat at x, whence *ii*). If *ii*) is true, then $\pi_*\mathcal{F} \otimes \mathscr{O}_{S,x}$ is free, hence *iii*) follows by shrinking S (and then T). Both implications *iii*) $\Rightarrow$ *iv*) and *iv*) $\Rightarrow$ *v*) are obvious. If *v*) is true, then by 2.13, $\mathcal{F}_{|T}$ is universally S-flat at y; therefore $\mathcal{F}$ is universally X-flat at y (2.12), that is, *vi*) is true; and *vi*) $\Rightarrow$ *i*) is obvious. □

(2.17) Corollary. *Let $Y \to X$ be a morphism of good k-analytic spaces, let $y \in Y$ such that $Y \to X$ is finite at y and let x be the image of y. Let $\mathcal{F}$ be a coherent sheaf on Y. If $y \in \mathsf{Supp}\ \mathcal{F}$ and if $\mathcal{F}$ is X-flat at y, the image of $\mathsf{Supp}\ \mathcal{F}$ on X is a neighborhood of x.*

Proof. Let us choose T and S as in *iii*) above; as $y \in \mathsf{Supp}\ \mathcal{F}$, the free $\mathscr{O}_S$-module $\pi_*\mathcal{F}_{|T}$ is of positive rank; therefore, the image of $\mathsf{Supp}\ \mathcal{F}$ contains S. □

(2.18) Corollary. *Let $Y \to X$ be a morphism of good k-analytic spaces, let $y \in Y$ such that $Y \to X$ is finite at y and let x be the image of y. Let*

$\mathcal{F}$ be a coherent sheaf on Y. If $y \in \mathsf{Supp}\ \mathcal{F}$ and if $\mathcal{F}$ is X-flat at y, then $\mathsf{dim}_x\ X = \mathsf{dim}_y\ \mathsf{Supp}\ \mathcal{F}$.

Proof. One can assume that $Y \to X$ is finite and that y is the only pre-image of x on Y. The image of $\mathsf{Supp}\ \mathcal{F}$ on Y is a Zariski-closed subset T of X, and one has $\mathsf{dim}_x\ T = \mathsf{dim}_y\ \mathsf{Supp}\ \mathcal{F}$ (0.15.2); on the other way, T is a neighborhood of x in X by the corollary above, hence $\mathsf{dim}_x\ T = \mathsf{dim}_x\ X$. □

Counter-examples

(2.19) Let r be a positive real number and let $f = \sum \alpha_i T^i$ be a power series with coefficients in k such that $|\alpha_i|r^i \xrightarrow[i \to +\infty]{} 0$ and such that $(|\alpha_i|s^i)_i$ is non-bounded as soon as $s > r$. We denote by $p : \mathbb{A}_k^{2,an} \to \mathbb{A}_k^{1,an}$ the first projection. Let X be the analytic domain of $\mathbb{A}_k^{2,an}$ defined by the inequality $|T_1| \leqslant r$, and let Y be the one-dimensional closed disc of radius r; note that $X = p^{-1}(Y)$, that is, X can be identified with $Y \times_k \mathbb{A}_k^{1,an}$. The map $\varphi := (\mathsf{Id}, f)$ from Y to X is a closed immersion; it induces an isomorphism between Y and the Zariski-closed subset $\varphi(Y)$ of X, endowed with its reduced structure; the converse isomorphism is nothing but $p_{|\varphi(Y)}$. Let y be the unique point of the Shilov boundary of Y, that is, the point given by the semi-norm $\sum a_i T \mapsto \max |a_i| r^i$; set $x = \varphi(y)$.

(2.20) Lemma. *Let T be reduced one-dimensional good analytic space and let $t \in T$ which is not a rigid point. The local ring $\mathscr{O}_{T,t}$ is a field.*

Proof. By cor. 1.12 one has $\mathsf{centdim}(T, t) + \mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{T,t} = \mathsf{dim}_t\ T$. As t is not rigid, $\mathsf{centdim}(T, t) > 0$; as T is one-dimensional, $\mathsf{dim}_t\ T \leqslant 1$. Therefore $\mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{T,t} = 0$; beeing reduced, $\mathscr{O}_{T,t}$ is thus a field. □

(2.21) Lemma. *The local ring $\mathscr{O}_{\mathbb{A}_k^{2,an},x}$ is a field.*

Proof. As the analytic space $\mathbb{A}_k^{2,an}$ is reduced, this is sufficient to prove that $\mathsf{dim}_{\mathsf{Krull}}\ \mathscr{O}_{\mathbb{A}_k^{2,an},x} = 0$, which is equivalent, in view of cor. 1.12, to the fact that $\mathsf{centdim}(\mathbb{A}_k^{2,an}, x) = 2$. Since x is not a rigid point (because $\mathscr{H}(x) = \mathscr{H}(y)$), $\mathsf{centdim}(\mathbb{A}_k^{2,an}, x) > 0$.

Assume that $\mathsf{centdim}(\mathbb{A}_k^{2,an}, x) = 1$. Then there exists an affinoid neighborhood V of x and an irreducible one-dimensional Zariski-closed subset Z of V which contains x.

Both $Z \cap X = Z \cap (V \cap X)$ and $\varphi(Y) \cap V = \varphi(Y) \cap (V \cap X)$ are purely one-dimensional Zariski-closed subsets of $V \cap X$ containing x. As x is not a rigid point, it belongs to a unique irreducible component of $(Z \cap X) \cup (\varphi(Y) \cap V)$; therefore it belongs to a unique irreducible component of $Z \cap X$, to a unique irreducible component of $\varphi(Y) \cap V$, and those two components coincide. One can hence shrink V so that $Z \cap X = \varphi(Y) \cap V$; by endowing Z with its reduced structure, this equality turns out to be an equality of Zariski-closed *subspaces* of $X \cap V$.

One has $\mathsf{d}(\mathscr{H}(y)/k) = 1$. If z is any point of Z such that $p(z) = y$, then the inequality $\mathsf{d}(\mathscr{H}(z)/k) \leqslant 1$ (due to the fact that Z is one-dimensional) forces $\mathsf{d}(\mathscr{H}(z)/\mathscr{H}(y))$ to be equal to zero. Therefore $p_{|Z}^{-1}(y)$ is purely zero-dimensional. In particular, $p_{|Z}$ is zero-dimensional at x; moreover, since x belongs to the topological interior of V in $\mathbb{A}_k^{2,an}$, the map $p_{|Z}$ is inner at x; prop. 3.1.4 of [2] then ensures that $p_{|Z}$ is finite at x; as $\mathscr{O}_{\mathbb{A}_k^{1,an},y}$ is a field by lemma 2.20, $p_{|Z}$ is flat at x. It follows from prop. 2.16 that one can shrink V so that there exists an affinoid neighborhood U of y in $\mathbb{A}_k^{1,an}$ such that $p(Z) \subset U$, and such that $p_{|Z} : Z \to U$ is finite and makes $\mathscr{O}_Z(Z)$ a free $\mathscr{O}_U(U)$-module of finite positive rank, say r (note that we thus have $p(Z) = U$). By restricting to Y, one sees that $p_{|Z\cap X} : (Z \cap X) \to (U \cap Y)$ is finite and makes $\mathscr{O}_{Z\cap X}(Z \cap X)$ a free $\mathscr{O}_{U\cap Y}(U \cap Y)$-module of rank r (remind that $X = p^{-1}(Y)$); we thus have $p(Z \cap X) = U \cap Y$.

As $p_{|\varphi(Y)}$ induces an isomorphism $\varphi(Y) \simeq Y$ whose converse ismorphism is φ, the image $p(\varphi(Y) \cap V)$ is an analytic domain of Y and $p_{|\varphi(Y)\cap V}$ induces an isomorphism $\varphi(Y) \cap V \simeq p(\varphi(Y) \cap V)$ whose converse isomorphism is $\varphi_{|p(\varphi(Y)\cap V)}$.

But $p(\varphi(Y) \cap V) = p(Z \cap X) = U \cap Y$. Therefore $p_{|Z\cap X}$ induces an isomorphism $(Z \cap X) \simeq (U \cap Y)$ whose inverse is $\varphi_{|U\cap Y}$. As $\mathscr{O}_{Z\cap X}(Z \cap X)$ is a free $\mathscr{O}_{U\cap Y}(U \cap Y)$-module of rank r, we have $r = 1$; otherwise said, p induces an isomorphism $Z \simeq U$. The converse isomorphism defines a section σ of the first projection $U \times_k \mathbb{A}_k^{1,an} \to U$; we have $\sigma_{|U\cap Y} = \varphi_{|U\cap Y}$. We can thus glue σ and φ to obtain a section of the first projection $(U \cup Y) \times_k \mathbb{A}_k^{1,an} \to (U \cup Y)$ which coincides with φ on Y, that is, an analytic function g on $U \cup Y$ which coincides with f on Y. As U is a neighborhood of y in $\mathbb{A}_k^{1,an}$, the analytic domain $U \cup Y$ of $\mathbb{A}_k^{1,an}$ contains a closed disc or radius $s > r$. The restriction of g to this disc can be written as a power series $\sum \beta_i T^i$ with $|\beta_i| s^i \xrightarrow[i\to+\infty]{} 0$. As $g_{|Y} = f$, one has $\beta_i = \alpha_i$ for every i. But by assumption $(|\alpha_i| s^i)_i$ is non-bounded, contradicton. Hence $\mathsf{centdim}(\mathbb{A}_k^{2,an}, x)$ can not be equal to one; therefore $\mathsf{centdim}(\mathbb{A}_k^{2,an}, x) = 2$. □

(2.22) Let W be a closed two-dimensional disc centered at the origin of $\mathbb{A}_k^{2,an}$ such that x belongs to the corresponding open polydisc. Note that $\mathscr{O}_{W,x}$ coincides with $\mathscr{O}_{\mathbb{A}_k^{2,an},x}$, hence is a field, and that $\varphi : Y \to \mathbb{A}_k^{2,an}$ goes through W. From now on, φ will denote the induced map $Y \to W$.

(2.23) As $\mathscr{O}_{W,x}$ is a field, φ is flat at y. *But it is not universally flat at y.* We will give two reasons for that.

- Let L be an analytic extension of k such that there exists an L-rational point y' on Y_L lying above y (e.g. $L = \mathscr{H}(y)$). As y' is a L-point, it belongs to $\mathsf{Int}\ Y_L/L$; therefore φ_L is finite at y' (it is even locally a closed immersion around y'). Then φ_L is not flat at y' for dimensional reasons (cor. 2.18).

- $Y \times_W (W \cap X) = Y$, and $Y \to W \cap X$ is a closed immersion; for the same kind of dimensional reasons as above, it is not flat at y.

(2.24) Let η be the generic point of $\mathscr{Y}$. As φ is flat at y, it follows from lemma 2.2 that the induced map $\mathscr{Y} \to \mathscr{W}$ is flat at η. Now, let ρ be a positive real number such that $\rho < r$. Let $\omega \in Y$ be the point that is given by the semi-norm $\sum a_i T^i \mapsto \max |a_i|\rho^i$. The image of ω on $\mathscr{Y}$ is η, but since $\omega \in \mathsf{Int}\, Y/k$, the morphism φ is finite at ω, hence is not flat at ω for dimensional reasons (cor.2.18).

(2.25) Let us now assume that $r \notin \sqrt{|k^*|}$. In this case, $\{y\}$ is an affinoid domain of Y (defined by the equality $|T| = r$); the corresponding k-affinoid algebra is nothing but k_r. Since $\mathscr{O}_{W,x}$ is a field, $\varphi_{|\{y\}}$ is flat. It follows from lemma 2.2 that $\mathsf{Spec}\, k_r \to \mathscr{W}$ is flat.

Let L be any analytic extension of k such that $r \in \sqrt{|L^*|}$ (e.g. $L = k_r$). The space $\mathscr{M}(L\widehat{\otimes}_k k_r)$ is strictly L-affinoid and non-empty; it has thus an L-rigid point, say t. By cor. 2.18, the morphism $\mathscr{M}(L\widehat{\otimes}_k k_r) \to W_L$ is not flat at t; it follows from th. 2.6 (for a direct and simpler proof, *cf.* 2.3.2) that $\mathsf{Spec}\, (L\widehat{\otimes}_k k_r) \to \mathscr{W}_L$ is not flat at the closed point of $\mathsf{Spec}\, (L\widehat{\otimes}_k k_r)$ that corresponds to t.

The non-necessarily good case

(2.26) Now let $Y \to X$ be a morphism of *non-necessarily good* k-analytic spaces and let $y \in Y$. Let $\mathcal{F}$ be a coherent sheaf on Y.

(2.26.1) From 2.12 , we deduce the equivalence of the following:

i) for all couples (V, U), where V is a good analytic domain of Y containing y and where U is a good analytic domain of X containing $\varphi(V)$, the coherent sheaf $\mathcal{F}_{|V}$ is universally U-flat at y;

ii) there exist a good analytic domain V of Y containing y and a good analytic domain U of X containing $\varphi(V)$ such that the coherent sheaf $\mathcal{F}_{|V}$ is universally U-flat at y.

(2.26.2) We will say that $\mathcal{F}$ is *universally X-flat at y* if it satisfies the equivalent assertions *i*) an *ii*); we will say that Y is *universally X-flat at y* if $\mathscr{O}_Y$ is, and that Y is *universally X-flat* if it is universally X-flat at all its points. Those definitions are compatible with the preceeding ones when Y and X are good.

(2.26.3) An immersion of an analytic domain is universally flat.

(2.26.4) Let $Y \to X$ be a morphism of k-analytic spaces, let V be an analytic doman of Y and let U be an analytic domain of X which contains the image of V. Let $\mathcal{F}$ be a coherent sheaf on Y and let $y \in V$. It follows immediatly from the definition that $\mathcal{F}$ is universally X-flat at y if and only if $\mathcal{F}_{|V}$ is universally U-flat at y.

(2.26.5) Universal flatness of a coherent sheaf at a point is preserved by base change and ground field extension; universal flatness of a map at a point is stable under composition.

(2.26.6) If X is any k-analytic space, the structure map $X \to \mathcal{M}(k)$ is universally flat.

Universally flat maps behave as expected

Let us begin with an immediate consequence of lemma 2.1.

(2.27) Lemma. *Let*
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & T \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$
be a commutative diagram of good analytic spaces, let $z \in Z$ and let t (resp. y) be its image on T (resp. Y). Let $\mathcal{F}$ be a coherent sheaf on Y and let $\mathcal{G}$ be its pull-back on Z. Suppose that T is X-flat at t and that Z is Y-flat at z. If $\mathcal{G}$ is T-flat at z then $\mathcal{F}$ is X-flat at y. □

(2.28) Lemma. *Let $Y \to X$ be a morphism of k-affinoid spaces and let L and F be two be an analytic extensions of k. Let X' be an F-affinoid space and let $X' \to X$ be a morphism; set $Y' = Y \times_X X' = Y_F \times_{X_F} X'$. Let y be a point on Y. Let u (resp. y') be a point of Y_L (resp. Y') lying above y. There exist a complete extension K of k, equipped with two isometric k-embeddings $F \hookrightarrow K$ and $L \hookrightarrow K$, and a point ω of*

$$Y'_K := Y' \times_F K \simeq Y \times_X X'_K \simeq Y_K \times_{X_K} X'_K \simeq Y_L \times_{X_L} X'_K \simeq Y_F \times_{X_F} X'_K$$

lying above both y' and u.

Proof. Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{A}'$ be the respective algebras of analytic functions on X, Y and X'. The points u and y' furnish a couple of characters

$$(\mathscr{B}\widehat{\otimes}_k L \to \mathscr{H}(u), \mathscr{B}\widehat{\otimes}_{\mathscr{A}}\mathscr{A}' \to \mathscr{H}(y')),$$

the restriction of every of which to $\mathscr{B}$ goes through $\mathscr{B} \to \mathscr{H}(y)$. The Banach algebra $\mathscr{H}(u)\widehat{\otimes}_{\mathscr{H}(y)}\mathscr{H}(y')$ is non-zero (a result by Gruson ensures that it contains $\mathscr{H}(u) \otimes_{\mathscr{H}(y)} \mathscr{H}(y')$, [19], th. 1, 4); there exists therefore an analytic field K and a bounded homomorphism $\mathscr{H}(u)\widehat{\otimes}_{\mathscr{H}(y)}\mathscr{H}(y') \to K$ ([1], th. 1.2.1), which makes K an analytic extension of both $\mathscr{H}(u)$ and $\mathscr{H}(y')$.

One thus get a new couple of characters

$$(\mathscr{B}\widehat{\otimes}_k L \to K, \mathscr{B}\widehat{\otimes}_{\mathscr{A}}\mathscr{A}' \to K)$$

whose restriction to $\mathscr{B}$ coincide; that couple induces tautologically a character $\mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{B}\widehat{\otimes}_k L \to K$, which extends canonically to a character

$$(\mathscr{A}'\widehat{\otimes}_F K)\widehat{\otimes}_{\mathscr{A}_K}\mathscr{B}\widehat{\otimes}_k K \to K.$$

The latter defines a point ω on $Y_K \times_{X_K} X'_K$ lying by construction above both y' and u. □

(2.29) Proposition. *Let*
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & T \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$
be a commutative diagram of k-analytic spaces, let $z \in Z$ and let t (resp. y) be its image on T (resp. Y). Let $\mathcal{F}$ be a coherent sheaf on Y, and let $\mathcal{G}$ be its pull-back on Z. Suppose that T is universally X-flat at t, that Z is universally Y-flat at z and that $\mathcal{G}$ is universally T-flat at z; under those assumptions $\mathcal{F}$ is universally X-flat at y.

Proof. One immediatly reduces to the case where all spaces are affinoid. Let X' be a good analytic space and let $X' \to X$ be a morphism. We set $Y' = Y \times_X X'$ and so on. Let y' be a point on Y' lying above y, and let z' be a point on Z' lying above both z and y'; denote by t' the image of z' on T'. Since $\mathcal{G}$ is universally T-flat at z, the sheaf $\mathcal{G}'$ is T'-flat at z'. Since T (resp. Z) is universally X-flat at t (resp. universally Y-flat at z), T' (resp. Z') is X'-flat at t' (resp. Y'-flat at z'). Lemma 2.27 above now implies that $\mathcal{F}'$ is X'-flat at y'. $\square$

(2.30) Proposition. *Let $Y \to X$ be a morphism of k-analytic spaces and let L be an analytic extension of k. Let $y \in Y$ and let $\mathcal{F}$ be a coherent sheaf on Y. Let u be a point of Y_L lying above y. Suppose that the pullback $\mathcal{F}_L$ of $\mathcal{F}$ on Y_L is universally X_L-flat at u; the coherent sheaf $\mathcal{F}$ is then universally X-flat at y.*

Proof. One can assume that both Y and X are affinoid. Let X' be a good F-analytic space for some analytic extension F of k. We set $Y' = Y \times_X X'$. Let y' be a point on Y' lying above y; we will show that the pullback $\mathcal{F}'$ of $\mathcal{F}$ on Y' is X'-flat at y; by shrinking X', one can assume that it is F-affinoid.

By lemma 2.28 there exists an analytic extension K of both F and L and a point ω on $Y'_K := Y_K \times_{X_K} X'_K$ lying above both u and y'. Let $\mathcal{F}'_K$ be the pre-image of $\mathcal{F}$ on $Y'_K \simeq Y_K \times_{X_K} X'_K \simeq Y_L \times_{X_L} X'_K$. By universal flatness of $\mathcal{F}_L$ at u the coherent sheaf $\mathcal{F}'_K$ is X'_K-flat at ω. Applying cor. 2.27 above to the diagram $\begin{array}{ccc} Y'_K & \longrightarrow & X'_K \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & X' \end{array}$ (which is possible thanks to 0.13.3) immediatly gives the X'-flatness of $\mathcal{F}'$ at y'. $\square$

(2.31) Proposition. *Let $Y \to X$ be a morphism between k-analytic spaces, let $y \in Y$ and let x be its image on X. Let L be an analytic extension of k, let X' be an L-analytic space and let $X' \to X$ be a morphism. Let y' be a pre-image of y on $Y' := Y \times_X X'$ and let x' denote the image of y' on X'. Let $\mathcal{L}$ be a coherent sheaf on X'.*

i) If $\mathcal{F} \to \mathcal{E}$ is a linear map of coherent sheaves on Y which is injective at y, and if $\mathcal{L}$ is universally X_L-flat at x', then $\mathcal{F} \boxtimes \mathcal{L} \to \mathcal{E} \boxtimes \mathcal{L}$ is injective at y'.

ii) If $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{E} \to 0$ is a sequence of coherent sheaves on Y which is exact at y and if $\mathcal{E}$ is universally X-flat at y, then

$$0 \to \mathcal{G} \boxtimes \mathcal{L} \to \mathcal{F} \boxtimes \mathcal{L} \to \mathcal{E} \boxtimes \mathcal{L} \to 0$$

is exact at y'.

Proof. For both assertions one can assume that X, Y, X' and Y' are affinoid.

(2.31.1) *Proof of i).* As $\mathcal{L}$ is universally X_L-flat at x', its pre-image on Y' is Y_L-flat at y'; since Y_L is flat over Y, this implies that Y' is Y-flat at y'. In other words, $\mathcal{L} \otimes \mathscr{O}_{Y',y'}$ is a flat $\mathscr{O}_{Y,y}$-module. By assumption $\mathcal{F} \otimes \mathscr{O}_{Y,y} \to \mathcal{E} \otimes \mathscr{O}_{Y,y}$ is injective. Tensoring with the flat $\mathscr{O}_{Y,y}$-module $\mathcal{L} \otimes \mathscr{O}_{Y',y'}$ then yields to the injectivity of the arrow $(\mathcal{F} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to (\mathcal{E} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'}$, as required.

(2.31.2) *Proof of ii).* As X' is affinoid, it can be identified to a Zariski-closed subspace of $X_L \times_L \mathbb{D}$ where $\mathbb{D}$ is some closed polydisc over the field L. Right-exactness of the tensor product ensures that

$$\mathcal{G} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{F} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{E} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to 0$$

is exact. Since $X_L \times_L \mathbb{D} \to X_L$ is universally flat (2.14), it now follows from assertion $i)$ already proven that $\mathcal{G} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{F} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'}$ is injective; hence

$$0 \to \mathcal{G} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{F} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{E} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to 0$$

is exact.

As X' is a Zariski-closed subspace of $X_L \times_L \mathbb{D}$, the local ring $\mathscr{O}_{Y',y'}$ is naturally isomorphic to $\mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \otimes_{\mathscr{O}_{X_L \times_L \mathbb{D}, x'}} \mathscr{O}_{X',x'}$. Therefore the sequence

$$0 \to (\mathcal{G} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to (\mathcal{F} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to (\mathcal{E} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to 0$$

is simply deduced from the exact sequence

$$0 \to \mathcal{G} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{F} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to \mathcal{E} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'} \to 0$$

by applying the functor $\otimes_{\mathscr{O}_{X_L \times_L \mathbb{D}, x'}} \mathcal{L} \otimes \mathscr{O}_{X',x'}$. As $\mathcal{E}$ is universally X-flat at y, the $\mathscr{O}_{X_L \times_L \mathbb{D}, x'}$-module $\mathcal{E} \otimes \mathscr{O}_{Y_L \times_L \mathbb{D}, y'}$ is flat; it follows then immediatly from the $\mathsf{Tor}_\bullet$'s exact sequence that

$$0 \to (\mathcal{G} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to (\mathcal{F} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to (\mathcal{E} \boxtimes \mathcal{L}) \otimes \mathscr{O}_{Y',y'} \to 0$$

is exact. □

(2.32) Lemma. *Let $Y \to X$ be a morphism between k-analytic spaces, let $y \in Y$ and let x be its image on X. Let $\mathcal{F} \to \mathcal{G}$ be a linear map between coherent sheaves on Y. If $\mathcal{G}_{|Y_x} \to \mathcal{F}_{|Y_x}$ is an isomorphism at y and if $\mathcal{F}$ is universally X-flat at y, then $\mathcal{G} \to \mathcal{F}$ is an isomorphism at y.*

Proof. We may assume that both Y and X are k-affinoid. Let L be any analytic extension of k such that X_L has an L-rational point x' lying over x, and let y' be any pre-image of y on $(Y_L)_{x'}$. By replacing the ground field k with L, the spaces Y and X with Y_L and X_L, the points y and x with y' and x', and the sheaves $\mathcal{F}$ and $\mathcal{G}$ with $\mathcal{F}_L$ and $\mathcal{G}_L$, we can thanks to 0.13.3reduce to the case where x is a k-point; let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}_{X,x}$.

As $\mathcal{G}_{|Y_x} \to \mathcal{F}_{|Y_x}$ is an isomorphism at y, the map $\mathcal{G} \otimes \mathscr{H}(y) \to \mathcal{F} \otimes \mathscr{H}(y)$ is an isomorphism, and is in particular surjective; thus $\mathcal{G} \to \mathcal{F}$ is surjective at y. It remains to show that it is injective at y; for that purpose, let us denote by $\mathcal{K}$ the kernel of the arrow $\mathcal{G} \otimes \mathscr{O}_{Y,y} \to \mathcal{F} \otimes \mathscr{O}_{Y,y}$.

The sequence

$$0 \to \mathcal{K} \to \mathcal{G} \otimes \mathscr{O}_{Y,y} \to \mathcal{F} \otimes \mathscr{O}_{Y,y} \to 0$$

is exact. Since $\mathcal{F}$ is (universally) X-flat at y, the $\mathscr{O}_{X,x}$-module $\mathcal{F} \otimes \mathscr{O}_{Y,y}$ is flat. Therefore, the sequence

$$0 \to \mathcal{K}/\mathfrak{m} \to \mathcal{G} \otimes \mathscr{O}_{Y,y}/\mathfrak{m} \to \mathcal{F} \otimes \mathscr{O}_{Y,y}/\mathfrak{m} \to 0$$

is exact. But as x is a k-point, $\mathscr{O}_{Y,y}/\mathfrak{m} \simeq \mathscr{O}_{Y_x,y}$. Hence the sequence

$$0 \to \mathcal{K}/\mathfrak{m} \to \mathcal{G} \otimes \mathscr{O}_{Y_x,y}/ \to \mathcal{F} \otimes \mathscr{O}_{Y_x,y} \to 0$$

is exact. As $\mathcal{G}_{|Y_x} \to \mathcal{F}_{|Y_x}$ is an isomorphism at y, this implies that $\mathcal{K}/\mathfrak{m} = 0$; but then Nakayama's lemma (applied to the finitely generated $\mathscr{O}_{Y,y}$-module $\mathcal{K}$) forces $\mathcal{K}$ to be zero, which ends the proof. □

(2.33) Lemma. *Let $Y \to X$ be a universally flat morphism between k-analytic spaces, let $y \in Y$ and let x be its image on X. One has the equality*

$$\mathsf{dim}_y\, Y = \mathsf{dim}_y\, Y_x + \mathsf{dim}_x\, X.$$

Proof. We immediately reduce to the case where Y and X are affinoid. By extending the ground field, we may assume that y is a k-point (hence, so is x).

The local ring $\mathscr{O}_{Y,y}$ is a flat $\mathscr{O}_{X,x}$-algebra; if me denote by $\mathfrak{m}_x$ the maximal ideal of $\mathscr{O}_{X,x}$, the quotient $\mathscr{O}_{Y,y}/\mathfrak{m}_x$ is nothing but $\mathscr{O}_{Y_x,y}$ (because x is a rigid point). One has therefore the equality

$$\mathsf{dim}_{\mathrm{Krull}}\, \mathscr{O}_{Y,y} = \mathsf{dim}_{\mathrm{Krull}}\, \mathscr{O}_{Y_x,x} + \mathsf{dim}_{\mathrm{Krull}}\, \mathscr{O}_{X,x},$$

that is,

$$\mathsf{dim}_y\, Y = \mathsf{dim}_y\, Y_x + \mathsf{dim}_X\, x = \mathsf{dim}_y\, Y_x + d. \ \square$$

3 Quasi-smooth morphisms

Remindings about the sheaf of relative differentials

A reference for the results of this section is [2], §3.3.

(3.1) Let $Y \to X$ be a morphism between k-analytic spaces. The diagonal map $\delta : Y \to Y \times_X Y$ is G-locally an immersion; its *conormal sheaf* is a coherent sheaf on Y which is denoted[3] by $\Omega_{Y/X}$ and is called the *sheaf of relative (Kähler) differentials of Y over X*; there is a natural X-derivation $\mathrm{d} : \mathscr{O}_Y \to \Omega_{Y/X}$, through which any X-derivation with source $\mathscr{O}_Y$ (and with target a coherent sheaf on Y) goes uniquely. The formation of $\Omega_{Y/X}$ commutes to base change and ground field extension.

(3.1.1) If $\mathscr{A}$ is a k-affinoid algebra and if $\mathscr{Y} \to \mathscr{X}$ is a morphism between $\mathscr{A}$-schemes of finite type, the sheaf $\Omega_{\mathscr{Y}^{an}/\mathscr{X}^{an}}$ is isomorphic to $(\Omega_{\mathscr{Y}/\mathscr{X}})^{an}$.

(3.1.2) Let

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

be a diagram in the category of k-analytic spaces. One has a natural exact sequence

$$g^*\Omega_{Y\to X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0.$$

[3]Berkovich denotes if by $\Omega_{Y_{\mathrm{G}}/X_{\mathrm{G}}}$; for the sake of simplicity, and according to our general conventions, we have decided to simply denote it by $\Omega_{Y/X}$.

(3.1.3) Let X be a k-analytic space. For every n the sheaf $\Omega_{\mathbb{A}^n_X/X}$ is free with basis $\mathrm{d}T_1, \ldots, \mathrm{d}T_n$.

(3.1.4) Let $Y \to X$ be a morphism of k-analytic spaces, let $f_1, \ldots, f_r$ be analytic functions on Y and let Z be the Zariski-closed subset of X defined by the sheaf of ideals $(f_i)_i$; let us denote by ι the closed immersion $Z \hookrightarrow Y$. The sheaf $\Omega_{Z/X}$ is then naturally isomorphic to $\iota^*\Omega_{Y/X}/(\iota^*\mathrm{d}f_i)_i$.

(3.2) Let X be a k-analytic space and let $x \in X$. The reader may find proofs of the following facts in [15] (lemma 6.2, prop. 6.3, prop. 6.6).

(3.2.1) One has the inequality $\mathsf{dim}_{\mathscr{H}(x)}\, \Omega_{X/k} \otimes \mathscr{H}(x) \geqslant \mathsf{dim}_x\, X$.

(3.2.2) The following are equivalent:

1) $\mathsf{dim}_{\mathscr{H}(x)}\, \Omega_{X/k} \otimes \mathscr{H}(x) = \mathsf{dim}_x\, X$;

2) X is geometrically regular at x, that is, for every analytic extension L of k and every point y of X_L lying above x, the space X_L is regular at y.

Moreover if $\mathscr{H}(x) = k$ then 1) and 2) hold if and only if X is regular at x.

(3.2.3) Assume that 1) and 2) hold, and let d be the dimension of X at x. There exists a purely d-dimensional open neighborhood U of $x \in X$ such that the coherent sheaf $\Omega_{U/k} = (\Omega_{X/k})_{|U}$ is free of rank d; note that this implies that 1) and 2) hold at every point of U.

Quasi-smoothness: definition and first properties

(3.3) Lemma. *Let X be a k-analytic space, let n be an integer and let V be an affinoid domain of $\mathbb{A}^n_X$. Let Y be a Zariski-closed subset of V, let I be the corresponding ideal of the ring of analytic functions on V, and let $g_1, \ldots, g_r$ be a generating family of I. For every $z \in Y$ let $s(z)$ denote the rank of the family $(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1)$ in the vector space $\Omega_{V/X} \otimes \mathscr{H}(z)$.*

i) For every $z \in V$ the $\mathscr{H}(z)$-dimension of $\Omega_{Y/X} \otimes \mathscr{H}(z)$ is equal to $n - s(z)$; in particular,

$$\mathsf{dim}_{\mathscr{H}(z)}\, \Omega_{Y/X} \otimes \mathscr{H}(z) \geqslant n - r.$$

ii) Assume that there exists $y \in Y$ with $s(y) = r$ (which is equivalent, by i), to the fact that $\Omega_{Y/X} \otimes \mathscr{H}(y)$ is $n - r$ dimensional). Then:

- *every generating family of I has cardinality at least r ;*
- *there exists an affinoid neighborhood U of y in V such that $U \cap Y \to X$ is purely of relative dimension $n - r$, and such that $s(z) = r$ for every $z \in U \cap Y$; in particular, every fiber of $(U \cap Y) \to X$ is geometrically regular.*

Proof. Let us first prove i). By 3.1.3, the $\mathscr{H}(z)$-vector space $\Omega_{V/X} \otimes \mathscr{H}(Z)$ is n-dimensional; and it follows from 3.1.4 that $\Omega_{Y/X} \otimes \mathscr{H}(z)$ is naturally isomorphic to

$$\Omega_{V/X} \otimes \mathscr{H}(z)/(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1),$$

whence i).

Now let us come to assertion ii). If $(h_1, \ldots, h_t)$ is a a generating family of I, applying i) to it yields the inequality $n - r \geqslant n - t$, that is, $t \geqslant r$, as required.

Let $x \in X$. Being an affinoid domain of $\mathbb{A}^{n,\mathrm{an}}_{\mathscr{H}(x)}$, the fiber V_x is normal and purely n-dimensional. As the ideal of Y_x in V_x is generated by r functions, it follows from the *Hauptidealsatz* that the Krull codimension of any irreducible component of Y_x in V_x is at most r. Therefore, the dimension of such a component is at least $n - r$, and it follows that $\mathsf{dim}_z\ (Y \to X) \geqslant n - r$ for every $z \in Y$.

By upper-semi-continuity of the rank of the fibers of a given coherent sheaf, there exists an affinoid neighborhood U of y in V such that $\Omega_{Y/X} \otimes \mathscr{H}(z)$ is of dimension bounded by $n - r$ for every $z \in U \cap Y$; not that the latter dimension is then actually *equal* to $n - r$ in view of i). Let $z \in U \cap Y$ and let x be its image on X. The $\mathscr{H}(x)$-analytic space Y_x if of dimension *at least* $n - r$ at z; and $\Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{H}(z) = \Omega_{Y/X} \otimes \mathscr{H}(z)$ is of dimension $n - r$. We thus deduce from 3.2.1 that Y_x is of dimension $n - r$ at z, which ends the proof (the claim about geometric regularity comes from 3.2.2). $\square$

(3.4) Definition. Let $Y \to X$ be a morphism of k-analytic spaces and let $y \in Y$. Let W be an affinoid domain of Y containing y, let $n \in \mathbb{N}$ and let V be an affinoid domain of $\mathbb{A}^n_X$ such that $W \to X$ goes through a closed immersion $W \hookrightarrow V$; let us denote by I the ideal defining the latter (in the ring of analytic functions on V), and set $r = n - \mathsf{dim}_{\mathscr{H}(y)}\ \Omega_{Y/X} \otimes \mathscr{H}(y)$. We will say that the diagram $W \hookrightarrow V \subset \mathbb{A}^n_X$ is a *Jacobian presentation of* $Y \to X$ *at* y if I can be generated by r elements.

(3.4.1) If it is the case, it follows from lemma 3.3 that r is the minimal cardinality of a generating family of I, that $Y \to X$ is of dimension $n - r$ at y, and that the fiber of $Y \to X$ containing y is geometrically regular at y.

(3.4.2) Lemma 3.3 also ensures that there exists an affinoid neighborhood V' of y inside V such that $W \times_V V' \hookrightarrow V'$ is a Jacobian presentation of $Y \to X$ at *every of its points*, and is purely of relative dimension $n - r$ over X.

(3.5) Definition. Let $Y \to X$ be a morphism of $k$analytic spaces and let $y \in Y$. We will say that $Y \to X$ is *quasi-smooth* at y if there exists a Jacobian presentation of $Y \to X$ at y. If this is the case, the fiber of $Y \to X$ containing y is geometrically regular at y.

A morphism $Y \to X$ is said to be *quasi-smooth* if it is quasi-smooth at every point of Y.

(3.6) We will say that a morphism of k-analytic spaces $Y \to X$ is *quasi-étale* at a point y of Y if it is quasi-smooth of relative dimension zero at y; and that it is *quasi-étale* if it is quasi-étale at every point of Y.

(3.7) We will say that a k-analytic space X is quasi-smooth (resp. quasi-étale) at a given point $x \in X$ if $X \to \mathscr{M}(k)$ is.

(3.8) Some remarks about the chosen terminology.

(3.8.1) Berkovich has defined ([2], §3) the notions of an étale and a smooth map. We will see below (cor. 3.20 of prop. 3.18, rem. 3.21): that a map is étale at a given point if and only if it is quasi-étale and inner at that point ; and that a map between *good* k-analytic spaces is smooth at a given point if and only if it is quasi-smooth and inner at that point (for some comments about the need of a goodness assumption, see rem. 3.21).

(3.8.2) There is already a notion of quasi-étale morphism, which was defined by Berkovich ([3], §3); we will see below that his definition is equivalent to ours (3.22).

(3.8.3) In [15], §6 , an analytic space geometrically was said to be quasi-smooth (*quasi-lisse* in french) at x if it is geometrically regular at x; this turns out to be coherent with our current definition of quasi-smoothness (cor. 3.15 *infra*.)

(3.8.4) If $|k^*| \neq \{1\}$, if Y and X are strictly k-analytic spaces and if y is a rigid point of Y, quasi-smoothness of $Y \to X$ at y is nothing but *rig-smoothness* of $Y \to X$ at y; we nevertheless have chosen to use 'quasi-smooth' instead of 'rig-smooth' to be coherent with the terminology 'quasi-étale'.

(3.9) Before giving some examples, and some basic properties of quasi-smooth morphisms, let us make a technical remark we will use several times in the sequel: it follows immediatly from Gerritzen-Grauert theorem (more precisely for Temkin's version of it for Berkovich spaces, [26], th. 3.1) that if X is an analytic space and it Y is a Zariski-closed subspace of X then Y can be G-covered by affinoid domains of the kind $V \cap Y$ with V beeing an affinoid domain of X

(3.10) Basic properties and examples. Let $Y \to X$ be a morphism between k-analytic spaces.

(3.10.1) If $Y = \mathbb{A}^n_X$, then Y is quasi-smooth over X of relative dimension n: for every $y \in \mathbb{A}^n_X$ and every affinoid domain W of $\mathbb{A}^n_X$ containing y, the diagram $W \simeq W \subset \mathbb{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y.

(3.10.2) Let $y \in Y$, let V be an analytic domain of Y containing y, and let U be an analytic domain of X containing the image of V. Then $Y \to X$ is quasi-smooth at y if and only if so is $V \to U$.

Indeed, let us first assume that $V \to U$ is quasi-smooth at y. Then if $Z \hookrightarrow T \subset \mathbb{A}^n_U$ is a Jacobian presentation of $V \to U$ at y, it follows immediatly from the definition that $Z \hookrightarrow T \subset \mathbb{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y.

Conversely, let us assume that $Y \to X$ is quasi-smooth at y, and let us choose a Jacobian presentation $Z \hookrightarrow T \subset \mathbb{A}^n_X$ of $Y \to X$ at y; the image of y in $\mathbb{A}^n_X$ belongs to $\mathbb{A}^n_U$. Let T' be an affinoid domain of $\mathbb{A}^n_U$ which contains the image of y. The fiber product $Z' := Z \times_T T'$ is an affinoid domain of Y which contains y, and $Z' \to T'$ is a closed immersion.

By 3.9 there exists an affinoid domain T'' of T' containing the image of y such that $Z'' := Z' \times_{T'} T''$ is included in $V \cap Z'$. It follows from the construction that $Z'' \hookrightarrow T'' \subset \mathbb{A}^n_U$ is a Jacobian presentation of $V \to U$ at y.

(3.10.3) The morphism Id_X is quasi-mooth (3.10.1 with $n = 0$); it follows by 3.10.2 that if Y is an analytic domain of X then $Y \to X$ is quasi-smooth.

(3.10.4) *Behavior with respect to base change.* Let X' be an analytic space and let $X' \to X$ be a morphism. If $y \in Y$, if y' is a point of $Y' := Y \times_X X'$ lying over y, and if $Y \to X$ is quasi-smooth at y then $Y' \to X'$ is quasi-smooth at y'. Indeed, let $W \hookrightarrow V \subset \mathbb{A}^n_X$ be a Jacobian presentation of $Y \to X$ at y, and let V' (resp. W') denote the fiber product $V \times_X X'$ (resp. $W \times_X X'$). Let V'' be any affinoid domain of V' which contains the image of y' by the closed immersion $V' \hookrightarrow W'$; the fiber product $W'' := W' \times_{V'} V''$ is an affinoid domain of Y', and it is easily seen that $W'' \hookrightarrow V'' \subset \mathbb{A}^n_{X'}$ is a Jacobian presentation of $Y' \to X'$ at y'.

(3.10.5) *Behavior with respect to composition.* Let Z be an analytic space, let $Z \to Y$ be a morphism, let $z \in Z$ and let y be its image on Y. If $Z \to Y$ is quasi-smooth at z and if $Y \to X$ is quasi-smooth at y then the composite map $Z \to Y \to X$ is quasi-smooth at z.

Indeed, let $W \hookrightarrow V \subset \mathbb{A}^n_X$ be a Jacobian presentation of $Y \to X$ at y. As $Z \to Y$ is quasi-smooth at z, the map $Z \times_Y W \to W$ is quasi-smooth at z too by 3.10.2 or 3.10.4 above. Let $T \hookrightarrow S \subset \mathbb{A}^m_W$ be a Jacobian presentation of $Z \times_Y W \to W$ at z. As S is an affinoid domain of $\mathbb{A}^m_W \hookrightarrow \mathbb{A}^m_V$, it follows from 3.9 that there exists an affinoid domain S' of $\mathbb{A}^m_V$ such that $S' \cap \mathbb{A}^m_W$ is contained in S and contains the image of z; set $T' = T \times_S (S' \cap \mathbb{A}^m_W)$. The morphism $T' \hookrightarrow S'$ is equal to the composition of $T' \hookrightarrow S' \cap \mathbb{A}^m_W$ and $S' \cap \mathbb{A}^m_W \hookrightarrow S'$, hence is a a closed immersion. Beeing an affinoid domain of $\mathbb{A}^m_V$, which is itself an analytic domain of $\mathbb{A}^{n+m}_X$, the space S' is an affinoid domain of $\mathbb{A}^{n+m}_X$. Let d (resp. δ) be the dimension of $\Omega_{Y/X} \otimes \mathscr{H}(y)$ (resp. $\Omega_{Z/Y} \otimes \mathscr{H}(z)$). It follows from 3.1.2 that the dimension of $\Omega_{Z/X} \otimes \mathscr{H}(z)$ is bounded by $d + \delta$.

Now, as $W \hookrightarrow V \subset \mathbb{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y, the Zariski-closed subspace W of V can be defined by $n - d$ equations; hence the Zariski-closed subset $S' \cap \mathbb{A}^m_W$ of S' can also be defined by $n - d$ equations.

And as $T \hookrightarrow S \subset \mathbb{A}^m_W$ is a Jacobian presentation of $Z \times_Y W \to W$ at z, the Zariski-closed subspace T of S can be defined by $m - \delta$ equations; hence the Zariski-closed subset T' of S' can be defined by $m - \delta$ equations.

It follows that the Zariski-closed subspace T' of S' can be defined using $m + n - d - \delta$ equations. It follows then from lemma 3.3 i) that the dimension of $\Omega_{Z/X} \otimes \mathscr{H}(z)$ is at least equal to $d + \delta$. On the other hand, we have proven above that $\mathsf{dim}_{\mathscr{H}(z)}\ \Omega_{Z/X} \otimes \mathscr{H}(z) \leqslant d + \delta$, whence eventually the equality $\mathsf{dim}_{\mathscr{H}(z)}\ \Omega_{Z/X} \otimes \mathscr{H}(z) = d + \delta$; therefore $T' \hookrightarrow S' \subset \mathbb{A}^{n+m}_X$ is a Jacobian presentation of $Z \to X$ at z, and $Z \to X$ is quasi-smooth at z.

(3.10.6) Let $\mathscr{A}$ be a k-affinoid algebra and let $\mathscr{Y} \to \mathscr{X}$ be a morphism between $\mathscr{A}$-schemes of finite type. Let $y \in \mathscr{Y}^{an}$ and let $\mathbf{y}$ be its image on $\mathscr{Y}$. If $\mathscr{Y} \to \mathscr{X}$ is smooth at $\mathbf{y}$ then $\mathscr{Y}^{an} \to \mathscr{X}^{an}$ is quasi-smooth at y. Indeed, as $\mathscr{Y} \to \mathscr{X}$ is smooth at $\mathbf{y}$, there exists an integer n, an affine open neighborhood $\mathscr{V}$ of $\mathbf{y}$, and an affine open subset $\mathscr{U}$ of $\mathbb{A}^n_{\mathscr{X}}$ so that $\mathscr{V} \to \mathscr{X}$ goes through a closed immersion $\mathscr{V} \hookrightarrow \mathscr{U}$ whose ideal can be generated by r elements, where $r = n - \mathsf{dim}_{\kappa(\mathbf{y})}\ \Omega_{\mathscr{Y}/\mathscr{X}} \otimes \kappa(\mathbf{y})$. Now if U is any affinoid domain of $\mathscr{U}^{an}$ containing the image of y and if we set $V = \mathscr{V}^{an} \times_{\mathscr{U}^{an}} U$ then $V \hookrightarrow U \subset \mathbb{A}^n_{\mathscr{X}^{an}}$ is a Jacobian presentation of $\mathscr{Y}^{an} \to \mathscr{X}^{an}$ at y.

(3.10.7) By obvious relative dimension arguments, the claims 3.10.2-3.10.6 remain true with 'quasi-smooth' replaced by 'quasi-étale'.

(3.10.8) Let $Y \to X$ be a morphism between k-analytic spaces and let $y \in Y$. Assume that $Y \to X$ is quasi-smooth of relative dimension d at y. The point y

admits a compact neighborhood which can be written $V_1 \bigcup \ldots \bigcup V_n$ where the V_i's are affinoid domains of Y. Thanks to 3.10.2, the morphism $V_i \to X$ is quasi-smooth of relative dimension d at y for every i. It follows then from 3.4.2 that there exists for every i an affinoid neighborhood V_i' of y in V_i such that $V_i' \to X$ is quasi-smooth of relative dimension d over X. The union of the V_i''s is then a compact analytic domain which is a neighborhood of y, and $\bigcup V_i' \to X$ is quasi-smooth of relative dimension d at every point (we use again 3.10.2).

Therefore, the subset of Y which consists in points at which $Y \to W$ is quasi-smooth of relative dimension d is open; we will see at the end of the paper that it is even *Zariski*-open (th. 7.4).

(3.10.9) Let $Y \to X$ be a map between k-analytic spaces, let $y \in Y$ and let x be its image on X. Assume that $Y \to X$ is étale at y. Under this assumption, there exists an affinoid domain U of X containing x and an affinoid domain V of Y containing y such that $V \to X$ goes through a finite étale map $V \to U$. But saying that $V \to U$ is finite étale simply means that $\mathscr{V} \to \mathscr{U}$ is finite étale, which implies that $V \to U$ is quasi-étale (3.10.6 and 3.10.7); as a consequence, $Y \to X$ is quasi-étale at y by 3.10.2.

(3.10.10) Let $Y \to X$ be a map between k-analytic spaces and let $y \in Y$. Assume that $Y \to X$ is smooth at y. By definition, there exists an open neighborhood V of y in Y such that $V \to X$ goes through an étale map $V \to \mathbb{A}^n_X$ for some n. It follows from 3.10.9 above that $V \to \mathbb{A}^n_X$ is quasi-étale; since $\mathbb{A}^n_X \to X$ is quasi-smooth by 3.10.1, one deduces from 3.10.5 and 3.10.3 that $Y \to X$ is quasi-smooth at y.

Quasi-smoothness, universal flatness and geometric regularity

(3.11) Lemma. *Let $Y \to X$ be a morphism of k-analytic spaces, let $y \in Y$ and let f be an analytic function on Y; denote by Z the Zariski-closed subspace of Y with equation $f = 0$. Assume that :*

1) $Y \to X$ is quasi-smooth at y, and $\mathscr{O}_Y$ and $\Omega_{Y/X}$ are universally X-flat at y;
2) $\mathrm{d}f \otimes 1 \neq 0 \in \Omega_{Y/X} \otimes \mathscr{H}(y)$.

Then $Z \to X$ is quasi-smooth at y, and $\mathscr{O}_Z$ and $\Omega_{Z/X}$ are universally flat at y.

Proof. Let $W \hookrightarrow V \subset \mathbb{A}^n_X$ be a Jacobian presentation of $Y \to X$ at y. There exists a finite family $(g_1, \ldots, g_r)$ of analytic functions on V such that the ideal $(g_1, \ldots, g_r)$ defines the closed immersion $W \hookrightarrow V$, and such that the family $(\mathrm{d}g_i \otimes 1)_i$ of elements of the vector space $\Omega_{V/X} \otimes \mathscr{H}(y)$ is free. As

$$\Omega_{Y/X} \otimes \mathscr{H}(y) \simeq \Omega_{V/X} \otimes \mathscr{H}(y)/(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1),$$

the fact that $\mathrm{d}f \otimes 1$ is non-zero in $\Omega_{Y/X} \otimes \mathscr{H}(y)$ simply means that the family $(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1, \mathrm{d}f \otimes 1)$ is free. As the ideal $(g_1, \ldots, g_r, f)$ defines precisely the closed immersion $W \cap Z \hookrightarrow V$, the diagram $(W \cap Z) \hookrightarrow V \subset \mathbb{A}^n_X$ is a Jacobian presentation of $Z \to X$ at y, and $Z \to X$ is quasi-smooth at y.

Now let us come to universal flatness of $\mathscr{O}_Z$ and $\Omega_{Z/X}$ at y. We may assume that X, Y (and hence Z) are affinoid. Since both assumptions 1) and 2) remain true after any base change, it is sufficient to show that $\mathscr{O}_Z$ and $\Omega_{Z/X}$ are flat at y; this can be proven after having extended the scalars, hence we may assume that y is a k-point.

Let x be the image of y on X, and let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}_{X,x}$. Reducing modulo $\mathfrak{m}$ the exact sequences

$$\mathscr{O}_{Y,y} \xrightarrow{\times f} \mathscr{O}_{Y,y} \longrightarrow \mathscr{O}_{Z,y} \longrightarrow 0$$

and

$$\mathscr{O}_{Y,y} \xrightarrow{\times \mathrm{d}f} \Omega_{Y/X} \otimes \mathscr{O}_{Y,y} \longrightarrow \Omega_{Z/X} \otimes \mathscr{O}_{Z,y} \longrightarrow 0$$

yields to the exact sequences

$$\mathscr{O}_{Y_x,y} \xrightarrow{\times f} \mathscr{O}_{Y_x,y} \longrightarrow \mathscr{O}_{Z_x,y} \longrightarrow 0$$

and

$$\mathscr{O}_{Y_x,y} \xrightarrow{\times \mathrm{d}f} \Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Y_x,y} \longrightarrow \Omega_{Z_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Z_x,y} \longrightarrow 0 .$$

As $Y \to X$ is quasi-smooth at y, the fiber Y_x is geometrically regular at y, and $\Omega_{Y_x/\mathscr{H}(x)}$ is free at y. The regular local ring $\mathscr{O}_{Y_x,y}$ is in particular a domain. Since $\mathrm{d}f \otimes 1$ is a non-zero in $\Omega_{Y/X} \otimes \mathscr{H}(y) = \Omega_{Y_x/k} \otimes \mathscr{H}(y)$, the function f in non-zero in $\mathscr{O}_{Y_x,y}$; the latter being a domain, the sequence

$$0 \longrightarrow \mathscr{O}_{Y_x,y} \xrightarrow{\times f} \mathscr{O}_{Y_x,y} \longrightarrow \mathscr{O}_{Z_x,y} \longrightarrow 0$$

is exact.

As $\mathrm{d}f \otimes 1$ is non-zero in $\Omega_{Y/X} \otimes \mathscr{H}(y) = \Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{H}(y)$, it is *a fortiori* non-zero in $\Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Y_x,y}$. Together with the fact that the latter is a free module over the domain $\mathscr{O}_{Y_x,y}$, this implies the exactness of the sequence

$$0 \longrightarrow \mathscr{O}_{Y_x,y} \xrightarrow{\times \mathrm{d}f} \Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Y_x,y} \longrightarrow \Omega_{Z_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Z_x,y} \longrightarrow 0 .$$

As $\mathscr{O}_Y$ and $\Omega_{Y/X}$ are X-flat at y, the sequences

$$\mathscr{O}_{Y,y} \xrightarrow{\times f} \mathscr{O}_{Y,y} \longrightarrow \mathscr{O}_{Z,y} \longrightarrow 0$$

and

$$\mathscr{O}_{Y,y} \xrightarrow{\times \mathrm{d}f} \Omega_{Y/X} \otimes \mathscr{O}_{Y,y} \longrightarrow \Omega_{Z/X} \otimes \mathscr{O}_{Z,y} \longrightarrow 0$$

are respectively truncations of flat resolutions of the $\mathscr{O}_{X,x}$-modules $\mathscr{O}_{Z,y}$ and $\Omega_{Z,y} \otimes \mathscr{O}_{Z,y}$. Therefore, it follows from the exactness of

$$0 \longrightarrow \mathscr{O}_{Y_x,y} \xrightarrow{\times f} \mathscr{O}_{Y_x,y} \longrightarrow \mathscr{O}_{Z_x,y} \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{O}_{Y_x,y} \xrightarrow{\times \mathrm{d}f} \Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Y_x,y} \longrightarrow \Omega_{Z_x/\mathscr{H}(x)} \otimes \mathscr{O}_{Z_x,y} \longrightarrow 0$$

that $\mathsf{Tor}^{\mathscr{O}_{X,x}}(\mathscr{O}_{Z,y}, \mathscr{O}_{X,x}/\mathfrak{m}) = 0$ and $\mathsf{Tor}^{\mathscr{O}_{X,x}}(\Omega_{Z/X} \otimes \mathscr{O}_{Z,y}, \mathscr{O}_{X,x}/\mathfrak{m}) = 0$; but this implies that $\mathscr{O}_{Z,y}$ and $\Omega_{Z/X} \otimes \mathscr{O}_{Z,y}$ are flat $\mathscr{O}_{X,x}$-modules ([18], prop. 4.1), which ends the proof. □

(3.12) Corollary. *Let $Y \to X$ be a morphism of k-analytic spaces, and let $y \in Y$ such that $Y \to X$ is quasi-smooth at x. The sheaves $\mathscr{O}_Y$ and $\Omega_{Y/X}$ are universally X-flat at y.*

Proof. Let us choose a Jacobian presentation $W \hookrightarrow V \subset \mathbb{A}^n_X$ of $Y \to X$ at y, and set $r = n - \mathsf{dim}_{\mathscr{H}(y)} \Omega_{Y/X} \otimes \mathscr{H}(x)$. By definition of a nice presentation, there exists a family $(g_1, \ldots, g_r)$ of analytic functions on V such that the ideal $(g_1, \ldots, g_r)$ defines the closed immersion $W \hookrightarrow V$, and such that the family $(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1)$ of elements of $\Omega_{V/X} \otimes \mathscr{H}(y)$ is free.

For every $i \in \{0, \ldots, r\}$, denote by V_i the Zariski-closed subspace of V defined by the ideal $(g_1, \ldots, g_i)$; note that $V_0 = V$ and that $V_r = W$.

The map $V \to X$ is quasi-smooth at y; moreover, $\mathscr{O}_V$ and $\Omega_{V/X}$ are universally X-flat. Indeed, for every affinoid space Z, the analytic space $\mathbb{A}^n_Z$ is universally flat over Z, because $\mathbb{A}^n_{\mathscr{Z}}$ is flat over $\mathscr{Z}$ (2.13); hence $\mathbb{A}^n_X$ is universally flat over X, and so is V. And as $\Omega_{V/X}$ is a free $\mathscr{O}_V$-module (with basis $(\mathrm{d}T_i)_i$), it is universally flat over X too.

Now, lemma 3.11 ensures that if $i \leqslant r - 1$, if $V_i \to X$ is quasi-smooth at y and if $\mathscr{O}_{V_i}$ and $\Omega_{V_i/X}$ are universally X-flat at y, then V_{i+1} is quasi-smooth at y and $\mathscr{O}_{V_{i+1}}$ and $\Omega_{V_{i+1}/X}$ are universally X-flat at y; hence the corollary follows by induction. □

(3.13) Let $Y \to X$ is a morphism of k-*affinoid* spaces, let $y \in Y$ and let x be its image on X. Let us assume that Y_x is geometrically regular, and that $Y \to X$ is universally flat at y.

There exists $n \in \mathbb{N}$ so that the morphism $Y \to X$ goes through a closed immersion $Y \hookrightarrow \mathbb{D} \times_k X$, where $\mathbb{D}$ is a closed n-dimensional polydisc. Set $r = n - \mathsf{dim}_{\mathscr{H}(y)} \Omega_{Y/X} \otimes \mathscr{H}(y)$, and let I be the ideal of the ring of analytic functions on $\mathbb{D} \times_k X$ that defines the closed immersion $Y \hookrightarrow \mathbb{D} \times_k X$.

It follows from 3.1.4 that there exists $g_1, \ldots, g_r$ such that

$$\Omega_{Y/X} \otimes \mathscr{H}(y) = \Omega_{V/X} \otimes \mathscr{H}(y)/(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1);$$

note that this forces the family $(\mathrm{d}g_1 \otimes 1, \ldots, \mathrm{d}g_r \otimes 1)$ to be free. Let Z be the Zariski-closed subspace of $\mathbb{D} \times_k X$ defined by the ideal $(g_1, \ldots, g_r)$; by construction, Y is a Zariski-closed subspace of Z, and $Z \to X$ is quasi-smooth at y of relative dimension $n - r$.

As $Z \to X$ is quasi-smooth at y of relative dimension $n - r$, the $\mathscr{H}(x)$-space Z_x is geometrically regular at y of relative dimension $n - r$. In particular, there exists a connected affinoid neighborhood U of y in Z_x which is normal, connected and $n - r$ dimensional.

As Y_x is geometrically regular at y and as $\Omega_{Y_x/\mathscr{H}(x)} \otimes \mathscr{H}(y) = \Omega_{Y/X} \otimes \mathscr{H}(y)$ is $(n - r)$-dimensional, one can shrink U so that $(U \cap Y_x)$ is $n - r$ dimensional. Being a Zariski-closed subspace of the reduced, irreducible, $n - r$ dimensional space U, it coincides with U.

The natural surjection $\mathscr{O}_{Z,y} \to \mathscr{O}_{Y,y}$ is bijective. Indeed, we have proven above that $U = U \cap Y$, which implies that $\mathscr{O}_{Z_x,y} \to \mathscr{O}_{Y_x,y}$ is a bijection; and since $\mathscr{O}_Y$ is universally X-flat at y by assumption, it follows then from lemma 2.32 that $\mathscr{O}_{Z,y} \to \mathscr{O}_{Y,y}$ is bijective.

This bijectivity implies the existence of an affinoid neighborhood V of y in $\mathbb{D} \times_k X$ such that $V \cap Y \hookrightarrow V \cap Z$ is an isomorphism; note that $V \cap Y$ is an affinoid neighborhood of y in Y, and that $V \cap Y \hookrightarrow V \subset \mathbb{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y.

(3.14) Proposition. *Let $Y \to X$ be a morphism of k-analytic spaces, let $y \in Y$ and let x be its image on X.*

1) The following are equivalent:

i) $Y \to X$ is quasi-smooth at y.

ii) Y_x is geometrically regular at y, and $Y \to X$ is universally flat at x.

2) If moreover Y and X are good, then those properties hold if and only if there exists a Jacobian presentation $W \hookrightarrow V \subset \mathbb{A}^n_V$ of $Y \to X$ at y with W being an affinoid neighborhood *of y in Y.*

Proof. If 1) $Y \to X$ is quasi-smooth at x, we already know that Y_x is geometrically regular at y, and universal flatness of $Y \to X$ at y is part of corollary 3.12.

Assume now that $Y \to X$ is universally flat at y, and that Y_x is geometrically regular at y. In order to prove that $Y \to X$ is quasi-smooth at y, we may assume that both Y and X are k-affinoid.

But under that assumption, we have seen in 3.13 that there exists a Jacobian presentation $W \hookrightarrow V \subset \mathbb{A}^n_V$ of $Y \to X$ at y with W being an affinoid neighborhood of y in Y, which at the same time ends the proof of ii⇒i) and proves 2). □

(3.15) Corollary. *If X is a k-analytic space and if $x \in X$ then X is quasi-smooth at x if and only if it is geometrically regular at x.*

Proof. This is an immediate consequence of prop. 3.14 above, together with the fact that $X \to \mathscr{M}(k)$ is automatically universally flat. □

(3.16) Proposition. *Let $Y \to X$ be a morphism of k-analytic spaces, let $y \in Y$ and let d be the relative dimension of $Y \to X$ at y. If $Y \to X$ is quasi-smooth at y, then $\Omega_{Y/X}$ is free of rank d at y.*

Proof. We may assume that both Y and X are affinoid; let x be the image of y. As $Y \to X$ is quasi-smooth at y, the dimension of $\Omega_{Y/X} \otimes \mathscr{H}(y)$ is equal to d. Let us choose global forms $\omega_1, \ldots, \omega_d$ belonging to $\Omega_{Y/X}(Y)$ such that $(\omega_i \otimes 1)_i$ is a basis of $\Omega_{Y/X} \otimes \mathscr{H}(x)$. The ω_i's define a morphism $\mathscr{O}_Y^d \to \Omega_{Y/X}$. Since Y_x is geometrically regular at y, the sheaf $\Omega_{Y_x/\mathscr{H}(x)}$ is free of rank d at y by geometric regularity of Y_x at y; therefore, $\mathscr{O}^d_{Y_x} \to \Omega_{Y_x/\mathscr{H}(x)}$ is an isomorphism at y. Thanks to the fact that $\Omega_{Y/X}$ is universally X-flat at y (cor. 3.12), lemma 2.32 allows to conclude that $\mathscr{O}_Y^d \to \Omega_{Y/X}$ is an isomorphism at y. □

Links with étale and smooth morphisms

(3.17) Lemma. *Let $Y \to X$ be a morphism between k-analytic spaces and let $y \in Y$. Assume that $Y \to X$ is quasi-smooth at y of relative dimension d, let $\ell \leqslant d$ let $f_1, \ldots, f_\ell$ be analytic functions on Y such that $(\mathrm{d}f_i \otimes 1)_i$ is a basis of $\Omega_{Y/X} \otimes \mathscr{H}(y)$ (note that such functions always exists if Y is affinoid). The map $Y \to \mathbb{A}^\ell_X$ defined by the f_i's is quasi-smooth of relative dimension $d - \ell$ at y.*

Proof. One immediatly reduces to the case where Y and X are affinoid. Under that assumption map the map $Y \to \mathbb{A}^\ell_X$ goes through $\mathbb{D} \times_k X$ for some ℓ-dimensional compact polydisc $\mathbb{D}$, and it is sufficient to prove that $Y \to \mathbb{D} \times_k X$ is quasi-smooth of relative dimension $d - \ell$ at y.

Both spaces Y and $\mathbb{D} \times_k X$ are k-affinoid, hence $Y \to \mathbb{D} \times_k X$ goes through a closed immersion $Y \hookrightarrow \Delta \times_k \mathbb{D} \times_k X$ where Δ is a closed polydisc of, say, dimension δ.

By the choice of the f_i's, the $\mathscr{H}(y)$-vector space $\Omega_{Y/\mathbb{D}\times_k X} \otimes \mathscr{H}(y)$ has dimension $d - \ell$; therefore, there exist $\delta - d + \ell$ analytic functions $g_1, \ldots, g_{\delta - d + \ell}$ on $\Delta \times_k \mathbb{D} \times_k X$ which belong to the ideal defining the closed immersion $Y \hookrightarrow \Delta \times_k \mathbb{D} \times_k X$ and are such that $(\mathrm{d}g_j \otimes 1)_j$ is a free family of elements of $\Omega_{\Delta\times_k\mathbb{D}\times_k X/\mathbb{D}\times_k X} \otimes \mathscr{H}(y)$; it remains free when viewed as a family of vectors of $\Omega_{\Delta\times_k\mathbb{D}\times_k X/X} \otimes \mathscr{H}(y)$, because the former vector space is a quotient of the latter. Moreover, $Y \to X$ is by assumption quasi-smooth at y; by prop. 3.14, $Y \to X$ is universally flat at y, and Y_x is geometrically regular at y.

Therefore, 3.13 ensures that there exists an affinoid neighborhood V of y in $\Delta \times_k \mathbb{D} \times_k X$ such that the Zariski-closed subspace $Y \cap V$ of V is defined by the ideal $(g_j)_j$; since $(\mathrm{d}g_j \otimes 1)_j$ is a free family of elements $\Omega_{\Delta\times_k\mathbb{D}\times_k X/\mathbb{D}\times_k X} \otimes \mathscr{H}(y)$, the map $Y \to \mathbb{D} \times_k X$ is quasi-smooth of relative dimension $d - l$ at y. □

(3.18) Proposition. *Let $Y \to X$ be a morphism of* good *k-analytic spaces, and let $y \in Y$. The following are equivalent :*

i) there exists an affinoid neighborhood Y_0 of y in Y and a smooth X-space Z such that Y_0 is X-isomorphic to an affinoid domain of Z;
ii) $Y \to X$ is quasi-smooth at y.

Proof. As smooth morphisms and embedding of analytic domains are quasi-smooth, i)⇒ii). Let us now assume that ii) is true. In order to prove i), one may assume that Y is affinoid. Let d be the dimension of $\Omega_{Y/X} \otimes \mathscr{H}(y)$ and let $f_1, \ldots, f_d$ be analytic functions on Y such that the family $(\mathrm{d}f_i \otimes 1)_i$ is a basis of $\Omega_{Y/X} \otimes \mathscr{H}(y)$; let $\varphi : Y \to \mathbb{A}^d_X$ be the morphism induced by the f_i's. Thanks to lemma 3.17, the morphism φ is quasi-étale at y.

Let ξ be the image of y in $\mathbb{A}^n_X$. As φ is zero-dimensional at y, analytic Zariski's Main Theorem ensures that Y can be shrunken so that φ admits a factorization $Y \to T_0 \to T \to \mathbb{A}^d_X$ where T is finite étale over an open neighborhood U of ξ, where T_0 is an affinoid domain of T, and where $Y \to T_0$ is finite.

The finite morphism $Y \to T_0$ is étale at y. Indeed, let us consider the diagram

$$Y \times_U T \to T_0 \times_U T \to T \times_U T \to T.$$

As $Y \to U$ is quasi-étale at y, the arrow $Y \times_U T \to T$ is quasi-étale at every pre-image of y.

The map $T \times_U T \to T$ admits a canonical section σ; as $T \to U$ is finite étale, $\sigma(T)$ is an open and closed subset of $T \times_U T$, and $\sigma(T) \to T$ is an isomorphism. The pre-image S of $\sigma(T)$ inside $Y \times_U T$ is thus naturally isomorphic to Y in such a way that the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\simeq"] & \sigma(T) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & T \arrow[d] \\ Y \arrow[r] & T \arrow[r] & U \end{tikzcd}$$

commutes. Since $S \to T$ is quasi-étale at the unique pre-image of y on S, the morphism $Y \to T$ is quasi-étale at y; hence $Y \to T_0$ is quasi-étale at y too.

This implies that $Y \to T_0$ is flat at y and that $\Omega_{Y/T_0} \otimes \mathscr{H}(y) = 0$; the map $Y \to T_0$ being finite, those conditions exactly means that it is étale at y.

Conclusion. Let t be the image of y on T. The categories of finite étale covers of the germ (T_0, t) and (T, t) are naturally equivalent (both are equivalent to the category of finite étale $\mathscr{H}(t)$-algebra, [2] th. 3.4.1). Therefore there exists:

- an open neighborhood T_1 of t in T;
- a finite étale map $Z \to T_1$;
- an isomorphism between $Z \times_{T_1} T_1 \cap T_0$ and an open neighborhood Y_1 of y in Y.

The morphisms $Z \to T_1, T_1 \to U, U \to \mathbb{A}^n_X$ and $\mathbb{A}^n_X$ to X are smooth; hence $T' \to X$ is smooth. Now one can take Y_0 as being equal to any affinoid neighborhood of y inside Y_1. □

(3.19) *Remark.* In the strictly k-analytic case, such a result has already been proved by Berkovich ([4], rem. 9.7).

(3.20) Corollary. *Let $Y \to X$ be a morphism between* good *k-analytic spaces and let $y \in Y$. The following are equivalent:*

i) $Y \to X$ is smooth at y ;
ii) $Y \to X$ is quasi-smooth and boundaryless at y. □

(3.21) Remark. The author doesn't know if cor. 3.20 above is true without the goodness assumption. By looking carefully at what happens, the reader should be convinced that the main problem to face in the non-good case is the following: if $Y \to X$ be a morphism of analytic spaces and if $y \in Y$, there is no reason why they should exist analytic functions $f_1, \ldots, f_r$ defined in a *neighborhood* of y such that $\mathrm{d}f_1 \otimes 1, \ldots, \mathrm{d}f_r \otimes 1$ generate $\Omega_{Y/X} \otimes \mathscr{H}(y)$.

However, note that in the case where the relative dimension is zero, then cor. 3.20 is true without any goodness assumption: indeed, let us assume that $Y \to X$ is quasi-étale and boundaryless at y. Being zero-dimensional and boundaryless at y, it is finite at y; hence we can shrink Y and X so that $Y \to X$ is finite, and so that y is the only pre-image of x on Y. Now choose a compact analytic neighborhood of $x \in X$ which can be written $V_1 \cup \ldots \cup V_m$ where the V_i's are affinoid domains of X containing the image x of y. For every i the morphism $Y \times_X V_i \to V_i$ is finite; being quasi-étale at y, it is in particular flat and unramified at y, hence étale at y, which is the only pre-image of x. As a consequence, there exists an affinoid neighborhood W_i of x in V_i such that $Y \times_X W_i \to W_i$ is étale. If one sets $W = \bigcup W_i$ then W is a compact analytic neighborhood of x and $Y \times_X W \to W$ is étale, whence our claim.

(3.22) Compatibility with the previous definition of quasi-étaleness. Let $Y \to X$ be a morphism of k-analytic spaces, and let $y \in X$. The following are equivalent :

i) $Y \to X$ is quasi-étale at y in the sense of Berkovich ([3], §3);
ii) $Y \to X$ is quasi-étale at y in our sense.

We are going to prove it; in what follows, 'quasi-étale' will mean 'quasi-étale in our sense', and we will explicitely write 'quasi-étale in the sense of Berkovich' when needed.

So, let us assume i). There exists in particular an affinoid domain V of Y containing y such that V can be identified with an affinoid domain of an analytic space X' which is étale over X. As $V \hookrightarrow X'$ and $X' \to X$ are quasi-étale, $V \to X$ is quasi-étale; in particular, $V \to X$ is quasi-étale at y, and $Y \to X$ is therefore quasi-étale at y.

Let us now assume that $Y \to X$ is quasi-étale at y, and let x denote the image of y on X. Let us choose a compact analytic neighborhood of Y which is a finite union $\bigcup V_i$ of affinoid domains of Y containing y; we may assume that there exists for every i an affinoid domain U_i of X such that $V_i \to X$ goes through U_i. Fix i. As $Y \to X$ is quasi-étale at y, the morphism $V_i \to U_i$ is quasi-étale at y too (3.10.2). Hence it follows from lemma 3.18 that there exists an affinoid neighborhood V_i' of y in V_i and an étale U_i-space U_i' such that V_i' is isomorphic to an affinoid domain of U_i'. The categories of finite étale covers of the germs (X, x) and (U_i, x) are naturally equivalent (both are equivalent to the category of finite étale $\mathscr{H}(x)$-algebra, [2] th. 3.4.1). Therefore there exists an open neighborhood X_i of x in X and a finite étale morphism $X_i' \to X_i$ so that $X_i' \times_X U_i$ can be identified with an open neighborhood of y in U_i'; let us choose an affinoid neighborhood V_i'' of y in V_i' such that $V_i'' \subset X_i' \times_X U_i \subset X_i'$.
The union of the V_i'''s is a neighborhood of y, and for every i one can identify V_i'' with an affinoid domain of the X-étale space X_i'; therefore $Y \to X$ is quasi-étale at y in the sense of Berkovich.

The transfer of algebraic properties

(3.23) Proposition. *Let $Y \to X$ be a morphism of good k-analytic spaces, let $y \in Y$ and let x be its image on X. If $Y \to X$ is quasi-smooth at y, then* $\mathsf{Spec}\ \mathscr{O}_{Y,y} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ *is flat with geometrically regular fibers.*

Proof. If d denote the relative dimension of Y over X at y, lemma 3.17 ensures that $Y \to X$ goes through a map $Y \to \mathbb{A}^d_X$ which is quasi-étale at y; let u be the image of y on $\mathbb{A}^d_X$. As $Y \to \mathbb{A}^d_X$ is quasi-étale at y, it follows from prop. 3.18 above that there exist :

- an affinoid neighborhood U of u in $\mathbb{A}^d_X X$;
- a finite étale map $V \to U$;

• an affinoid neighborohood W of y in Y and an isomorphism between W and an affinoid domain of V.

Let us consider W as an affinoid domain of V (through the aforementioned automorphism). The morphism $\mathscr{O}_{Y,y} = \mathscr{O}_{W,y} \to \mathscr{O}_{V,y}$ is flat with geometrically regular fibers, and so is $\mathscr{O}_{V,y} \to \mathscr{O}_{U,u}$, because it is finite étale.

We will prove that $\mathscr{O}_{U,u} \to \mathscr{O}_{X,x}$ is flat with geometrically regular fibers, which will end the proof. The space $\mathbb{A}^d_X$ is universally flat over X (2.13 or 2.14), whence the flatness of $\mathscr{O}_{U,u} \to \mathscr{O}_{X,x}$.

Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_{X,x}$ and let F be a finite radicial extension of $\mathsf{Frac}\ \mathscr{O}_{X,x}/\mathfrak{p}$; we want to show that $\mathscr{O}_{U,u} \otimes_{\mathscr{O}_{X,x}} F$ is a regular scheme.

(3.23.1) *Reduction to the case where* $\mathfrak{p} = 0$ *and* $F = \mathsf{Frac}\ \mathscr{O}_{X,x}$. Let $\mathscr{A}$ be the algebra of analytic functions on X; as $\mathfrak{p}$ is finitely generated, we can shrink X (and U) so that there exists an ideal $\mathscr{I}$ of $\mathscr{A}$ satisfying the equality $I\mathscr{O}_{X,x} = \mathfrak{p}$. Hence by replacing X (resp. U) with its Zariski-closed subspace defined by $\mathscr{I}$ (resp. the pull-back of $\mathscr{I}$) we can assume that $\mathfrak{p} = 0$.

We may choose a finite generating family $(\lambda_1, \ldots, \lambda_r)$ of F over $\mathsf{Frac}\ \mathscr{O}_{X,x}$ having the following property : for every i there exists an integer n_i such that $\lambda_i^{p^{n_i}} \in \mathscr{O}_{X,x}$. Let B be the $\mathscr{O}_{X,x}$-subalgebra of F generated by the λ_i's. It is finitely presented (because $\mathscr{O}_{X,x}$ is noetherian) and radicial – that is, there exists m such that $b^{p^m} \in \mathscr{O}_{X,x}$ for every $b \in \mathsf{B}$; note also that by its very definition, B is a domain whose fraction field is nothing but F.

Therefore we may again shrink X and U so that there exists a finite, radicial Banach $\mathscr{A}$-algebra $\mathscr{B}$ with $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{O}_{X,x} \simeq \mathsf{B}$. Therefore if x' denotes the only pre-image of x on $\mathscr{M}(\mathscr{B})$ one has $\mathscr{O}_{\mathscr{M}(\mathscr{B}),x'} \simeq \mathsf{B}$. Hence by replacing X with $\mathscr{M}(\mathscr{B})$, x with x', U with $U \times_X \mathscr{M}(\mathscr{B})$ and u with its only pre-image on $U \times_X \mathscr{M}(\mathscr{B})$, we eventually reduce, as announced, to the case where $\mathfrak{p} = 0$ and $F = \mathsf{Frac}\ \mathscr{O}_{X,x}$.

(3.23.2) *Proof in the case where* $\mathfrak{p} = 0$ *and* $F = \mathsf{Frac}\ \mathscr{O}_{X,x}$. By faithful flatness of $\mathscr{O}_{X,x} \to \mathscr{O}_{\mathscr{X},\mathbf{x}}$, the local ring $\mathscr{O}_{\mathscr{X},\mathbf{x}}$ is a subring of $\mathscr{O}_{X,x}$; in particular, this is a domain, and the generic point η of $\mathsf{Spec}\ \mathscr{O}_{X,x}$ lies above the generic point ξ of $\mathsf{Spec}\ \mathscr{O}_{\mathscr{X},\mathbf{x}}$.

Let ζ be a point of $\mathsf{Spec}\ \mathscr{O}_{U,u}$ lying above η; our goal is now to prove that the generic fiber of $\mathsf{Spec}\ \mathscr{O}_{U,u} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ is regular at ζ; but the latter property is equivalent to regularity of $\mathsf{Spec}\ \mathscr{O}_{U,u}$ itself at ζ. We denote by $\mathbf{u}$ the image of u on $\mathbb{A}^d_{\mathscr{X}}$ and by $\mathbf{z}$ the image of ζ on $\mathsf{Spec}\ \mathscr{O}_{\mathbb{A}^d_{\mathscr{X}},\mathbf{x}}$. By construction, $\mathbf{z}$ is lying above ξ.

As $\mathscr{O}_{\mathscr{X},\mathbf{x}}$ is a domain and as ξ is the generic point of its spectrum, the local ring $\mathscr{O}_{\mathscr{X},\xi}$ is a field κ, and $\mathscr{O}_{\mathbb{A}^d_{\mathscr{X}},\mathbf{z}}$ is nothing but the local ring of $\mathbf{z}$ *inside its fiber*; since the latter fiber coincides with the regular scheme $\mathbb{A}^d_\kappa$, the ring $\mathscr{O}_{\mathbb{A}^d_{\mathscr{X}},\mathbf{z}}$ is regular. As $\mathsf{Spec}\ \mathscr{O}_{U,u} \to \mathsf{Spec}\ \mathscr{O}_{\mathbb{A}^d_{\mathscr{X}},\mathbf{x}}$ is flat with (geometrically) regular fibers, $\mathsf{Spec}\ \mathscr{O}_{U,u}$ is regular at ζ, which ends the proof. □

(3.24) Corollary. *Let $\varphi : Y \to X$ be a morphism between k-analytic spaces, let $y \in X$, let x be its image on X, and let $\mathcal{F}$ be a coherent sheaf on X. Let* P *be one the properties listed in 0.19.5, α), and let* Q *be one of those listed in 0.19.5, β).*

i) Assume that φ is universally flat at y. If Y satisfies P *at y, then so does X at x; if $\varphi^*\mathcal{F}$ satisfies* Q *at y, then so does $\mathcal{F}$ at x.*

ii) Assume moreover that φ is quasi-smooth at y. If X satisfies P *at x, then so does Y at y; if $\mathcal{F}$ satisfies* Q *at x, then so does $\varphi^*\mathcal{F}$ at y.*

Proof. For both assertions, we can assume that Y and X are affinoid; now $\mathsf{Spec}\ \mathscr{O}_{Y,y} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ is flat under the hypothesis of i), and is flat with (geometrically) regular fibers under the hypothesis of ii), in view of prop. 3.23. The corollary then follows immediately thanks to classical results of commutative algebra, most of which can be found with references in [15], §0.5.1 ; but note that for the validity of Q in assertion ii) one needs cor. 6.4.2 of [17], which is not mentioned in [15], but should have been because it is implicitely used at several places of this paper (thm. 3.1 and 3.4). □

4 Generic fibers in analytic geometry

Some technical preliminary lemmas

(4.1) Lemma. *Let K be an analytic field and let V be a subgroup of K. Assume that there exists $\rho \in]0;1[$ such that for every $\lambda \in K$ there exists $\mu \in V$ with $|\lambda - \mu| \leqslant \rho|\lambda|$. The group V is then dense in K.*

Proof. For every $\lambda \in K$ we choose an element $\varphi(\lambda)$ in the group V such that $|\lambda - \varphi(\lambda)| \leqslant \rho|\lambda|$. Now let $\lambda \in K$. Define inductively the sequence $(\lambda_i)_i$ by setting $\lambda_0 = 0$ and $\lambda_{i+1} = \lambda_i + \varphi(\lambda - \lambda_i)$. By induction, one sees that $\lambda_i \in V$ and that $|\lambda - \lambda_i| \leqslant \rho^i|\lambda|$ for every i; hence $\lambda_i \to \lambda$. □

(4.2) Lemma. *Let K be an analytic field with $|K^*|$ free of rank one, and let F be a complete subfield of K with $|F^*| \neq \{1\}$. Assume that the* classical *residue extension $\widetilde{F}^1 \hookrightarrow \widetilde{K}^1$ is finite; the field K is then a finite extension of F.*

Proof. The assumptions on the value groups ensures that $|K^*|/|F^*|$ is finite; hence the *graded* extension $\widetilde{F} \hookrightarrow \widetilde{K}$ is finite too. Let $\lambda_1, \ldots, \lambda_n$ be elements of K^* such that $(\widetilde{\lambda_i})_i$ is a basis of $\widetilde{K}$ over $\widetilde{F}$. Let us call V be the F-vector subspace of K generated by the λ_i's. Let $\lambda \in K$; there exist $a_1, \ldots, a_n \in F$ such that $\widetilde{\lambda} = \sum \widetilde{a_i}\widetilde{\lambda_i}$, which exactly means, if $\lambda \neq 0$, that $|\lambda - \sum a_i\lambda_i| < |\lambda|$. As $|K^*|$ is free of rank one, $|K^*| \cap]0;1[$ has a maximal element ρ. By the above, for every $\lambda \in K$ there exists $\mu \in V$ with $|\lambda - \mu| \leqslant \rho|\lambda|$. By lemma 4.1, V is dense in K. Since V is a finite dimensional F-vector space, it is complete; hence $V = K$. □

(4.3) Corollary. *Let F be a trivially valued field and let X be a non-empty, boundaryless F-space. There exists $x \in X$ such that $\mathscr{H}(x)$ is either a finite extension of F or a finite extension of F_r for some $r \in]0;1[$.*

Proof. Choose an arbitrary $s \in]0;1[$. As X_s is a non-empty, boundaryless space over the non-trivially valued field F_s, the analytic *Nullstellensatz* provides a point $y \in X_s$ with $\mathscr{H}(y)$ finite over k_s; let x be the image of y on X. Note that $\widetilde{\mathscr{H}(x)}^1$ is a subfield of $\widetilde{\mathscr{H}(y)}^1$, which is itself finite over $\widetilde{F_s}^1 = F$; hence $\widetilde{\mathscr{H}(x)}^1$ is finite over F.

If $|\mathscr{H}(x)^*| = \{1\}$ then as $\mathscr{H}(x) = \widetilde{\mathscr{H}(x)}^1$, it is finite over F and we are done.

If $|\mathscr{H}(x)^*| \neq \{1\}$ let $r \in |\mathscr{H}(x)^*| \cap]0;1[$, and let $\lambda \in \mathscr{H}(x)^*$ be an element with $|\lambda| = r$. The complete subfield E generated by λ over F in $\mathscr{H}(x)$ is isomorphic to F_r. As $\mathscr{H}(x)$ is a subfield of $\mathscr{H}(y)$, the non-trivial group $|\mathscr{H}(x)^*|$ is free of rank one; together with the fact that $\widetilde{E}^1 = \widetilde{F_r}^1 = F$ this implies, in view of lemma 4.2, that $\mathscr{H}(x)$ is a finite extension of $E \simeq F_r$. $\square$

(4.4) Lemma. *Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a k-free polyray and let $S_1, \ldots, S_n$ be elements of $k_{\mathbf{r}}$ such that $|S_i| = r_i$ for every i. The complete subfield $k(\widehat{S_1, \ldots, S_n})$ of $k_{\mathbf{r}}$ generated by the S_i's over k is equal to $k_{\mathbf{r}}$; in other words, $S_1, \ldots, S_n$ are coordinate functions of $k_{\mathbf{r}}$.*

Proof. Let $T_1, \ldots, T_n$ be coordinate functions of $k_{\mathbf{r}}$; note that there is a well-defined isometry $\varphi : \sum a_I \mathbf{T}^I \mapsto \sum a_I \mathbf{S}^I$ between $k_{\mathbf{r}}$ and its subfield $k(\widehat{S_1, \ldots, S_n})$.

For every i one can write $S_i = \alpha_i T_i + u_i$ where $\alpha_i \in k$ and $u_i \in k_{\mathbf{r}}$, and where $|\alpha_i| = 1$ and $|u_i| < r_i$. By replacing S_i with $\alpha_i^{-1} S_i$, we may assume that $\alpha_i = 1$ for all i. Therefore, there exists $\rho \in]0;1[$ such that $|T_i - S_i| \leqslant \rho |T_i|$ for every i; it follows immediately that $|\lambda - \varphi(\lambda)| < \rho|\lambda|$ for every $\lambda \in k_{\mathbf{r}}$. Lemma 4.1 then ensures that $k(\widehat{S_1, \ldots, S_n})$ is dense in $k_{\mathbf{r}}$; as it is complete, $k(\widehat{S_1, \ldots, S_n}) = k_{\mathbf{r}}$, as required. $\square$

(4.5) Lemma. *Let F be a field and let L be a finite, separable extension of $F((t))$. There exists a finite extension K of F such that $L \otimes_F K$ admits a quotient isomorphic to $K((\tau))$.*

Proof. Let us consider $F((t))$ as the completion of the function field of $\mathbb{P}^1_F$ at the origin. Krasner's lemma ensures that there exists a projective, normal, irreducible F-curve Y equipped with a finite, generically étale map to $\mathbb{P}^1_F$, such that L can be identified with the completion of $F(Y)$ at a closed point P lying above the origin. There exists a finite extension F_0 of F such that the normalization of $Y \times_F F_0$ is smooth. Now one can take for K the residue field of any point of this normalization lying above P. $\square$

(4.6) Lemma. *Let Y be a quasi-smooth k-analytic space and let $y \in Y$ be a point such that $\mathscr{H}(y) \simeq k_{r_1, \ldots, r_m}$ for some k-free polyray $(r_1, \ldots, r_m)$. Let $(g_1, \ldots, g_m)$ be analytic functions on Y such that $|g_i(y)| = r_i$ for every i. The $\mathrm{d}g_i \otimes 1$'s are then $\mathscr{H}(y)$-linearly independant elements of $\Omega^1_{Y/k} \otimes \mathscr{H}(y)$.*

Proof. One can assume that Y is k-affinoid and of pure dimension, say, n. Let V be the affinoid domain of Y defined as the simultaneous validity locus of the equalities $|g_i| = r_i$. Its k-affinoid structure factorizes through a $k_{r_1, \ldots, r_m}$-structure given by the g_i's, for which y is $k_{r_1, \ldots, r_m}$-rational. By 0.15, $\mathsf{dim}_{k_{r_1, \ldots, r_m}} V = n - m$. As Y is quasi-smooth, $\mathscr{O}_{Y,y}$ is regular; hence $\mathscr{O}_{V,y}$ is regular too (0.19.5). As y is $k_{r_1, \ldots, r_m}$-rational by lemma 4.4 above, V is smooth over $k_{r_1, \ldots, r_m}$ at y; therefore $\mathscr{H}(y) \otimes \Omega^1_{V/k_{r_1, \ldots, r_m}}$ is of dimension $n - m$. But the $\mathscr{H}(y)$-vector space $\Omega^1_{V/k} \otimes \mathscr{H}(y)$ is of dimension n and is equal to

$$\otimes \Omega^1_{V/k_{r_1, \ldots, r_m}} + \sum \mathscr{H}(y).(\mathrm{d}g_i \otimes 1),$$

whence the lemma. $\square$

Relative polydiscs inside relative smooth spaces

(4.7) Lemma. *Let X be a good k- analytic space, let $x \in X$, and let $n \in \mathbb{N}$. Let $m \leqslant n$ and let $\mathbf{r} = (r_1, \ldots, r_m)$ be a $\mathscr{H}(x)$-free polyray, and set $r_i = 0$ for $m < i \leqslant n$. Let ξ be the point of $\mathbb{A}^n_X$ lying above x and defined by the semi-norm*

$$\sum a_I \mathbf{T}^I \mapsto \max |a_I| \mathbf{r}^I$$

on the ring $\mathscr{H}(x)[\mathbf{T}]$, and let V be an open neighborhood of ξ in $\mathbb{A}^n_X$. Under those assumptions, V contains an open neighborhood of ξ of the form

$$U \times_k \mathbb{D}_1 \times_k \ldots \times_k \mathbb{D}_n$$

where U is an open neighborhood of x in X and where $\mathbb{D}_i$ is for every $i \leqslant m$ (resp. $i > m$) a one-dimensional open annulus (resp. disc) with coordinate function T_i.

Proof. Through a straightforward induction argument one immediately reduces to the case where $n = 1$; in that situation r is either zero, either an $\mathscr{H}(x)$-free positive number. Let X_0 be an affinoid neighborhood of x in X and set $\mathscr{A} = \mathscr{O}_X(X_0)$; let X'_0 be the topological interior of X_0 in X. By the explicit description of the topology of the analytification of an $\mathscr{A}$-scheme of finite type, there exist a finite family $P_1, \ldots, P_r$ of elements of $\mathscr{A}[T]$, and a finite family $I_1, \ldots, I_r$ of open subsets of $\mathbb{R}_+$ such that the open subset of $\mathbb{A}^1_{X_0}$ defined by the conditions $|P_j| \in I_j$ (for $j = 1, \ldots, r$) contains ξ and is included in V; set $P_j = \sum a_{i,j} T^i$.

(4.7.1) *The case where $r = 0$.* In that case one has $|P_j(\xi)| = |a_{0,j}(x)|$ for every j. There exists for every j an open neighborhood I'_j of $|a_{0,j}(x)|$ in I_j and a positive number R_j such that $|P_j(\eta)| \in I_j$ as soon as $|a_{0,j}(\eta)| \in I'_j$ and $|T(\eta)| < R_j$. Let us denote by U the set of points $y \in X'_0$ such that $|a_{0,j}(y)| \in I'_j$ for every j, and let R be any positive number smaller than all R_j's.

The product of U and of the open disc centered at the origin with radius R is then included in V and contains ξ, which ends the proof when $r = 0$.

(4.7.2) *The case where r is an $\mathscr{H}(x)$-free positive number.* In that case there exists for every j an index i_j such that $|a_{i_j,j}(x)|r^{i_j} > |a_{i,j}(x)|r^i$ for all $i \neq i_j$. One can find for every j two positive numbers S_j and R_j with $S_j < r < R_j$ and an open subset I'_j of I_j containing $|a_{i_j,j}(x)|r^{i_j}$ such that $|P_j(\eta)| \in I_j$ as soon as $|a_{i_j,j}(\eta)| \in r^{-i_j} I'_j$ and $S_j < |T(\eta)| < R_j$. Let us denote by U the set of points $y \in X'_0$ such that $|a_{i_j,j}(y)| \in r^{-i_j} I'_j$ for every j, and let R and S be two positive number such that $S < r < R$ and such that $S_j < S$ and $R < R_j$ for every j.

The product of U and of the open annulus with bi-radius (S, R) is then included in V and contains ξ, which ends the proof. $\square$

(4.8) Lemma. *Let X be a good k- analytic space, let $x \in X$ and let $n \in \mathbb{N}$; let $m \leqslant n$ and let $\mathbf{r} = (r_1, \ldots, r_m)$ be a $\mathscr{H}(x)$-free polyray. Let $Y \to X$ be a smooth morphism of relative dimension n, and assume that Y_x contains a point y with $\mathscr{H}(y) \simeq \mathscr{H}(x)_{\mathbf{r}}$. Under those assumptions, there exists an open subset V of Y which is X-isomorphic to $U \times_k \mathbb{D} \times_k \Delta$, where U is an open neighborhood of x in X, where $\mathbb{D}$ is a m-dimensional open poly-annulus, and where Δ is a $(n-m)$-dimensional open polydisc.*

Proof. Let us choose analytic functions $f_1, \ldots, f_m$ defined in the neighborhood of y such that $|f_i(y)| = r_i$ for every i. According to lemma 4.6 (which one applies to the $\mathscr{H}(x)$-space Y_x), the elements $\mathrm{d}f_1 \otimes 1, \ldots, \mathrm{d}f_m \otimes 1$ are linearly independant in $\Omega_{Y/X} \otimes \mathscr{H}(y)$; one can hence choose $f_{m+1}, \ldots, f_n$ in $\mathscr{O}_{Y,y}$ so that $\mathrm{d}f_1 \otimes 1, \ldots, \mathrm{d}f_m \otimes 1, \mathrm{d}f_{m+1} \otimes 1, \ldots, \mathrm{d}f_n \otimes 1$ is a *basis* of $\Omega_{Y/X} \otimes \mathscr{H}(y)$.

The X-morphism $Y \to \mathbb{A}^n_X$ induced by the f_i's is quasi-étale at y by lemma 3.17; as $Y \to X$ is boundaryless (it is smooth), $Y \to \mathbb{A}^n_X$ is boundaryless at y, and thus étale (remark 3.21). Lemma 4.4 ensures that the complete subfield of $\mathscr{H}(y)$ generated by the $f_i(y)$'s for $i = 1, \ldots, m$ is equal to $\mathscr{H}(y)$ itself. Therefore, if y' denotes the image of y on $\mathbb{A}^n_X$, one has $\mathscr{H}(y') = \mathscr{H}(y)$; as $Y \to \mathbb{A}^n_X$ is étale at y, this implies that $Y \to X$ induces an isomorphism between an open neighborhood of y in Y and an open neighborhood of y' in $\mathbb{A}^n_X$. We thus may reduce to the case where Y is an open subset of $\mathbb{A}^n_X$, and where the following holds: let p be the projection $\mathbb{A}^n_X \to \mathbb{A}^m_X$ defined by $(T_1, \ldots, T_m)$; the point $p(y) \in \mathbb{A}^{m,\mathrm{an}}_{\mathscr{H}(x)}$ is given by the semi-norm $\sum a_I \mathbf{T}^I \mapsto \max a_I \mathbf{r}^I$, and y is a $\mathscr{H}(p(y))$-rational point of the fiber $p^{-1}(y)$.

Let κ be the residue field of $\mathscr{O}_{\mathbb{A}^m_X, p(y)}$; by density of κ inside $\mathscr{H}(p(y))$, the fiber $p_{|Y}^{-1}(y)$ possesses a $\mathscr{H}(p(y))$-point z such that $T_i(z) \in \kappa$ for every integer $i \in \{m+1, \ldots, n\}$. Let V be an open neighborhood of $p(y)$ in $\mathbb{A}^m_X$ on which the $T_i(z)$'s are defined; translation by $(0, \ldots, 0, T_{m+1}(z), \ldots, T_{m+1}(z))$ identifies over V the space $Y \times_{\mathbb{A}^m_X} V$ with an open subset of $\mathbb{A}^n_X \times_{\mathbb{A}^m_X} V$ whose fiber over $p(y)$ contains the origin of $\mathbb{A}^{m,\mathrm{an}}_{\mathscr{H}(p(y))}$. It follows then from lemma 4.7 that there exists an open subset W of Y which is V-isomorphic to $V' \times \Delta$ where V' is an open neighborhood of $p(y)$ in V and where Δ is a $n-m$-dimensional open polydisc.

Now by applying one again lemma 4.7, but that time to the map $\mathbb{A}^m \to X$ and at the point $p(y)$, one sees that there exists an open neighborhood V'' of $p(y)$ in V' which is X-isomorphic to $U \times_k \mathbb{D}$ for some open neighborhood U of x in X and some m-dimensional open poly-annulus $\mathbb{D}$. Now $W \times_{V'} V''$ is an open subset of Y which is X-isomorphic to $U \times_k \mathbb{D} \times_k \Delta$, as required. $\square$

(4.9) Proposition. *Let $n \in \mathbb{N}$ and let $Y \to X$ be a smooth map of pure dimension n between good k-analytic spaces. Let $x \in X$, and let W be a non-empty open subset of Y_x. There exist :*

- *a flat locally finite morphism $X' \to X$ which can be chosen to be étale if $|\mathscr{H}(x)^*| \neq \{1\}$;*
- *a pre-image x' of x on X', such that the closed fiber of the morphism* $\mathsf{Spec}\ \mathscr{O}_{X',x'} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ *is reduced;*
- *an open subset V of $Y' := Y \times_X X'$ whose intersection with $Y'_{x'}$ is contained in $W_{\mathscr{H}(x')}$, and which is isomorphic to $X' \times_k \mathbb{D}$ where $\mathbb{D}$ is:*
    - ◇ *a n-dimensional open polydisc if W has a $\mathscr{H}(x)$-rigid point, which is always the case if $|\mathscr{H}(x)^*| \neq \{1\}$ or if $n = 0$;*
    - ◇ *the product of an open annulus by an $(n-1)$-dimensional open polydisc if W has a no $\mathscr{H}(x)$-rigid point, which can occur only if $\mathscr{H}(x)$ is trivially valued and $n > 0$.*

Proof. By replacing Y with an open subset of Y whose intersection with Y_x is equal to W, we may assume that $W = Y_x$. By assumption, $Y_x \neq \emptyset$.

(4.9.1) *Let us assume that* $|\mathscr{H}(x)^*| \neq \{1\}$. Let us choose $y \in Y_x$. As $Y \to X$ is smooth, there exists a neighborhood Z of y in Y such that $Z \to X$ goes through an étale map $Z \to \mathbb{A}^n_X$; the image of Z on $\mathbb{A}^n_X$ is an open subset U of the latter, and U_x is non-empty. Let $\mathbb{K}$ be the completion of an algebraic closure of $\mathscr{H}(x)$. Since $|\mathscr{H}(x)^*| \neq \{1\}$, the analytic *Nullstellensatz* ensures that $U_x(\mathbb{K}) \neq \emptyset$; as the separable closure of $\mathscr{H}(x)$ in $\mathbb{K}$ is dense (again because $|\mathscr{H}(x)^*| \neq \{1\}$), the exists $z \in U_x$ with $\mathscr{H}(z)$ a finite, separable extension of $\mathscr{H}(x)$. Now let us choose a pre-image t of z on Z; as $Z \to U$ is étale, $\mathscr{H}(t)$ is a finite, separable extension of $\mathscr{H}(z)$, and hence a finite, separable extension of $\mathscr{H}(x)$ too.

The categories of finite étale covers of the germ (X, x) and of finite étale $\mathscr{H}(x)$-algebras being naturally equivalent ([2], th. 3.4.1), there exists an étale morphism $X' \to X$ and a pre-image x' of x on X such that t has a pre-image t' on $Y' := Y \times_X X'$ with $\mathscr{H}(t') = \mathscr{H}(x')$. This implies, thanks to lemma 4.8, that one can shrink X' around x' so that Y' possesses an open subset which is X'-isomorphic to the product of X' by a n-dimensional open polydisc. This ends the proof in the case where $|\mathscr{H}(x)^*| \neq \{1\}$.

(4.9.2) *The case where* $|\mathscr{H}(x)^*| = \{1\}$. Let us first assume that Y_x has an $\mathscr{H}(x)$-rigid point y. As $\mathscr{H}(x)$ is trivially valued, it coincides with the residue field κ of $\mathscr{O}_{X,x}$. Therefore, there exists a finite, flat, local $\mathscr{O}_{X,x}$-algebra A with $A \otimes_{\mathscr{O}_{X,x}} \kappa \simeq \mathscr{H}(y)$. One can find a locally finite, flat map $X' \to X$ and a pre-image x' of x on X' such that $\mathscr{O}_{X',x'} \simeq A$; note that $\mathscr{H}(x') \simeq \mathscr{H}(y)$ and that the closed fiber of $\mathsf{Spec}\ \mathscr{O}_{X',x'} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ is reduced. By construction, y has a pre-image y' on $Y' := Y \times_X X'$ lying above x' and satisfying $\mathscr{H}(y') = \mathscr{H}(x')$. This implies, thanks to lemma 4.8, that one can shrink X' around x' so that Y' possesses an open subset which is X'-isomorphic to the product of X' by a n-dimensional open polydisc. This ends the proof in the case where $|\mathscr{H}(x)^*| = \{1\}$ and where Y_x has a rigid point.

Let us now assume that Y_x has no $\mathscr{H}(x)$-rigid point. In that case, there exists $t \in Y_x$ and $r \in]0;1[$ such that $\mathscr{H}(t)$ is a finite extension of $\mathscr{H}(x)_r$ (lemma 4.3). Thanks to lemma 4.5, there exists a finie $\mathscr{H}(x)$-extension F such that $F \otimes_{\mathscr{H}(x)_r} \mathscr{H}(t)$ admits a quotient isomorphic to F_s for some $s \in]0;1[$. As $\mathscr{H}(x)$ is trivially valued, it coincides with the residue field of $\mathscr{O}_{X,x}$. Therefore, there exists a finite, flat, local $\mathscr{O}_{X,x}$-algebra A with $A \otimes_{\mathscr{O}_{X,x}} \kappa \simeq F$. One can find a locally finite, flat map $X' \to X$ and a pre-image x' of x on X' such that $\mathscr{O}_{X',x'} \simeq A$; note that $\mathscr{H}(x') \simeq F$ and that the closed fiber of $\mathsf{Spec}\ \mathscr{O}_{X',x'} \to \mathsf{Spec}\ \mathscr{O}_{X,x}$ is reduced. By construction, y has a pre-image y' on $Y' := Y \times_X X'$ lying above x' and such that $\mathscr{H}(y')$ is isomorphic to $\mathscr{H}(x')_s$ for some $s \in]0;1[$. This implies, thanks to lemma 4.8, that one can shrink X' around x' so that Y' possesses an open subset which is X'-isomorphic to the product of X', of a $n-1$-dimensional open polydisc, and of an open annulus, which ends the proof. □

Let us give now some consequences of the latter proposition. Note that the first one, corollary 4.10, has already been proven by Berkovich ([2], cor. 3.7.4), by using a decomposition of smooth morphisms in 'elementary' curve fibrations, whose existence comes from the semi-stable reduction theorem.

(4.10) Corollary. *Any quasi-smooth, boundaryless map is open.*

Proof. Let $Y \to X$ be a quasi-smooth, boundaryless map. To prove that it is open, one may argue G-locally on X, hence assume that X is good. Now Y is also good because $Y \to X$ is boundaryless, and it follows from cor. 3.20 that $Y \to X$ is smooth. Its openness then follows immediatly from proposition 4.9 above together with openness of flat, locally finite morphisms. □

(4.11) *Remark.* The fact that a smooth map is open (remind that in the good case, smooth is equivalent to quasi-smooth and boundaryless by cor. 3.20) has already be proved by Berkovich ([2], cor. 3.7.3), using his 'elementary fibrations' (*loc. cit.*, def. 3.7.1), whose existence is established with the help of the semi-stable reduction theorem.

(4.12) Corollary. *Let $Y \to X$ be a smooth morphism between good k-analytic spaces and let $x \in X$ with $|\mathscr{H}(x)^*| \neq \{1\}$. If $Y_x \neq \emptyset$, there exists an étale morphism $X' \to X$ whose image contains x and an X-map $X' \to Y$.* □

(4.13) Corollary. *Let n and d be two integers and let $\varphi : Y \to X$ be a quasi-smooth morphism between k-analytic spaces. Assume that X is purely d-dimensional, and that φ is of pure relative dimension n. Then Y is purely $n + d$-dimensional.*

Proof. One can argue G-locally on Y and X ; therefore, one can assume firs, that X and Y are good, and then (prop. 3.18) that Y is an analytic domain of a smooth X-space of pure relative dimension n. Eventually (by replacing Y with the latter), we reduce to the case where φ itself is smooth. Now if V is any non-empty open subset of Y, corollary 4.10 above ensures that $\varphi(V)$ is an open subset of X. It is therefore of dimension d. By 0.15.1, $\dim V = n + d$; by 0.16, Y is purely of dimension $n + d$. □

Generic fibers

(4.14) Proposition. *Let $Y \to X$ be a map between good k-analytic spaces. Let $y \in Y$, let x be its image on X. Assume that $Y \to X$ is smooth at y and that $\mathscr{O}_{X,x}$ is artinian. Let Z be a Zariski-closed subset of Y which contains a neighborhood of y in Y_x. Under those assumptions Z is a neighborhood of y in Y.*

Proof. The required property being purely topological, one may assume that X is reduced; in that case, $\mathscr{O}_{X,x}$ is a field, and is in particular normal. Thus we may shrink X so that it is itself normal. Now the X-quasi-smooth space Y is normal too, in view of prop. 3.23; by shrinking Y (and Z, accordingly) we eventually reduce to the case where Y is connected, hence irreducible, and where Z is the zero-locus of a finite family $(f_1, \ldots, f_n)$ of analytic functions on Y. We will prove that Z contains a non-empty open subset of Y, which will force it to coincide with Y, and end the proof.

Thanks to prop. 4.9, there exists a flat, locally finite map $X' \to X$, a point x' on X' lying above x, and a k-analytic space $\mathbb{D}$ such that:

- $\mathscr{O}_{X',x'}$ is a field ;

- $\mathbb{D}$ is an open polydisc, or the product of an open polydisc by a one-dimensional open annulus;

- if one sets $Y' = Y \times_X X'$ and $Z' = Z \times_X X'$, there exists an open subset V of Y which is X'-isomorphic to $\mathbb{D} \times_k X'$ and such that $V_{x'} \subset Z_{x'}$.

As $\mathscr{O}_{X',x'}$ is a field, we may shrink X' so that it is connected and normal.

Let us still denote by $f_1, \ldots, f_n$ the pull-backs of the f_i's on Y'. Analytic function on $V \simeq \mathbb{D} \times_k X'$ consist in power series $\sum a_I \mathbf{T}^I$ where the a_I' s are analytic functions on X'. For any j, let us write $f_{j|V} = \sum a_{I,j} \mathbf{T}^I$. By construction, $V_{x'} \subset Z'_{x'}$. Therefore $a_{I,j}(x') = 0$ for every (I, j). As $\mathscr{O}_{X',x'}$ is a field, $a_{I,j}$ vanishes for every (I, j) in a neighborhood of x' in the normal, connected space X'; therefore $a_{I,j} = 0$ for every (I, j). This implies that $V \subset Z'$; hence Z' contains a non-empty open subset of Y'. As $Y' \to Y$ is flat, locally finite it is open by cor. 2.17. Therefore Z contains a non-empty open subset of Y, which ends the proof. □

(4.15) Theorem. *Let $Y \to X$ be a boundaryless map between good k-analytic spaces. Let $y \in Y$ and let x be its image on X. Assume that $\mathscr{O}_{X,x}$ is a field. Then* $\mathsf{Spec}\ \mathscr{O}_{Y_x,y} \to \mathsf{Spec}\ \mathscr{O}_{Y,y}$ *is flat ; its fibers are* CI*, and regular if char.* $k = 0$.

Proof. Let n be the relative dimension of $Y \to X$ at y. We will argue by induction on n; let us begin with some preparation.

According to the th. 4.6 of [14], one can shrink Y around y such that $Y \to X$ goes through a map $Y \to \mathbb{A}^n_X$ which is zero-dimensional at y. By assumption, $Y \to X$ has no boundary; it implies that $Y \to \mathbb{A}^n_X$ has no boundary ; hence, it is finite at y ([1], prop. 3.1.4). Denote by t the image of y on $\mathbb{A}^n_X$. One can shrink Y around y so that it is finite over an affinoid neighborhood V of t in $\mathbb{A}^n_X$, and so that y is the only preimage of t in Y. Let $\mathscr{A}$ (resp. $\mathscr{B}$) be the algebra of analytic functions on V (resp. Y). Then $\mathscr{O}_{Y,y} = \mathscr{B} \otimes_{\mathscr{A}} \mathscr{O}_{V,t}$ and $\mathscr{O}_{Y_x,y} = \mathscr{B} \otimes_{\mathscr{A}} \mathscr{O}_{V_x,t}$; hence $\mathscr{O}_{Y_x,y} = \mathscr{O}_{V_x,t} \otimes_{\mathscr{O}_{V,t}} \mathscr{O}_{Y,y}$. Then it is sufficient to prove that $\mathsf{Spec}\ \mathscr{O}_{V_x,t} \to \mathsf{Spec}\ \mathscr{O}_{V,t}$ is flat and that its fibers are CI, and regular if char. $k = 0$.

(4.15.1) As $\mathscr{O}_{X,x}$ is a field, it is regular; hence $\mathscr{O}_{V,t}$ is regular by lemma 3.23.

(4.15.2) The local ring $\mathscr{O}_{V_x,t}$ is regular by 0.20.

(4.15.3) The ring $\mathscr{O}_{V,t}$ is in particular reduced; it implies that if f is a non-zero element of it, its zero-locus (which is a Zariski-closed subset of a suitable neighborhood of t) contains no neighborhood of t in V. Thanks to the prop. 4.14, this zero-locus contains no neighborhood of t in V_x.

(4.15.4) Let us now go back to the proof by induction on n. If $n = 0$ then $t = x$, and both $\mathscr{O}_{V,t} = \mathscr{O}_{X,x}$ and $\mathscr{O}_{V_x,t} = \mathscr{H}(x)$ are fields; hence we are done. Assume that $n > 0$ and that the theorem has been proved for any integer stictly lower than n.

(4.15.5) Let us first prove the flatness claim. Let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}_{V,t}$. By [18], Th. 5.6, it is sufficient to prove that $\mathscr{O}_{V_x,t}/\mathfrak{m}^d$ is a flat $\mathscr{O}_{V,t}/\mathfrak{m}^d$-algebra for any $d > 0$. We distinguish two cases.

- *The first case.* If $\mathfrak{m} = 0$, then $\mathscr{O}_{V,t}$ is a field and we are done.

- *The second case.* Suppose $\mathfrak{m} \neq \{0\}$ and let d be a positive integer. Due to remark 3), the Zariski-closed subspace Z defined in a neighborhood of t by the finitely generated ideal $\mathfrak{m}^d$ contains no neighborhood of t in V_x ; therefore $Z \to X$ is of dimension strictly lower than n at t. By induction, $\mathscr{O}_{Z_x,t}$ is a flat $\mathscr{O}_{Z,t}$-algebra. But $\mathscr{O}_{Z_x,t}$ (resp. $\mathscr{O}_{Z,t}$) is nothing but $\mathscr{O}_{V_x,t}/\mathfrak{m}^d$ (resp. $\mathscr{O}_{V,t}/\mathfrak{m}^d$).

(4.15.6) Now, let us proof that any fiber of $\mathsf{Spec}\ \mathscr{O}_{V_x,t} \to \mathsf{Spec}\ \mathscr{O}_{V,t}$ is CI, and regular if char. $k = 0$. Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_{V,t}$. We distinguish two cases.

- *The first case.* If $\mathfrak{p} = 0$, then the fiber of $\mathsf{Spec}\ \mathscr{O}_{V_x,t}$ over $\mathfrak{p}$ is the spectrum of a localization of $\mathscr{O}_{V_x,t}$; but the latter is regular (4.15.2), hence we are done.

- *The second case.* Suppose $\mathfrak{p} \neq \{0\}$. By 4.15.3, the Zariski-closed subspace Z defined in a neighborhood of t by the finitely generated ideal $\mathfrak{p}$ contains no neighborhood of t in V_x ; therefore $Z \to X$ is of dimension strictly lower than n at t. The fiber of $\mathsf{Spec}\ \mathscr{O}_{V_x,t}$ over $\mathfrak{p}$ is nothing but the generic fiber of $\mathsf{Spec}\ \mathscr{O}_{Z_x,t} \to \mathsf{Spec}\ \mathscr{O}_{Z,t}$. By the induction hypothesis, the latter is CI, and regular if char. $k = 0$. $\square$

(4.16) Remarks. Let us give some counter-examples which show that th. 4.15 is probably not far from being optimal.

(4.16.1) *One can not expect in general flatness of* $\mathsf{Spec}\ \mathscr{O}_{Y_x,y} \to \mathsf{Spec}\ \mathscr{O}_{Y,y}$ *if* $y \in \partial Y/X$. Indeed, let $r > 0$, let and let $f = \sum \alpha_i T^i$ be a power series with coefficients in k such that $|\alpha_i|r^i \xrightarrow[i \to +\infty]{} 0$ and such that $(|\alpha_i|s^i)_i$ is non-bounded as soon as $s > r$. Let V be the analytic domain of $\mathbb{A}_k^{2,\mathrm{an}}$ defined by the condition $|T_1| = r$. There is a natural closed immersion $\varphi : \mathbb{D} \to V$ given by (Id, f), where $\mathbb{D}$ is the closed disc of radius r; let x denote the image under φ of the unique point of the Shilov boundary of $\mathbb{D}$.

Lemma 2.21 ensures that $\mathscr{O}_{\mathbb{A}_k^{2,\mathrm{an}},x}$ is a field. The fiber of $V \hookrightarrow \mathbb{A}_k^{2,\mathrm{an}}$ at x is nothing but $\mathscr{M}(\mathscr{H}(x))$, and $\mathscr{O}_{V_x,x}$ is then simply the field $\mathscr{H}(x)$. As x lies on a one-dimensional Zariski-closed subset (namely, $\varphi(\mathbb{D})$), of the purely two-dimensional space V, the local ring $\mathscr{O}_{V,x}$ can *not* be a field (cor. 1.12). As a consequence, $\mathsf{Spec}\ \mathscr{H}(x) \to \mathsf{Spec}\ \mathscr{O}_{V,x}$ is not flat.

(4.16.2) *One can not expect in general regularity of the fibers of the morphism* $\mathsf{Spec}\ \mathscr{O}_{Y_x,y} \to \mathsf{Spec}\ \mathscr{O}_{Y,y}$ *if k is of positive characteristic.* Indeed, let us give the following counter-example which was communicated to the author by M. Temkin. Assume that k is a non-algebraically closed field of char. $p > 0$ and that $|k^*| \neq \{1\}$. Let k^a be an algebraic closure of k, and let k^s be the separable closure of k inside k^a. Let $(k_n)_{n\in\mathbb{N}}$ be an increasing sequence of subfields of k^s which are finite over k and whose union is equal to k^s. For any n, the complement of a finite union of proper k-vector subspaces of the k-Banach space k_n is a dense subset of it. Therefore there exists a sequence (λ_n) of elements of k^s and a decreasing sequence (r_n) of positive numbers such that:

i) for any n, one has $k[\lambda_n] = k_n$;

ii) for any n and any conjugate element μ of λ_n in k^a, one has $\mu = \lambda_n$ or $|\mu - \lambda_n| > r_n$;

iii) for any $m > n$, one has $|\lambda_m - \lambda_n| < r_n$;

iv) $r_n \to 0$ as $n \to \infty$.

For any n let us denote by $\mathbb{D}_n$ (resp. D_n) the affinoid domain of $\mathbb{A}^{1,an}_{k_n}$ defined by the inequality $|T - \lambda_n| \leqslant r_n$ (resp. the closed disc of $\widehat{k^a}$ of center λ_n and radius r_n). It follows from *i)* and *ii)* that the natural map $\mathbb{A}^{1,an}_{k_n} \to \mathbb{A}^{1,an}_k$ induces an isomorphism beween $\mathbb{D}_n$ and an affinoid domain Δ_n of $\mathbb{A}^{1,an}_k$. It follows from *iii)* and *iv)* that (D_n) is a decreasing sequence of closed discs whose intersection consists in a single element $\lambda \in \widehat{k^a}$. Let $x \in \mathbb{A}^{1,an}_k$ be the point that corresponds to λ. We have $x \in \bigcap \Delta_n$. Therefore, k_n embedds into $\mathscr{H}(x) \subset \widehat{k^a}$ for every n. Hence $\mathscr{H}(x)$ is a closed subfield of $\widehat{k^a}$ containing k^s; the latter being dense in $\widehat{k^a}$, $\mathscr{H}(x) = \widehat{k^a}$; in particular, x is not a rigid point.

Let φ be the (finite, flat) morphism $\mathbb{A}^{1,an}_k \to \mathbb{A}^{1,an}_k$ induced by the morphism from $k[T]$ to itself that sends T to T^p and let y be the unique preimage of x by φ. As x in non-rigid, y is non-rigid; both local rings $\mathscr{O}_{\mathbb{A}^{1,an}_k,x}$ and $\mathscr{O}_{\mathbb{A}^{1,an}_k,y}$ are thus fields (lemma 2.20). Now $\mathscr{O}_{\varphi^{-1}(x),y} = \mathscr{H}(x)[\tau]/(\tau^p - T(x))$. Since $\mathscr{H}(x) = \widehat{k^a}$, the local ring $\mathscr{O}_{\varphi^{-1}(x),y}$ is non-reduced, and in particular, non-regular.

5 Dévissages à la Raynaud-Gruson

Most of this section is inspired by the first part of Raynaud-Gruson's work on flatness ([22])). Prop. 5.6, prop. 5.7 and th. 5.10 are the respective analogs of lemma 2.2, th. 2.1 and prop. 1.2.3 of *loc. cit.*

Universal injectivity

(5.1) Definition. Let $Y \to X$ be a morphism between k-analytic spaces, and let $\mathcal{E} \to \mathcal{F}$ be a linear map between coherent sheaves on Y. Let $y \in Y$. We will say that $\mathcal{E} \to \mathcal{F}$ is *X-universally injective at y* if for every analytic space X', for every morphism $X' \to X$, and for every point y' lying above y on $Y' := Y \times_X X'$, the map $\mathcal{E}' \to \mathcal{F}'$ is injective at y', where $\mathcal{E}'$ and $\mathcal{F}'$ are the respective inverse images of $\mathcal{E}$ and $\mathcal{F}$ on Y'.

(5.2) The following facts come straightforwardly from the definition and from 0.19.4.

(5.2.1) Universal injectivity is preserved by base change and ground field extension.

(5.2.2) Let $Y \to X$ be a morphism between k-analytic spaces, and let $\mathcal{E} \to \mathcal{F}$ be a linear map between coherent sheaves on Y. Let V be an analytic domain of Y, and let U be an analytic domain of X which contains the image of V. For any $u \in V$, the map $\mathcal{E} \to \mathcal{F}$ is X-universally injective at y if and only if $\mathcal{E}_{|V} \to \mathcal{F}_{|V}$ is U-universally injective at y.

(5.2.3) Definition 5.1 of universal injectivity is equivalent to the same one in which one (apparently) weakens the condition by taking for X' a good space, or even an affinoid one.

(5.3) Remark. We could also have defined the notion of universal surjectivity (resp. bijectivity) at a point in the same way, and obtained analogous properties; but as one sees after having reduced to the good case, universal surjectivity (resp. universal bijectivity) at a point is simply equivalent to surjectivity (resp. bijectivity) at that point, because of Nakayama's lemma (resp. for obvious reasons).

Universal injectivity and universal flatness

(5.4) Let T be a k-analytic space, let U be a *Zariski*-open subset of T and let $t \in T$; set $F = T - U$. We immediately see that the following are equivalent :

i) $t \in \overline{U}^T$;

ii) U intersects at least one of the irreducible components of T that contain t;

iii) there exists an irreducible component Z of T which contains t and such that $\mathsf{dim}_k\,(F \cap Z) < \mathsf{dim}_k\, Z$.

This has two straightforward consequences:

(5.4.1) If S is an analytic domain in T containing t, then $t \in \overline{U}^T$ if and only if $t \in \overline{(U \cap S)}^S$.

(5.4.2) If L is an analytic extension of k and if s is a point of T_L lying above t, then $t \in \overline{U}^T \iff s \in \overline{U_L}^{T_L}$.

(5.5) Standard notations. We will often have to consider the following situation: T and X are affinoid k-spaces, $T \to X$ is a quasi-smooth morphism, t is a point of T and x is its image on X. Then we will systematically denote by $\mathbf{t}$ (resp. $\mathbf{t}_x$) the image of t on $\mathscr{T}$ (resp. $\mathscr{T}_x$), by $\mathbf{x}$ the image of x on $\mathscr{X}$, by $\mathfrak{m}$ the maximal ideal of $\mathscr{O}_{\mathscr{X},\mathbf{x}}$ and by z an arbitrary point on T whose image $\mathbf{z}_x$ on $\mathscr{T}_x$ is the generic point of the connected component of $\mathbf{t}_x$ (as T_x is a quasi-smooth, hence geometrically regular, $\mathscr{H}(x)$-analytic space, $\mathscr{T}_x$ is a regular scheme by GAGA); note that $\mathscr{O}_{\mathscr{T}_x,\mathbf{z}_x}$ is the fraction field of the regular local ring $\mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x}$. The image of z on $\mathscr{T}$ will be denoted by $\mathbf{z}$.

Let us make some remarks about those notations.

(5.5.1) One can choose such a z on any given neighborhood U on t in T_x; indeed, let n be the dimension of T_x at x and let V be the intersection of U and of the connected component of t in T_x; then the dimension of V at x is also n, and as a consequence, it exists z in V satisfying $\mathsf{d}(\mathscr{H}(z)/\mathscr{H}(x)) = n$, which proves the claim.

(5.5.2) Any Zariski-open subset of T containing t has a non-empty intersection with the connected component of t in T_x, hence contains z; as a consequence, $\mathbf{t} \in \overline{\{\mathbf{z}\}}^{\mathscr{T}}$ and $\mathscr{O}_{\mathscr{T},\mathbf{z}}$ is a localization of $\mathscr{O}_{\mathscr{T},\mathbf{t}}$.

(5.6) Proposition. *Let $T \to X$ be a quasi-smooth morphism between k-analytic spaces. Let $\mathcal{L}$ be a free $\mathscr{O}_T$-module of finite rank and let $\mathcal{N}$ be a coherent sheaf on T. Let $t \in T$, and let x be its image on X. Let $\mathcal{L} \to \mathcal{N}$ be a map such that $t \in \overline{\mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})}^{T_x}$; the following are equivalent:*

i) the map $\mathcal{L} \to \mathcal{N}$ is X-universally injective at t;

ii) the map $\mathcal{L} \to \mathcal{N}$ is injective at t;

iii) $t \in \overline{\mathsf{Bij}(\mathcal{L} \to \mathcal{N})_x}^{T}$.

Proof. We first begin with a remark. Using 5.4.2 and the fact that the bijectivity locus can only *increase* by any base change, we see that the property *iii*) is universal, that is, remains true after any base change. Then, it is sufficient to prove that *ii*) $\iff$ *iii*). We can assume that T and X are affinoid, and we use the standard notations (5.5).

(5.6.1) *Let us prove that ii) $\Rightarrow$ iii).* Assume that *ii*) holds. It means that the arrow $\mathcal{L} \otimes \mathscr{O}_{T,t} \to \mathcal{N} \otimes \mathscr{O}_{T,t}$ is injective. Therefore:

α) $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{t}} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{t}}$ is injective by 0.13.5;

β) $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}$ is injective by α) and 5.5.2.

By assumption, $t \in \overline{\mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})}^{T_x}$; hence $z \in \mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})$ and $\mathcal{L} \otimes \mathscr{H}(z) \to \mathcal{N} \otimes \mathscr{H}(z)$ is an isomorphism. Therefore:

γ) $\mathcal{L} \otimes \kappa(\mathbf{z}) \to \mathcal{N} \otimes \kappa(\mathbf{z})$ is an isomorphism too;

δ) $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}$ is surjective by γ) and Nakayama's lemma;

ε) $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}$ is bijective by δ) and β).

By ε), $z \in \mathsf{Bij}(\mathcal{L} \to \mathcal{N})$; by the choice of z, one has $t \in \overline{\mathsf{Bij}(\mathcal{L} \to \mathcal{N})_x}^{T}$.

(5.6.2) *Let us now show that iii) $\Rightarrow$ ii).* Assume that *iii*) holds. By 0.13.3, we can prove *ii*) after having replaced k by any of its analytic extensions; it allows us to assume that x is a k-point. As *iii*) holds, $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \simeq \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}$.

Thanks to 0.13.5, it is sufficient to prove that the top horizontal arrow of the following commutative diagram

$$\begin{array}{ccc} \mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{t}} & \longrightarrow & \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{t}} \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} & \longrightarrow & \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \end{array}$$

is an injection; the bottom horizontal arrrow beeing an isomorphism, it is enough to establish the injectivity of the left vertical arrow. The $\mathscr{O}_T$-module $\mathcal{L}$ is free of finite rank; therefore, we have reduced the problem to proving that the map $\mathscr{O}_{\mathscr{T},\mathbf{t}} \to \mathscr{O}_{\mathscr{T},\mathbf{z}}$ is injective.

Let S be the multiplicative subset of $\mathscr{O}_{\mathscr{T},\mathbf{t}}$ wich consists in all elements a such that $a(\mathbf{z}) \neq 0$; then $\mathscr{O}_{\mathscr{T},\mathbf{z}} = \mathsf{S}^{-1}\mathscr{O}_{\mathscr{T},\mathbf{t}}$ (5.5.2). Let a in S. Since $a(\mathbf{z}) \neq 0$, the image of a in $\kappa(\mathbf{z}_x)$ is non-zero. But $\kappa(\mathbf{z}_x)$ coincides with $\mathscr{O}_{\mathscr{T}_x,\mathbf{z}_x}$, that is, with $\mathsf{Frac}\ \mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x}$. Therefore the image of a in the domain $\mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x}$ is non-zero, and hence is not a zero divisor.

As x is a k-point, $\mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x}$ is nothing but $\mathscr{O}_{\mathscr{T},\mathbf{t}}/\mathfrak{m}$. It then follows from [16], chapt. 0, §10.2.4 that for any $a \in \mathsf{S}$, the multiplication by a in $\mathscr{O}_{\mathscr{T},\mathbf{t}}$ is injective; as a consequence, the localization map $\mathscr{O}_{\mathscr{T},\mathbf{t}} \to \mathscr{O}_{\mathscr{T},\mathbf{z}}$ is injective. $\square$

(5.7) Proposition. *We keep the assumptions and notations of the proposition above. Let $\mathcal{P}$ be the cokernel of $\mathcal{L} \to \mathcal{N}$. The following are equivalent:*

i) $\mathcal{N}$ is universally X-flat at t;

ii) $\mathcal{L} \to \mathcal{N}$ is injective at t and $\mathcal{P}$ is universally X-flat at t.

If moreover $t \in \mathsf{Int}\ T/X$, then the following are equivalent:

i′) $\mathcal{N}$ is X-flat at t;

ii′) $\mathcal{L} \to \mathcal{N}$ is injective at t, and $\mathcal{P}$ is X-flat at t.

Proof. We can assume that T and X are affinoid; we will use the standard notations (5.5). Let us first make some observations.

(5.7.1) The quasi-smoothness of $T \to X$ implies hat $\mathscr{O}_T$ is universally X-flat at t; as $\mathcal{L}$ is free, it is universally X-flat at t too.

(5.7.2) By prop. 5.6 above,

$$\mathcal{L} \to \mathcal{N} \text{ is injective at } t \iff \mathcal{L} \to \mathcal{N} \text{ is universally injective at } t$$

$$\iff t \in \overline{\mathsf{Bij}(\mathcal{L} \to \mathcal{N})_x}^{T}.$$

(5.7.3) By using 5.7.1 and the first equivalence of 5.7.2, it is easily seen that if $\mathcal{L} \to \mathcal{N}$ is injective at t, then $\mathcal{N}$ is X-flat (resp. universally X-flat) at t if and only if $\mathcal{P}$ is X-flat (resp. universally X-flat) at t; in order to prove the proposition, it is then sufficient to show that if $\mathcal{N}$ is universally X-flat at t (resp. if $\mathcal{N}$ is X-flat at t and if $t \in \mathsf{Int}\ T/X$), then $\mathcal{L} \to \mathcal{N}$ is injective at t; moreover, as far as the first implication is concerned, we can prove it after having extended the ground field (0.13.3), which allows to reduce to the case where t is a k-point; in this particular case, $t \in \mathsf{Int}\, T/X$. Hence it is fact sufficient for both our purposes to prove that if $t \in \mathsf{Int}\, T/X$ and if $\mathcal{N}$ is X-flat at t, then $\mathcal{L} \to \mathcal{N}$ is injective at t. Using the second equivalence of 5.7.2, we eventually reduce the problem to proving the following claim: *if $t \in \mathsf{Int}\, T/X$ and if $\mathcal{N}$ is X-flat at t, then $t \in \overline{\mathsf{Bij}(\mathcal{L} \to \mathcal{N})_x}^T$* ; that is what we are going to do.

(5.7.4) From now on, we assume that $t \in \mathsf{Int}\ T/X$ and that $\mathcal{N}$ is X-flat at t. We will use the standard notations (5.5); we can shrink X such that $\mathfrak{m}$ is generated by global functions $(f_1, \ldots, f_r)$ on X, and such that $\mathfrak{m}\mathscr{O}_{X,x}$ is the maximal ideal of $\mathscr{O}_{X,x}$, that is, such that $\mathscr{O}_{X,x}/\mathfrak{m}$ is a field; since $\mathsf{Int}\, T/X$ is an open subset of T, we can choose z such that it also belongs to $\mathsf{Int}\, T/X$ (5.5.1).

Let $\mathscr{R}$ be the kernel of $\mathcal{L} \to \mathcal{N}$. As $t \in \overline{\mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})}^{T_x}$, the point z belongs to $\mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})$, and one has thus $\mathcal{P} \otimes \mathscr{H}(z) = 0$. It follows from Nakayama's lemma that $\mathcal{P} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} = 0$; we thus have an exact sequence

$$0 \to \mathscr{R} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \to \mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}} \to 0.$$

By assumption, $\mathcal{N} \otimes \mathscr{O}_{T,t}$ is $\mathscr{O}_{X,x}$-flat. By 0.13.5, the $\mathscr{O}_{\mathscr{X},\mathbf{x}}$-module $\mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{t}}$ is therefore flat; by 5.5.2, the $\mathscr{O}_{\mathscr{X},\mathbf{x}}$-module $\mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}$ is also flat. Hence

$$0 \to \mathscr{R} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}/\mathfrak{m} \to \mathcal{L} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}/\mathfrak{m} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{T},\mathbf{z}}/\mathfrak{m} \to 0$$

is exact.

Let Y be the closed analytic subspace of X defined by the f_i's, and let S be the fiber product $T \times_X Y$. The exact sequence above is nothing but

$$0 \to \mathscr{R} \otimes \mathscr{O}_{\mathscr{S},\mathbf{z}} \to \mathcal{L} \otimes \mathscr{O}_{\mathscr{S},\mathbf{z}} \to \mathcal{N} \otimes \mathscr{O}_{\mathscr{S},\mathbf{z}} \to 0.$$

By construction, $\mathscr{O}_{Y,x} = \mathscr{O}_{X,x}/\mathfrak{m}$ is a field and $z \in \mathsf{Int}\, S/Y$. But th. 4.15 then tells us that $\mathscr{O}_{S_x,z}$ is a flat $\mathscr{O}_{S,z}$-algebra; using 0.13.5, it implies that the $\mathscr{O}_{\mathscr{S},\mathbf{z}}$-algebra $\mathscr{O}_{S_x,z}$ is flat, whence we deduce the exactness of

$$0 \to \mathscr{R} \otimes \mathscr{O}_{S_x,z} \to \mathcal{L} \otimes \mathscr{O}_{S_x,z} \to \mathcal{N} \otimes \mathscr{O}_{S_x,z} \to 0,$$

that is, of

$$0 \to \mathscr{R} \otimes \mathscr{O}_{T_x,z} \to \mathcal{L} \otimes \mathscr{O}_{T_x,z} \to \mathcal{N} \otimes \mathscr{O}_{T_x,z} \to 0.$$

As $t \in \overline{\mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})}^{T_x}$, the point z belongs to $\mathsf{Bij}(\mathcal{L}_{|T_x} \to \mathcal{N}_{|T_x})$. Therefore $\mathscr{R} \otimes \mathscr{O}_{T_x,z} = 0$; hence $\mathscr{R} \otimes \mathscr{H}(z) = 0$. By Nakayama's lemma, $\mathscr{R} \otimes \mathscr{O}_{T,z} = 0$; this exactly means that $t \in \overline{\mathsf{Bij}(\mathcal{L} \to \mathcal{N})_x}^T$. $\square$

(5.8) Lemma. *Let Y, T and X be good k-analytic spaces, let $Y \to X$ be a morphism which can be written as a composition*

$$Y \xrightarrow{\pi} T \longrightarrow X$$

where π is finite. *Let $y \in Y$, let t and x be its images on T and X, and let $\mathcal{F}$ be a coherent sheaf on Y. Then:*

i) if $\pi_*\mathcal{F}$ is X-flat (resp. universally X-flat) at t, then $\mathcal{F}$ is X-flat (resp. universally X-flat) at y;

ii) if y is the only pre-image of t on Y and if $\mathcal{F}$ is X-flat (resp. universally X-flat) at y, then $\pi_*\mathcal{F}$ is X-flat (resp. universally X-flat) at t.

Proof. Let X' be a good analytic space over an analytic extension of k, let $X' \to X$ be a map, set $T' = T \times_X X'$, $Y' = T \times_X X'$ and $Z' = Z \times_X Z'$; denote by $\mathcal{F}'$ the inverse image of $\mathcal{F}$ on Y' and by π' the morphism $Z' \to T'$ induced by π. The inverse image of $\pi_*\mathcal{F}$ on T' is then $\pi'_*\mathcal{F}'$. Let t' be a point on T' lying above t, and let x' be the image of t' on X'. Let $y_1, \ldots, y_r$ be the pre-images of t' on Y'. The $\mathscr{O}_{X',x'}$-module $(\pi'_*\mathcal{F}') \otimes \mathscr{O}_{T',t'}$ beeing nothing but $\bigoplus \mathcal{F}' \otimes \mathscr{O}_{Y',y_i}$, it is flat if and only if all $\mathcal{F}' \otimes \mathscr{O}_{Y',y_i}$'s are flat. Both assertions follow straightforwardly from those facts. □

Dévissages: definition and existence

If Y is a k-analytic space and if $\mathcal{F}$ is a coherent sheaf on Y, the unique coherent sheaf on $\mathsf{Supp}\,\mathcal{F}$ that induces $\mathcal{F}$ will be also denoted by $\mathcal{F}$, if there is no risk of confusion.

(5.9) Definition. Let $Y \to X$ be a morphism between good k-analytic spaces. Let $\mathcal{F}$ be a coherent sheaf on Y, let $y \in \mathsf{Supp}\,\mathcal{F}$ and let x be its image on X. Let r be a positive integer, and let $n_1 > n_2 > \ldots > n_r$ be a decreasing sequence of positiver integers. A *Γ-strict X-dévissage of $\mathcal{F}$ at y in dimensions* $n_1, \ldots, n_r$ is a list of data $V, \{T_i, \pi_i, t_i, \mathcal{L}_i, \mathcal{P}_i\}_{i \in \{1,\ldots,r\}}$, where:

1) V is a Γ-strict affinoid neighborhood of y in Y;

2) T_i is for any i a Γ-strict k-affinoid domain of a smooth X-space of pure relative dimension n_i and t_i is a point of T_i lying over x;

3) for any i, $\mathcal{L}_i$ and $\mathcal{P}_i$ are coherent $\mathscr{O}_{T_i}$-modules and $\mathcal{L}_i$ is free;

4) $t_i \in \mathsf{Supp}\,\mathcal{P}_i$ if $i < r$, and $\mathcal{P}_r = 0$;

5a) π_1 is a finite map $\mathsf{Supp}\,\mathcal{F}_{|V} \to T_1$ through which $\mathsf{Supp}\,\mathcal{F}_{|V} \to X$ goes, and such that we have $\pi_1^{-1}(t_1) = \{y\}$ *set-theoretically*;

5b) π_i is for any $i \in \{2, \ldots, r\}$ a finite map from $\mathsf{Supp}\,\mathcal{P}_{i-1}$ to T_i through which $\mathsf{Supp}\,\mathcal{P}_{i-1} \to X$ goes and such that we have $\pi_i^{-1}(t_i) = \{t_{i-1}\}$ *set-theoretically*;

6a) $\mathcal{L}_1$ is endowed with a map $\mathcal{L}_1 \to \pi_{1*}\mathcal{F}_{|V}$ whose cokernel is $\mathcal{P}_1$ and such that $t_1 \in \overline{\mathsf{Bij}(\mathcal{L}_{1|T_{1,x}} \to (\pi_{1*}\mathcal{F}_{|V})_{|T_{1,x}})}^{T_{1,x}}$;

6b) for any $i \in \{2, \ldots, r\}$, $\mathcal{L}_i$ is endowed with a map $\mathcal{L}_i \to \pi_{i*}\mathcal{P}_{i-1}$ whose cokernel is $\mathcal{P}_i$ and such that $t_i \in \overline{\mathsf{Bij}((\mathcal{L}_i)_{|T_{i,x}} \to (\pi_{i*}\mathcal{P}_{i-1})_{|T_{i,x}})}^{T_{i,x}}$.

The following commutative diagram of pointed spaces will hopefully make things easier to understand; at the beginning of every line, we have put the corresponding exact sequence of coherent sheaves (they live on the space T_i that lies on the line).

$$\mathcal{L}_1 \to \pi_{1*}\mathcal{F}_{|V} \to \mathcal{P}_1 \to 0$$

$$\mathcal{L}_2 \to \pi_{2*}\mathcal{P}_1 \to \mathcal{P}_2 \to 0$$

$$\vdots$$

$$\mathcal{L}_{r-1} \to \pi_{r-1,*}\mathcal{P}_{r-2} \to \mathcal{P}_{r-1} \to 0$$

$$\mathcal{L}_r \to \pi_{r*}\mathcal{P}_{r-1} \to \mathcal{P}_r = 0$$

$$\begin{tikzcd}
 & & & & (\mathsf{Supp}\ \mathcal{F}_{|V}, y) \arrow[r, hook] \arrow[d, "\pi_1"] & (V, y) \arrow[dddd] \\
 & & & (\mathsf{Supp}\ \mathcal{P}_1, t_1) \arrow[r, hook] \arrow[d, "\pi_2"] & (T_1, t_1) \arrow[dddr] & \\
 & & (\mathsf{Supp}\ \mathcal{P}_2, t_2) \arrow[r, hook] & (T_2, t_2) \arrow[ddrr] & & \\
(\mathsf{Supp}\ \mathcal{P}_{r-1}, t_{r-1}) \arrow[r, hook] \arrow[d, "\pi_r"] & (T_{r-1}, t_{r-1}) \arrow[drrrr] & & & & \\
(T_r, t_r) \arrow[rrrrr] & & & & & (X, x)
\end{tikzcd}$$

(5.10) Theorem. *Let $Y \to X$ be a map between good k-analytic spaces, let $\mathcal{F}$ be a coherent module on Y, let $y \in \mathsf{Supp}\ \mathcal{F}$ and let x be its image on X. Assume that the germ (Y, y) is Γ-strict. Let $c = \mathsf{codepth}_{\mathscr{O}_{Y_x,y}}\ \mathcal{F} \otimes \mathscr{O}_{Y_x,y}$ and let $n = \mathsf{dim}_y\ (\mathsf{Supp}\ \mathcal{F})_x$. Then $c \leqslant n$ and there exists a Γ-strict X-dévissage of $\mathcal{F}_{|V}$ at y in dimensions belonging to $[n - c\ ;\ n]$.*

Proof. We have $c \leqslant \mathsf{dim}_{\mathrm{Krull}}\ \mathscr{O}_{(\mathsf{Supp}\ \mathcal{F})_x,y} \leqslant \mathsf{dim}_y\ (\mathsf{Supp}\ \mathcal{F})_x = n$ (the second inequality is due to corollary 1.12).

(5.10.1) According to the corollary 4.7 of [14], there exist an affinoid neighborhood Z of y in $\mathsf{Supp}\ \mathcal{F}$, an affinoid domain T of a smooth X-space of pure relative dimension n, and a finite map $\pi : Z \to T$ through which $Z \to X$ goes. Let us set $t = \pi(y)$. We can first assume, by shrinking T, that y is the only pre-image of t. By 1.9.3, the germ (T, t) is Γ-strict; then 0.30.2 allows to shrink Z and T such that both are Γ-strict (and still k-affinoid). By replacing T by a small enough Laurent neighborhood of t whose definition only involves scalars belonging to Γ, one eventually can assume that $Z = \mathsf{Supp}\ \mathcal{F}_{|V}$ for a suitable Γ-strict affinoid neighborhood V of y in Y. To simplify the notations, we will write $\mathcal{F}$ instead of $\mathcal{F}_{|V}$ in the remining part of the proof; this should not cause any confusion.

(5.10.2) The map $Z \to T$ is finite and Z_x is of dimension n at y; it follows that $\mathsf{dim}_t\ (\mathsf{Supp}\ \pi_*\mathcal{F})_x = n$. As T_x is of pure dimension n, $(\mathsf{Supp}\ \pi_*\mathcal{F})_x$ contains the connected component of t in T_x.

(5.10.3) As $\pi^{-1}(t) = \{y\}$, $\pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t} = \mathcal{F} \otimes \mathscr{O}_{Y_x,y}$; therefore

$$\mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t} = \mathsf{depth}_{\mathscr{O}_{Y_x,y}}\ \mathcal{F} \otimes \mathscr{O}_{Y_x,y}$$

([16], chapt. 0, §16.4.8).

(5.10.4) As $\mathsf{dim}_y\ Z_x = \mathsf{dim}_t\ T_x = n$, it follows from cor. 1.12 and lemma 1.13 that $\mathscr{O}_{Z_x,y}$ and $\mathscr{O}_{T_x,t}$ have the same Krull dimension, say δ. By 5.10.2, $\mathsf{dim}_{\mathrm{Krull}}\ \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t} = \delta$.

As a consequence, and thanks to 5.10.3,

$$\mathsf{codepth}_{\mathscr{O}_{T_x,t}}\ \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t} = \mathsf{codepth}_{\mathscr{O}_{Y_x,y}}\ \mathcal{F} \otimes \mathscr{O}_{Y_x,y} = c.$$

We are now going to argue by induction on c.

(5.10.5) *Assume that* $c = 0$. Then $\pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t}$ is a finitely generated module of codepth 0 and of maximal Krull dimension over the regular local ring $\mathscr{O}_{T_x,t}$. It is thus free ([16], chapt. 0, 17.3.4). Let $(f_i)_{1 \leqslant i \leqslant r}$ be a family of analytic functions on the affinoid space T such that $(f_i(t))$ is a basis of $\pi_*\mathcal{F} \otimes \mathscr{H}(t)$; set $\mathcal{L} = \mathscr{O}_T^r$ and consider the map $\mathcal{L} \to \pi_*\mathcal{F}$ which sends $(a_1, \ldots, a_r)$ to $\sum a_i f_i$. By Nakayama's lemma, this map is surjective at t; moreover, its restriction to T_x is *bijective* at t, because its stalk at t is a surjective map between free modules of the same finite rank over $\mathscr{O}_{T_x,t}$. Hence by suitably shrinking T (and all other data), one can assume that $\mathcal{L} \to \pi_*\mathcal{F}$ is surjective, and that its restriction to T_x is bijective. We get this way a Γ-strict X-dévissage of $\mathcal{F}$ at y in dimension n.

(5.10.6) *Suppose now that $c > 0$, and that the theorem has been proved in codepth $< c$.* We will use the standard notations (5.5). By 5.10.2, the vector space $\pi_*\mathcal{F} \otimes \kappa(\mathbf{z}_x)$ is of positive dimension; let us call it r. Let $(f_i)_{1\leqslant i\leqslant r}$ be a family of analytic functions on T such that $(f_i(\mathbf{z}_x))_i$ is a basis of the $\kappa(\mathbf{z}_x)$-vector space $\pi_*\mathcal{F} \otimes \kappa(\mathbf{z}_x)$. Set $\mathcal{L} = \mathscr{O}_T^r$ and consider the map $\mathcal{L} \to \pi_*\mathcal{F}$ which sends $(a_1, \ldots, a_r)$ to $\sum a_i f_i$. By construction, it induces an isomorphism on a Zariski neighborhood of $\mathbf{z}_x$ in $\mathscr{T}_x$. The scheme $\mathscr{T}_x$ being regular and the coherent sheaf $\mathcal{L}$ being free, $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x} \to \mathcal{L} \otimes \mathscr{O}_{\mathscr{T}_x,\mathbf{z}_x}$ is injective; those facts imply the injectivity of $\mathcal{L} \otimes \mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x} \to \pi_*\mathcal{F} \otimes \mathscr{O}_{\mathscr{T}_x,\mathbf{t}_x}$.

Therefore, $t \in \overline{\mathsf{Bij}(\mathcal{L}_{|T_x} \to \pi_*\mathcal{F}_{|T_x})}^{T_x}$, and $\mathcal{L}\otimes\mathscr{O}_{T_x,t} \to \pi_*\mathcal{F}\otimes\mathscr{O}_{T_x,t}$ is injective in view of 0.13.5. The arrow $\mathcal{L} \otimes \mathscr{O}_{T_x,t} \to \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t}$ is thus not surjective, because if it were, it would then be bijective and the codepth of $\pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t}$ would be equal to zero ($\mathscr{O}_{T_x,t}$ is regular), which would contradict the fact that $c > 0$. Therefore, if we set $\mathcal{P} = \mathsf{Coker}\ (\mathcal{L} \to \pi_*\mathcal{F})$, then t lies on $\mathsf{Supp}\ \mathcal{P}$. Moreover, $(\mathsf{Supp}\ \mathcal{P})_x$ is included in $T_x - \mathsf{Bij}(\mathcal{L}_{|T_x} \to \pi_*\mathcal{F}_{|T_x})$ by Nakayama's lemma; hence $\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x < n$.

Therefore there exists an analytic function ξ on T_x whose zero-locus contains no neighborhood of t and which is such that $\xi\mathcal{P}_{|T_x} = 0$; the image of ξ in $\mathscr{O}_{T_x,t}$ is a non-zero element of the annihilator of $\mathcal{P} \otimes \mathscr{O}_{T_x,t}$; as a consequence, $\mathsf{dim}_{\mathrm{Krull}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t} < \delta$. We thus have $\mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t} < \delta$; moreover, since $r > 0$ and since $\mathscr{O}_{T_x,t}$ is regular, $\mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \mathcal{L} \otimes \mathscr{O}_{T_x,t} = \delta$. Let κ be the residue field of $\mathscr{O}_{T_x,t}$. Using [16], chapt. 0, cor. 16.4.4 and the $\mathsf{Ext}^\bullet(\kappa, .)$'s exact sequence associated with

$$0 \to \mathcal{L} \otimes \mathscr{O}_{T_x,t} \to \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t} \to \mathcal{P} \otimes \mathscr{O}_{T_x,t} \to 0,$$

we see that

$$\mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t} = \mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t} = \delta - c.$$

But $\mathsf{dim}_{\mathrm{Krull}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t} < \delta$; therefore, $\mathsf{codepth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t} < c$. Now, by the induction hypothesis, $\mathcal{P}$ admits a Γ-strict X-dévissage at t, in dimensions belonging to

$$I := [\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x - \mathsf{codepth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t}\ ;\ \mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x].$$

We will now show that $I \subset [n - c\ ;\ n[$; by shrinking suitably V, Z, and T, the dévissage of $\mathcal{P}$ together with $V, T, \pi, \mathcal{L}, \mathcal{P}, \mathcal{L} \to \pi_*\mathcal{F}$, will then provide a Γ-strict dévissage of $\mathcal{F}$ at y in dimensions belonging to $[n - c\ ;\ n]$.

(5.10.7) *Proof of the inclusion $I \subset [n - c\ ;\ n[$.* As $\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x < n$, I is strictly bounded above by n.

Let us now prove that it is bounded below by $n - c$. The term

$$\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x - \mathsf{codepth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t}$$

can be rewritten as

$$\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x - \mathsf{dim}_{\mathrm{Krull}}\ \mathscr{O}_{(\mathsf{Supp}\ \mathcal{P})_x,t} + \mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t}.$$

By cor. 1.12, we have the equalities

$$\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x - \mathsf{dim}_{\mathrm{Krull}}\ \mathscr{O}_{(\mathsf{Supp}\ \mathcal{P})_x,t} = \mathsf{centdim}\ (T_x, t)$$

and $\mathsf{dim}_y\ Z_x - \mathsf{dim}_{\mathrm{Krull}}\ \mathscr{O}_{Z_x,y} = \mathsf{centdim}\ (Z_x, y)$.

Since $Z \to T$ is finite, $\mathsf{centdim}\ (T_x, t) = \mathsf{centdim}\ (Z_x, y)$ by lemma 1.13. Moreover, it follows from 5.10.6 that

$$\mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t} = \mathsf{depth}_{\mathscr{O}_{T_x,t}}\ \pi_*\mathcal{F} \otimes \mathscr{O}_{T_x,t},$$

and 5.10.3 ensures that the latter coincides with $\mathsf{depth}_{\mathscr{O}_{Y_x,y}}\ \mathcal{F} \otimes \mathscr{O}_{Y_x,y}$. Fitting all those facts together, we obtain that $\mathsf{dim}_t\ (\mathsf{Supp}\ \mathcal{P})_x - \mathsf{codepth}_{\mathscr{O}_{T_x,t}}\ \mathcal{P} \otimes \mathscr{O}_{T_x,t}$ is equal to $\mathsf{dim}_y\ (\mathsf{Supp}\ \mathcal{F})_x - \mathsf{dim}_{\mathrm{Krull}}\ \mathscr{O}_{(\mathsf{Supp}\ \mathcal{F})_x,y} + \mathsf{depth}_{\mathscr{O}_{Y_x,y}}\ \mathcal{F} \otimes \mathscr{O}_{Y_x,y}$, that is, to $n - c$; it ends the proof of the theorem. $\square$

Boundaryless flatness is automatically universal

(5.11) Let $Y \to X$ be a map between good k-analytic spaces and let $\mathcal{F}$ be a coherent module on Y. Let y be a point of Y. We will say that $\mathcal{F}$ is *X-extendable at y* if there exist an affinoid neighborhood W of y in Y, an isomorphism between W and an affinoid domain of a *boundaryless* X-space W', and a coherent sheaf $\mathcal{G}$ on W' such that $\mathcal{F}_{|W} \simeq \mathcal{G}_{|W}$. Note that if $y \in \mathsf{Int}\ Y/X$, then $\mathcal{F}$ is X-extendable at y.

(5.12) Let $Y \to X$ be a map between good k-analytic spaces and let $\mathcal{F}$ be a coherent module on Y. Let $y \in Y$. We want to give some criteria for $\mathcal{F}$ to be universally X-flat at y, and to use them to show that in the boundaryless (and, more generally, extendable) case, X-flatness at y is automatically universal; this fact had already been proved by Berkovich, using a completely different method, in some unpublished notes.

(5.12.1) If $y \notin \mathsf{Supp}\ \mathcal{F}$ then $\mathcal{F}$ is universally X-flat at y.

(5.12.2) Assume that $y \in \mathsf{Supp}\ \mathcal{F}$, let $c = \mathsf{codepth}_{\mathscr{O}_{Y_x,y}}\ \mathcal{F} \otimes \mathscr{O}_{Y_x,y}$ and let $n = \mathsf{dim}_y\ (\mathsf{Supp}\ \mathcal{F})_x$. Thanks to theorem 5.10, there exists an X-dévissage (we don't care about Γ-strictness here) of $\mathcal{F}$ at y in dimensions belonging to $[n-c; n]$. Let $V, \{T_i, \pi_i, t_i, \mathcal{L}_i, \mathcal{P}_i\}_{i \in \{1,\ldots,r\}}$ be the corresponding list of data.

(5.12.3) Theorem. *We work with the notations introduced above. The following are equivalent:*

i) $\mathcal{F}$ is universally X-flat at y;

ii) the arrow $\mathcal{L}_1 \to \pi_{1*}\mathcal{F}_{|V}$ is injective at t_1 and for any $i \geqslant 2$, the arrow $\mathcal{L}_i \to \pi_{i*}\mathcal{P}_{i-1}$ is injective at t_i.

If moreover $\mathcal{F}$ is X-extendable at y (e.g. $y \in \mathsf{Int}\ Y/X$), then both propositions above also are equivalent to the following third one:

iii) $\mathcal{F}$ is is X-flat at y.

Proof. Thanks to lemma 5.8, X-flatness (resp. X-universal flatness) of $\mathcal{F}$ at y is equivalent to that of $\pi_{1*}\mathscr{F}_{|V}$ at t_1 ; for the same reason, if $i \leqslant r-1$, then X-flatness (resp. X-universal flatness) of $\mathcal{P}_i$ at t_i is equivalent to that of $\pi_{i+1*}\mathcal{P}_i$ at t_{i+1}. Hence the equivalence of i) and ii), and the equivalence of i), ii) and iii) *when* $y \in \mathsf{Int}\, Y/X$, follow from a repeated application of proposition 5.7, once one has remarked that since $\mathcal{P}_r = 0$, it is universally X-flat at t_r.

It remains to show that $iii) \Rightarrow i)$ as soon as $\mathcal{F}$ is X-extendable at y. Let we assume that $\mathcal{F}$ is X-flat at Y, that Y is an affinoid domain of a boundaryless X-space Y', and that $\mathcal{F}$ is the restriction to Y of a coherent sheaf $\mathcal{G}$ on Y'; one is going to prove that $\mathcal{F}$ is X-universally flat at y.

By assumption, $\mathcal{F} \otimes \mathscr{O}_{Y,y} = \mathcal{G} \otimes \mathscr{O}_{Y,y}$ is $\mathscr{O}_{X,x}$-flat. Since $\mathscr{O}_{Y,y}$ is a (faithfully) flat $\mathscr{O}_{Y',y}$-algebra (0.13.4), $\mathcal{G} \otimes \mathscr{O}_{Y',y}$ is $\mathscr{O}_{X,x}$-flat. By the boundaryless case already established, $\mathcal{G}$ is universally X-flat at y. Therefore, $\mathcal{F}$ is universally X-flat at y (2.12). $\square$

(5.12.4) Remark. If properties i) and ii) are satisfied, it turns out that $\mathcal{L}_r \to \pi_{r*}\mathcal{P}_{r-1}$ (or $\mathcal{L}_1 \to \pi_{1*}(\mathcal{F}_{|V})$ if $r = 1$) is *bijective at* t_r, because it is injective by ii), and surjective since its cokernel $\mathcal{P}_r$ is zero.

Thanks to theorem 5.12.3above, it is possible to slightly improve theorem 2.6.

(5.12.5) Theorem. *Let $Y \to X$ be a morphism between k-affinoid spaces and let Z be a Zariski-closed subspace of Y such that $Z \to X$ is finite. Let $y \in Z$ and let* $\mathbf{y}$ *be its image on $\mathscr{Y}$. Let $\mathcal{F}$ be a coherent sheaf on Y. If $\mathcal{F}$ is $\mathscr{X}$-flat at* $\mathbf{y}$*, then it is universally X-flat at x.*

Proof. Theorem 2.6 tells us that $\mathcal{F}$ is X-flat at y. As $Z \to X$ is finite, $Y \to X$ is inner at y; in view of theorem 5.12.3, $\mathcal{F}$ is universally X-flat at y. $\square$

(5.13) Let $Y \to X$ be a morphism of k-analytic spaces, let $y \in Y$, let x be its image on X and let $\mathcal{F}$ be a coherent sheaf on Y. Let P be one of the properties listed in 0.19.5 a) (resp. 0.19.5 b), except flatness). We will say that $Y \to X$ (resp. that $\mathcal{F}$) *satisfies* P *at* y (resp. *satisfies* P *over* X *at* y) if it is universally flat at y (resp. if it is X-universally flat at y) and if Y_x (resp. $\mathcal{F}_{|Y_x}$) satisfies P at y.

About the CM property

(5.14) Lemma. *Let $Y \to T$ be a finite morphism and let $T \to X$ be a quasi-smooth morphism. Let $\mathcal{F}$ be a coherent sheaf on Y which is universally T-flat. The sheaf $\mathcal{F}$ is* CM *over X.*

Proof. One can assume that Y, T and X are k-affinoid. The universal X-flatness of $\mathcal{F}$ is clear. Let $x \in X$. One will show that $\mathcal{F}_{|Y_x}$ is CM. Let $y \in Y_x$ and let t be its image on T_x. The ring $\mathscr{O}_{T_x,t}$ is regular by quasi-smoothness of T_x, hence CM. The morphism $\mathsf{Spec}\ \mathscr{O}_{Y_x,y} \to \mathsf{Spec}\ \mathscr{O}_{T_x,t}$ is finite, hence has zero-dimensional fibers; the finite module $\mathcal{F} \otimes \mathscr{O}_{Y_x,y}$ is $\mathscr{O}_{T_x,t}$-flat by universal T-flatness of $\mathcal{F}$. It follows then from prop. 6.4.1 of [17] that $\mathcal{F} \otimes \mathscr{O}_{Y_x,y}$ is CM. $\square$

This lemma admits kind of a converse claim.

(5.15) Theorem. *Let $Y \to X$ be a morphism of good k-analytic spaces and let $y \in Y$. Assume that (Y, y) is Γ-strict. Let n be an integer. Let $\mathcal{F}$ be a coherent sheaf on Y such that $y \in \mathsf{Supp}\ \mathcal{F}$. Assume that $\mathcal{F}$ is* CM *and of relative dimension n over X at y. There then exist:*

- *a Γ-strict k-affinoid neighborhood V of y in Y, which is included in the* CM *locus of $\mathcal{F}$ over X;*

- *a Γ-strict k-affinoid domain T of a smooth X-space of pure relative dimension n;*

- *a finite X-morphism $\pi : \mathsf{Supp}\ \mathcal{F}_{|V} \to T$ with respect to which $\mathcal{F}_{|V}$ is T-flat.*

Proof. Let us call x the image of y on X; we then have $\mathsf{codepth}\ \mathcal{F} \otimes \mathscr{O}_{Y_x, y} = 0$. By th. 5.10, $\mathcal{F}$ admits a full Γ-strict X-dévissage $V, T, \pi, t, \mathcal{L}, \mathcal{P} = 0$ at y in dimension n. As $\mathcal{F}$ is universally X-flat at y, th. 5.12.3 and remark 5.12.4 ensure that $\mathcal{L} \to \pi_* \mathcal{F}_{|V}$ is bijective at t. We can hence shrink the data so that $\pi_* \mathcal{F}_{|V}$ is a free $\mathscr{O}_T$-module. Lemma 5.14 above then ensures that V is included in the CM-locus of $\mathcal{F}$ over X. □

(5.16) Remark. Let $Y \to X$ be a morphism of k-analytic spaces and let $\mathcal{F}$ be a coherent sheaf on Y which is universally X-flat. Let U be the CM locus of $\mathcal{F}$ over X ; it follows from thm. 5.15 that U is an open subset of Y.

Now let $x \in X$. As $\mathcal{F}$ is universally flat, the intersection $U \cap Y_x$ is the CM-locus of $\mathcal{F}_{|Y_x}$. It is a Zariski-open subset if Y_x (*cf.*[15], th. 4.4), which is dense. Indeed, to show it one can assume that Y is affinoid. Now if $\mathbf{y}$ is the generic point of an irreducible component of $\mathscr{Y}_x$ then $\mathcal{F}_{|Y_x} \otimes \mathscr{O}_{\mathscr{Y}, \mathbf{y}}$ is CM because $\mathscr{O}_{\mathscr{Y}, \mathbf{y}}$ is zero-dimensional; this fact, together with GAGA, *cf.*0.20), implies our claim.

6 Images of maps

The zero-dimensional CM case

(6.1) Proposition. *Let Y be a Γ-strict compact k-analytic space, let X be a separated k-analytic space and let $\varphi : Y \to X$ be a morphism. Let $\mathcal{F}$ be a coherent sheaf on Y which is* CM *and purely zero-dimensional over X. The image $\varphi(\mathsf{Supp}\ \mathcal{F})$ is a Γ-strict compact analytic domain of X.*

Proof. We can replace Y by the support of $\mathcal{F}$, that is, we can assume that $\mathsf{Supp}\ \mathcal{F} = Y$. We will first reduce to a particular case.

(6.1.1) Since Y is compact, this allows us to argue G-locally on Y.

- By 1.10.2, *one can assume that X is Γ-strict and k-affinoid.*

- Thanks to cor. 5.15, one can assume that there exist a Γ-strict k-affinoid quasi-étale X-space T, and a factorization of $Y \to X$ through a finite map $\pi : Y \to T$ such that $\pi_* \mathcal{F}$ is a non-zero free $\mathscr{O}_T$-module. The latter condition implies that $\pi(Y) = T$. Replacing Y by T, *one can assume that* $Y \to X$ *is quasi-étale.*

- We can suppose that the quasi-étale map $Y \to X$ can be written as a composition $Y \hookrightarrow X' \to X$ where $Y \hookrightarrow X'$ identifies Y with a Γ-strict compact analytic domain of X', where X' is connected and where $X' \to X$ goes through a finite étale map from X' to a connected Γ-strict affinoid domain Z of X. Let X'' be by a connected finite Galois covering of Z going through X'. One can replace X by Z and Y by its preimage on X''; the union of all Galois conjugates of Y is a Γ-strict compact analytic domain of X'' whose image on X coincides with that of Y.

  We can eventually assume that X is a Γ-strict k-affinoid space and that Y is a Galois-invariant Γ-strict compact analytic domain of a finite Galois cover X'' of X.

(6.1.2) Let $y \in Y$ and let $x = \varphi(y)$. Let (U, x) be the smallest analytic domain of (X, x) through which $(Y, y) \to (X, x)$ goes (1.9.2 and 1.9.4). It is Γ-strict, and its reduction $\widetilde{(U,x)}^{\Gamma}$ its the image of $\widetilde{(Y,y)}^{\Gamma}$ on $\mathbb{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$. As Y is Galois-invariant, the pre-image of $\widetilde{(U,x)}^{\Gamma}$ on $\mathbb{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}}$ is precisely $\widetilde{(Y,y)}^{\Gamma}$. It means that the map $(Y, y) \to (U, x)$ is boundaryless (0.30.5). Being quasi-étale, it is étale. Therefore, there exist a Γ-strict compact analytic neighborhood W of y in Y, and a Γ-strict compact analytic neighborhood V of x in U such that φ induces a finite étale map $W \to V$. As a consequence, $\varphi(W)$ is a finite union of connected components of V; in particular, it is a Γ-strict compact analytic domain of X. Thanks to the compactness of Y, this ends the proof. $\square$

Existence of CM multisections: the local case

(6.2) Theorem. *Assume that $|k^*| \neq \{1\}$, and let $\varphi : Y \to X$ be a morphism of k-analytic spaces, with Y being strict and X separated. Let $y \in Y$, let x be its image on X and let $\mathcal{F}$ be a coherent sheaf on Y which is universally X-flat at y. Denote by Z the CM locus of $\mathcal{F}$ over X and by (U, x) the smallest analytic domain of (X, x) through which $(\mathsf{Supp}\,\mathcal{F}, y) \to (X, x)$ goes ((1.9.2 and 1.9.4).There exist $r \geqslant 1$, zero-dimensional maps*

$$\psi_1 : X_1 \to X, \ldots, \psi_r : X_r \to X,$$

and X-morphisms

$$\sigma_1 : X_1 \to Z \cap \mathsf{Supp}\,\mathcal{F}, \ldots, \sigma_r : X_r \to Z \cap \mathsf{Supp}\,\mathcal{F}$$

such that:

i) for every j, the space X_j is compact and strictly k-analytic and the point x has a unique pre-image x_j on X_j;

ii) for every j, the coherent sheaf $\sigma_j^*\mathcal{F}$ is CM *over X, and $\psi_j(X_j)$ is thus a compact strictly k-analytic domain of X (prop. 6.1);*

iii) one has $(U,x) = \bigcup(\psi_j(X_j),x)$.

Moreover :

α) if $Y \to X$ is quasi-smooth at y and if $\mathcal{F} = \mathscr{O}_Y$, the ψ_j's can be chosen to be quasi-étale;

β) if $Y \to X$ is boundaryless at y (which implies that $(U,x) = (X,x)$, see 1.9.3) and if the germs (Y,y) and (X,x) are good one can take $r = 1$, and ψ_1 inner, hence finite, at x_1.

Proof. By replacing Y with $\mathsf{Supp}\,\mathcal{F}$, we may assume that $Y = \mathsf{Supp}\,\mathcal{F}$.

(6.2.1) Let us first reduce to the case where both Y and X are strictly k-affinoid, using the fact that all assertions involved are local on Y and X.

We begin with assertion β), that is, under the assumption that Y and X are good ant that $Y \to X$ is inner at y. As we have noticed, this implies in view of 1.9.3 that $(U,x) = (X,x)$; moreover, strictness of (Y,y) implies that of (X,x) by 1.9.2. Thus we can skrink Y and X so that both are strictly k-affinoid.

Let us now come to the other assertions (so, we don't assume anymore neither Y and X to be good nor φ to be inner at y). By replacing Y with a compact neighborhood of y, one can assume that it is compact. Now, as X is separated, $\varphi(Y)$ is contained in a compact, strictly analytic domain X_0 of X (1.10.2); as $(U,x) \subset (X_0,x)$ one can replace X with X_0, and hence reduce to the case where X itself is strict. The point x has thus a neighborhood in X which is a finite union of strictly affinoid domains containing x; all assertions involved being G-local on the germ (X,x), one can assume that Xitself is strictly k-affinoid. The point y has a neighborhood in Y which is a finite union of strictly affinoid domains containing it; all assertions involved being G-local on the germ (Y,y), one can assume that Y itself is strictly k-affinoid.

Caution: as the proof will involve from now on only strictly k-analytic spaces, it will be sufficient to consider *non-graded* reductions; therefore, in order to simplify notations, we will write *all this proof along* $\widetilde{(Y,y)}, \widetilde{k}$, and so on... instead of $\widetilde{(Y,y)}^1, \widetilde{k}^1$, and so on.

(6.2.2) Let $\mathscr{A}$ (resp. $\mathscr{B}$) be the algebra of analytic functions on X (resp. Y). Let $\mathscr{A}^\circ$ be the subring of $\mathscr{A}$ that consists in functions whose spectral semi-norm is bounded by 1, and let $\mathscr{A}^{\circ\circ}$ be the ideal of $\mathscr{A}^\circ$ that consists in functions whose spectral semi-norm is *strictly* bounded by 1, and let $\widetilde{\mathscr{A}}$ be the quotient $\mathscr{A}^\circ/\mathscr{A}^{\circ\circ}$; we define $\mathscr{B}^\circ, \mathscr{B}^{\circ\circ}$ and $\widetilde{\mathscr{B}}$ in a similar way; both $\widetilde{k}$-algebras $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ are finitely generated. By Temkin's definition of the reduction of an analytic germ, one has $\widetilde{(X,x)} = \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{\widetilde{\mathscr{A}}\}$ and $\widetilde{(Y,y)} = \mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{\mathscr{B}}\}$. Let $f_1, \ldots, f_n$ be elements of $\mathscr{B}^\circ$ whose images generate the $\widetilde{k}$-algebra $\widetilde{\mathscr{B}}$. One has also obviously $\widetilde{(Y,y)} = \mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}\}$.

(6.2.3) If φ is inner at y then $\widetilde{(Y,y)}$ is equal to the pre-image of $\widetilde{(X,x)}$ in $\mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$; in other words, $\mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{\mathscr{B}}\} = \mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{\mathscr{A}}\}$, which means that $\widetilde{\mathscr{B}}$ is integral over $\widetilde{\mathscr{A}}$.

(6.2.4) Let B be the sub-$\widetilde{\mathscr{H}(x)}$-algebra of $\widetilde{\mathscr{H}(y)}$ generated by the $\widetilde{f_i(y)}$'s; note that if φ is inner at y then B is a field, because in that case $\widetilde{f_i(y)}$ is algebraic over $\widetilde{\mathscr{H}(x)}$ for every i (6.2.3). By th. 1.4, there exist finitely many closed points $\mathsf{y}_1, \ldots, \mathsf{y}_m$ of $\mathsf{Spec}\, B$ such that

$$\widetilde{(U,x)} = \bigcup p_j \left(\mathbb{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_n(y)}(\mathsf{y}_j)\}\right),$$

where p_j denotes for every j the natural map $\mathbb{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}} \to \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$. For every j, set $\mathsf{U}_j = p_j \left(\mathbb{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_n(y)}(\mathsf{y}_j)\}\right) \subset \mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$; thanks to prop. 1.3, U_j is open and quasi-compact. There exist compact strictly strictly analytic domains $U_1, \ldots, U_m$ of X which contain x and satisfy the equalities $\widetilde{(U_j,x)} = \mathsf{U}_j$ for $j = 1, \ldots, m$.

The inner case. If φ is inner at y then as B is a field, $m = 1$ and y_1 is the only point of $\mathsf{Spec}\, B$. It follows that $\mathbb{P}_{\kappa(\mathsf{y}_1)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_1), \ldots, \widetilde{f_n(y)}(\mathsf{y}_1)\}$ is nothing but $\mathbb{P}_{B/\widetilde{k}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}\}$. But since $\mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}\}$ is, under our innerness assumption, the pre-image of $\widetilde{(X,x)}$ in $\mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$, the open subset $\mathbb{P}_{B/\widetilde{k}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}\}$ of $\mathbb{P}_{B/\widetilde{k}}$ is the pre-image of $\widetilde{(X,x)}$ in $\mathbb{P}_{B/\widetilde{k}}$. One thus has

$$\mathbb{P}_{\kappa(\mathsf{y}_1)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_1), \ldots, \widetilde{f_n(y)}(\mathsf{y}_1)\} = p_1^{-1}(\widetilde{(X,x)}).$$

(6.2.5) *We fix an integer j belonging to* $\{1, \ldots, m\}$. Let R be the subring of $\mathscr{O}_{X,x}$ consisting in functions f such that $|f(x)| \leqslant 1$. For any $i \in \{1, \ldots, n\}$, let P_i be a polynomial of $R[T_1, \ldots, T_i]$, which is monic in T_i and which is such that $\widetilde{P_i}(\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_{i-1}(y)}(\mathsf{y}_j), T)$ is the minimal polynomial of $\widetilde{f_i(y)}(\mathsf{y}_j)$ over $\widetilde{\mathscr{H}(x)}[\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_{i-1}(y)}(\mathsf{y}_j)]$ (by $\widetilde{P_i}$ we denote of course the image of P_i under the natural map $R[T_1, \ldots, T_i] \to \widetilde{\mathscr{H}(x)}[T_1, \ldots, T_i]$). Let Z be a strictly affinoid neighborhood of x in X on which all the coefficients of the P_i's are defined. Let Ω be the open subset of $Y \times_X Z$ defined as the simultaneous validity locus of the inequalities

$$|P_1(f_1)| < 1, |P_2(f_1, f_2)| < 1, \ldots, |P_n(f_1, \ldots, f_n)| < 1.$$

(6.2.6) *Claim:* $\Omega_x \neq \emptyset$. Indeed, suppose that $\Omega_x = \emptyset$. Let J be the set of integers $i \in \{1, \ldots, n\}$ such that $|P_i(f_1(y), \ldots, f_i(y))| = 1$. For every $i \in J$, let Y_i be the affinoid domain of Y_x defined by the condition $|P_i(f_1, \ldots, f_i)| = 1$. Under our assumption, the union of the Y_i's for $i \in J$ is a neighborhood of y in Y_x. We thus have $\widetilde{(Y_x, y)} = \bigcup\limits_{i \in J} \widetilde{(Y_i, y)}$. Let us describe both terms of this equality.

- By prop. 4.6 of [25],[4] $\widetilde{(Y_x, y)} = \mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{\mathscr{H}(x)}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}\}$.

[4] The latter proposition a priori only concerns morphisms of k-analytic spaces, but while reading its proof, one sees that thanks to prop. 3.1 *v*) of [25], it extends to k-morphisms between analytic spaces over arbitrary complete extensions of k; hence one can apply it to the diagram $Y \to X \leftarrow \mathscr{M}(\mathscr{H}(x))$.

- If i is any element of J, then $\widetilde{(Y_i, y)}$ is equal to

$$\mathbb{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{\mathscr{H}(x)}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}, \widetilde{P}_i(\widetilde{f_1(y)}, \ldots, \widetilde{f_i(y)}), \widetilde{P}_i(\widetilde{f_1(y)}, \ldots, \widetilde{f_i(y)})^{-1}\}.$$

There exists a valuation $\langle . \rangle$ on $\widetilde{\mathscr{H}(y)}$ which is trivial on $\widetilde{\mathscr{H}(x)}$ and whose ring $\mathscr{O}_{\langle . \rangle}$ dominates $\mathscr{O}_{\mathsf{Spec}\ A, y_j}$. As $\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}$ belong to A, they belong to $\mathscr{O}_{\langle . \rangle}$; as $\widetilde{P}_i(\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_i(y)}(\mathsf{y}_j)) = 0$ for all i, the element $\widetilde{P}_i(\widetilde{f_1(y)}, \ldots, \widetilde{f_i(y)})$ belongs to the maximal ideal of $\mathscr{O}_{\langle . \rangle}$ for all i.

It follows now from the explicit descriptions of $\widetilde{(Y_x, y)}$ and of the $\widetilde{(Y_i, y)}$'s we have just given that $\langle . \rangle$ belongs to $\widetilde{(Y_x, y)}$ but not to $\bigcup\limits_{i \in J} \widetilde{(Y_i, y)}$, contradiction.

(6.2.7) As $\Omega_x \neq \emptyset$, it follows from remark 5.16 that there exists a point ω on Ω_x lying on Z. From th. 5.15, one deduces the existence of a strictly affinoid neighborhood V of ω in $\Omega \cap Z$ such that $V \to X$ admits a factorization $V \to W \to X$, where W is a strictly affinoid domain of a smooth X-space T, and where $V \to W$ is a finite map with respect to which $\mathcal{F}_{|V}$ is W-flat; if φ is quasi-smooth at y and if $\mathcal{F} = \mathscr{O}_Y$, one can suppose that $V = W$.

Let ϖ be the image of ω on W. By cor. 2.17, the image of $\mathsf{Int}\ V/Y$ on W contains an open neighborhood Ξ of ϖ. As Ξ_x is a non-empty strictly $\mathscr{H}(x)$-analytic space, it has an $\mathscr{H}(x)$-rigid point, which automatically belongs to $\mathsf{Int}\ W_x/T_x$. It implies the existence of an open subset of T whose fiber at x is non-empty and is included in Ξ_x. Applying prop. 4.9 to this open subset provides an étale X-space X' and an X-morphism $X' \to T$ whose image intersects Ξ_x. We fix a pre-image x' of x in X' whose image on T belongs to Ξ and is denoted by w. We choose a pre-image v of w in $\mathsf{Int}\ V/Y$. We denote by W' the analytic domain $W \times_T X'$ of X', and by V' the fiber product $V \times_W W'$. We chose a point v' on V' lying above both v and x'. We then have the following commutative diagram of pointed spaces,

$$\begin{tikzcd}
 & (V, v) \arrow[r, hook] \arrow[d] & (Z, v) \arrow[dd, bend left] \\
(V', v') \arrow[ur] \arrow[d] & (W, w) \arrow[r, hook] & (T, w) \arrow[d] \\
(W', x') \arrow[ur] \arrow[r, hook] & (X', x') \arrow[ur] \arrow[r] & (X, x)
\end{tikzcd}$$

in which both squares are cartesian. Since $X' \to X$ is étale, it is boundaryless. As a consequence, $X' \to T$, $W' \to W$ and $V' \to V$ are boundaryless.

As $v \in \mathsf{Int}\ V/Y$, the germ (V, v) coincides with (Y, v); in other words, we have $\widetilde{(V, v)} = \mathbb{P}_{\widetilde{\mathscr{H}(v)}/\widetilde{k}}\{\widetilde{f_1(v)}, \ldots, \widetilde{f_n(v)}\}$. Since $V' \to V$ is boundaryless, 0.30.5 therefore provides the equality $\widetilde{(V', v')} = \mathbb{P}_{\widetilde{\mathscr{H}(v')}/\widetilde{k}}\{\widetilde{f_1(v')}, \ldots, \widetilde{f_n(v')}\}$ (one still writes f_i for the pull-back of f_i into the ring of functions on V').

By choice of V, the point v belongs to Ω_x. We hence have for every i the inequality $|P_i(f_1(v), \ldots, f_i(v))| < 1$. It implies that $\widetilde{P}_i(\widetilde{f_1(v')}, \ldots, \widetilde{f_i(v')}) = 0$ for every i. By the very definition of the P_i's, it follows that there exists an $\widetilde{\mathscr{H}(x)}$-isomorphism between $\kappa(\mathsf{y}_j)$ and $\widetilde{\mathscr{H}(x)}[\widetilde{f_1(v')}, \ldots, \widetilde{f_n(v')}]$ which sends $\widetilde{f_i(y)}(\mathsf{y}_j)$ to $\widetilde{f_i(v')}$ for any i. The image of $\widetilde{(V',v')}$ inside $\mathbb{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ coincides therefore with $p_j\left(\mathbb{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_n(y)}(\mathsf{y}_j)\}\right)$, that is, with U_j. As a consequence, (U_j, x) is the smallest analytic domain of (X,x) through which $(V',v') \to (X,x)$ goes.

The morphism $V' \to X$ is purely zero-dimensional. The space W' is quasi-étale, and in particular smooth, over X, and the pull-back of $\mathcal{F}$ to V' is flat over W', because $\mathcal{F}_{|V}$ is flat over W. Lemma 5.14 then ensures that the pull-back of $\mathcal{F}$ to V' is CM over X. Moreover, if $Y \to X$ is quasi-smooth and if $\mathcal{F} = \mathscr{O}_Y$, then V' is quasi-étale over X (because in this situation $V = W$).

The inner case. If φ is inner at y it follows from 6.2.4 that $j = 1$ and that

$$\mathbb{P}_{\kappa(\mathsf{y}_1)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_1), \ldots, \widetilde{f_n(y)}(\mathsf{y}_1)\} = p_1^{-1}(\widetilde{(X,x)}).$$

Therefore $\widetilde{(V',v')}$ is the pre-image of $\widetilde{(X,x)}$ in $\mathbb{P}_{\widetilde{\mathscr{H}(v')}/\widetilde{k}}$, which exactly means that $(V',v') \to (X,x)$ is boundaryless.

(6.2.8) Conclusion As $V' \to X$ is zero-dimensional, there exists a compact strictly k-analytic neighborhood X_j of v' in V' such that v' is the only pre-image of x inside X_j. To emphasize the dependance on j, let us denote now by x_j the point v', by ψ_j the natural map $X_j \to X$, and by σ_j the natural X-map $X_j \to Z$.

Let us fix $j \in \{1, \ldots, m\}$. From what has been done in 6.2.7, one deduces the following: ψ_j is zero-dimensional; the coherent sheaf $\sigma_j^*\mathcal{F}$ is CM over X; and the smallest analytic domain of (X,x) through which (X_j, x_j) goes is (U_j, x). If $\mathcal{F} = \mathscr{O}_Y$ and if φ is quasi-smooth at y, then ψ_j is quasi-étale; if φ is inner at y then $j = 1$ and ψ_1 is inner, hence finite, at x_1.

As the coherent sheaf $\sigma_j^*\mathcal{F}$ is CM over X (and has support X_j because $\mathcal{F}$ has support Y), prop. 6.1 ensures that $\psi_j(X_j)$ is an analytic domain of X. We can shrink X_j so that $\psi_j(X_j) \subset U_j$, whence the inclusion $(\psi_j(X_j), x) \subset (U_j, x)$, and eventually the equality $(\psi_j(X_j), x) \subset (U_j, x)$ by minimality of (U_j, x).

Since (U, x) is the union of the (U_j, x)'s (6.2.4, the data $(X_j, \psi_j, \sigma_j)_j$ satisfy the conclusions of the theorem. □

Existence of CM multisections: the global case

(6.3) Theorem. *Assume that $|k^*| \neq \{1\}$. Let Y be a compact, strictly k-analytic space and let X be a separated k-analytic space. Let $\varphi : Y \to X$ be a morphism and let $\mathcal{F}$ be a universally X-flat coherent sheaf on Y. Denote by Z the CM locus of $\mathcal{F}$ over X. There exist a compact, strictly k-analytic space X', a zero-dimensional map $\psi : X' \to X$ and an X-morphism $X' \to Z \cap \mathsf{Supp}\ \mathcal{F}$ such that the following hold:*

i) $\sigma^*\mathcal{F}$ is CM over X, which implies that $\psi(X')$ is a compact strictly analytic domain of X (prop. 6.1).

ii) one has $\varphi(Y) = \psi(X')$.

If moreover $Y \to X$ is quasi-smooth and $\mathcal{F} = \mathscr{O}_Y$, then ψ can be chosen to be quasi-étale.

Proof. By replacing Y with $\mathsf{Supp}\,\mathcal{F}$ we may assume that $\mathsf{Supp}\,\mathcal{F} = Y$. Let $y \in Y$. Using th. 6.2, and setting $X_y := \coprod X_j, \psi_y = \coprod \psi_j$ and $\sigma_y = \coprod \sigma_j$, one gets the existence of a compact strictly k-analytic space X_y, which is quasi-étale if $\mathcal{F} = \mathscr{O}_Y$, of a zero-dimensional map $\psi_y : X_y \to X$, and of an X-map $\sigma_y : X_y \to Z$, such that the following are satisfied:

• $\sigma_y^* \mathcal{F}$ is CM over X, which forces $\psi_y(X_y)$ to be a compact strictly k-analytic domain of X by prop. 6.1;

• $(\psi_y(X_y), x) = (U_y, x)$, where (U_y, x) is the smallest analytic domain of (X, x) through which $(Y, y) \to (X, x)$ goes.

As $(Y, y) \to (X, x)$ goes through (X_y, x), there exists a compact neighborhood V_y of y in Y such that $\varphi(V_y) \subset \psi_y(X_y)$.

By compactness of Y there exist finitely many points $y_1, \ldots, y_n$ on Y such that the V_{y_i}'s cover Y. Now set $X' = \coprod X_{y_i}, \psi = \coprod \psi_{y_i}$, and $\sigma = \coprod \sigma_{y_i}$. By construction, i) is satisfied and ψ is quasi-étale as soon as $\mathcal{F} = \mathscr{O}_Y$ and φ is quasi-smooth; it thus remains to show ii).

For every i, one has $\varphi(V_{y_i}) \subset \psi_{y_i}(X_{y_i})$. As the V_{y_i}'s cover Y this implies that $\varphi(Y) \subset \psi(X')$; but the existence of σ provides the converse inclusion, whence ii). □

Images of maps: the compact case

The following theorem contains, and extends, the celebrated result by Raynaud that tells that if $\varphi : Y \to X$ is a flat morphism between affinoid rigid spaces, then $\varphi(Y)$ is a finite union of affinoid domains of X (*cf.* [8], cor. 5.11). But note that our proof is different – we don't use any formal geometry.

(6.4) Theorem. *Let Y be a Γ-strict k-analytic space, let X be a k-analytic space and let $\varphi : Y \to X$ be a compact morphism. Let $\mathcal{F}$ be a coherent sheaf on Y which is universally X-flat. If X is Γ-strict, or if X is separated and if Y is paracompact, then $\varphi(\mathsf{Supp}\,\mathcal{F})$ is a closed Γ-strict analytic domain of X.*

Proof. By replacing Y by $\mathsf{Supp}\,\mathcal{F}$ we reduce to the case where $\mathsf{Supp}\,\mathcal{F} = Y$. Since φ is compact, $\varphi(Y)$ is a closed subset of X. We are now going to reduce to the case where both Y and X are Γ-strict and k-affinoid.

(6.4.1) The case where X is Γ-strict. One can check the result G-locally on X; it allows to assume that X is Γ-srtict and k-affinoid. affinoid. In this case, Y is compact, hence admits a finite covering by Γ-strict, affinoid domains; one therefore immediatly reduces to the case where Y is also Γ-strict and affinoid.

(6.4.2) The case where X is separated and where Y is paracompact. Since Y is paracompact, it admits a locally finite covering (Y_i) by compact Γ-strict analytic domains. If we prove that $\varphi(Y_i)$ is a Γ-strict compact analytic domain of X for any i, then $(\varphi(Y_i))$ will be by compactness of φ a locally finite covering of $\varphi(Y)$ by Γ-strict compact analytic domains of X. As X is separated, this will imply that $\varphi(Y)$ is itself a Γ-strict analytic domain of X. Hence we reduced to the case where Y is compact. Thanks to 1.10.2, one can then assume X is compact and Γ-strict, and even, since one can check the result G-locally on X, that it is affinoid and Γ-strict. And as Y admits a finite covering by Γ-strict k-affinoid domains, one eventually reduces to the case where Y is also Γ-strict and k-affinoid.

(6.4.3) The proof in the case where both Y and X are Γ-strict and k-affinoid. Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a k-free polyray such that the r_i's belong to Γ, such that $|k_{\mathbf{r}}^*| \neq \{1\}$ and such that $X_{\mathbf{r}}$ and $Y_{\mathbf{r}}$ are strictly $k_{\mathbf{r}}$-analytic; let $\mathfrak{s} : X \to X_{\mathbf{r}}$ be the Shilov section. Thanks to theorem 6.3, $\varphi_{\mathbf{r}}(Y_{\mathbf{r}})$ is a compact strictly $k_{\mathbf{r}}$-analytic domain of $X_{\mathbf{r}}$. The subset $\varphi(Y)$ of X is nothing but $\mathfrak{s}^{-1}(\varphi_{\mathbf{r}}(Y_{\mathbf{r}}))$. By the Gerritzen-Grauert theorem, $\varphi_{\mathbf{r}}(Y_{\mathbf{r}})$ is a finite union of strictly $k_{\mathbf{r}}$-rational domains; it is then easily seen (using the explicit formula for $\mathfrak{s}$, see the proof of the lemma 2.4 of [13]) that $\mathfrak{s}^{-1}(\varphi_{\mathbf{r}}(Y_{\mathbf{r}}))$ is itself a finite union of rational domains whose definitions only involve scalars which belong to Γ; therefore, it is a compact Γ-strict analytic domain of X. □

(6.5) Theorem. *Let n and d be two integers, let Y be a Γ-strict k-analytic space, and let φ be a compact morphism from Y to a normal k-analytic space X. Assume that X is purely d-dimensional, that φ is purely n-dimensional, and that Y is purely $(n+d)$-dimensional. If X is Γ-strict, or if X is separated and if Y is paracompact, then $\varphi(Y)$ is a Γ-strict closed k-analytic domain of X.*

Proof. We reduce exactly as at the beginning of the proof of cor. 6.4 to the case where both Y and X are Γ-strict k-affinoid spaces. By compactness, one can argue locally on Y. Hence cor. 4.7 of [14] and 1.9.3 allow to suppose that there exist a factorization $Y \to T \to X$ where $Y \to T$ is finite, where T is Γ-strict and k-affinoid, and where $T \to X$ is quasi-smooth of pure dimension n. Thanks to 0.15.1 and to th. 6.4, T is purely $(n+d)$-dimensional. As $Y \to T$ is finite, the image of Y on T is a Zariski-closed subset Z of T of pure dimension $n+d$. This implies that Z is a union of irreducible components of T. Since X is normal and since $T \to X$ is quasi-smooth, T is normal (cor. 3.24) Therefore Z is a union of connected components of T, hence is a Γ-strict affinoid domain of T. By th. 6.4, the image of Z on X, which coincides with that of Y, is a compact Γ-strict analytic domain of X. □

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(6.6) Theorem. *Let $\varphi : Y \to X$ be a morphism between k-analytic spaces and let $\mathcal{F}$ be a coherent sheaf on Y which is universally X-flat. Let y be a point of $\mathsf{Supp}\,\mathcal{F}$ at which φ is inner, and let x be its image on X. The image $\varphi(\mathsf{Supp}\,\mathcal{F})$ is a neighborhood of x.*

Proof. By replacing Y by a compact analytic neighborhood of y, one can assume that Y is compact. It follows then from th. 6.4 that $\varphi(\mathsf{Supp}\,\mathcal{F})$ is an analytic domain U of X. As φ is inner at y, $\varphi_{|\mathsf{Supp}\,\mathcal{F}}$ is inner at y too. Therefore x belongs to $\mathsf{Int}\,U/X$, that is, to the *topological* interior of U in X, whence the result. □

(6.7) Remark. The openness of flat, boundaryless morphisms between good k-analytic spaces has already been proved by Berkovich, in a slightly different way, in its unpublished notes we have already mentioned.

(6.8) Theorem. *Let n and d be two integers and let $\varphi : Y \to X$ be a morphism between k-analytic spaces. Assume that X is normal and of pure dimension d, that φ is of pure relative dimension n, and that Y is of pure dimension $n+d$. Let y be a point of Y at which φ is inner, and let x be its image on X. The image $\varphi(Y)$ is a neighborhood of x.*

Proof. By replacing Y by a compact analytic neighborhood of y, one can assume that Y is compact. It follows then from th. 6.5 that $\varphi(Y)$ is an analytic domain U of X. As φ is inner at y, x belongs to $\mathsf{Int}\ U/X$, that is, to the *topological* interior of U in X, whence the result. $\square$

7 Universal flatness and quasi-smoothness loci

We will now apply Kiehl's method, introduced in [20] to show the Zariski-openness of the flatness locus of a *complex analytic* morphism, to prove that it also holds in the non-Archimedean setting.

(7.1) Theorem. *Let $Y \to X$ be a morphism of k-analytic spaces and let $\mathcal{F}$ be a coherent sheaf on Y. The universal X-flatness locus of $\mathcal{F}$ is a Zariski-open subset of Y.*

Proof. One can assume that both Y and X are affinoid (2.26.4). Then $Y \to X$ goes through a closed immersion $\iota : Y \hookrightarrow X \times_k \mathbb{D}$ for some closed polydisc $\mathbb{D}$. We can replace Y by $X \times_k \mathbb{D}$ and $\mathcal{F}$ by $\iota_* \mathcal{F}$; hence we reduce to the case where $Y \to X$ is universally flat.

Let p be the first projection $Y \times_X Y \to Y$. We call U the universal X-flatness locus of $\mathcal{F}$; we call V (resp. $\mathscr{W}$) the universal Y-flatness locus (resp. the $\mathscr{Y}$-flatness locus) of $p^*\mathcal{F}$ with respect to the *second* projection $Y \times_X Y \to Y$ (resp. $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y} \to \mathscr{Y}$).

Let $y \in Y$ and let $z \in p^{-1}(y)$. If $y \in U$, then $z \in V$, by the very definition of universal flatness. If $z \in V$, it follows from the universal flatness of $Y \to X$ and from prop. 2.29 that $y \in U$. Let $\sigma : Y \to Y \times_X Y$ be the diagonal immersion; by what we have just seen, $U = \sigma^{-1}(V)$. If Z denotes the diagonal of $Y \times_X Y$, it is then sufficient to prove that $V \cap Z$ is a Zariski-open subset of Z.

By th. 5.12.5, the intersection $V \cap Z$ is nothing but the pre-image of $\mathscr{W} \cap \mathscr{Z}$ under the canonical map $Z \to \mathscr{Z}$. Both $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}$ and $\mathscr{Y}$ are noetherian schemes, and $\mathscr{Z}$ is a Zariski-closed subscheme of $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}$ which is of finite type over $\mathscr{Y}$ through the second projection. A theorem by Kiehl ([20], Satz 1) then asserts that $\mathscr{W} \cap \mathscr{Z}$ is a Zariski-open subset of $\mathscr{Z}$, which ends the proof. $\square$

Thanks to this theorem, we recover the well-known fact that in the rigid setting, global algebraic flatness implies global universal analytic flatness.

(7.2) Corollary. *Let $\mathscr{A} \to \mathscr{B}$ be a morphism of strictly k-affinoid algebras, and let $Y \to X$ be the corresponding morphism of k-affinoid spaces. Let M be a* *finitely generated $\mathscr{B}$-module, and let $\mathcal{F}$ be the corresponding coherent sheaf on Y. Then $\mathcal{F}$ is universally X-flat if and only if M is a flat $\mathscr{A}$-module.*

Proof. If $\mathcal{F}$ is universally X-flat, it is in particular X-flat, and lemma 2.2 then ensures that M is $\mathscr{A}$-flat. Conversely, assume that M is $\mathscr{A}$-flat. Then by th. 5.12.5, $\mathcal{F}$ is universally X-flat at any rigid point of Y. Let Z be the set of points of Y at which $\mathcal{F}$ is *not* universally X-flat. By th. 7.1 above, Z is a Zariski-closed subset of Y, and we have just seen that it contains no rigid point. By the analytic (resp. algebraic) Nullstellensatz if $|k^*| \neq \{1\}$ (resp. $|k^*| = \{1\}$), we conclude that $Z = \emptyset$. □

(7.3) Corollary. *Let $\mathfrak{Y} \to \mathfrak{X}$ be a topologically finitely presented morphism between topologically finitely presented* Spf *k^o-formal schemes, and let $\mathcal{F}$ be a coherent sheaf on $\mathfrak{Y}$ which is $\mathfrak{X}$-flat. Then the associated coherent sheaf $\mathcal{F}_\eta$ on $\mathfrak{Y}_\eta$ is universally $\mathfrak{X}_\eta$-flat.*

Proof. One can assume that both $\mathfrak{Y}$ and $\mathfrak{X}$ are affine formal schemes; let us call B and A the corresponding topologically finitely presented k^{o}-algebras, and let M be the finitely presented B-module associated with $\mathcal{F}$. By assumption, M is a flat A-module; therefore, $\mathsf{M} \otimes_{k^{\mathrm{o}}} k$ is a flat $\mathsf{A} \otimes_{k^{\mathrm{o}}} k$-module. In view of the preceeding corollary, this implies that $\mathcal{F}_\eta$ is universally $\mathfrak{X}_\eta$-flat. □

(7.4) Theorem. *Let $Y \to X$ be a morphism between k-analytic spaces, and let $d \in \mathbb{N}$. The set $\mathscr{E}$ of point of Y at which $Y \to X$ is quasi-smooth of relative dimension d is a Zariski-open subset of Y.*

Proof. We can argue G-locally on Y, which allows to assume that it is affinoid. A point $y \in Y$ lies on $\mathscr{E}$ if and only if the three following conditions are fulfilled:

i) $\mathsf{dim}_y\ (Y \to X) = d$;
ii) $\mathsf{dim}_{\mathscr{H}(y)}\ \Omega_{Y/X} \otimes \mathscr{H}(y) = d$;
iii) $Y \to X$ is universally flat at y.

The validity loci of the i), ii), iii) are all Zariski-constructible: for i), it follows from upper-semi-continuity of the relative dimension ([14], th. 4.6); for ii), from the fact that the pointwise rank of a coherent sheaf on a noetherian scheme is a constructible function; and for iii), this is a direct consequence of th. 7.1.

Moreover, $\mathscr{E}$ is open by 3.10.8. Being at the same time open and constructible, it is Zariski-open ([2], cor. 2.6.6). □

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